A Note on the $M/M/\infty$ Queue in Random Environment

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Abstract

We show that, for the $M/M/\infty$ queue in a Markovian random environment, the joint probability distribution function (pdf) of the two variables: random phase and number of customers in the system, is equal to the product of the corresponding marginal pdfs *if and only if* the *ratio* of arrival rate to service rate is the *same* for all phases. We explicitly derive this joint pdf. Furthermore, for the general case, we calculate the conditional and overall mean queue lengths.

1 Introduction

Steady-state $M/M/*$ queues in random environment have been long studied in the literature. Early models considered the $M/M/1$ queue with an underlying n-phase continuous-time Markov chain (MC) as its random environment. That is, when in phase (environment) i, the system acts as a $M(\lambda_i)/M(\mu_i)/1$ queue with arrival and service rate λ_i and μ_i, respectively. In Yechiali and Naor [8], and recently in Gupta, Wolf, Harchol-Balter and Yechiali [4], the case with $n = 2$ was analyzed. Yechiali [9] studied the case where $2 \leq n < \infty$. Neuts [6] formulated Markovian random environment systems via a matrix-geometric approach and studied both the $M/M/1$ and the $M/M/c$ queues. However, when considering the $M/M/\infty$ queue, he indicated that such a system is 'surprisingly resistent to analytic solution' and advocated a truncated brute force numerical solution. O'cinneide and Purdue [7] further studied the n-phase $M/M/\infty$ queue and showed that a necessary and sufficient condition for stability is that at least one of the service rates should be positive. For the special case of $M/M/\infty$ queue with a Markov modulated arrival process, Keilson and Servi [5] showed that a decomposition property holds and provided an explicit solution. Recently, Baykel-Gursoy and Xiau [1] discussed the 2-phase $M/M/\infty$ queue and provided explicit solution, using Kumar functions. Lately, D'Auria [2] considered the $M/M/\infty$ queue in a special 2-phase (on-off) environment where service stops upon system failure (no change in the arrival rate) and derived the tail of the number of customers in the system for the case where the distribution of the off periods is heavy tailed. Furthermore, in [3] the author considered a quasi-Markovian random environment and developed a recursive formula that allows to compute all the factorial moments for the number of customers in the system in steady state.

We note that the $M/M/\infty$ queue in a random environment may also serve as a model for related transportation systems where the $M/M/\infty$ queue represents a delay line such as a highway, whose crossing time by vehicles may be affected by random road conditions such as weather or an accident (see e.g. [1]).

In this Note we consider the $M/M/\infty$ queue under a (general) n-phase continuous time MC random environment and show that the joint probability distribution function (pdf) of the environment phase and the number of customers in the system is equal to the product of the corresponding marginal pdfs *if and only if* the *ratio* of arrival rate to service rate is the *same* for all phases. We explicitly derive this joint probability distribution function. For the general case with arbitrary arrival and service rates in each phase we derive a linear set of n equations determining the system's conditional mean queue sizes, given its underlying environmental phase. The mean total queue size readily follows.

2 The model and balance equations

Consider an M/M/∞ type queue operating in 'random environment' for which the underlying process is an n-dimensional continuous-time MC. That is, when the process is in phase i, the system operates as an $M(\lambda_i)/M(\mu_i)/\infty$ queue, with Poisson arrival rate λ_i and service rate μ_i by each server.

The duration of time the MC (and the system) stays in phase i is an exponentially distributed random variable with mean $1/\eta_i$. When the system ends its sojourn period in phase i, it jumps (instantaneously) to phase j with probability q_{ij} $(i, j = 1, 2, \ldots, n)$, $\sum_{j=1}^{n} q_{ij} = 1 \ \forall i$. We denote the phase-transition matrix of the underlying MC by $Q \equiv [q_{ij}]$, and assume w.l.o.g. that $q_{ii} = 0 \ \forall i$.

A stochastic process $\{U(t), X(t)\}$ describes the system's state at time t as follows: $U(t)$ denotes the phase in which the system operates at time t, while $X(t)$ counts the number of customers present in the system at that time. The system is said to be in state (i, m) if it is in phase i, and there are m customers in the system. Accordingly, let p_{im} be the steady-state probability of the system in state (i, m). That is, $p_{im} = P(U(t) = i, X(t) = m) \ \forall t \geq 0$, $1 \leq i \leq n$, $m = 0, 1, 2, \ldots$.

Figure 1 below depicts a transition-rate diagram of the described queueing system.

The steady-state balance equations are given as follows:

For $i = 1, 2, \ldots, n$ and $m = 0$,

$$(\lambda_i + \sum_{j=1}^{n} \eta_i q_{ij}) p_{i0} = \mu_i p_{i1} + \sum_{j=1}^{n} \eta_j q_{ji} p_{j0} \tag{1}$$

For $i = 1, 2, \ldots, n$; $m \geq 1$,

$$(\lambda_i + m\mu_i + \sum_{j=1}^{n} \eta_i q_{ij}) p_{im} = \lambda_i p_{i,m-1} + (m+1)\mu_i p_{i,m+1} + \sum_{j=1}^{n} \eta_j q_{ji} p_{jm} \tag{2}$$

Define $\eta_{ij} = \eta_i q_{ij}$ and $\lambda_i = 0$ for $m < 0$. Then equations (1) and (2) can be written as

$$(\lambda_i + m\mu_i + \eta_i) p_{im} = \lambda_i p_{i,m-1} + (m+1)\mu_i p_{i,m+1} + \sum_{j=1}^{n} \eta_{ji} p_{jm} \quad \forall i = 1, 2, \ldots, n; \quad m \geq 0 \tag{3}$$

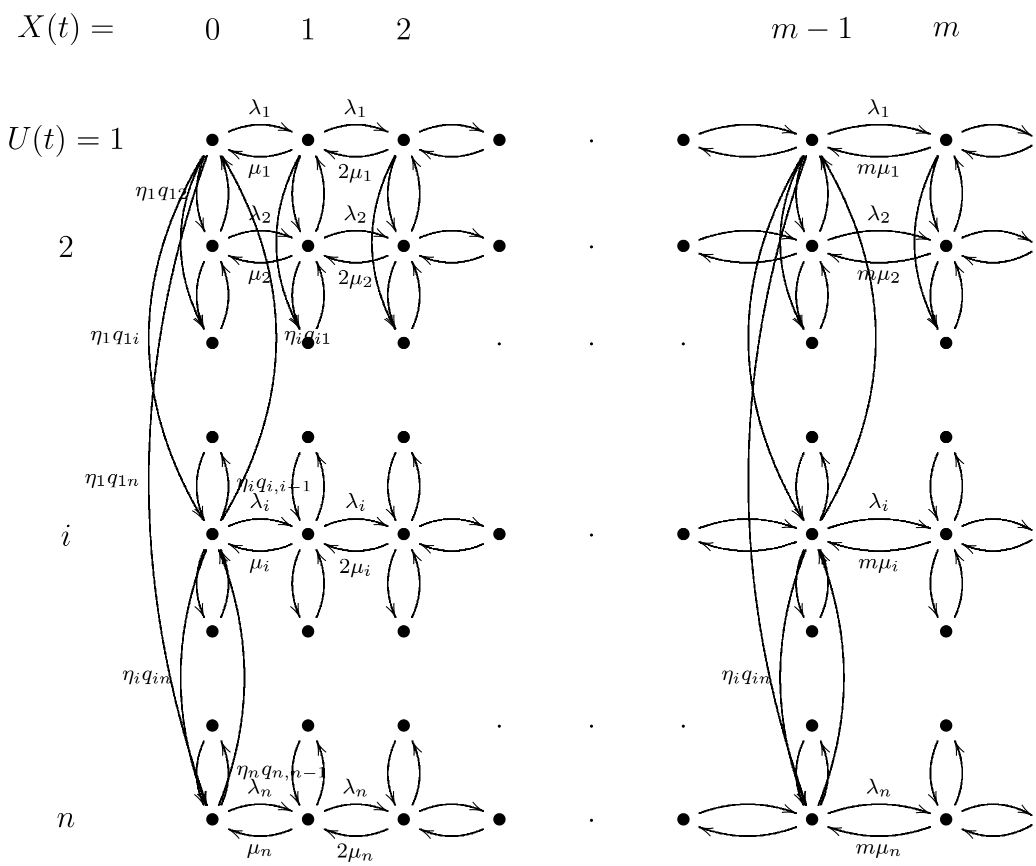

Figure 1: A transition-rate diagram of an infinite-server queueing system in random environment. Transitions between phases are shown only for $X(t) = 0$ and $X(t) = m - 1$.

Summing equation (3) over all m gives

$$\eta_i p_{i\bullet} = \sum_{j=1}^{n} \eta_{ji} p_{j\bullet} \tag{4}$$

where $p_{i\bullet} = \sum_{m=0}^{\infty} p_{im}$ and $\sum_{i=1}^{n} p_{i\bullet} = 1$.

The limit probabilities of the underlying Markov chain Q, describing just the random environment process, are $\pi_j = P(U(t) = j)$. The π_j's satisfy $\sum_{j=1}^{n} \pi_j = 1$ and $\pi_j = \sum_{i=1}^{n} \pi_i q_{ij}$, and are independent of the values of $\{\lambda_i\}$, $\{\mu_i\}$ and $\{\eta_i\}$. The proportion of time the system resides in phase i is given by

$$p_{i\cdot} = \frac{\dfrac{\pi_i}{\eta_i}}{\displaystyle\sum_{k=1}^{n} \frac{\pi_k}{\eta_k}} = \frac{\pi_i E[\text{Sojourn time in phase } i]}{\displaystyle\sum_{k=1}^{n} \pi_k E[\text{Sojourn time in phase } k]} \tag{5}$$

Evidently, the $\{p_{i\cdot}\}$ are independent of $\{\lambda_i\}$ and $\{\mu_i\}$.

3 Stability

Intuitively, the $M/M/\infty$ queue is inheritly stable. Indeed, O'cinneide and Purdue [7] showed that a necessary and sufficient condition for stability of a $M/M/\infty$ queue in a random environment is that "in at least one of the environments the service rate should be positive". Nevertheless, we now show *explicitly* that, under the above condition, our system regulates itself for *any* set of arrival and service rates.

Summing equation (3) over all i and canceling terms gives

$$\sum_{i=1}^{n} \lambda_i p_{im} = \sum_{i=1}^{n} (m+1)\mu_i p_{i,m+1} \quad m = 0, 1, 2, \ldots \tag{6}$$

Summing equation (6) over all m yields

$$\sum_{m=0}^{\infty} \sum_{i=1}^{n} \lambda_i p_{im} = \sum_{m=1}^{\infty} \sum_{i=1}^{n} m\mu_i p_{im} \tag{7}$$

After defining

$$\hat{\mu} = \sum_{i=1}^{n} \sum_{m=0}^{\infty} m\mu_i p_{im} = \sum_{i=1}^{n} \mu_i \sum_{m=0}^{\infty} m p_{im} = \sum_{i=1}^{n} \mu_i E[L_i]$$

where $E[L_i] = \displaystyle\sum_{m=0}^{\infty} m p_{im}$, and

$$\hat{\lambda} = \sum_{i=1}^{n} \sum_{m=0}^{\infty} \lambda_i p_{im} = \sum_{i=1}^{n} \lambda_i p_{i\cdot}$$

we get, using equation (7),

$$\begin{aligned}\hat{\mu} - \hat{\lambda} &= \sum_{i=1}^{n}\sum_{m=0}^{\infty} m\mu_i p_{im} - \sum_{i=1}^{n}\sum_{m=0}^{\infty}\lambda_i p_{im} \\ &= \sum_{m=1}^{\infty}\sum_{i=1}^{n} m\mu_i p_{im} - \sum_{i=1}^{n}\sum_{m=0}^{\infty}\lambda_i p_{im} \\ &= \sum_{m=0}^{\infty}\sum_{i=1}^{n}\lambda_i p_{im} - \sum_{i=1}^{n}\sum_{m=0}^{\infty}\lambda_i p_{im} = 0\end{aligned} \tag{8}$$

That is, in contrast with the $M/M/1$ queue in random environment, where stability holds if and only of $\hat{\mu} \equiv \sum_{i=1}^{n} \mu_i p_{i\bullet} > \hat{\lambda}$ (see e.g. Yechiali [9]), the $M/M/\infty$ queue in random environment is always stable.

4 Generating functions and mean queue sizes

We now use (partial) generating functions to express the unknown set of probabilities $\{p_{im}\}$.

Let

$$G_i(z) = \sum_{m=0}^{\infty} z^m p_{im} \quad i = 1, 2, \ldots, n \tag{9}$$

be the (partial) generating function of phase i. Multiplying both sides of equation (3) by z^m and summing over all m yield a system of n differential equations in the n unknowns $G_i(z)$:

$$\mu_i(1-z)G_i'(z) = (\lambda_i(1-z) + \eta_i)G_i(z) - \sum_{j=1}^{n} \eta_{ji} G_j(z) \quad i = 1, 2, \ldots, n \tag{10}$$

Equation (10) can be written as a matrix differential equation:

$$A(z)G'(z) = B(z)G(z)$$

where the matrices $A(z)$ and $B(z)$ are given by

$$A(z) = \begin{pmatrix} \mu_1(1-z) & 0 & 0 & \ldots & 0 \\ 0 & \mu_2(1-z) & 0 & \ldots & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & 0 & 0 & \ldots & \mu_n(1-z) \end{pmatrix}$$

and

$$B(z) = \begin{pmatrix} \lambda_1(1-z)+\eta_1 & -\eta_{21} & -\eta_{31} & \ldots & -\eta_{n1} \\ -\eta_{12} & \lambda_2(1-z)+\eta_2 & -\eta_{32} & \ldots & -\eta_{n2} \\ . & . & & & \\ . & & . & & \\ . & & & . & \\ -\eta_{1n} & -\eta_{2n} & -\eta_{3n} & \ldots & \lambda_n(1-z)+\eta_n \end{pmatrix}$$

and $G(z)$ is a n-dimensional column vector: $G(z) = (G_1(z), G_2(z), \ldots, G_n(z))^T$.

Note that $A(z)$ is singular at $z = 1$. However, for $0 \le z < 1$, the above can be written as

$$G'(z) = C(z)G(z)$$

where

$$
C(z) = A^{-1}(z)B(z)
$$

$$
= \begin{pmatrix}
\frac{\lambda_1(1-z)+\eta_1}{\mu_1(1-z)} & \frac{-\eta_{21}}{\mu_1(1-z)} & \frac{-\eta_{31}}{\mu_1(1-z)} & \cdots & \frac{-\eta_{n1}}{\mu_1(1-z)} \\
\frac{-\eta_{12}}{\mu_2(1-z)} & \frac{\lambda_2(1-z)+\eta_2}{\mu_2(1-z)} & \frac{-\eta_{32}}{\mu_2(1-z)} & \cdots & \frac{-\eta_{n2}}{\mu_2(1-z)} \\
. & . & & & \\
. & & . & & \\
. & & & . & \\
\frac{-\eta_{1n}}{\mu_n(1-z)} & \frac{-\eta_{2n}}{\mu_n(1-z)} & \frac{-\eta_{3n}}{\mu_n(1-z)} & \cdots & \frac{\lambda_n(1-z)+\eta_n}{\mu_n(1-z)}
\end{pmatrix} \tag{11}
$$

Apparently, there is no simple analytic solution to the above set. Indeed, Neuts ([6], page 274) states the following: "We note that the infinite-server queue $M/M/\infty$ in a Markovian environment is surprisingly resistent to analytic solution... Brute force numerical solution of a truncated version of the birth-and-death equations enables one to solve this model for a wide range of parameter values in spite of the lack of a mathematically elegant solution." However, we can calculate mean queue sizes as follows (see also [7] and [3]):

Differentiating equation (10) yields

$$
-\mu_i G_i'(z) + \mu_i(1-z)G_i''(z) = -\lambda_i G_i(z) + (\lambda_i(1-z)+\eta_i)G_i'(z) - \sum_{j=1}^{n} \eta_{ji} G_j'(z), \quad i = 1, 2, \ldots, n \tag{12}
$$

By setting $z = 1$ we get

$$
-\mu_i E[L_i] = -\lambda_i G_i(1) + \eta_i E[L_i] - \sum_{j=1}^{n} \eta_{ji} E[L_j] \tag{13}
$$

where $G_i'(1) = E[L_i] = \sum_{m=0}^{\infty} m p_{im}$.

Also, $G_i(1) = \sum_{m=0}^{\infty} p_{im} = p_{i\bullet}$. Then, from equation (13),

$$(\eta_i + \mu_i)E[L_i] - \sum_{j=1;j\neq i}^{n} \eta_{ji}E[L_j] = \lambda_i p_{i\bullet} \quad i = 1, 2, \ldots, n \tag{14}$$

The set (14) can be written in a matrix equation form

$$DE[\underline{\mathrm{L}}] = \underline{\mathrm{b}} \tag{15}$$

where the matrix D is given by

$$D = \begin{pmatrix} \eta_1 + \mu_1 & -\eta_{21} & -\eta_{31} & \ldots & -\eta_{n1} \\ -\eta_{12} & \eta_2 + \mu_2 & -\eta_{32} & \ldots & -\eta_{n2} \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ -\eta_{1n} & -\eta_{2n} & -\eta_{3n} & \ldots & \eta_n + \mu_n \end{pmatrix}$$

and the column vectors $E[\underline{\mathrm{L}}]$ and $\underline{\mathrm{b}}$ are given by $E[\underline{\mathrm{L}}] = (E[L_1], \ldots, E[L_n])^T$ and $\underline{\mathrm{b}} = (\lambda_1 p_{1\bullet}, \ldots, \lambda_n p_{n\bullet})^T$. Therefore, the solution of the system is given by

$$E[\underline{\mathrm{L}}] = D^{-1}\underline{\mathrm{b}} \tag{16}$$

The expected value of the total number of customers in the system is $E[L] = \sum_{i=1}^{n} E[L_i]$, and the mean sojourn time of an arbitrary customer is, by Little's law, $E[W] = \frac{1}{\hat{\lambda}}E[L]$, where $\hat{\lambda} = \sum_{i=1}^{n} \lambda_i p_{i\bullet}$.

Examples: a. When $n = 1$, i.e., when the system shrinks to a single phase, then $D = \Delta(\eta + \mu) = (\eta_1 + \mu_1)$, where $\Delta(a)$ is the diagonal matrix of a, for $a = (a_1, a_2, \ldots, a_n)^T$. For the single phase we have $\eta_{11} = \eta_1 = 0$ and $p_{1\bullet} = 1$. Using (15) we get

$$(\eta_1 + \mu_1)E[L_1] = \lambda_1 p_{1\bullet} \tag{17}$$

which leads to

$$E[L_1] = \frac{\lambda_1}{\mu_1} \tag{18}$$

Indeed, for an $M(\lambda_1)/M(\mu_1)/\infty$ system, the mean queue size is $E[L_1] = \dfrac{\lambda_1}{\mu_1}$.

b. When $n = 2$, equation (15) leads to

$$\begin{pmatrix} \eta_1 + \mu_1 & -\eta_{21} \\ -\eta_{12} & \eta_2 + \mu_2 \end{pmatrix} \begin{pmatrix} E[L_1] \\ E[L_2] \end{pmatrix} = \begin{pmatrix} \lambda_1 p_{1\bullet} \\ \lambda_2 p_{2\bullet} \end{pmatrix} \tag{19}$$

Multiplying both sides by the inverse matrix of D we get

$$\begin{pmatrix} E[L_1] \\ E[L_2] \end{pmatrix} = \frac{1}{(\eta_1 + \mu_1)(\eta_2 + \mu_2) - \eta_{12}\eta_{21}} \begin{pmatrix} \eta_2 + \mu_2 & \eta_{21} \\ \eta_{12} & \eta_1 + \mu_1 \end{pmatrix} \begin{pmatrix} \lambda_1 p_{1\bullet} \\ \lambda_2 p_{2\bullet} \end{pmatrix} \tag{20}$$

Hence, for $n = 2$,

$$\begin{pmatrix} E[L_1] \\ E[L_2] \end{pmatrix} = \frac{1}{(\eta_1 + \mu_1)(\eta_2 + \mu_2) - \eta_{12}\eta_{21}} \begin{pmatrix} \lambda_1 p_{1\bullet}(\eta_2 + \mu_2) + \lambda_2 p_{2\bullet}\eta_{21} \\ \lambda_2 p_{2\bullet}(\eta_1 + \mu_1) + \lambda_1 p_{1\bullet}\eta_{12} \end{pmatrix} \tag{21}$$

5 The case where $\lambda_i/\mu_i = c$ for all i

Of special interest is the case when the ratios between the arrival rate and the service rate, $\dfrac{\lambda_i}{\mu_i}$, are the same for all phases. We will show that if $\dfrac{\lambda_i}{\mu_i} = c$ for every phase i, then the system possesses properties of a standard $M/M/\infty$ queue, and an explicit simple solution can be derived. We state the following:

Theorem 1

$$p_{im} = p_{i\bullet} p_{\bullet m} = p_{i\bullet} e^{-c} \frac{c^m}{m!} \quad (i = 1, 2, \dots, n; \quad m = 0, 1, 2, \dots)$$

if and only if, for every i,

$$\frac{\lambda_i}{\mu_i} = c \tag{22}$$

where $p_{i\bullet}$ *is given by equation (5), and* $p_{\bullet m} = \sum_{i=1}^{n} p_{im}$.

Before proceeding with a formal proof we note that having the same ratio of $\frac{\lambda_i}{\mu_i}$ in all phases is probabilistically equivalent to scaling the time differently when the system stays in different phases. This scaling does not change the distribution of the *number* of jobs in the $M/M/\infty$ queue.

Proof: The proof will be carried out in three steps via a sequence of lemmas.

Lemma 2 *If* $\frac{\lambda_i}{\mu_i} = c$*, then*

$$p_{im} = p_{i\bullet} e^{-c} \frac{c^m}{m!} \quad i = 1, \ldots, n; \quad m = 0, 1, 2, \ldots \tag{23}$$

Proof: Assume $\frac{\lambda_i}{\mu_i} = c \; \forall i$. Adding same terms to both sides of (4) we write

$$(\lambda_i + m\mu_i + \eta_i) p_{i\bullet} = m\mu_i p_{i\bullet} + \lambda_i p_{i\bullet} + \sum_{j=1}^{n} \eta_{ji} p_{j\bullet} \tag{24}$$

Multiplying by c^m and using the assumption $\frac{\lambda_i}{\mu_i} = c$ yields

$$(\lambda_i + m\mu_i + \eta_i) p_{i\bullet} c^m = m\lambda_i p_{i\bullet} c^{m-1} + \mu_i p_{i\bullet} c^{m+1} + \sum_{j=1}^{n} \eta_{ji} p_{j\bullet} c^m \tag{25}$$

Dividing by $m!$ and multiplying by e^{-c} leads to

$$(\lambda_i + m\mu_i + \eta_i) p_{i\bullet} e^{-c} \frac{c^m}{m!} = \lambda_i p_{i\bullet} e^{-c} \frac{c^{m-1}}{(m-1)!} + (m+1)\mu_i p_{i\bullet} e^{-c} \frac{c^{m+1}}{(m+1)!} + \sum_{j=1}^{n} \eta_{ji} p_{j\bullet} e^{-c} \frac{c^m}{m!} \tag{26}$$

Setting $p_{im} = p_{i\bullet} e^{-c}\frac{c^m}{m!}$ in (26) leads to the steady-state balance equation (3). Since equations (3) and (4) possess a unique solution, then $p_{im} = p_{i\bullet} e^{-c}\frac{c^m}{m!}$ is the one, and the proof is complete. ■

Lemma 3 *If* $\frac{\lambda_i}{\mu_i} = c$*, then* $p_{im} = p_{i\bullet} p_{\bullet m}$ *for* $i = 1, 2, \ldots, n$*;* $m \geq 0$.

Proof: Assume $\frac{\lambda_i}{\mu_i} = c$. Then, using lemma 2 and summing equation (23) over all i leads to

$$p_{\bullet m} = e^{-c}\frac{c^m}{m!} \quad \forall m \tag{27}$$

Substituting (27) in (23) completes the proof. ■

Lemma 4 *If* $p_{im} = p_{i\bullet} p_{\bullet m}$*,* $i = 1, \ldots, n$*;* $\forall m$*, then, for all* $m \geq 0$*,*

$$p_{\bullet m} = e^{-\frac{\lambda_i}{\mu_i}}\frac{(\frac{\lambda_i}{\mu_i})^m}{m!}, \quad i = 1, \ldots, n$$

and

$$\frac{\lambda_i}{\mu_i} = c, \quad i = 1, \ldots, n$$

Proof: Substituting $p_{im} = p_{i\bullet} p_{\bullet m}$ in equation (3) gives

$$(\lambda_i + m\mu_i + \eta_i) p_{i\bullet} p_{\bullet m} = \lambda_i p_{i\bullet} p_{\bullet, m-1} + (m+1)\mu_i p_{i\bullet} p_{\bullet, m+1} + \sum_{j=1}^{n} \eta_{ji} p_{j\bullet} p_{\bullet m} \tag{28}$$

By using equation (5) and the definition $\eta_{ij} = \eta_i q_{ij}$ we get

$$(\lambda_i + m\mu_i + \eta_i)\frac{\pi_i}{\eta_i} p_{\bullet m} = \lambda_i \frac{\pi_i}{\eta_i} p_{\bullet, m-1} + (m+1)\mu_i \frac{\pi_i}{\eta_i} p_{\bullet, m+1} + \sum_{j=1}^{n} \eta_j q_{ji} \frac{\pi_j}{\eta_j} p_{\bullet m} \tag{29}$$

Multiplying both sides of (29) by η_i and using $\pi_j = \sum_{i=1}^{n} \pi_i q_{ij}$ yields

$$(\lambda_i + m\mu_i + \eta_i)\pi_i p_{\bullet m} = \lambda_i \pi_i p_{\bullet, m-1} + (m+1)\mu_i \pi_i p_{\bullet, m+1} + \eta_i \pi_i p_{\bullet m} \tag{30}$$

Dividing by π_i leads to

$$(\lambda_i + m\mu_i)p_{\bullet m} = \lambda_i p_{\bullet,m-1} + (m+1)\mu_i p_{\bullet,m+1} \tag{31}$$

Equations (31) are the steady-state balance equations of a standard $M/M/\infty$ queue. That is, the marginal distribution of L, given $U(t) = i$, is Poissonian, namely,

$$p_{\bullet m} = e^{-\frac{\lambda_i}{\mu_i}} \frac{(\frac{\lambda_i}{\mu_i})^m}{m!} \tag{32}$$

Since, by Lemma 3, $p_{\bullet m}$ is independent of the phase i, we must have that $\dfrac{\lambda_i}{\mu_i} = c$ for all i. This completes the proof. ■

Lemmas 3 and 4 now complete the proof of Theorem 1. ■

Corollary 5 *If* $\dfrac{\lambda_i}{\mu_i} = c$, $i = 1, \ldots, n$, *then the mean total number of customers in the system is given by* $E[L] = c$

Proof: Assume $\dfrac{\lambda_i}{\mu_i} = c$, $i = 1, \ldots, n$. From (27),

$$p_{\bullet m} = e^{-c}\frac{c^m}{m!} \quad \forall m$$

Thus,

$$E[L] = \sum_{m=0}^{\infty} m p_{\bullet m} = \sum_{m=0}^{\infty} m e^{-c}\frac{c^m}{m!} = c$$

■

6 Extreme cases of η_i

We now investigate two extreme cases relating to the values of the η_i's.

a. Consider the case where, for some i, $\eta_i \to 0$, but $\eta_j > 0\ \forall j \neq i$. Then using equation (5)

$$p_{i\bullet} = \frac{\dfrac{\pi_i}{\eta_i}}{\dfrac{\pi_i}{\eta_i} + \displaystyle\sum_{k=1;k\neq i}^{n} \dfrac{\pi_k}{\eta_k}} = \frac{\pi_i}{\pi_i + \eta_i \displaystyle\sum_{k=1;k\neq i}^{n} \dfrac{\pi_k}{\eta_k}} \xrightarrow[\eta_i \to 0]{} 1$$

Similarly, $p_{j\bullet} \xrightarrow[\eta_i \to 0]{} 0$ for every $j \neq i$.

Indeed, when $\eta_i \to 0$ the system (almost) always stays in phase i, and the proportion of time it stays in another phase tends to 0.

b. Suppose $\eta_i \to \infty$, while $\eta_j > 0\ \forall j \neq i$. Again, using equation (5), we have

$$p_{i\bullet} = \frac{\dfrac{\pi_i}{\eta_i}}{\dfrac{\pi_i}{\eta_i} + \displaystyle\sum_{k=1;k\neq i}^{n} \dfrac{\pi_k}{\eta_k}} = \frac{\pi_i}{\pi_i + \eta_i \displaystyle\sum_{k=1;k\neq i}^{n} \dfrac{\pi_k}{\eta_k}} \xrightarrow[\eta_i \to \infty]{} 0$$

$$p_{j\bullet} = \frac{\dfrac{\pi_j}{\eta_j}}{\dfrac{\pi_i}{\eta_i} + \displaystyle\sum_{k=1;k\neq i}^{n} \dfrac{\pi_k}{\eta_k}} \xrightarrow[\eta_i \to \infty]{} \frac{\dfrac{\pi_j}{\eta_j}}{\displaystyle\sum_{k=1;k\neq i}^{n} \dfrac{\pi_k}{\eta_k}} \quad j \neq i$$

That is, the proportion of time the system stays in phase i tends to 0 and the system behaves as if it consists of only $(n-1)$ phases.

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