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Source: *Operations Research*, Vol. 43, No. 5 (Sep. - Oct., 1995), pp. 879-884
Published by: INFORMS
Stable URL: http://www.jstor.org/stable/172041

ONE-ATTRIBUTE SEQUENTIAL ASSIGNMENT MATCH PROCESSES IN DISCRETE TIME

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(Received February 1990; revisions received February 1991, June 1992; accepted August 1993)

We consider a sequential matching problem where M offers arrive in a random stream and are to be sequentially assigned to N waiting candidates. Each candidate, as well as each offer, is characterized by a random attribute drawn from a known discrete-valued probability distribution function. An assignment of an offer to a candidate yields a (nominal) reward $R > 0$ if they match, and a smaller reward $r \leq R$ if they do not. Future rewards are discounted at a rate $0 \leq \alpha \leq 1$. We study several cases with various assumptions on the problem parameters and on the assignment regime and derive optimal policies that maximize the total (discounted) reward. The model is related to the problem of donor-recipient assignment in live organ transplants, studied in an earlier work.

We use the term sequential assignment match processes (SAMP) to describe models where, typically, N "candidates" are waiting to be matched with M random "offers" arriving sequentially in time. Assignments are made one at a time, and once an offer is assigned (or rejected) it is unavailable for future assignments. Each candidate, as well as each arriving offer, is characterized by a fixed-length vector of random attributes $X = (X_1, X_2, \ldots, X_p)$. These vectors are drawn from a known, discrete-valued, joint probability distribution function. An attribute may be thought of as human blood-type, sex, preference in music, certain enzyme or antigen possession, and so on, each having several possible outcomes. The candidates' attributes are known in advance, while each offer's attributes are revealed only upon arrival. When an offer is assigned to a candidate the two vectors are matched, and the higher the compatibility the bigger the reward realized by this assignment. "Compatibility" here is measured by *counting* matching attributes. In this way, under any reasonable criterion, there are at most $p + 1$ possible match levels, where p is the attributes' vector length. The objective is to find assignment policies that maximize total expected reward, both for discounted and undiscounted cases.

The motivation for SAMP lies in the useful, yet difficult, problem of optimal donor-recipient assignment in live-organ transplants. The decision whether to transplant an organ (e.g., a kidney) that becomes available depends on the degree of histocompatibility between the donor ("offer") and the recipient ("candidate"). One relevant criterion is the compatibility in the so-called HL-A antigen system. Basically, one counts the number of antigens of the donor that are not possessed by the recipient: There is an A-match when all donor antigens are possessed by the recipient, a B-match when only one antigen of the donor is not matched by the recipient, and so forth. With each match level a value is associated, such as the odds for a successful operation. This value is the "reward" of assigning a given offer to a waiting candidate. An important aspect of the problem is that live organs must be assigned within a short time after arrival or else become unusable. Further description of the problem may be found in David and Yechiali (1985), where the concentration was on the *single* candidate case. In that study an appropriate time-dependent stopping problem was defined, and optimal assignment policies were derived under several assumptions on the arrival process and on the decay properties of the lifetime distribution of the candidate. To bring the analysis closer to reality it is desirable to treat the case of *many* candidates competing for the randomly arriving offers. Thus, each time a graft becomes available, the problem is to select the candidate (if at all) to which a transplant is performed.

The present work deals accordingly with a *multicandidate* problem, with some prescribed number of candidates. It is assumed that p, the attribute vector length, equals 1. Thus, each assignment of an offer to a candidate yields simply a reward R if they match, or a smaller reward $r \leq R$ if they do not. The case $p = 1$ is relevant to problems that resemble matchmaking, such as technological choice, or the typical "match-mismatch" sequential assignment processes in biology (DNA replication). In the transplant application it refers to considering only one antigen possession. Previously we studied a version

Subject classifications: Dynamic programming/optimal control: sequential assignment, match processes. Health care: live-organ transplants.
Area of review: Stochastic Processes and their Applications.

0030-364X/95/4305-0879 $01.25

of the problem in infinite horizon and with *simultaneous arrival* of candidates and offers (David and Yechiali 1990).

SAMP relates to the well known sequential stochastic assignment problem introduced by Derman, Lieberman and Ross (1972) (DLR). It is essentially the reward structure that makes the difference between our model and that of DLR. In the latter type of problem, each offer and candidate carries a numerical value, and the reward from assigning an offer with value x to a candidate with value y is a continuous, often multiplicative, function $r(x, y)$. Naturally, the optimal policies for SAMP admit a different form from those obtained for DLR-like models.

Section 1 starts with a general case where the number of offers exceeds the number of candidates (i.e., $M > N$), and each candidate *must* be assigned an offer. There is a constant discount rate, tantamount to a fixed-rate decay of the lifetime distribution of the entire process. The main theorem states that in the case of *distinct* candidates, the individually optimal (myopic) policy, in which candidates with rarer attributes are given higher priority, is also socially optimal. Rejecting an offer for any candidate in a group is optimal if and only if it is optimal to reject it in the case of a single candidate (cf. Righter 1987 for an analogous property in context of a DLR-like sequential assignment problem). Section 2 discusses special cases and further properties, and in Section 3 we relax the constraint that each candidate should be assigned an offer. In the latter case, the optimal assignment policies are all of a control-limit type. Summary tables of the various results are presented.

1. THE GENERAL CASE $M \geq N$ WITH DISCOUNTING

Our model deals with a one-dimensional, typically nonnumerical variable X that represents a certain attribute with $Q \leq \infty$ possible outcomes. Let $\{P(X = x_i)\}_{i=1}^{Q}$ be the probability distribution of the attributes for the whole population, candidates, and offers.

The reward from assigning an offer with a realization x_j to a candidate with a realization x_i is a bivariate symmetric function:

$$R(x_i, x_j) = \begin{cases} R & \text{if } x_i = x_j \text{ (a match)} \\ r & \text{if } x_i \neq x_j \text{ (a mismatch),} \end{cases}$$

where R and r are nonnegative reals, $R \geq r$. Offers arrive sequentially, and each offer's value is observed upon arrival. The offers are independent of each other, and are assigned (or rejected) one at a time. We assume that $M \geq N$, future rewards are discounted at a fixed rate $0 \leq \alpha \leq 1$, and each candidate *must* be assigned an offer.

Denote by $V_{N,M}(f_1, f_2, \ldots, f_N)$ the maximal expected discounted total reward when there are N attribute realizations $a_1, a_2, \ldots, a_N$ of N waiting candidates (all taken from the distribution of X). M is the number of offers, and $f_1, \ldots, f_N$ are the respective frequencies $P(X = a_1), \ldots, P(X = a_N)$ of the N realizations. Without loss of generality assume $f_1 \leq f_2 \leq \cdots \leq f_N$, such that candidate number 1 possesses the rarest attribute among all N candidates. The symbol $\bar{f}$ will stand for the probability that a random offer will mismatch all of the waiting candidates (if the candidates are distinct, $\bar{f} = 1 - \sum_{i=1}^{N} f_i$). For a single candidate with realization a_i, the expected reward attained when a random arriving offer is assigned to it is $V_{1,1}(f_i) = E[R(a_i, X)] = f_i R + (1 - f_i) r \equiv \xi_i$. Using the notation $(\mathbf{f})$ for $(f_1, \ldots, f_{N+1})$ and $(\mathbf{f}_{-i})$ for $(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{N+1})$, the optimality equations are:

$$V_{N+1,M+1}(\mathbf{f})|X_1 = \max \begin{cases} R + \alpha V_{N,M}(\mathbf{f}_{-i}) | \{X_1 = a_i\} & \text{(match)} \\ r + \alpha \max\limits_{k} V_{N,M}(\mathbf{f}_{-k}) & \text{(a mismatch)} \\ \alpha V_{N+1,M}(\mathbf{f}) & \text{(rejection)} \end{cases} \quad (1)$$

for $M > N$. A similar formula holds for the case $M = N$ where only the maximum of the first two expressions applies.

Now, any assignment policy π gives rise to a transformation from the set of all possible offer-streams $\mathbf{x} = (x_1, x_2, \ldots, x_M)$ to the set of $M!$ permutations of $(1, 2, \ldots, M)$. However, not all such transformations are legitimate policies. If $\pi(\mathbf{x})_i$ denotes the candidate selected in round i (possibly a dummy candidate signifying rejection), then we must have $\pi(\mathbf{x})_i = \pi(\mathbf{y})_i$ if $x_k = y_k$ for all $1 \leq k \leq i$. Arguing that as far as the given candidate attributes $a_1, \ldots, a_N$ are concerned, all other possible attributes realized by the offers can be treated as one, we see that there are only a finite number of relevant (nonrandomized) policies.

To find the optimal assignment policy we need the following lemma.

Lemma 1. *It is optimal to assign a match whenever possible.*

Proof. Assume that at a certain stage the arriving offer can be matched with one of the candidates. If instead, in a given policy, this option is *not* exercised, there is an immediate loss of either $(R - r)$ (the offer is assigned to a nonmatching candidate) or R (rejection). While this may allow for an additional future match of one of the currently nonmatched candidates, the present value of the future gain is at best $\alpha(R - r)$. The net gain is thus nonpositive with probability one.

Lemma 1 implies that for any $1 \leq i, j \leq N + 1$,

$$(R - r) + \alpha(V_{N,M}(\mathbf{f}_{-i}) - V_{N,M}(\mathbf{f}_{-j})) \geq 0$$

(a match with a_i is better than any mismatch with a_j), and

$$R + \alpha(V_{N,M}(\mathbf{f}_{-i}) - V_{N+1,M}(\mathbf{f})) \geq 0$$

(a match with a_i is better than rejection). Also, the optimality equations (1) reduce to

$$V_{N+1,M+1}(\mathbf{f})|\{X_1 = a_i\} = R + \alpha V_{N,M}(\mathbf{f}_{-i})$$

and

$$V_{N+1,M+1}(\mathbf{f})|\{X_1 \neq a_i \text{ for all } i\}$$

$$= \max\begin{cases} r + \alpha \max_i V_{N,M}(\mathbf{f}_{-i}) \\ \alpha V_{N+1,M}(\mathbf{f}). \end{cases} \tag{2}$$

The question is whether to reject a mismatch or to assign it to one of the candidates. In the case of a single candidate ($N = 1$, $M > 1$) it is easy to see that the optimal stopping rule is myopic and independent of time, namely one stops if and only if stopping is better than waiting for the next offer and then stopping. Thus, a_i rejects a mismatch if $\alpha\xi_i \geq r$ (in which case we call a_i *persistent*) and accepts a mismatch if $\alpha\xi_i < r$ (*nonpersistent*). In the first case,

$$\begin{aligned} V_{1,M}(f_1) &= f_1 R + (1 - f_1) f_1 \alpha R + \ldots \\ &= f_1 R \frac{1 - \alpha^{M-1}(1 - f_1)^{M-1}}{1 - \alpha(1 - f_1)} \\ &\quad + (1 - f_1)^{M-1}\alpha^{M-1}\xi_1. \end{aligned}$$

In the second case, $V_{1,M}(f_1) = \xi_1$. Both expressions are monotone in f_1, and they are equal when f_1 satisfies $\alpha\xi_1 = r$, namely $f_1 = (1 - \alpha)/\alpha \cdot r/(R - r) \equiv \varphi$.

Lemma 2. *For $M \geq N + 1$ and any $a_1, \ldots, a_N$*

$$V_{N+1,M}(\mathbf{f}) \leq V_{1,M}(f_1) + V_{N,M}(f_2, \ldots, f_{N+1}).$$

Proof. Let π^* denote an *optimal* policy for the problem with M offers and candidate attribute vector $(a_1, \ldots, a_{N+1})$. For the same problem, let π' denote the policy which assigns a_1 whenever π^* does, and rejects it otherwise. Similarly, let π'' denote the policy which assigns one of the candidates in $\{a_2, \ldots, a_{N+1}\}$ whenever π^* does, and rejects it otherwise. Thus:

$$\begin{aligned} V_{N+1,M}(\mathbf{f}) &= V^{\pi'}_{1,M}(f_1) + V^{\pi''}_{N,M}(f_2, \ldots, f_{N+1}) \\ &\leq V_{1,M}(f_1) + V_{N,M}(f_2, \ldots, f_{N+1}). \end{aligned}$$

The following two theorems will be proved simultaneously by induction on M.

Theorem 1. (Monotonicity). *Let $N \leq M$. Let $\mathbf{f} = (f_1, \ldots, f_N)$, $\bar{\mathbf{f}} = (\bar{f}_1, \ldots, \bar{f}_N)$. Suppose that $f_i \leq \bar{f}_i$ and $f_k = \bar{f}_k$ for $k \neq i$. Then $V_{N,M}(\mathbf{f}) \leq V_{N,M}(\bar{\mathbf{f}})$.*

Note that monotonicity of $V_{N,M}(\cdot)$ in each argument implies that $\max_i V_{N,M}(\mathbf{f}_{-i}) = V_{N,M}(\mathbf{f}_{-1})$ when $f_1 \leq f_2 \leq \cdots \leq f_{N+1}$.

Theorem 2. (Optimal policy). *If $N < M$ and $a_1, \ldots, a_N$ are distinct, then the optimal policy is to assign a match when possible, and to reject a mismatch or assign it to a_1, depending on whether $\alpha\xi_1 \geq r$ or $\alpha\xi_1 < r$.*

Proof of Theorems 1 and 2. The first part of Theorem 2 is Lemma 1. For $M = 1$ Theorem 1 is immediate, and for $M = 2$ so is Theorem 2 (a single candidate case). Assume then that Theorem 1 holds for $M - 1$ (for any $N \leq M - 1$) and Theorem 2 holds for M (for any $N < M$). We first prove Theorem 1 for M. Let $N < M$ and $f_1 \geq \varphi$. Then $\alpha\xi_1 \geq r$. So by Theorem 2

$$\begin{aligned} V_{N,M}(\mathbf{f}) &= \sum_{k=1}^{N} f_k[R + \alpha(V_{N-1,M-1}(\mathbf{f}_{-k})] \\ &\quad + \bar{f}\alpha V_{N,M-1}(\mathbf{f}) \\ &= \sum_{k=1}^{N} f_k[R + \alpha(V_{N-1,M-1}(\mathbf{f}_{-k}) - V_{N,M-1}(\mathbf{f}))] \\ &\quad + \alpha V_{N,M-1}(\mathbf{f}). \end{aligned}$$

(The number of candidates determines the dimension of $\mathbf{f}$; now it is N). Differentiate the right-hand side with respect to f_i. We get

$$\begin{aligned} &\sum_{k=1}^{N} \left(\frac{\partial}{\partial f_i} f_k\right)[R + \alpha(V_{N-1,M-1}(\mathbf{f}_{-k}) - V_{N,M-1}(\mathbf{f}))] \\ &\quad + \sum_{k=1}^{N} f_k \alpha\left(\frac{\partial}{\partial f_i} V_{N-1,M-1}(\mathbf{f}_{-k}) - \frac{\partial}{\partial f_i} V_{N,M-1}(\mathbf{f})\right) \\ &\quad + \frac{\partial}{\partial f_i} \alpha V_{N,M-1}(\mathbf{f}) = [R + \alpha(V_{N-1,M-1}(\mathbf{f}_{-i}) \\ &\quad - V_{N,M-1}(\mathbf{f}))] + \sum_{k=1}^{N} f_k \alpha\left[\frac{\partial}{\partial f_i} V_{N-1,M-1}(\mathbf{f}_{-k})\right] \\ &\quad + \bar{f}\alpha\left[\frac{\partial}{\partial f_i} V_{N,M-1}(\mathbf{f})\right]. \end{aligned}$$

The first brackets in the right-hand side are nonnegative by Lemma 1. The second brackets are zero for $i = k$ and nonnegative for $i \neq k$, by the induction hypothesis (Theorem 1). By the same hypothesis the last brackets are also nonnegative. This shows monotonicity in f_i. Now if $N = M$ rejection is not permitted and if $f_1 < \varphi$ rejection is not optimal (Theorem 2). In both cases we get

$$\begin{aligned} V_{N,M}(\mathbf{f}) &= \sum_{k=1}^{N} f_k(R + \alpha(V_{N-1,M-1}(\mathbf{f}_{-k})) \\ &\quad + \bar{f}(r + \alpha V_{N-1,M-1}(\mathbf{f}_{-1})) \\ &= \sum_{k=1}^{N} f_k[(R - r) + \alpha V_{N-1,M-1}(\mathbf{f}_{-k}) \\ &\quad - V_{N-1,M-1}(\mathbf{f}_{-1}))] \\ &\quad + \alpha V_{N-1,M-1}(\mathbf{f}_{-1}) + r. \end{aligned}$$

Differentiate and use similar arguments to show again the monotonicity of $V_{N,M}(\cdot)$ in each f_i. Now we will prove Theorem 2 for $M + 1$. The case $N = 1$ was discussed separately. Let the number of candidates be $N + 1$, $N < M$. Equation 2 now transforms to

$$V_{N+1,M+1}(\mathbf{f})|\{X_1 \neq a_i \text{ for all } i\}$$

$$= \max\{r + \alpha V_{N,M}(\mathbf{f}_{-1}),\ \alpha V_{N+1,M}(\mathbf{f})\}.$$

Write the right-hand side as $\max\{I, II\}$. By Lemma 2 we have

$$\alpha(V_{N+1,M}(\mathbf{f}) - V_{N,M}(\mathbf{f}_{-1})) \leq \alpha V_{1,M}(\mathbf{f}_1) = \alpha\xi_1,$$

and thus $\alpha\xi_1 < r$ gives $I > II$. We complete the proof by showing that $\alpha\xi_1 \geq r \Rightarrow II \geq I$. Proving $V_{N+1,M}(\mathbf{f}) - V_{N,M}(\mathbf{f}_{-1}) \geq \xi_1$ is enough because multiplying by α and using $\alpha\xi_1 \geq r$ implies $II \geq I$. For distinct candidates, with $\bar{f} = 1 - \sum_{i=1}^{N+1} f_i$,

$$\begin{aligned} V_{N+1,M}(\mathbf{f}) &= \sum_{i=1}^{N+1} f_i(R + V_{N,M-1}(\mathbf{f}_{-i})) \\ &\quad + \bar{f}V_{N+1,M}(\mathbf{f})|\{X_1 \neq a_i \text{ for all } i\}) \\ &\geq \sum_{i=1}^{N+1} f_i(R + \alpha V_{N,M-1}(\mathbf{f}_{-i})) \\ &\quad + \bar{f}(r + \alpha V_{N,M-1}(\mathbf{f}_{-1})), \end{aligned} \tag{3}$$

where the inequality follows from (1). Also,

$$\begin{aligned} V_{N,M}(\mathbf{f}_{-1}) &= \sum_{i=2}^{N+1} f_i(R + \alpha V_{N-1,M-1}(\mathbf{f}_{-1,-i})) \\ &\quad + (\bar{f} + f_1)\alpha V_{N,M-1}(\mathbf{f}_{-1}), \end{aligned} \tag{4}$$

($\bar{f} + f_1$ is the probability of a mismatch in the -1 case. In such a case, we reject by hypothesis because $\xi_2 \geq \xi_1$ and thus candidate 2 is persistent). Combine (3) and (4) to get

$$\begin{aligned} &V_{N+1,M}(\mathbf{f}) - V_{N,M}(\mathbf{f}_{-1}) \\ &\geq \sum_{i=1}^{N+1} f_i(R + \alpha V_{N,M-1}(\mathbf{f}_{-i})) + \bar{f}r \\ &\quad - \sum_{i=2}^{N+1} f_i(R + \alpha V_{N-1,M-1}(\mathbf{f}_{-1,-i})) - f_1\alpha V_{N,M-1}(\mathbf{f}_{-1}) \\ &= f_1R + \bar{f}r + \sum_{i=2}^{N+1} f_i[\alpha(V_{N,M-1}(\mathbf{f}_{-i}) - V_{N-1,M-1}(\mathbf{f}_{-1,-i}))]. \end{aligned}$$

However, by the induction hypothesis ($II \geq I$) the brackets are at least r, so that

$$\begin{aligned} &V_{N+1,M}(\mathbf{f}) - V_{N,M}(\mathbf{f}_{-1}) \\ &\geq f_1R + \bar{f}r + \sum_{i=2}^{N+1} f_ir = f_1R + (1 - f_1)r = \xi_1. \end{aligned}$$

To summarize, the candidate with the rarest attribute is offered an offer that does not match any candidate, and (in the distinct candidate case) that candidate's decision is myopic. The combination of individually optimal (myopic) policies, in which candidates with rarer attributes are given higher priority, is socially optimal.

2. SPECIAL CASES AND FURTHER COMMENTS

2.1. The Case $M = N$

If $M = N$ (no rejections), Lemma 1 and Theorem 1 imply the optimality of the following policy for any α, r, and R, and *any* candidates $a_1, \ldots, a_N$.

Intuitive Policy. If an offer matches one or more of the candidates, it is assigned to one of them. Otherwise it is assigned to a candidate with the rarest attribute.

The intuitive policy implies some simpler recursion formulas for the continuous reward functions V_N:

$$\begin{aligned} V_1(f_1) &= \xi_1, \\ V_N(\mathbf{f}) &= \sum_{k=1}^{N} f_k(R + \alpha V_{N-1}(\mathbf{f}_{-k})) \\ &\quad + \bar{f}(r + \alpha V_{N-1}(\mathbf{f}_{-1})) \\ &= \sum_{k=1}^{N} f_k[R - r + \alpha V_{N-1}(\mathbf{f}_{-k}) - \alpha V_{N-1}(\mathbf{f}_{-1})] \\ &\quad + \alpha V_{N-1}(\mathbf{f}_{-1}) + r, \end{aligned}$$

where we assume for convenience that candidates are distinct.

Stochastic Optimality. The intuitive policy is also *stochastic* optimal. We outline the proof. First, write the optimality equations for the problem of stochastically maximizing the *number of successes*, where a success means a well-matched pair (i.e., assuming $R = 1$, $r = 0$). Denote by $V_{n,k}$ the maximum probability of at least k successes out of n candidates, and we have (where, for simplicity, we assume distinct candidates and no discounting),

$$V_{n,k}(\mathbf{f}) = \sum_{i=1}^{n} f_iV_{n,k}(\mathbf{f})|\{X_1 = a_i\} + \left(1 - \sum_{i=1}^{n} f_i\right)V_{n,k}(\mathbf{f})| \{X_1 \neq a_i \text{ for all } i\},$$

where

$$\begin{aligned} &V_{n,k}(\mathbf{f})|\{X_1 = a_i\} \\ &= \max\{V_{n-1,k-1}(\mathbf{f}_{-i}), \max_{j \neq i} V_{n-1,k}(\mathbf{f}_{-j})\}, \\ &V_{n,k}(\mathbf{f})|\{X_1 \neq a_i \text{ for all } i\} \\ &= \max_i V_{n-1,k}(\mathbf{f}_{-i}), \end{aligned} \tag{5}$$

with $V_{n-k} = 0$ for all $k > n$, $V_{n,0} = 1$ for all $n \geq 0$.

Now, let $V^I_{n,k}$ be the probability of at least k successes out of n when applying the intuitive policy. Then,

$$V^I_{n,k}(\mathbf{f}) = \sum_{i=1}^{n} f_iV^I_{n-1,k-1}(\mathbf{f}_{-i}) + \left(1 - \sum_{i=1}^{n} f_i\right)V^I_{n-1,k}(\mathbf{f}_{-1}) \tag{6}$$

with $V^I_{n,0} = 1$ for $n \geq 0$, and $V^I_{n,k} = 0$ for $k > n$. Furthermore, by induction (David and Yechiali 1989), the constants $V^I_{n,k}$ defined recursively in (6) satisfy

$$\begin{aligned} V^I_{n-1,k-1}(\mathbf{f}_{-i}) &\geq V^I_{n-1,k}(\mathbf{f}_{-j}) \quad \text{for all } 1 \leq i, j \leq n, \\ V^I_{n-1,k}(\mathbf{f}_{-1}) &\geq V^I_{n-1,k}(\mathbf{f}_{-j}) \quad \text{for all } 1 \leq j \leq n, \end{aligned}$$

where $f_1 \leq f_2 \leq \cdots \leq f_n$. Thus, the $V^I_{n,k}$'s satisfy (5) and the intuitive policy stochastically maximizes the number of successes. Now, for *any* R and r, if Y_π is the (random) number of successes under a given policy π, and U_π is the (random) reward incurred by π, then $U_\pi = (R - r)Y_\pi + Nr$, which is linearly increasing in Y. Stochastic optimality of the intuitive policy thus follows.

2.2. Special Cases With $M > N$

If $M > N$ and $\alpha = 1$ or $r = 0$, it is clear that the following policy is optimal.

M-Intuitive Policy. As long as the number of remaining offers exceeds that of the remaining candidates, assign only a match. As soon as $M = N$, follow the intuitive policy.

We now rephrase Theorem 2 in the following form.

Theorem 2. (Alternative formulation). *Suppose that at some stage there are N distinct candidates with frequencies $f_1 \leq \cdots \leq f_N$, and M future arrivals ($M \geq N$). Let the discount factor be $0 \leq \alpha \leq 1$, and the rewards $R \geq r \geq 0$. Let $\varphi = (1 - \alpha)/\alpha \cdot r/(R - r)$ (a constant independent of M or N). If i^*, $0 \leq i^* \leq N$, is such that $0 = f_0 < f_1 \leq f_2 \leq \cdots \leq f_{i^*} \leq \varphi < f_{i^*+1} \leq \cdots \leq f_N$, then the optimal rule is to cpply an intuitive policy to candidates $a_1, a_2, \ldots, a_i^*$, and an M-intuitive policy to the other candidates.*

The cases $r = 0$, or $\alpha = 1$, where the optimal policy is M-intuitive, now follow as special cases with $\varphi = 0$. We refer to the policy stated in Theorem 2 as the combined-singles (CS) policy.

The Issue of Distinct Candidates. Theorem 2 (with $\alpha < 1$ and $r > 0$) requires the candidates to be distinct, while this requirement is not necessary in the other cases mentioned. Can the restriction be dropped in Theorem 2 as well? The answer is *negative.*

Example. Let $N = 2$, $M = 3$, $a_1 = a_2$, and $f_1 = f_2 = f$. A straightforward computation shows that if $\varphi < f$, but $(\alpha f/(1 + \alpha f))f < \varphi$, then a_1 or a_2, although persistent as singles, must be assigned a mismatch and thus the CS policy is not optimal.

Informally, the presence of another identical candidate, to whom a future matching offer would be assigned, "interferes" with the future prospects of a given persistent candidate. It therefore might be better at some stage to assign it a mismatch.

3. THE CASE WHERE NOT ALL CANDIDATES MUST BE ASSIGNED

The assumption in classical sequential assignment problems that each candidate should be assigned an offer is natural when dealing with "jobs" and "workers." However, relaxing that constraint may only improve *total expected* reward. Furthermore, in applications such as the transplant assignment, candidates may "leave the system" (die), so that assignment to all candidates is not guaranteed anyway. Therefore, in this section we no longer assume that every candidate should be assigned an offer, and we also allow for the case where $M < N$.

To avoid repetition, we state that Lemma 1 and the monotonicity of $V_N(\mathbf{f})$ still hold. Thus, upon a mismatch, we focus on the candidate with the smallest frequency f_1, for possible assignment. Monotonicity then calls for a control-limit policy, with a control φ on f_1. We therefore consider the following rule.

φ-Intuitive Policy. *For any N, M and α, let $f_1 \leq f_2 \leq \cdots \leq f_N$. If an offer matches one of the candidates assign it. If not, assign it to a_1 provided $f_1 < \varphi$, or else reject it. (φ is to be calculated, typically a function of the f_i's.)*

Remarks. The intuitive policy is the φ-intuitive policy (φI) with $\varphi = 1$. The M-intuitive policy is φI with $\varphi = 0$. The combined-singles policy is φI with $\varphi = (1 - \alpha)/\alpha \cdot r/(R - r)$, independent of the f_i's.

It is left to calculate the control for the various cases. We assume that the candidates are distinct.

3.1. The Case $M > N$

Proposition 1. *For $M > N$, $\varphi = (1 - \alpha)/\alpha\, \bar{r}$, where $\bar{r} = r/(R - r)$.*

Arguing informally, let us consider a mismatch. Then, if a_1 is nonpersistent (i.e., $f_1 < \varphi$) the presence of other candidates cannot turn him persistent. On the other hand, if a_1 is persistent, then in the previous model, with possibly forced assignments in the future, the offer should be rejected (Theorem 2). It is all the more so when assignments are not forced, as in the present case. The formal proof follows by induction, as in Theorem 2. (Note that for $N = 1$ the two models coincide.)

3.2. The Case $M \leq N$

The case $M = 1$ is trivial. For $M = 2$ we have the following proposition.

Proposition 2. *For $M = 2$ and any $N \geq 2$, $\varphi = \bar{r}/\alpha$, independent of the f_i's.*

Proof. On a mismatch we compare I: $r + \alpha V_{N-1,M=1}(f_2, \ldots, f_N)$, and II: $\alpha V_{N,M=1}(\mathbf{f})$. But naturally

$$V_{N,1}(\mathbf{f}) = \left(\sum_{i=1}^{N} f_i\right) \cdot R + \left(1 - \sum_{i=1}^{N} f_i\right) r.$$

Table I
Optimal Policies for the Case Where All Candidates Should Be Assigned

Case	M Versus N	Candidates $a_1, \ldots, a_N$	Discount Factor α	Second Prize r	Optimal Policy
1	$M = N$	Any	$\alpha = 1$	Any	I
2	$M = N$	Any	Any	Any	I
3	$M > N$ (or $M = \infty$)	Any	$\alpha = 1$	Any	MI
4	"	Any	Any	$r = 0$	MI
5	"	Distinct	Any	$r > 0$	CS
6	"	Not distinct	Any	$r > 0$	φI

I = Intuitive; MI = M-Intuitive; CS = Combined Singles; φI = φ-Intuitive.

Table II
Optimal Policies When Candidates May Be Left Unassigned

Case	M Versus N	Candidates	Discount Factor	Optimal Policy	Remarks
1	$M > N$ (Cases 3–6 of Table I)			In all cases same as Table I	
2	$M = N$	Any	$\alpha = 1$	φI	Compare with Table I
3	$M = N$	Any	Any	φI	
4	$M \leq N$	Any	Any	φI	
5	$N \geq M = 2$	Distinct	Any	φI	$\varphi = \frac{1}{\alpha} \frac{r}{R - r}$
6	$N \geq M = 3$	Distinct	Any	φI	Formulas for φ are given in Proposition 3

Subtracting II from I we get

$$r + \alpha\left[\left(\sum_{i=2}^{N} f_i\right)R + \left(1 - \sum_{i=2}^{N} f_i\right)r - \left(\sum_{i=1}^{N} f_i\right)R - \left(1 - \sum_{i=1}^{N} f_i\right)r\right]$$
$$= r(1 + \alpha f_1) - \alpha f_1 R.$$

Equating this to zero and letting $\bar{r} = r/(R - r)$ we get the stated result.

Note that for $M = N = 2$ the intuitive-policy is no longer optimal.

Proposition 3. *For $M = 3$, $N \geq 3$, let*

$$F^+ = 1 + \alpha \sum_{i=2}^{N} f_i,\ F^- = 1 - \alpha \sum_{i=2}^{N} f_i.$$

Then

$$\varphi = \begin{cases} \dfrac{\bar{r}}{\alpha F^+}(F^+ - \alpha) & \textit{if } f_1 \leq \dfrac{f_2}{F^+}(1 - \alpha F^-) \\ \dfrac{\bar{r}}{\alpha F^+} - \dfrac{\alpha f_2 F^-}{F^+} & \textit{otherwise.} \end{cases}$$

The result for this case emerges, similar to Proposition 2, by comparing terms and direct computation.

It is evident that the calculations of the controls become more difficult with larger values of M.

Tables I and II summarize the results for the optimal policies under the various assumptions.

ACKNOWLEDGMENT

We sincerely thank the associate editor for a very careful review and two referees who contributed to the contents and flow of this paper.

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