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THE ISRAELI QUEUE WITH PRIORITIES

Nir Perel and Uri Yechiali

Department of Statistics and Operations Research, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel

□ *We consider a 2-class, single-server, preemptive priority queueing model in which the high-priority customers form a classical $M/M/1$ queue, while the low-priority customers form the so-called Israeli Queue with at most N different groups and unlimited-size batch service. We provide an extensive probabilistic analysis and calculate key performance measures. Special cases are analyzed and numerical examples are presented and discussed.*

Keywords Matrix geometric; Polling; Priority queues; Probability generating functions; The Israeli queue.

Mathematics Subject Classification Primary 60K25; Secondary 90B22.

1. INTRODUCTION

The "Israeli Queue" model was introduced in Boxma, van der Wal, and Yechiali[3], when studying an N-queue, single-server polling system with unlimited-size batch service (see also Ref.[4]) governed by the following server's visit-order rule: After completion of a visit at a queue, the next queue to be served is the one where its first customer in line has been waiting in the system for the longest time. That is, the criterion for selecting the next queue to visit and serve is an age-based one. This type of service discipline was termed the *Israeli Queue*, illustrated vividly as follows: A new arriving customer may find in the system up to N groups, where each group is headed by a "leader." This new arrival looks for a "friend" among all group leaders in the system. If he ("he" stands for "she" as well) finds such a leader, he joins him and his group and waits with all the group's members to be served in a batch mode. That is, the whole group

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Address correspondence to Uri Yechiali, Department of Statistics and Operational Research, School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel; E-mail: uriy@post.tau.ac.il

is served at one service period, while the service duration is not affected by the size of the batch. For example, this queue discipline represents a physical waiting line for buying tickets to a movie, theater, or rock-concert performance. A new arrival that finds a friend already standing in line joins him and his group. When the "leader" reaches the cashier, he buys tickets for the entire group. It is assumed that the buying process is (almost) not affected by the number of tickets purchased. Recently, Perel and Yechiali[16] extended this model to the case where there is no bound on the number of different groups that can be present simultaneously in the system. They studied single-server models with finite and infinite number of groups, as well as models with multiple servers, and derived various performance measures.

Unlimited batch-service has been studied by van der Wal and Yechiali[20] when analyzing a computer tape-reading problem in a system where large amounts of information are stored on tapes. Requests for data stored on one of these tapes arrive randomly, and in order to read the data, the tape has to be mounted, read, and then dismounted. If there are several requests to be read from a tape, they all can be read in (more or less) the same time, thus suggesting a modeling as a batch-service with unlimited batch size. Unlimited batch-service models were considered in the literature as application to videotex, telex, and TDMA systems (see, e.g., Ref.[1], Ref.[10], and Ref.[13]). In addition, Van Oyen and Teneketzis[19] formulated a central database system and an Automated Guided Vehicle (AGV) as a polling system with an infinite capacity batch service.

There are systems in which a single server handles two streams of arrivals: primary (important jobs) and secondary (less important jobs). The primary jobs are processed individually one by one, while the less important jobs can be served simultaneously. For example, in a production scenario, a painting spray booth can process simultaneously a very large number of items.

Consequently, in this article, we consider a single-server preemptive priority queueing system with two classes of customers: VIP (class 1, high priority) and regular (class 2, low priority). The VIP customers form a classical infinite-buffer $M/M/1$ queue, while the customers of class 2 form the so called Israeli Queue with batch service and at most N groups, where the service time of a group is exponentially distributed. That is, a lower-priority customer, upon arrival, looks for a friend among the class-2 group leaders (each one heading a class-2 group) present in the system. We assume that the probability for an arriving class-2 customer to know a group leader standing in line is p, with the same p for all groups. Thus, if the number of low-priority groups in the system is k $(1 \leq k \leq N-1)$, then an arriving class-2 customer will join the ith group $(i \leq k)$ with probability $(1-p)^{i-1}p$ or will create a new class-2 group with probability $(1-p)^k$. If N low-priority groups are present, then an arriving class-2 customer will join the Nth group if he does not know any of the first $N-1$ group leaders. This occurs with probability $(1-p)^{N-1}$. We can also assume, due to the memoryless property of the exponential distribution, that an arriving class-2 customer can join (with probability p) a class-2 group that is being served. As indicated, the class-2 groups are served in an unlimited-size batch mode. That is, the whole group is served at one service period, independent of its size. We assume that VIP customers have preemptive priority over class 2, implying that a service of a class-2 group is immediately interrupted by an arrival of a VIP customer.

Recently, He and Chavoushi[11] studied a queueing model with customer interjections, where customers are distinguished between normal and interjecting. All customers join a single queue. A normal customer joins the queue at its end, while an interjecting customer tries to cut into the queue following a geometric distribution. The waiting times of normal customers and of interjecting customers are studied.

Queueing models with priorities have been studied extensively in the literature and have wide applications in computer systems, communication networks, health-care systems, production systems, and in many other aspects of real life. For studies on priority queues, we refer the reader to Cobham[6], White and Christie[21], Conway, Maxwell, and Miller[7], Miller[14], Kella and Yechiali[12], Cidon and Sidi[5], Takagi [18], Zhang and Shi[22], and the many references there.

Other works related to ours are Drekic and Grassmann[8], Drekic and Woolford[9], Sivasamy[17] and Bitran and Caldentey[2]. In Ref.[8], a two-class, single-server, preemptive priority queueing model is studied, where the low-priority source population is finite. The steady-state distribution of the number of class-1 and class-2 customers in the system is determined by applying the method of generalized eigenvalues. Furthermore, Ref.[9] analyzes a two-class, single-server, preemptive priority queueing model with low-priority balking customers, where the decision to balk or not depends on the queue length. Two specific forms of balking behavior are considered, and the method of generalized eigenvalues is used in order to derive the steady-state distribution of the number of customers in the system. A priority model with batch service is studied in Ref.[17] where the higher-priority units are served in batches of a *finite* size, according to a certain bulk service rule. In Ref.[2], a 2-dimensional preemptive priority queueing system with state-dependent arrivals is studied. It is assumed that if at time t there are $N_i(t)$ class-i customers in the system, then an arriving class-i customer joins the system with probability $Di(N_i(t)),\ i=1,2,$ or leaves with the complementary probability. Using truncation, the steady-state joint distribution for the two queue lengths is derived, and the waiting times and busy period are characterized. We note that, in our model, a class-2 customer always joins the queue, either to an existing group or by forming a new one (the last in the groups' line).

The article is organized as follows: In Section 2, we employ Probability Generating Functions (PGFs) to analyze the model. This requires the calculation of certain boundary probabilities. In Section 3, we derive various performance measures, such as the mean number of low-priority groups in the system, the covariance between the number of high- and low-priority customers, sojourn times of a class-2 group leader and of an arbitrary class-2 customer, and the mean size of a class-2 group. In Section 4, we use Matrix Geometric methods to further analyze the system, and in Section 5 we present some extreme cases and numerical results.

2. THE MODEL

2.1. Model Description

We consider the general model described in the Introduction, namely, two classes of customers, VIP and regular, where the VIP have preemptive priority over the regular customers and where, among each class, the VIP follow the $M/M/1$ queue, while the regular customers follow the Israeli Queue regime with at most N groups.

We assume that the arrival stream of VIP (ordinary) customers follows a homogeneous Poisson process with rate λ_1 (λ_2), while service time is exponentially distributed with rate μ_1 (μ_2). Let $L_1(t)$ be the total number of VIP customers in the system at time t, and $L_2(t)$ the number of class-2 *groups* in the system at time t. For $i = 1, 2$, let $L_i = \lim_{t\to\infty} L_i(t)$, and $P_{mn} = \mathbb{P}(L_1 = m, L_2 = n)$, for $m \geq 0$ and $0 \leq n \leq N$. A transition rate diagram of the two-dimensional process (L_1, L_2) is depicted in Figure 1. Define the marginal probabilities,

$$\mathbb{P}(L_1 = m) = P_{m\bullet} = \sum_{n=0}^{N} P_{mn}, \quad m \geq 0,$$

$$\mathbb{P}(L_2 = n) = P_{\bullet n} = \sum_{m=0}^{\infty} P_{mn}, \quad 0 \leq n \leq N.$$

Clearly, since the VIP customers are not affected by the ordinary customers, they form a regular $M/M/1$ queue for which, for all $m \geq 0$ (where $\rho = \frac{\lambda_1}{\mu_1}$),

$$P_{m\bullet} = \rho^m(1 - \rho).$$

For a stable system, we have $\rho < 1$.

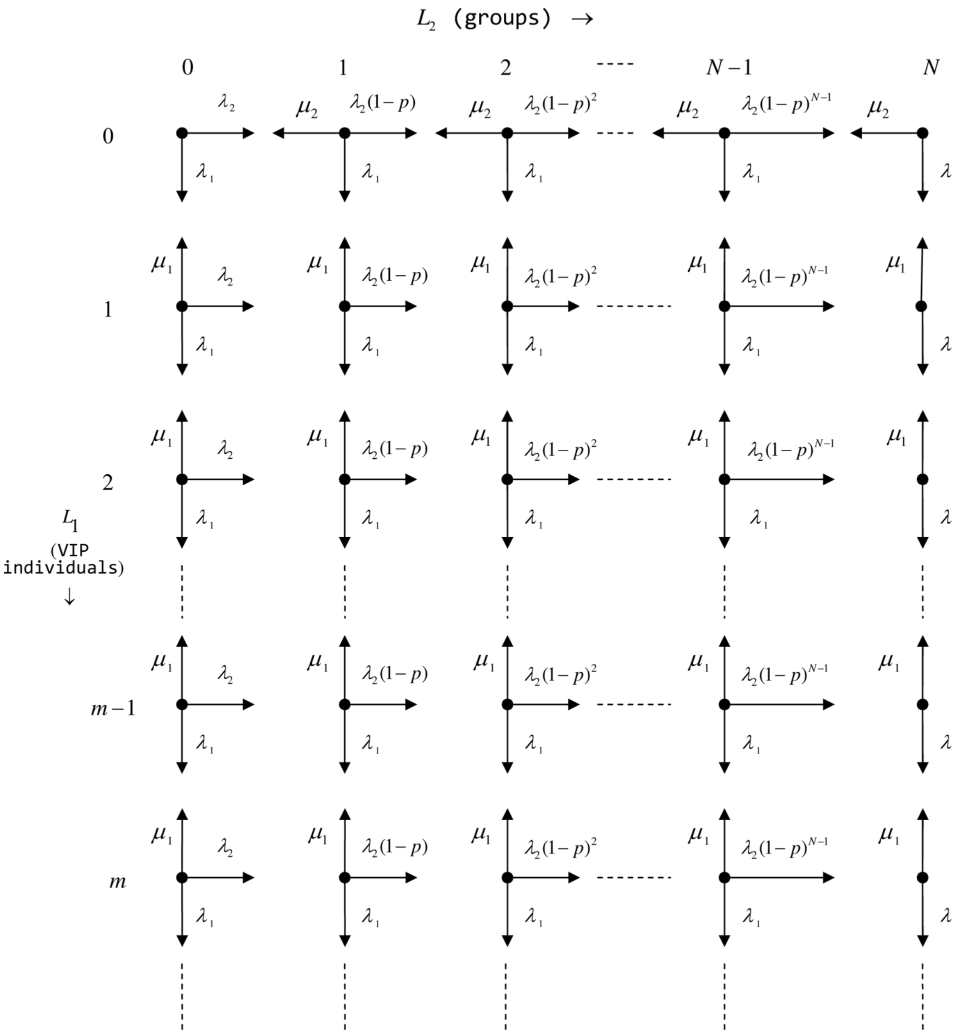

FIGURE 1 Transition rate diagram of (L_1, L_2).

2.2. Balance Equations and Generating Functions

For $n = 0$, the following relations hold,

$$(\lambda_1 + \lambda_2)\, P_{00} = \mu_1 P_{10} + \mu_2 P_{01}, \tag{2.1}$$

$$(\lambda_1 + \lambda_2 + \mu_1)\, P_{m0} = \lambda_1 P_{m-1,0} + \mu_1 P_{m+1,0}, \quad m \geq 1. \tag{2.2}$$

Define the marginal PGF of L_1: $F_n(w) = \sum_{m=0}^{\infty} P_{mn} w^m$, for all $0 \leq n \leq N$. Then, multiplying equation (2.2) by w^m and summing over m together with

(2.1) gives

$$((\lambda_1 w - \mu_1)(1-w) + \lambda_2 w)\, F_0(w) = -\mu_1(1-w)P_{00} + \mu_2 w P_{01}. \tag{2.3}$$

Moreover, for $1 \le n \le N-1$ we get

$$\left(\lambda_1 + \lambda_2(1-p)^n + \mu_2\right) P_{0n} = \lambda_2(1-p)^{n-1} P_{0,n-1} + \mu_1 P_{1n} + \mu_2 P_{0,n+1}, \tag{2.4}$$

$$\begin{aligned} &\left(\lambda_1 + \lambda_2(1-p)^n + \mu_1\right) P_{mn} \\ &\quad = \lambda_1 P_{m-1,n} + \lambda_2(1-p)^{n-1} P_{m,n-1} + \mu_1 P_{m+1,n}, \quad m \ge 1. \end{aligned} \tag{2.5}$$

Multiplying equation (2.5) by w^m and summing over m together with (2.4) leads to

$$\begin{aligned} &\left((\lambda_1 w - \mu_1)(1-w) + \lambda_2(1-p)^n w\right) F_n(w) - \lambda_2(1-p)^{n-1} w F_{n-1}(w) \\ &\quad = -\mu_1(1-w)P_{0n} + \mu_2(P_{0,n+1} - P_{0n})w, \quad 1 \le n \le N-1. \end{aligned} \tag{2.6}$$

Last, for $n = N$ we have

$$(\lambda_1 + \mu_2)\, P_{0N} = \lambda_2(1-p)^{N-1} P_{0,N-1} + \mu_1 P_{1N}, \tag{2.7}$$

$$(\lambda_1 + \mu_1)\, P_{mN} = \lambda_1 P_{m-1,N} + \lambda_2(1-p)^{N-1} P_{m,N-1} + \mu_1 P_{m+1,N}, \quad m \ge 1. \tag{2.8}$$

Multiplying equation (2.8) by w^m and summing over m together with (2.7) yields

$$\begin{aligned} &(\lambda_1 w - \mu_1)\,(1-w) F_N(w) - \lambda_2(1-p)^{N-1} w F_{N-1}(w) \\ &\quad = -\mu_1(1-w)P_{0N} - \mu_2 w P_{0N}. \end{aligned} \tag{2.9}$$

Define

$$\begin{aligned} &\alpha_n(w) = (\lambda_1 w - \mu_1)(1-w) + \lambda_2(1-p)^n w, \quad \text{for } 0 \le n \le N-1, \\ &\alpha_N(w) = (\lambda_1 w - \mu_1)(1-w). \end{aligned}$$

The set of equations (2.3), (2.6), and (2.9) can be written as

$$A(w) \cdot \vec{F}(w) = \vec{b}(w), \tag{2.10}$$

where

$$A(w) = \begin{pmatrix} \alpha_0(w) & 0 & \cdots & \cdots & \cdots & 0 \\ -\lambda_2 w & \alpha_1(w) & 0 & 0 & \cdots & \vdots \\ 0 & -\lambda_2(1-p)w & \alpha_2(w) & 0 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -\lambda_2(1-p)^{N-2}w & \alpha_{N-1}(w) & 0 \\ 0 & \cdots & 0 & 0 & -\lambda_2(1-p)^{N-1}w & \alpha_N(w) \end{pmatrix},$$

$\vec{F}(w) = (F_0(w), F_1(w), \ldots, F_N(w))^T$ is a column vector (of size $N+1$) of the desired PGFs, and

$$\vec{b}(w) = \begin{pmatrix} b_0(w) \\ b_1(w) \\ \vdots \\ b_n(w) \\ \vdots \\ b_{N-1}(w) \\ b_N(w) \end{pmatrix} = \begin{pmatrix} -\mu_1 P_{00}(1-w) + \mu_2 P_{01} w \\ -\mu_1 P_{01}(1-w) + \mu_2(P_{02} - P_{01})w \\ \vdots \\ -\mu_1 P_{0,n}(1-w) + \mu_2(P_{0,n+1} - P_{0,n})w \\ \vdots \\ -\mu_1 P_{0,N-1}(1-w) + \mu_2(P_{0N} - P_{0,N-1})w \\ -\mu_1 P_{0N}(1-w) - \mu_2 P_{0N} w \end{pmatrix}.$$

Note that for $0 \le n \le N-1$, $\alpha_n(w)$ has 2 roots given by

$$w_{1,2}^{(n)} = \frac{\lambda_1 + \lambda_2(1-p)^n + \mu_1 \pm \sqrt{(\lambda_1 + \lambda_2(1-p)^n + \mu_1)^2 - 4\lambda_1\mu_1}}{2\lambda_1}. \tag{2.11}$$

It is easy to verify that the smaller root satisfies $0 < w_1^{(n)} < 1$ and $w_2^{(n)} > 1$. As for $\alpha_N(w)$, its roots are $w_1^{(N)} = 1$ and $w_2^{(N)} = \frac{\mu_1}{\lambda_1}$, which is greater than 1 if the system is stable.

In general, to obtain the PGFs $F_n(w)$, one uses Cramer's rule. That is, for all $0 \le n \le N$

$$F_n(w) = \frac{|A_n(w)|}{|A(w)|},$$

where $|A|$ is the determinant of a matrix A, and $A_n(w)$ is the matrix obtained from $A(w)$ by replacing the nth column by the vector $\vec{b}(w)$. This leads to an expression for $F_n(w)$ in terms of the $N+1$ unknown probabilities, $P_{00}, P_{01}, \ldots, P_{0N}$ appearing in $\vec{b}(w)$. However, in our case, the PGFs can be calculated iteratively, using the structure of the matrix $A(w)$. Specifically, from equation (2.3), we immediately have that

$$F_0(w) = \frac{b_0(w)}{\alpha_0(w)}.$$

Rewriting equation (2.6) in terms of $\alpha_n(w)$ and $b_n(w)$ gives

$$F_n(w) = \frac{b_n(w) + \lambda_2(1-p)^{n-1} w F_{n-1}(w)}{\alpha_n(w)}. \tag{2.12}$$

Iterating (2.12) yields, for $0 \le n \le N$, a closed form expression for $F_n(w)$, as follows:

$$F_n(w) = \frac{\sum_{j=0}^{n} b_{n-j}(w)(\lambda_2 w)^j (1-p)^{(n2)\text{-}(n-j2)} \left(\prod_{i=0}^{n-j-1} \alpha_i(w)\right)}{\prod_{i=0}^{n} \alpha_i(w)}, \tag{2.13}$$

where $\prod_{i=0}^{-1}(\cdot) \triangleq 1$.

To calculate the boundary probabilities $P_{00}, P_{01}, \ldots, P_{0N}$, we utilize the N roots $w_1^{(n)}$, $n = 0, 1, \ldots, N-1$. Note that, since $F_n(w)$ is a (partial) probability generating function defined for all $0 \le w \le 1$, each root of $|A(w)|$ is a root of $|A_n(w)|$. Specifically, substituting $w_1^{(0)}$ in $F_0(w)$ gives a relation between P_{00} and P_{01}. Substituting $w_1^{(1)}$ in $F_1(w)$ gives a relation between P_{00}, P_{01} and P_{02}, and so on. The last substitution is $w_1^{(N-1)}$ in $F_{N-1}(w)$, which relates $P_{00}, P_{01}, \ldots, P_{0N}$. At last, we get N equations relating $N+1$ unknowns. The additional needed equation is

$$P_{00} + P_{01} + \cdots + P_{0N} = P_{0\bullet} = 1 - \rho. \tag{2.14}$$

Applying vertical "cuts" on Figure 1 yields, for $0 \le n \le N-1$,

$$\lambda_2(1-p)^n P_{\bullet n} = \mu_2 P_{0,n+1}. \tag{2.15}$$

That is, once the boundary probabilities P_{00}, P_{01}, ... , P_{0N} are calculated, the marginal probabilities of the queue length of class-2 groups is obtained as follows:

$$P_{\bullet n} = F_n(1) = \mathbb{P}(L_2 = n) = \frac{\mu_2 P_{0,n+1}}{\lambda_2(1-p)^n}, \quad 0 \le n \le N-1,$$
$$P_{\bullet N} = F_N(1) = \mathbb{P}(L_2 = N) = 1 - \sum_{n=0}^{N-1} P_{\bullet n}. \tag{2.16}$$

3. PERFORMANCE MEASURES

3.1. Mean Queue Lengths and Waiting Times

As for class 1, all performance measures are well known, since VIP customers form a regular $M/M/1$ queue. In particular, $\mathbb{E}[L_1] = \lambda_1/(\mu_1 - \lambda_1)$.

Utilizing equation (2.16), the mean total number of class-2 groups in the system is given by

$$\mathbb{E}[L_2] = \sum_{n=0}^{N} nP_{\bullet n}.$$

Let $\hat{\lambda}_2$ denote the mean rate in which new class-2 groups are formed. Then, $\hat{\lambda}_2 = \lambda_2 \sum_{n=0}^{N-1} P_{\bullet n}(1-p)^n$. Define W_2 to be the total sojourn time of a class-2 group *leader* in the system. W_2 can be looked upon as the lifetime in the system of an arbitrary group, that is, the duration of time from the moment of a group's initiation until the moment when it departs the system. Then, from Little's Law,

$$\mathbb{E}[W_2] = \frac{\mathbb{E}[L_2]}{\hat{\lambda}_2}. \tag{3.1}$$

3.2. Covariance of L_1 and L_2

Given $\mathbb{E}[L_i]$ for $i = 1, 2$, we need to calculate $\mathbb{E}[L_1 L_2]$. We have

$$\mathbb{E}[L_1 L_2] = \sum_{n=0}^{N} \sum_{m=0}^{\infty} mnP_{mn} = \sum_{n=0}^{N} nF_n'(1). \tag{3.2}$$

Differentiating $F_n(w)$ in equation (2.12) for $0 \leq n \leq N-1$ and substituting $w = 1$ results in the following recurrence relation:

$$\begin{aligned} F_n'(1) &= \frac{F_{n-1}'(1)}{1-p} + \frac{\mu_1 P_{0n} \lambda_2 (1-p)^n - \mu_2 P_{0,n+1}(\mu_1 - \lambda_1)}{(\lambda_2(1-p)^n)^2} \\ &= \frac{F_{n-1}'(1)}{1-p} + \frac{\mu_1 P_{0n} - (\mu_1 - \lambda_1) P_{\bullet n}}{\lambda_2 (1-p)^n}. \end{aligned} \tag{3.3}$$

Iterating (3.3) yields

$$F_n'(1) = \frac{1}{\lambda_2(1-p)^n} \sum_{j=0}^{n} \left(\mu_1 P_{0,j} - (\mu_1 - \lambda_1) P_{\bullet j}\right), \quad 0 \leq n \leq N-1. \tag{3.4}$$

Now, to calculate $F_N'(1)$, we use the fact that the VIP queue is a regular $M(\lambda_1)/M(\mu_1)/1$ system for which

$$\sum_{n=0}^{N} F_n'(1) = \mathbb{E}[L_1] = \frac{\lambda_1}{\mu_1 - \lambda_1}.$$

This implies,

$$
\begin{aligned}
F_N'(1) &= \frac{\lambda_1}{\mu_1-\lambda_1} - \sum_{n=0}^{N-1} F_n'(1) \\
&= \frac{\lambda_1}{\mu_1-\lambda_1} - \sum_{n=0}^{N-1} \frac{1}{\lambda_2(1-p)^n} \sum_{j=0}^{n} \left(\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}\right) \\
&= \frac{\lambda_1}{\mu_1-\lambda_1} - \sum_{j=0}^{N-1} \left(\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}\right) \sum_{n=j}^{N-1} \frac{1}{\lambda_2(1-p)^n} \\
&= \frac{\lambda_1}{\mu_1-\lambda_1} - \sum_{j=0}^{N-1} \left(\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}\right) \frac{(1-p)^{1-N} - (1-p)^{1-j}}{\lambda_2 p} \\
&= \frac{\lambda_1}{\mu_1-\lambda_1} - \frac{1}{\lambda_2 p(1-p)^{N-1}} \left(-\mu_1 P_{0N} + (\mu_1-\lambda_1)P_{\bullet N}\right) \\
&\quad + \sum_{j=0}^{N-1} \frac{\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}}{\lambda_2 p(1-p)^{j-1}} \\
&= \frac{\lambda_1}{\mu_1-\lambda_1} + \sum_{j=0}^{N} \frac{\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}}{\lambda_2 p(1-p)^{j-1}}. \qquad (3.5)
\end{aligned}
$$

Substituting in equation (3.2) the expressions for $F_n'(1)$, $n = 0, 1, \ldots N-1$, and for $F_N'(1)$ given in equations (3.4) and (3.5), respectively, results in

$$
\begin{aligned}
\mathbb{E}[L_1 L_2] &= \sum_{n=0}^{N-1} n F_n'(1) + N F_N'(1) \\
&= \sum_{n=0}^{N-1} \frac{n}{\lambda_2(1-p)^n} \sum_{j=0}^{n} \left(\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}\right) + \frac{N\lambda_1}{\mu_1-\lambda_1} \\
&\quad + N \sum_{j=0}^{N} \frac{\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}}{\lambda_2 p(1-p)^{j-1}} \\
&= \sum_{j=0}^{N-1} \left(\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}\right) \sum_{n=j}^{N-1} \frac{n}{\lambda_2(1-p)^n} + \frac{N\lambda_1}{\mu_1-\lambda_1} \\
&\quad + N \sum_{j=0}^{N} \frac{\mu_1 P_{0,j} - (\mu_1-\lambda_1)P_{\bullet j}}{\lambda_2 p(1-p)^{j-1}}. \qquad (3.6)
\end{aligned}
$$

Using the fact that

$$\sum_{n=j}^{N-1} \frac{n}{(1-p)^n} = \frac{Np-1}{p^2(1-p)^{N-1}} - \frac{jp-1}{p^2(1-p)^{j-1}},$$

equation (3.6) translates to

$$\mathbb{E}[L_1 L_2] = \frac{N\lambda_1}{\mu_1 - \lambda_1} + \sum_{j=0}^{N} \frac{\mu_1 P_{0,j} - (\mu_1 - \lambda_1) P_{\bullet j}}{\lambda_2 p^2 (1-p)^{j-1}} ((N-j)p + 1). \tag{3.7}$$

3.3. The Position of a New Class-2 Arriving Customer

Let X denote the position (group's number) that a new class-2 arrival enters to. The distribution of X is given by

$$\mathbb{P}(X = k) = P_{\bullet,k-1}(1-p)^{k-1} + \sum_{n=k}^{N} P_{\bullet n}(1-p)^{k-1}p, \quad k = 1, \ldots, N-1,$$
$$\mathbb{P}(X = N) = P_{\bullet,N-1}(1-p)^{N-1} + P_{\bullet N}(1-p)^{N-1}.$$

From the above, we get

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{N-1} \left(P_{\bullet,k-1}(1-p)^{k-1} + \sum_{n=k}^{N} P_{\bullet n}(1-p)^{k-1}p \right) k \\
&\quad + NP_{\bullet,N-1}(1-p)^{N-1} + NP_{\bullet N}(1-p)^{N-1}p + NP_{\bullet N}(1-p)^N \\
&= \sum_{k=1}^{N} P_{\bullet,k-1}(1-p)^{k-1}k + \sum_{k=1}^{N}\sum_{n=k}^{N} P_{\bullet n}(1-p)^{k-1}pk + NP_{\bullet N}(1-p)^N,
\end{aligned}$$

which leads to

$$\mathbb{E}[X] = \sum_{n=1}^{N} P_{\bullet n} \sum_{k=1}^{n} (1-p)^{k-1}pk + \sum_{n=0}^{N-1} P_{\bullet n}(1-p)^n(n+1) + P_{\bullet N}(1-p)^N N.$$

With some algebraic manipulations and the use of equation (2.15), we get

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{N} P_{\bullet n} \left(\frac{1-(1-p)^n(1+np)}{p} \right) \\
&\quad + \frac{1}{\lambda_2} \sum_{n=0}^{N-1} (n+1)\mu_2 P_{0,n+1} + P_{\bullet N}(1-p)^N N
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p}(1 - P_{\bullet 0}) - \left(\frac{\mu_2}{\lambda_2} \sum_{n=1}^{N-1} nP_{0,n+1} + NP_{\bullet N}(1-p)^N \right. \\
&\quad \left. + \frac{\mu_2}{\lambda_2 p} \sum_{n=1}^{N-1} P_{0,n+1} + \frac{1}{p} P_{\bullet N}(1-p)^N \right) \\
&\quad + \frac{\mu_2}{\lambda_2} \sum_{n=0}^{N-1} nP_{0,n+1} + \frac{\mu_2}{\lambda_2} \sum_{n=0}^{N-1} P_{0,n+1} + NP_{\bullet N}(1-p)^N \\
&= \frac{1}{p} - \frac{\mu_2}{\lambda_2 p} P_{01} - \frac{\mu_2}{\lambda_2 p} \sum_{n=1}^{N-1} P_{0,n+1} + \frac{\mu_2}{\lambda_2} \sum_{n=0}^{N-1} P_{0,n+1} - \frac{1}{p} P_{\bullet N}(1-p)^N \\
&= \frac{1}{p} \left(1 - \frac{\mu_2}{\lambda_2}(1 - \rho - P_{00})(1-p) - P_{\bullet N}(1-p)^N \right). \qquad (3.8)
\end{aligned}
$$

3.4. Sojourn Times

Let Θ_m $(m = 1, 2, \dots)$ denote the time from the first moment when there are m VIP customers in the system until the first moment thereafter that no VIP customers are present. Clearly, since the order of service does not affect the length of Θ_m, it follows that $\Theta_m = \sum_{i=1}^{m} \Theta_{1,i}$, where $\Theta_{1,i} \sim \Theta_1$, all $\Theta_{1,i}$ are independent, and Θ_1 is the busy period in a standard $M/M/1$ queue. Hence, the Laplace Stieltjes Transform (LST) of Θ_m is given by

$$
\widetilde{\Theta}_m(s) = \mathbb{E}\left[e^{-s\Theta_m}\right] = \left(\frac{\lambda_1 + \mu_1 + s - \sqrt{(\lambda_1 + \mu_1 + s)^2 - 4\lambda_1\mu_1}}{2\lambda_1} \right)^m = \left(\widetilde{\Theta}_1(s)\right)^m, \qquad (3.9)
$$

and

$$
\mathbb{E}[\Theta_m] = -\widetilde{\Theta}'_m(s)|_{s=0} = \frac{m}{\mu_1 - \lambda_1}. \qquad (3.10)
$$

Now, suppose there are no VIP customers in the system and there are $n \geq 1$ class-2 groups. Let C_n be the time until these n groups are served and leave the system. Since the waiting times of these n groups are not affected by the stream of new class-2 arrivals, C_n may also be considered as the "draining time" of the system if, when there are n class-2 groups present, the arrival stream of VIP customers is stopped. Therefore,

$$
C_n = Exp(\lambda_1 + \mu_2) + \begin{cases} C_{n-1} & \text{w.p. } \dfrac{\mu_2}{\lambda_1 + \mu_2} \\ \Theta_1 + C_n & \text{w.p. } \dfrac{\lambda_1}{\lambda_1 + \mu_2} \end{cases}, \qquad (3.11)
$$

where $Exp(\lambda_1 + \mu_2)$ is an exponential random variable with parameter $(\lambda_1 + \mu_2)$. The lower expression in (3.11) is a consequence of the fact that a new arrival of a VIP customer opens a class-1 busy period Θ_1, and class-2 arrivals during Θ_1 do not affect the waiting times of the class-2 customers already present in the system.

From equation (3.11), we get that the LST of C_n, denoted by $\widetilde{C}_n(s)$, satisfies the following recursive relation

$$\widetilde{C}_n(s) = \frac{\mu_2}{\lambda_1 + \mu_2 + s}\widetilde{C}_{n-1}(s) + \frac{\lambda_1}{\lambda_1 + \mu_2 + s}\widetilde{\Theta}_1(s)\widetilde{C}_n(s), \quad 0 \le n \le N. \quad (3.12)$$

Iteration of equation (3.12) leads to

$$\widetilde{C}_n(s) = \left(\frac{\mu_2}{\mu_2 + \lambda_1(1 - \widetilde{\Theta}_1(s)) + s}\right)^n = \left(\widetilde{C}_1(s)\right)^n, \quad (3.13)$$

where

$$\widetilde{C}_1(s) = \frac{\mu_2}{\mu_2 + \lambda_1(1 - \widetilde{\Theta}_1(s)) + s}.$$

From (3.13), we get

$$\mathbb{E}[C_n] = -\widetilde{C}_n'(s)|_{s=0} = \frac{n\mu_1}{\mu_2(\mu_1 - \lambda_1)}. \quad (3.14)$$

A probabilistic interpretation to equation (3.14) arises from rewriting it as

$$\mathbb{E}[C_n] = n\left(\frac{1}{\mu_2} + \frac{\lambda_1}{\mu_2} \cdot \frac{1}{\mu_1 - \lambda_1}\right),$$

where $\frac{1}{\mu_2}$ is the mean service time of a class-2 group and $\frac{\lambda_1}{\mu_2}$ is the mean number of VIP arrivals (each one causing an interruption) during a service time of a class-2 group. Each such interruption generates a class-1 busy period with mean $\frac{1}{\mu_1 - \lambda_1}$.

In addition, assume there are m VIP customers and n class-2 groups in the system, and denote by $D_{m,n}$ (with LST $\widetilde{D}_{m,n}(s)$) the time until the service of all those n class-2 groups is completed. Then

$$\widetilde{D}_{m,n}(s) = \widetilde{\Theta}_m(s)\widetilde{C}_n(s) = \left(\widetilde{\Theta}_1(s)\right)^m \left(\widetilde{C}_1(s)\right)^n, \quad (3.15)$$

and from equations (3.10) and (3.14),

$$\mathbb{E}[D_{m,n}] = \mathbb{E}[\Theta_m] + \mathbb{E}[C_n] = \frac{1}{\mu_1 - \lambda_1}\left(m + \frac{n\mu_1}{\mu_2}\right). \quad (3.16)$$

We now calculate the LST of the sojourn time of a class-2 group leader in the system, denoted by $\widetilde{W}_2(s)$. When the system is in state (m, n), $m \geq 0$ and $n = 0, 1, \ldots, N-1$, an arriving class-2 customer becomes a group leader (in the $(n+1)$ position) with probability $(1-p)^n$. The LST of his sojourn time is therefore $\widetilde{D}_{m,n+1}(\cdot)$. Using (3.15), we get

$$\begin{aligned}
\widetilde{W}_2(s) &= \frac{\sum_{m=0}^{\infty}\sum_{n=0}^{N-1} P_{mn}(1-p)^n \widetilde{D}_{m,n+1}(s)}{\sum_{n=0}^{N-1} P_{\bullet n}(1-p)^n} \\
&= \frac{\sum_{n=0}^{N-1}(1-p)^n \widetilde{C}_{n+1}(s)\sum_{m=0}^{\infty} P_{mn}\widetilde{\Theta}_m(s)}{\sum_{n=0}^{N-1} P_{\bullet n}(1-p)^n} \\
&= \frac{\sum_{n=0}^{N-1}(1-p)^n (\widetilde{C}_1(s))^{n+1} F_n(\widetilde{\Theta}_1(s))}{\sum_{n=0}^{N-1} P_{\bullet n}(1-p)^n},
\end{aligned} \tag{3.17}$$

where the numerator is, in fact, the probability for an arriving class-2 customer to become a group leader.

Let W_2^a denote the total sojourn time in the system of an *arbitrary* class-2 customer, with LST $\widetilde{W}_2^a(s)$. Assume the system is in state (m, n), for $m \geq 0$ and $n = 0, 1, \ldots, N-1$. Then, either (i) an arriving class-2 customer will join the kth group $(k = 1, 2, \ldots, n)$ with probability $(1-p)^{k-1}p$ and the LST of his total sojourn time in the system will be $\widetilde{D}_{m,k}(s)$, or (ii) an arriving class-2 customer will form a new group (with probability $(1-p)^n$) so that the LST of his total sojourn time in the system will be $\widetilde{D}_{m,n+1}(s)$. In addition, if the system is in state (m, N), an arriving class-2 customer will join the Nth group if he does not know any of the first $N-1$ group leaders (with probability $(1-p)^{N-1}$). Combining all of the arguments above, using equation (3.15) and the definition of $F_n(\cdot)$ leads to

$$\begin{aligned}
\widetilde{W}_2^a(s) &= \sum_{m=0}^{\infty}\left[\sum_{n=0}^{N} P_{mn}\sum_{k=1}^{n}(1-p)^{k-1}p\widetilde{D}_{mk}(s) + P_{mN}(1-p)^N \widetilde{D}_{mN}(s)\right. \\
&\qquad\left. + \sum_{n=0}^{N-1} P_{mn}(1-p)^n \widetilde{D}_{m,n+1}(s)\right] \\
&= \sum_{n=0}^{N}\sum_{k=1}^{n}(1-p)^{k-1}p\widetilde{C}_k(s)F_n(\widetilde{\Theta}_1(s)) + (1-p)^N \widetilde{C}_N(s)F_N(\widetilde{\Theta}_1(s)) \\
&\qquad + \sum_{n=0}^{N-1}(1-p)^n \widetilde{C}_{n+1}(s)F_n(\widetilde{\Theta}_1(s)).
\end{aligned} \tag{3.18}$$

Substituting in (3.18) the expression of $\widetilde{C}_n(s)$ given in (3.13) gives

$$\begin{aligned}\widetilde{W}_2^a(s) &= \sum_{n=0}^{N} \frac{p\widetilde{C}_1(s)\left(1-\left((1-p)\widetilde{C}_1(s)\right)^n\right)}{1-(1-p)\widetilde{C}_1(s)} F_n(\widetilde{\Theta}_1(s)) \\ &\quad + \left((1-p)\widetilde{C}_1(s)\right)^N F_N(\widetilde{\Theta}_1(s)) \\ &\quad + \widetilde{C}_1(s)\sum_{n=0}^{N-1}\left((1-p)\widetilde{C}_1(s)\right)^n F_n(\widetilde{\Theta}_1(s)). \qquad (3.19)\end{aligned}$$

To calculate $\mathbb{E}[W_2]$, the expected total sojourn time in the system of a class-2 group leader, one can differentiate equation (3.17), or use equations (3.16) and (3.4). We get

$$\begin{aligned}\mathbb{E}[W_2] &= \frac{\sum_{m=0}^{\infty}\sum_{n=0}^{N-1} P_{mn}(1-p)^n \mathbb{E}[D_{m,n+1}]}{\sum_{n=0}^{N-1} P_{\bullet n}(1-p)^n} \\ &= \frac{\sum_{m=0}^{\infty}\sum_{n=0}^{N-1} P_{mn}(1-p)^n \left(m+\frac{(n+1)\mu_1}{\mu_2}\right)}{(\mu_1-\lambda_1)\sum_{n=0}^{N-1} P_{\bullet n}(1-p)^n} \\ &= \frac{\lambda_2}{(\mu_1-\lambda_1)\hat{\lambda}_2}\left[\sum_{n=0}^{N-1}\left((1-p)^n\sum_{m=0}^{\infty} mP_{mn} + \frac{\mu_1}{\mu_2}(n+1)(1-p)^n\sum_{m=0}^{\infty}P_{mn}\right)\right] \\ &= \frac{\lambda_2}{(\mu_1-\lambda_1)\hat{\lambda}_2}\left[\sum_{n=0}^{N-1}\left((1-p)^n F_n'(1) + \frac{\mu_1}{\mu_2}(n+1)(1-p)^n P_{\bullet n}\right)\right].\end{aligned}$$

Using the expression for $F_n'(1)$ given in equation (3.4) yields

$$\begin{aligned}\mathbb{E}[W_2] &= \frac{\lambda_2}{(\mu_1-\lambda_1)\hat{\lambda}_2}\left[\sum_{n=0}^{N-1}\left(\frac{1}{\lambda_2}\sum_{j=0}^{n}\left[\mu_1 P_{0,j}-(\mu_1-\lambda_1)P_{\bullet j}\right]\right.\right. \\ &\qquad\left.\left. +\frac{\mu_1}{\mu_2}(n+1)(1-p)^n P_{\bullet n}\right)\right] \\ &= \frac{\lambda_2}{(\mu_1-\lambda_1)\hat{\lambda}_2}\left[\frac{1}{\lambda_2}\sum_{j=0}^{N-1}(N-j)\left[\mu_1 P_{0,j}-(\mu_1-\lambda_1)P_{\bullet j}\right]\right. \\ &\qquad\left. +\frac{\mu_1}{\mu_2}\sum_{n=0}^{N-1}(n+1)(1-p)^n P_{\bullet n}\right]. \qquad (3.20)\end{aligned}$$

By using the relation $\lambda_2(1-p)^n P_{\bullet n} = \mu_2 P_{0,n+1}$, and after some algebra, equation (3.20) becomes

$$\mathbb{E}[W_2] = \frac{\sum_{j=0}^{N} jP_{\bullet j}}{\hat{\lambda}_2} = \frac{\mathbb{E}[L_2]}{\hat{\lambda}_2}, \tag{3.21}$$

which is indeed the result obtained by Little's Law in (3.1). In a similar manner, $\mathbb{E}[W_2^a]$ is derived as follows:

$$\mathbb{E}[W_2^a] = \sum_{m=0}^{\infty}\left[\sum_{n=0}^{N} P_{mn}\sum_{k=1}^{n}(1-p)^{k-1}p\mathbb{E}[D_{m,k}] + P_{mN}(1-p)^N\mathbb{E}[D_{m,N}] + \sum_{n=0}^{N-1} P_{mn}(1-p)^n\mathbb{E}[D_{m,n+1}]\right]. \tag{3.22}$$

Using equation (3.16) and the fact that $\sum_{n=0}^{N} F_n'(1) = \mathbb{E}[L_1] = \frac{\lambda_1}{\mu_1-\lambda_1}$, equation (3.22) becomes, after some algebra,

$$\mathbb{E}[W_2^a] = \frac{1}{\mu_1-\lambda_1}\left(\frac{\lambda_1}{\mu_1-\lambda_1} + \frac{\mu_1}{\mu_2 p}\sum_{n=0}^{N} P_{\bullet n}(1-(1-p)^n(1+np)) + \frac{N\mu_1}{\mu_2}(1-p)^N P_{\bullet N} + \frac{\mu_1}{\mu_2}\sum_{n=0}^{N-1}(n+1)(1-p)^n P_{\bullet n}\right). \tag{3.23}$$

Further manipulations on equation (3.23) yield

$$\mathbb{E}[W_2^a] = \frac{1}{\mu_1-\lambda_1}\left(\frac{\lambda_1}{\mu_1-\lambda_1} + \frac{\mu_1}{\mu_2 p} + \frac{\mu_1}{\lambda_2}(1-\rho-P_{00})\left(1-\frac{1}{p}\right) - \frac{\mu_1}{\mu_2 p}(1-p)^N P_{\bullet N}\right). \tag{3.24}$$

Rearranging terms in equation (3.24) leads to

$$\mathbb{E}[W_2^a] = \frac{1}{\mu_1-\lambda_1}\cdot\frac{\lambda_1}{\mu_1-\lambda_1} + \left(\frac{1}{\mu_2} + \frac{\lambda_1}{\mu_2}\cdot\frac{1}{\mu_1-\lambda_1}\right)\frac{1}{p} \times\left(1 - \frac{\mu_2}{\lambda_2}(1-\rho-P_{00})(1-p) - P_{\bullet N}(1-p)^N\right), \tag{3.25}$$

or, equivalently, using (3.10), (3.14), and (3.8),

$$\mathbb{E}[W_2^a] = \mathbb{E}[\Theta_1]\mathbb{E}[L_1] + \mathbb{E}[C_1]\mathbb{E}[X]. \tag{3.26}$$

Indeed,

$$W_2^a = \sum_{i=1}^{L_1} \Theta_{1,i} + \sum_{i=1}^{X} C_{1,i},$$

where $\Theta_{1,i} \sim \Theta_1$ and $C_{1,i} \sim C_1$ for all i. As L_1 and Θ_1, and X and C_1 are independent, (3.26) follows.

3.5. The Mean Size of a Class-2 Batch

We wish to calculate the mean size of a class-2 group that has completed service. We define the following random variables:

- $D^{(k)}$ = size of the class-2 group standing in the kth position ($k = 1, 2, \ldots, N$) at the moment of service completion ($k = 1$ refers to the group that leaves the system).
- $D_i^{(k)}$ = size of the group standing in the kth position at the moment of service completion, given that it was formed in the ith position ($i \geq k$).
- $\xi^{(k)}$ = number of customers who joined the kth class-2 group ($k = 1, 2, \ldots, N$) during a class-2 service duration (including VIP interruptions), assuming that the kth group exists.

To derive $\mathbb{E}[D^{(k)}]$, we condition on the position in which the kth group (moving to the $k-1$-st position) was created. From the definitions above, and from the fact that $\lambda_2 P_{\bullet,i-1}(1-p)^{i-1} = \mu_2 P_{0i}$, we have

$$D_i^{(k)} = 1 + \sum_{m=k}^{i} \xi^{(m)} \quad \text{w.p. } \frac{P_{\bullet,i-1}(1-p)^{i-1}}{\sum_{j=k}^{N} P_{\bullet,j-1}(1-p)^{j-1}} = \frac{P_{0i}}{\sum_{j=k}^{N} P_{0j}}. \tag{3.27}$$

Since the mean number of class-2 arrivals during a service period of a class-2 group is $\lambda_2 \mathbb{E}[C_1]$, we have, using (3.14),

$$\begin{aligned}
\mathbb{E}\left[\xi^{(k)}\right] &= \frac{\lambda_2 \mu_1}{\mu_2(\mu_1 - \lambda_1)}(1-p)^{k-1} p, \quad 1 \leq k \leq N-1, \\
\mathbb{E}\left[\xi^{(N)}\right] &= \frac{\lambda_2 \mu_1}{\mu_2(\mu_1 - \lambda_1)}(1-p)^{N-1}.
\end{aligned} \tag{3.28}$$

Combining all of the above leads to

$$\begin{aligned}
\mathbb{E}\left[D^{(k)}\right] &= \sum_{i=k}^{N} \mathbb{E}\left[D_i^{(k)}\right] \frac{P_{0i}}{\sum_{j=k}^{N} P_{0j}} = \sum_{i=k}^{N} \left(1 + \sum_{m=1}^{i-k+1} \mathbb{E}\left[\xi^{(i+1-m)}\right]\right) \frac{P_{0i}}{\sum_{j=k}^{N} P_{0j}} \\
&= 1 + \sum_{i=k}^{N-1} \frac{P_{0i}}{\sum_{j=k}^{N} P_{0j}} \sum_{m=1}^{i-k+1} \frac{\lambda_2 \mu_1}{\mu_2(\mu_1 - \lambda_1)}(1-p)^{i-m} p
\end{aligned}$$

$$
\begin{aligned}
&+\frac{P_{0N}}{\sum_{j=k}^{N} P_{0j}}\Bigg(\sum_{m=1}^{N-k}\frac{\lambda_2\mu_1}{\mu_2(\mu_1-\lambda_1)}(1-p)^{N-1-m}p \\
&\qquad\qquad + \frac{\lambda_2\mu_1}{\mu_2(\mu_1-\lambda_1)}(1-p)^{N-1}\Bigg) \\
&= 1+\frac{\lambda_2\mu_1}{\mu_2(\mu_1-\lambda_1)}\left(\sum_{i=k}^{N-1}\frac{P_{0i}}{\sum_{j=k}^{N} P_{0j}}\left((1-p)^{k-1}-(1-p)^{i}\right)\right. \\
&\qquad\qquad \left. +\frac{P_{0N}}{\sum_{j=k}^{N} P_{oj}}(1-p)^{k-1}\right) \\
&= 1+\frac{\lambda_2\mu_1}{\mu_2(\mu_1-\lambda_1)}\left((1-p)^{k-1}-\sum_{i=k}^{N-1}\frac{P_{0i}(1-p)^{i}}{\sum_{j=k}^{N} P_{0j}}\right).
\end{aligned}
\tag{3.29}
$$

In particular, $\mathbb{E}[D^{(1)}]$, the mean size of a group completing service, is

$$
\begin{aligned}
\mathbb{E}[D^{(1)}] &= 1+\frac{\lambda_2\mu_1}{\mu_2(\mu_1-\lambda_1)}\left(1-\sum_{i=1}^{N-1}\frac{P_{0i}(1-p)^{i}}{\sum_{j=1}^{N} P_{0j}}\right) \\
&= 1+\frac{\lambda_2\mu_1}{\mu_2(\mu_1-\lambda_1)}\left(1-\sum_{i=1}^{N-1}\frac{P_{0i}(1-p)^{i}}{1-\rho-P_{00}}\right).
\end{aligned}
$$

Furthermore, the expected total number of class-2 customers in the system *at the moment of service completion* of a class-2 group, denoted by $\mathbb{E}[L_2^{total}]$, is derived from

$$
\mathbb{E}\left[L_2^{total}\right] = \sum_{n=1}^{N} P_{\bullet n}\sum_{k=1}^{n}\mathbb{E}[D^{(k)}].
$$

4. MATRIX GEOMETRIC METHOD

An alternative approach to analyze this model is by constructing a finite Quasi Birth and Death (QBD) process, with $N+1$ phases and infinite number of levels. State (m, n) indicates that there are m different VIP customers and n class-2 groups in the system, $m \geq 0$, $0 \leq n \leq N$. The infinitesimal generator of the QBD is denoted by Q, and is given by

$$
Q = \begin{pmatrix}
B & A_0 & \mathbf{0} & \mathbf{0} & \cdots \\
A_2 & A_1 & A_0 & \mathbf{0} & \cdots \\
\mathbf{0} & A_2 & A_1 & A_0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},
$$

where B, A_0, A_1, and A_2 are all square matrices of order $N+1$, as follows:

$$B = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_2 & 0 & \cdots & \cdots & 0 \\ \mu_2 & -(\lambda_1+\lambda_2(1-p)+\mu_2) & \lambda_2(1-p) & 0 & \cdots & \vdots \\ 0 & \mu_2 & -(\lambda_1+\lambda_2(1-p)^2+\mu_2) & \lambda_2(1-p)^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu_2 & -(\lambda_1+\lambda_2(1-p)^{N-1}+\mu_2) & \lambda_2(1-p)^{N-1} \\ 0 & 0 & 0 & 0 & \mu_2 & -(\lambda_1+\mu_2) \end{pmatrix},$$

$A_0 = \lambda_1 I,$

$$A_1 = \begin{pmatrix} -(\lambda_1+\lambda_2+\mu_1) & \lambda_2 & 0 & \cdots & \cdots & 0 \\ 0 & -(\lambda_1+\lambda_2(1-p)+\mu_1) & \lambda_2(1-p) & 0 & \cdots & \vdots \\ 0 & 0 & -(\lambda_1+\lambda_2(1-p)^2+\mu_1) & \lambda_2(1-p)^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -(\lambda_1+\lambda_2(1-p)^{N-1}+\mu_1) & \lambda_2(1-p)^{N-1} \\ 0 & 0 & 0 & 0 & 0 & -(\lambda_1+\mu_1) \end{pmatrix},$$

and $A_2 = \mu_1 I$, where I is the identity matrix of order $N+1$.

Define the matrix $A = A_0 + A_1 + A_2$. We get

$$A = \begin{pmatrix} -\lambda_2 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ 0 & -\lambda_2(1-p) & \lambda_2(1-p) & 0 & \cdots & \vdots \\ 0 & 0 & -\lambda_2(1-p)^2 & \lambda_2(1-p)^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\lambda_2(1-p)^{N-1} & \lambda_2(1-p)^{N-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is the infinitesimal generator of the "birth" process of class-2 groups, when there is at least one VIP customer in the system. Recall that the service of class-2 customers (groups) is represented only in the matrix B, since they receive service only when there are no VIP customers in the system. Let $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_N)$ be the stationary vector of the matrix A, i.e., $\vec{\pi}A = \vec{0}$ and $\vec{\pi} \cdot \vec{e} = 1$ (where $\vec{e}$ is a column vector with all entries equal to 1). Then, an immediate result is that $\vec{\pi} = (\underbrace{0, 0, \ldots, 0}_{N \text{ times}}, 1)$. The stability condition given in Neuts[15],

$$\vec{\pi} A_0 \vec{e} < \vec{\pi} A_2 \vec{e},$$

translates here into $\lambda_1 < \mu_1$ (see end of section 2.1).

Define for all $m \geq 0$ the steady-state probability vector $\vec{P}_m = (P_{m0}, P_{m1}, \ldots, P_{mN})$. Then (see Ref.[15]),

$$\vec{P}_m = \vec{P}_0 R^m, \quad m \geq 0,$$

where R is the minimal non-negative solution of the matrix quadratic equation

$$A_0 + RA_1 + R^2 A_2 = 0. \tag{4.1}$$

The vector $\vec{P}_0$ is derived by solving the following linear system,

$$\begin{aligned} \vec{P}_0(B + RA_2) &= \vec{0}, \\ \vec{P}_0 \cdot \vec{e} &= 1 - \rho. \end{aligned} \tag{4.2}$$

4.1. Calculation of the Rate Matrix *R*

We denote the elements of R as r_{ij}, for $i, j = 0, 1, \ldots, N$. Since A_0 and A_2 are diagonal matrices, and A_1 consists of the main diagonal and the one above it (all other elements are 0), it follows that R is an upper triangular matrix. A similar case in which R was in this form was studied in Ref.[23]. It follows that

$$\begin{aligned} \left[R^2\right]_{ii} &= r_{ii}^2, \quad i = 0, 1, \ldots, N, \\ \left[R^2\right]_{ij} &= \sum_{k=i}^{j} r_{ik} r_{kj}, \quad i < j. \end{aligned}$$

Rewriting equation (4.1) for the elements on the main diagonal leads to

$$\lambda_1 - r_{ii}(\lambda_1 + \lambda_2(1-p)^i + \mu_1) + \mu_1 r_{ii}^2 = 0, \quad i = 0, 1, \ldots, N-1, \tag{4.3}$$

$$\lambda_1 - r_{NN}(\lambda_1 + \mu_1) + r_{NN}^2 \mu_1 = 0. \tag{4.4}$$

Since R is the minimal non-negative solution of (4.1), from (4.3) and (4.4) we get

$$r_{ii} = \frac{\lambda_1 + \lambda_2(1-p)^i + \mu_1 - \sqrt{(\lambda_1 + \lambda_2(1-p)^i + \mu_1)^2 - 4\lambda_1\mu_1}}{2\mu_1}, \quad i = 0, 1, \ldots, N-1, \tag{4.5}$$

$$r_{NN} = \frac{\lambda_1}{\mu_1} = \rho. \tag{4.6}$$

It is easy to see that the elements on the main diagonal of R and the roots $w_1^{(n)}$ of the polynomials $\alpha_n(w)$ discussed in Section 2 satisfy the relation

$$r_{ii} = \frac{\lambda_1}{\mu_1} w_1^{(i)}, \quad 0 \le i \le N.$$

The other elements of R (for $i < j$) satisfy

$$\lambda_2(1-p)^{j-1} r_{i,j-1} - (\lambda_1 + \lambda_2(1-p)^j + \mu_1) r_{ij} + \mu_1 \sum_{k=i}^{j} r_{ik} r_{kj} = 0. \tag{4.7}$$

From equation (4.7), we can recursively calculate the other elements of the matrix R, in the following order. We first compute the elements on the first diagonal above the main diagonal. More precisely, let $j = i + 1$ in (4.7). We then have

$$\lambda_2(1-p)^i r_{ii} - (\lambda_1 + \lambda_2(1-p)^{i+1} + \mu_1) r_{i,i+1} + \mu_1 (r_{ii} r_{i,i+1} + r_{i,i+1} r_{i+1,i+1}) = 0,$$

which gives

$$r_{i,i+1} = \frac{\lambda_2(1-p)^i r_{ii}}{\lambda_1 + \lambda_2(1-p)^{i+1} + \mu_1 - \mu_1(r_{ii} + r_{i+1,i+1})}.$$

We continue and calculate the next diagonal, that is, we assume that $j = i + 2$. Equation (4.7) then becomes

$$\begin{aligned} &\lambda_2(1-p)^{i+1} r_{i,i+1} - (\lambda_1 + \lambda_2(1-p)^{i+2} + \mu_1) r_{i,i+2} \\ &\quad + \mu_1 (r_{ii} r_{i,i+2} + r_{i,i+1} r_{i+1,i+2} + r_{i,i+2} r_{i+2,i+2}) = 0, \end{aligned}$$

or,

$$r_{i,i+2} = \frac{\lambda_2(1-p)^{i+1} r_{i,i+1} + \mu_1 r_{i,i+1} r_{i+1,i+2}}{\lambda_1 + \lambda_2(1-p)^{i+2} + \mu_1 - \mu_1(r_{ii} + r_{i+2,i+2})}.$$

In general, the $(i, i+k)$th element of R is given by

$$r_{i,i+k} = \frac{\lambda_2(1-p)^{i+k-1} r_{i,i+k-1} + \mu_1 \sum_{n=1}^{k-1} r_{i,i+n} r_{i+n,i+k}}{\lambda_1 + \lambda_2(1-p)^{i+k} + \mu_1 - \mu_1(r_{ii} + r_{i+k,i+k})}, \quad k = 1, 2, \ldots, N - i, \tag{4.8}$$

where we define $\sum_{n=1}^{0} (\cdot) = 0$.

By using this procedure, we bypass the need to numerically solve the system of non-linear equations given in (4.1), and convergence issues become irrelevant.

5. SPECIAL CASES AND NUMERICAL RESULTS

5.1. Special Cases

5.1.1. *A Single Class-2 Group*

Consider the case where $N = 1$. That is, at most one class-2 group can be formed in the system. An arriving class-2 customer immediately enters this group if it exists, or becomes its leader if it does not exist. So, $N = 1$ is equivalent to $p = 1$. We present the calculation of the steady-state probabilities P_{mn} $(m \geq 0,\ n = 0, 1)$ by using both the PGFs and the Matrix Geometric methods.

First, writing the balance equations and using the PGFs lead to

$$F_0(w) = \frac{\mu_2 w P_{01} - \mu_1(1-w)P_{00}}{(\lambda_1 w - \mu_1)(1-w) + \lambda_2 w}, \tag{5.1}$$

and

$$(\lambda_1 w - \mu_1)(1-w)F_1(w) - \lambda_2 w F_0(w) = -\mu_1(1-w)P_{01} - \mu_2 w P_{01}, \tag{5.2}$$

which are equivalent to equations (2.3) and (2.9). Since the denominator of (5.1) vanishes at $w = w_1^{(0)} = \frac{\lambda_1+\lambda_2+\mu_1-\sqrt{(\lambda_1+\lambda_2+\mu_1)^2-4\lambda_1\mu_1}}{2\lambda_1}$ (see equation 2.11), we obtain P_{00} and P_{01} by solving the equations

$$\mu_2 w_1^{(0)} P_{01} - \mu_1(1-w_1^{(0)})P_{00} = 0,$$
$$P_{00} + P_{01} = 1 - \rho = 1 - \frac{\lambda_1}{\mu_1}.$$

The solution is

$$P_{00} = \frac{\mu_2(\mu_1 - \lambda_1)w_1^{(0)}}{\mu_1\left[(\mu_2 - \mu_1)w_1^{(0)} + \mu_1\right]}, \quad P_{01} = \frac{(\mu_1 - \lambda_1)(1 - w_1^{(0)})}{\mu_1(1 - w_1^{(0)}) + \mu_2 w_1^{(0)}}. \tag{5.3}$$

Now, to find $P_{\bullet 0}$ and $P_{\bullet 1}$ we have

$$\lambda_2 P_{\bullet 0} = \mu_2 P_{01},$$
$$P_{\bullet 0} + P_{\bullet 1} = 1.$$

Clearly, $\mathbb{E}[L_2] = P_{\bullet 1}$ and $\mathbb{E}[W_2] = \frac{\mathbb{E}[L_2]}{\hat{\lambda}_2}$, where $\hat{\lambda}_2 = \lambda_2 P_{\bullet 0}$.

Furthermore, since $X \equiv 1$ in this case, we have:

$$\mathbb{E}[W_2^a] = \mathbb{E}[\Theta_1]\mathbb{E}[L_1] + \mathbb{E}[C_1]\mathbb{E}[X] = \frac{1}{\mu_1 - \lambda_1} \cdot \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\mu_1}{\mu_2(\mu_1 - \lambda_1)},$$

and

$$\mathbb{E}[D^{(1)}] = 1 + \frac{\lambda_2 \mu_1}{\mu_2(\mu_1 - \lambda_1)} = 1 + \lambda_2 \cdot \frac{\lambda_1}{\mu_2} \cdot \frac{1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2}.$$

Indeed, $\mathbb{E}[D^{(1)}]$ consists of the first class-2 arrival, the class-2 arrivals during the service time of VIP customers, and the class-2 arrivals during the service time of the class-2 group.

A second approach of analysis is via the Matrix Geometric method. Following the notations and calculations presented in Section 4, we get

$$B = \begin{pmatrix} -(\lambda_1 + \lambda_2) & \lambda_2 \\ \mu_2 & -(\lambda_1 + \mu_2) \end{pmatrix}, \quad A_0 = \lambda_1 I,$$
$$A_1 = \begin{pmatrix} -(\lambda_1 + \lambda_2 + \mu_1) & \lambda_2 \\ 0 & -(\lambda_1 + \mu_1) \end{pmatrix}, \quad A_2 = \mu_1 I.$$

The matrix R is given by

$$R = \begin{pmatrix} \frac{\lambda_1+\lambda_2+\mu_1-\sqrt{(\lambda_1+\lambda_2+\mu_1)^2-4\lambda_1\mu_1}}{2\mu_1} & \frac{\lambda_1-\lambda_2-\mu_1+\sqrt{(\lambda_1+\lambda_2+\mu_1)^2-4\lambda_1\mu_1}}{2\mu_1} \\ 0 & \frac{\lambda_1}{\mu_1} \end{pmatrix}.$$

Solving the system in (4.2) gives the results in (5.3), and the rest of the steady-state probabilities are derived from

$$\vec{P}_m = \vec{P}_0 R^m, \quad m \geq 0,$$

where $\vec{P}_m = (P_{m0}, P_{m1})$.

5.1.2. *No Friends Among Class-2 Customers*

We assume that $p = 0$ and $N > 1$. That is, in such a world of strangers, an arriving class-2 customer that finds the system in state $(\bullet, n)$, for $0 \leq n \leq N - 1$ immediately enters the $n + 1$-st position. If the system is in state $(\bullet, N)$, the customer is not lost and joins the group in the Nth position. So, a group with more than one customer can be built only in the last position (the Nth), and while this group steps forward in line toward the server, new class-2 arrivals cannot join it.

An analysis via PGFs yields similar results to those obtained in Section 2, with the following modifications:

$$F_n(w) = \frac{\sum_{j=0}^{n} b_{n-j}(w)(\lambda_2 w)^j\, (\alpha(w))^{n-j}}{(\alpha(w))^{n+1}}, \quad \text{for } 0 \leq n \leq N - 1,$$

$$F_N(w) = \frac{\sum_{j=0}^{N} b_{N-j}(w)(\lambda_2 w)^j (\alpha(w))^{N-j}}{(\alpha(w))^N \alpha_N(w)}, \tag{5.4}$$

where $b_n(w)$ are the same as in previous sections, and

$$\alpha(w) = (\lambda_1 w - \mu_1)(1-w) + \lambda_2 w,$$
$$\alpha_N(w) = (\lambda_1 w - \mu_1)(1-w).$$

The calculation of the boundary probabilities $P_{00}, P_{01}, \ldots, P_{0N}$ appearing in $b_j(w)$, $j = 0, 1, \ldots, N$, is done in a similar manner as in Section 2. The only difference is that only a single root is utilized, $w_1^{(0)} = \frac{\lambda_1+\lambda_2+\mu_1-\sqrt{(\lambda_1+\lambda_2+\mu_1)^2-4\lambda_1\mu_1}}{2\lambda_1}$. The substitution of this root in the denominator of $F_n(w)$, for $n = 0, 1, \ldots, N-1$, together with equation (2.14), provides us with $N+1$ equations relating between the boundary probabilities. Once P_{0n}, $n = 0, 1, \ldots, N$, are known, the marginal distribution of L_2 is derived from

$$\lambda_2 P_{\bullet n} = \mu_2 P_{0,n+1}, \quad n = 0, 1, \ldots, N-1,$$

and $\sum_{n=0}^{N} P_{\bullet n} = 1$. Now, $\mathbb{E}[L_2] = \sum_{n=0}^{N} nP_{\bullet n}$ and $\mathbb{E}[W_2] = \frac{\mathbb{E}[L_2]}{\hat{\lambda}_2}$, where $\hat{\lambda}_2 = \lambda_2 \sum_{n=0}^{N-1} P_{\bullet n}$.

Furthermore,

$$\mathbb{E}[X] = \sum_{n=0}^{N-1} P_{\bullet n}(n+1) + NP_{\bullet N} = \mathbb{E}[L_2] + 1 - P_{\bullet N},$$

and $\mathbb{E}[W_2^a]$ is derived from (3.26).

The expression for $\mathbb{E}[D^{(1)}]$ is identical to the one given in the previous extreme case, when $N = 1$. This follows since class-2 customers can join other class-2 customers only in the last group, and no other jobs can join it during its progress down the line.

An analysis via the Matrix Geometric method is similar to the one presented in Section 4.

5.2. Numerical Results

In Tables 1–3, we present some numerical results for each one of the cases $N = 5$, $N = 10$, and $N = 15$, for different values of λ_1, λ_2, μ_1, μ_2, and p. We calculate the measures introduced in previous sections. Note that $r(L_1, L_2)$ stands here for the correlation coefficient between L_1 and L_2 (number of VIP customers against number of class-2 groups).

The tables exhibit that, as expected, $\mathbb{E}[L_2]$, $\mathbb{E}[W_2]$, $\mathbb{E}[W_2^a]$, and $\mathbb{E}[X]$ decrease monotonically when p increases. In addition, as p increases, $\mathbb{E}[D^{(1)}]$ ascends, and $\mathbb{E}[L^{total}]$ decreases, since the mean size of served

TABLE 1 Numerical results for $N = 5$

$\lambda_1 = 2,\ \mu_1 = 3,\ \mu_2 = 2$		$\hat{\lambda}_2$	$\mathbb{E}[L_2]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_2^a]$	$\mathbb{E}[D^{(1)}]$	$\mathbb{E}\left[L^{total}\right]$	$r(L_1, L_2)$	$\mathbb{E}[X]$
$\lambda_2 = 3$	$p = 0.01$	0.6598	4.6144	6.9937	9.1222	3.6178	18.5020	0.2700	4.7481
	$p = 0.1$	0.6555	4.4506	6.7896	7.8852	3.8034	15.6031	0.3000	3.9235
	$p = 0.3$	0.6371	3.8546	6.0504	5.8774	4.1872	11.1145	0.3704	2.5849
	$p = 0.6$	0.5806	2.4833	4.2775	4.3050	4.6158	7.1984	0.4246	1.5366
	$p = 0.9$	0.4989	1.3567	2.7195	3.6389	5.1724	5.2178	0.4057	1.0926
$\lambda_2 = 5$	$p = 0.01$	0.6654	4.8042	7.2196	9.2678	6.3047	32.1411	0.2340	4.8452
	$p = 0.1$	0.6644	4.7118	7.0918	8.0429	6.4808	26.8430	0.2646	4.0286
	$p = 0.3$	0.6585	4.3116	6.5479	6.0291	6.8933	18.6870	0.3421	2.6860
	$p = 0.6$	0.6276	2.9845	4.7551	4.3713	7.3568	11.9090	0.4228	1.5808
	$p = 0.9$	0.5589	1.5723	2.8133	3.6481	8.0139	8.4837	0.4058	1.0987
$\lambda_2 = 8$	$p = 0.01$	0.6664	4.8910	7.3388	9.3190	10.5734	53.3000	0.2000	4.8793
	$p = 0.1$	0.6662	4.8379	7.2615	8.1024	10.7276	44.3099	0.2302	4.0683
	$p = 0.3$	0.6647	4.5822	6.8937	6.0999	11.1414	30.3136	0.3105	2.7332
	$p = 0.6$	0.6509	3.4227	5.2577	4.4130	11.6345	19.0396	0.4185	1.6087
	$p = 0.9$	0.6007	1.7735	2.9519	3.6542	12.3237	13.3857	0.4053	1.1028

batches grows (larger-size groups leave the system). An interesting observation is that in some cases $\mathbb{E}[W_2] < \mathbb{E}[W_2^a]$, meaning that the expected sojourn time in the system of a class-2 group leader is smaller than that of an arbitrary class-2 customer. This result is somewhat counter intuitive. However, it occurs in two situations: when p is small and when p is large. In the first case new groups are formed fast, and L_2 reaches its maximal value, N, quite fast. Therefore, when p is small, a new class-2 arrival does not know any of the present group leaders and will join the last group, spending more time in the system than a class-2 customer who becomes a group leader. The second case seems to be a result of the finiteness of the number of groups.

TABLE 2 Numerical results for $N = 10$

$\lambda_1 = 2,\ \mu_1 = 3,\ \mu_2 = 2$		$\hat{\lambda}_2$	$\mathbb{E}[L_2]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_2^a]$	$\mathbb{E}[D^{(1)}]$	$\mathbb{E}[L^{total}]$	$r(L_1, L_2)$	$\mathbb{E}[X]$
$\lambda_2 = 3$	$p = 0.01$	0.6655	9.5629	14.3471	16.0721	3.6113	35.0865	0.2603	9.3814
	$p = 0.1$	0.6655	8.7898	13.2077	11.2830	4.0145	24.7811	0.3087	6.1887
	$p = 0.3$	0.6456	5.1689	8.0067	6.2427	4.3289	12.9022	0.3836	2.8284
	$p = 0.6$	0.5807	2.4998	4.3043	4.3064	4.6173	7.2151	0.4238	1.5376
	$p = 0.9$	0.4989	1.3567	2.7195	3.6389	5.1724	5.2178	0.4572	1.0926
$\lambda_2 = 5$	$p = 0.01$	0.6666	9.7886	14.6830	16.2517	6.3190	61.2881	0.2346	9.5012
	$p = 0.1$	0.6666	9.4137	14.1218	11.5957	6.7435	42.2392	0.2916	6.3971
	$p = 0.3$	0.6627	6.4667	9.7576	6.5231	7.2000	22.2722	0.3832	3.0154
	$p = 0.6$	0.6279	3.0346	4.8324	4.3744	7.3637	11.9658	0.4229	1.5829
	$p = 0.9$	0.6666	1.5723	2.8133	3.6481	8.0139	8.4837	0.4203	1.0987
$\lambda_2 = 8$	$p = 0.01$	0.6666	9.8839	14.8259	16.3082	10.5915	102.2750	0.2038	9.5388
	$p = 0.1$	0.6666	9.6870	14.5305	11.7018	10.9874	69.3388	0.2657	6.4679
	$p = 0.3$	0.6661	7.5257	11.2976	6.6806	11.6269	36.3851	0.3815	3.1204
	$p = 0.6$	0.6514	3.5401	5.4349	4.4186	11.6559	19.1880	0.4223	1.6124
	$p = 0.9$	0.6007	1.7735	2.9519	3.6542	12.3237	13.3857	0.4201	1.1028

TABLE 3 Numerical results for $N = 15$

$\lambda_1 = 2,\ \mu_1 = 3,\ \mu_2 = 2$		$\hat{\lambda}_2$	$\mathbb{E}[L_2]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_2^a]$	$\mathbb{E}[D^{(1)}]$	$\mathbb{E}[L^{total}]$	$r(L_1, L_2)$	$\mathbb{E}[X]$
$\lambda_2 = 3$	$p = 0.01$	0.6666	14.4949	21.7425	22.6602	3.3776	47.2886	0.2494	13.7735
	$p = 0.1$	0.6663	12.3595	18.5498	13.0680	4.2239	31.0610	0.3063	7.3786
	$p = 0.3$	0.6456	5.2121	8.0572	6.2467	4.3319	12.9439	0.3818	2.8312
	$p = 0.6$	0.5807	2.4998	4.3043	4.3064	4.6173	7.2151	0.4238	1.5376
	$p = 0.9$	0.4989	1.3567	2.7195	3.6390	5.1724	5.2178	0.4238	1.0926
$\lambda_2 = 5$	$p = 0.01$	0.6666	14.7742	22.1613	22.8956	6.3560	89.1376	0.2378	13.9304
	$p = 0.1$	0.6666	13.7569	20.6356	13.6121	7.0146	52.6509	0.3074	7.7414
	$p = 0.3$	0.6628	6.6089	9.9710	6.5360	7.2159	22.4781	0.3800	3.0240
	$p = 0.6$	0.6279	3.0346	4.8324	4.3744	7.3637	11.9658	0.4229	1.5829
	$p = 0.9$	0.5589	1.5723	2.8133	3.6481	8.0139	8.4837	0.4229	1.0987
$\lambda_2 = 8$	$p = 0.01$	0.6666	14.8765	22.3148	22.9549	10.6135	148.7310	0.2072	13.9700
	$p = 0.1$	0.6666	14.3656	21.5484	13.7963	11.2736	85.3069	0.2940	7.8642
	$p = 0.3$	0.6661	7.9218	11.8917	6.7085	11.6779	37.0417	0.3799	3.1390
	$p = 0.6$	0.6514	3.5409	5.4350	4.4186	11.6559	19.1880	0.4223	1.6124
	$p = 0.9$	0.6007	1.7735	2.9519	3.6542	12.3237	13.3857	0.4223	1.1028

Numerical results for the matrix R and for the vector $\vec{P}_0$ are presented below, for $N = 5$ and $N = 10$, with parameters values $\lambda_1 = 2$, $\mu_1 = 3$, $\lambda_2 = 5$, $\mu_2 = 2$, and $p = 0.1$. Note that the elements r_{ij}, $(i, j = 0, 1, \ldots, 4)$ of R when $N = 5$ are identical to those when $N = 10$. This follows from equation (4.8).

For $N = 5$, we get

$$R = \begin{pmatrix} 0.2137 & 0.1306 & 0.0833 & 0.0557 & 0.0391 & 0.0378 \\ 0 & 0.2268 & 0.1334 & 0.0825 & 0.0541 & 0.0487 \\ 0 & 0 & 0.2401 & 0.1357 & 0.0815 & 0.0669 \\ 0 & 0 & 0 & 0.2537 & 0.1376 & 0.1005 \\ 0 & 0 & 0 & 0 & 0.2674 & 0.1704 \\ 0 & 0 & 0 & 0 & 0 & 0.6667 \end{pmatrix},$$

$$\vec{P}_0 = \left(0.00113,\ 0.00359,\ 0.01100,\ 0.03137,\ 0.08294,\ 0.20330\right).$$

For $N = 10$, we get

$$R = \begin{pmatrix} 0.2137 & 0.1306 & 0.0833 & 0.0557 & 0.0391 & 0.0287 & 0.0219 & 0.0172 & 0.0137 & 0.0112 & 0.0136 \\ 0 & 0.2268 & 0.1334 & 0.0825 & 0.0541 & 0.0374 & 0.0272 & 0.0206 & 0.0160 & 0.0128 & 0.0152 \\ 0 & 0 & 0.2401 & 0.1357 & 0.0815 & 0.0523 & 0.0357 & 0.0257 & 0.0193 & 0.0149 & 0.0173 \\ 0 & 0 & 0 & 0.2537 & 0.1376 & 0.0802 & 0.0504 & 0.0339 & 0.0242 & 0.0181 & 0.0201 \\ 0 & 0 & 0 & 0 & 0.2674 & 0.1391 & 0.0786 & 0.0484 & 0.0322 & 0.0228 & 0.0242 \\ 0 & 0 & 0 & 0 & 0 & 0.2814 & 0.1402 & 0.0768 & 0.0464 & 0.0305 & 0.0302 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2954 & 0.1407 & 0.0748 & 0.0443 & 0.03986 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3094 & 0.1409 & 0.0726 & 0.0570 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3235 & 0.1406 & 0.0918 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3376 & 0.1753 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6667 \end{pmatrix},$$

$$\vec{P}_0 = \big(0.0000282, 0.0000896, 0.0002743, 0.0007822, 0.0020676, 0.0050676, 0.0115328, 0.0244097, 0.0481318, 0.0885722, 0.1523773\big).$$

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