# ON OPTIMAL BALKING RULES AND TOLL CHARGES IN THE $GI/M/1$ QUEUING PROCESS

**Uri Yechiali**

*New York University, New York, N.Y.*

(Received November 1, 1969)

This paper considers a $GI/M/1$ queuing process with an associated linear cost-reward structure and stationary balking process, and, based on a probabilistic analysis of the system, it derives optimal joining rules for an individual arrival, as well as for the entire community of customers. For the infinite-horizon, average-reward criterion, it shows that, among all stationary policies, the optimal strategies are control-limit rules of the form: join if and only if the queue size is not greater than some specific number. However, it finds that, in general, exercising self-optimization does not optimize public good. Accordingly, the paper explores the idea of controlling the queue size by levying tolls—thus achieving the system's over-all-optimal economic performance. Finally, it analyzes a 'competition' model in which customers face a service agency that is a profit-making organization, and shows it to be similar to the monopoly model of price theory.

WE CONSIDER a $GI/M/1$ queuing system with a stationary balking process. Our main objectives are three: (1) to find optimal balking rules for arriving customers—a point of view that has been generally overlooked; (2) to find optimal toll charges as a measure for controlling the queue size—an idea discussed qualitatively by LEEMAN,[13, 14] and SAATY,[17] and applied to the $M/M/1$ queue by Naor;[16] and (3) to find optimal service charges for a profit-making service agency in a 'competition' model between the station and the customers.

Consider a $GI/M/1$ queuing process where customers arrive at instants $\tau_0, \tau_1, \tau_2, \cdots, \tau_n, \cdots$ and the interarrival times $\tau_{n+1}-\tau_n (n=0, 1, 2, \cdots)$ have common distribution $H(\cdot)$ with finite mean $1/\lambda$. The arrivals that join the queue are served by a single server according to the first-come, first-served discipline, and the service times are exponentially distributed with parameter $\mu$. Denote by $\eta(t)$ the queue size (customers waiting and/or being served) at instant $t$, and let $\eta_n=\eta(\tau_n-0)$. We say that the system is in state $i$ at the $n$th step if $\eta_n=i$. Now, suppose that an arrival who finds the system in state $i$ is *not* obliged to join the queue and may decide to balk. Thus, generally speaking, we regard the system as being 'observed' at instants $\tau_0, \tau_1, \tau_2, \cdots$ to be in some state $i\epsilon I=\{0, 1, 2, \cdots\}$, and whenever the system is observed in state $i\epsilon I$ an action $k$ from a set $K_i$ of actions is taken, where, for the balking model, we restrict ourselves to only two actions at each state, i.e., an arrival can either join or balk.

Denote by $k=1$ the action of joining and by $k=0$ the action of balking. Then $K_i=\{0, 1\}$ for all $i\epsilon I$.

As noted, our objective will be to find optimal joining and balking rules for arriving customers (under the long-run average-reward criterion). However, we assume that the only information available to a newly arriving customer is the current state of the system. This assumption, together with the Markovian property of the service times, amounts to considering only the so-called stationary Markovian policies.[6] Let $\{\Delta_n\}(n=0, 1, 2, \cdots)$ denote the sequence of successive decisions made by the arriving customers, where $\Delta_n=0$ or 1 according to whether the $n$th customer balks or joins the system, respectively. Let $D_{ik}$ denote the stationary conditional probability of taking action $k\epsilon K_i$ when state $i$ is observed. That is, $D_{ik}=P(\Delta_n=k|\eta_n=i)$ $(n=0, 1, 2, \cdots)$. Since $k=0$ or 1 for all $K_i$ let $D_{i1}=D_i$ and let $D_{i0}=1-D_i$. The queuing and decision process $\{\eta_n, \Delta_n\}$ $(n=0, 1, 2, \cdots)$ belongs to the set of processes generally known as Markovian decision processes.[5, 6] Denote the transition probabilities of this process by:

$$q_{ij}(k)=P\{\eta_{n+1}=j|\eta_n=i, \quad \Delta_n=k\}. \qquad (i, j=0, 1, 2, \cdots; k=0, 1) \quad (1)$$

The $\{q_{ij}(1)\}$'s are given by[11,19]

$$\begin{aligned} &q_{ij}(1)=a_{i+1-j}, \; i+1\geqq j\geqq 1, \\ &q_{ij}(1)=0, \; j>i+1, \\ &q_{i0}(1)=\textstyle\sum_{k=i+1}^{k=\infty} a_k=1-\sum_{k=0}^{k=i} a_k=r_i, \qquad (i=0, 1, 2, \cdots) \end{aligned} \quad (2)$$

where

$$a_k=\int_0^\infty e^{-\mu v}[(\mu v)^k/k!]dH(v). \qquad (k=0, 1, 2, \cdots) \quad (3)$$

It is also easy to see that

$$\begin{aligned} &q_{ij}(0)=q_{i-1, j}(1) \quad \text{for} \quad i=1, 2, \cdots, \qquad j=0, 1, 2, \cdots, \\ &q_{00}(0)=1. \end{aligned} \quad (4)$$

For any given set of joining probabilities, $\{D_i\}$, the sequence of random variables $\{\eta_n\}$, forms a homogeneous Markov chain (imbedded at instants of arrival) with transition probabilities

$$p_{ij}=\textstyle\sum_{k\epsilon K_i} q_{ij}(k)D_i=q_{ij}(1)D_i+q_{ij}(0)(1-D_i). \qquad (i, j=0, 1, 2, \cdots) \quad (5)$$

Let $\pi_i=\lim_{n\to\infty} P\{\eta_n=i\}(i=0, 1, 2, \cdots)$. These limits always exist and they are all nonnegative.[7]

It is well known[19] that, for an ergodic chain, the $\{\pi_i\}$ form a distribution function and are uniquely determined by the system of linear equations:

$$\pi_j = \sum_{i\epsilon I} \pi_i P_{ij} \quad \text{for} \quad j\epsilon I, \quad \text{and} \quad \sum_{j\epsilon I} \pi_j = 1. \qquad (6)$$

A detailed analysis of problems of recurrence and transiency for the general balking process may be found in reference 9, where the process is imbedded at instants of joining rather than at instants of arrival. Additional results about the relations between the limiting probabilities and mean queue sizes of the above two imbedded Markov chains may be found in reference 21.

## THE $GI/M/1/n$ QUEUE

SUPPOSE NOW that the service facility has a limited waiting room of size $n$. That is, there could be at most $n+1$ customers in the system including the one in service. An arrival who finds more than $n$ customers ahead of him balks with probability one. Suppose also that customers who find the system in state $i \leqq n$ join the queue with positive probability. In summary, this situation is equivalent to having $\{D_i\}$ such that $0 < D_i \leqq 1$ for $i=0, 1, 2, \cdots, n$ and $D_i = 0$ for all $i \geqq n+1$. We denote this queuing process by $GI/M/1/n$.

The transition matrix of the $GI/M/1/n$ queue (imbedded at instants of arrival) is of order $n+2$ and is mainly derived from (5). The only modification is made on the $(n+2)$nd row. We have:

$$\begin{aligned} p_{ij} &= q_{ij}(1) D_i + q_{ij}(0)(1-D_i), \\ & \qquad (i=0, 1, \cdots, n; j=0, 1, \cdots, n+1) \quad (7) \\ p_{n+1,j} &= q_{n+1,j}(0). \qquad (j=0, 1, \cdots, n+1) \end{aligned}$$

The finite Markov chain thus defined is irreducible and aperiodic, and therefore has a single ergodic class (independent of the relative values of $\lambda$ and $\mu$) with $n+2$ positive recurrent states. We may say that the process 'regulates' itself by forcing ultimate balking whenever the queue size is beyond a given limit.

Let $I_n = \{0, 1, 2, \cdots, n, n+1\}$. Denote by

$$\pi_i(n) = \lim_{m\to\infty} P\{\eta_m = i\} \qquad (i\epsilon I_n) \quad (8)$$

the limiting probabilities of the imbedded $GI/M/1/n$ queuing process. Clearly, the terms of $\{\pi_i(n)\}$ are all positive and satisfy (6) with $I_n$ replacing $I$.

Moreover, FINCH[8] has shown that the terms of $\{\pi_i(n)\}$ are given by

$$\pi_i(n) = q_{n+1-k} / [\textstyle\sum_{j=0}^{n+1} q_j], \qquad (i\epsilon I_n) \quad (9)$$

where the terms of $\{q_k\}(k=0, 1, 2, \cdots)$ can be obtained successively from:

$$q_0=1, \qquad q_k=a_0\beta_{k+1}q_{k+1}+\sum_{r=0}^{k-1}[a_{r+1}\beta_{k-r}+a_r(1-\beta_{k-r})]q_{k-r}+a_kq_0, \quad (k=0, 1, 2, \cdots)$$

with $\beta_k=D_{n+1-k}$ for $k=0, 1, 2, \cdots, n+1$.

We now state a theorem that will be useful later on, the proof of which may be found in reference 21.

THEOREM 1. *For the stationary process, the conditional distribution function of the waiting time W (time from arrival until the start of service) of an arbitrary customer, given that he joins the queue, is given by*

$$F_W(x)=1-[1/\sum_{i=0}^{i=n}D_i\pi_i(n)][\sum_{j=1}^{j=n}D_j\pi_j(n)(\sum_{k=0}^{j-1}e^{-\mu x}(\mu x)^k/k!)], \tag{10}$$

*and the conditional expected waiting time is*

$$EW=[\sum_{j=1}^{j=n}jD_j\pi_j(n)]/[\mu\sum_{i=0}^{i=n}D_i\pi_i(n)]. \tag{11}$$

*Remark 1.* Equations (10) and (11) may be extended for the $GI/M/1$ queue as well. It is easy to see that the only modification needed is to replace $n$ by $\infty$ in the summation signs for $i$ and $j$.

### *The Effect of the Capacity n on the System*

We now consider $n$ as a controllable parameter and examine its effect on the limiting probabilities and the mean queue size of the process. Define a special balking rule called *deterministic control limit rule n* and denoted by $R_n^D$ as: $R_n^D=\{D_i:D_i=1,\ i\leqq n;\ D_i=0,\ i>n\}$. This rule corresponds to setting the waiting-room capacity at level $n$ and balking if and only if the system is full. We have the following:

LEMMA 1. *For any $n\geqq 0$, rules $R_n^D$ and $R_{n+1}^D$, and every $i\epsilon I_n$,*

$$\pi_i(n)-\pi_{i+1}(n+1)=\pi_i(n)\pi_0(n+1). \tag{12}$$

This is an immediate result of relation (9).

Let $L(n)$ denote the mean queue size of the imbedded Markov chain when rule $R_n^D$ is employed. Using (12), we obtain the following result readily:

LEMMA 2. $L(n+1)=[1-\pi_0(n+1)][L(n)+1]$.

The proof of Lemma 3 and an inductive proof of Theorem 2 [using relations (6) and (7)] may also be found in reference 21:

LEMMA 3. *For any $n\geqq 0$ and rules $R_n^D$ and $R_{n+1}^D$, we have*

$$\pi_0(n)>\pi_0(n+1). \tag{13}$$

THEOREM 2. *For any $n\geqq 0$ and rules $R_n^D$ and $R_{n+1}^D$, $L(n+1)>L(n)$.*

COROLLARY 1. $1-\pi_0(n+1)>\pi_0(n+1)L(n)$.

This is an immediate consequence of Lemma 2 and Theorem 2.

## OPTIMAL CUSTOMERS' BALKING RULES

WE DIFFER NOW from many studies of queuing systems for which either it is assumed that every arrival joins the queue with probability one, or it is supposed that a specific balking rule is given in advance: We will be interested in *finding* optimal balking and joining rules for the customers in the $GI/M/1$ queuing process. When saying 'optimal' we mean optimal according to some economic criterion, and for this reason we will subsequently impose a cost structure on the system and define several objective functions. The problems we will be concerned with are: (i) When should an individual customer join the queue? What is the structure of his optimal joining or balking rules, and how are these rules affected by the cost parameters? (ii) What is the structure of the optimal policy when the customers are organized in order to achieve public optimization—that is, long-run average net benefit per individual—and how can such an optimal policy be found? (iii) What is the relation between these two procedures? Or, does self optimization bring public or social optimization?

### *The Cost Structure*

We make the following assumptions for the cost structure:

(A) We suppose that, upon successful completion of service, the customer obtains a nonnegative finite *reward* of $G$ monetary units. In fact, the existence of such a reward is what attracts customers to the counter.

(B) There is a finite service charge $\theta$ that has to be paid (to the service station) by every customer who passes through service. We denote by $g=G-\theta$ the *net reward* of a customer who has been served.

(C) There are two types of costs associated with the two different decisions that can be made by a newly-arriving customer: (i) If he joins the queue, then there are waiting-time losses incurred at the rate of $c$ monetary units per customer ($\infty > c \geqq 0$) for every unit time spent at the system. (ii) If the customer decides to balk, that is, not join the line, then a penalty of $l$ monetary units is incurred ($\infty > l \geqq 0$).

We note here that, because of the Markovian properties of the model, and since join-or-balk decisions are made only at times of arrival, reneging would never be optimal, i.e., once joining, a customer leaves the system only after his service is completed.

(D) As is usually supposed in queuing models, we assume that the set of costs $\{G, \theta, c, l\}$ is the same for *all* customers.

(E) In order to eliminate trivialities, we assume that $g-(c/\mu) \geqq -l$. This last assumption will be clarified later.

### *The Decision Process*

As was pointed out above, our process is a special Markovian decision process. In general, a *policy* $R$ for controlling the system is a set of functions $\{D_k^R(H_{m-1}, \eta_m)\}$, $m=0, 1, \cdots$, where $H_m=\{\eta_0, \Delta_0, \cdots, \eta_m, \Delta_m\}$, $D_k^R(\cdot)\geqq 0$, and $\sum_{k\epsilon K_{\eta_m}} D_k^R(H_{m-1}, \eta_m)=1$; and where $D_k^R(H_{m-1}, \eta_m)$ is to be interpreted as the probability of implementing decision $k$ at time $m$ given the *history* $H_{m-1}$ and the present state $\eta_m$. However, in all the preceding sections we have assumed that $D_k^R(H_{m-1}, \eta_m|\eta_m=i)=D_{ik}$ for every $m=0, 1, \cdots$, and thus we have obtained the Markov chains represented by (5) or (7).

Let $W_m$, $m=0, 1, \cdots$, denote the reward obtained at time $\tau_m$, defined as follows:

$$W_m=w_{ik} \quad \text{if} \quad \eta_m=i \quad \text{and} \quad \Delta_m=k. \qquad (k\epsilon K_i;\ i\epsilon I) \quad (14)$$

Given a policy $R$ and an initial state $\eta_0=i$, then the sequence $\{W_m\}$, $m=0, 1, 2, \cdots$, is a stochastic process, for which the expected reward at instant $\tau_m$ is

$$E_R W_m=\sum_j \sum_k w_{jk} P_R\{\eta_m=j,\ \Delta_m=k|\eta_0=i\}, \qquad (15)$$

where $E_R$ and $P_R$ denote the expectation and probability under the policy $R$. Let

$$\phi_{R,T}(i)=[1/(T+1)]\sum_{m=0}^{m=T} E_R W_m,$$

i.e., $\phi_{R,T}(i)$ is the average expected reward incurred by the system up to time $\tau_T$, given that $\eta_0=i$ and $R$ is the policy controlling the system. Let

$$\phi_R(i)=\lim_{T\to\infty} \sup \phi_{R,T}(i).$$

*For public optimization our problem is that of finding $R$ to maximize $\phi_R(i)$ for all $i$.*

For what follows, it is convenient to consider several subclasses of the general class $C$ which contains *all* policies of the form $\{D_k^R(H_{m-1}, \eta_m)\}$. We first consider the class of *stationary Markovian* policies that uses, at each point in time (e.g., instant of arrival), only the state of the system at that instant as a basis for making a decision. We denote this class by $C_S$. In our case, these rules are represented by the $D_{ik}$'s introduced before, and we recall that, since $K_i=\{0, 1\}$ for every $i\epsilon I$, it suffices to specify $D_{i1}=D_i$ for all $i\epsilon I$.

Next, we define, for any fixed $k$, a class $S_k$ of all stationary policies such that $1\geqq D_i>0$ for all $i\leqq k$ and $D_i=0$ for all $i>k$. We refer to $S_k$ as the class of *stationary control-limit rules* of order $k$. The class of control-limit rules of infinite order will be denoted by $S$; that is, $R\epsilon S$ if and only if

$$R=\{D_i\colon D_i>0 \quad \text{for all} \quad i=0, 1, \cdots\}.$$

Thus, the (extended) class of all control-limit rules of all orders is defined by

$$C_{CL}=(\cup_{k\epsilon I} S_k)\cup S, \quad \text{where} \quad I=\{0, 1, \cdots\}.$$

Note that, if $k$ is specified, then the system in its steady-state equilibrium can occupy only the $k+2$ states $0, 1, \cdots, k+1$. That is, $\pi_i=0$ for all $i>k+1$, and thus the effective state space is $I_k=\{0, 1, \cdots, k+1\}$.

Another subclass of $C_S$ is the class $C_D$ of nonrandomized policies, where $D_{ik}=0$ or 1 for all $i$ and $k$.

As a last class, we consider a subclass of $C_D$ which is also a subclass of $C_{CL}$. This is the class of all *deterministic* control-limit rules. We denote this class by $C_{DCL}$. By a (finite) rule $R_k^D \epsilon C_{DCL}$ we mean

$$R_k^D=\{D_i: D_i=1,\ i\leqq k;\ D_i=0,\ i>k\}$$

for some $k$, and by $R_\infty^D \epsilon C_{DCL}$ we mean

$$R_\infty^D=\{D_i: D_i=1 \quad \text{for all} \quad i=0, 1, \cdots\}.$$

Note that for each $k$ there is one and only one deterministic control-limit rule $R_k^D$, and hence $C_{DCL}=\{R_k^D,\ k=0, 1, \cdots\}\cup R_\infty^D$.

We now proceed to find optimal balking procedures for our queuing model.

### *Customer Self-Optimization*

We consider now the decision problem of an individual customer who arrives at the counter and ponders whether to join or not. The problem of interest is thus to find a set of joining probabilities $\{D_k^R(\cdot)\}$ such that the customer's expected net benefit will be maximized. Suppose a customer arrives and finds $i (i=0, 1, \cdots)$ customers ahead of him. If he balks, he incurs a penalty $l$ and consequently his net benefit is $-l$. If he decides to join the queue, he will have to wait for a time period equal to his own service time and the service times of the $i$ customers ahead of him, and only then will he obtain the reward $g$. Recall that a customer who joins the line never leaves before service completion. Since the service time is exponential with parameter $\mu$, the customer's expected total waiting time in the system is $(i+1)/\mu$, and therefore his expected net benefit is $g-(c/\mu)(i+1)$. Thus, for each $i\epsilon I$ and every history $H_{m-1}$ and time $m=0, 1, \cdots$, we wish to find $\{D_k^R(H_{m-1}, i)\}$, $k=0, 1$, so as to maximize

$$\{D_1^R(H_{m-1}, i)\cdot[g-(c/\mu)(i+1)]-[1-D_1^R(H_{m-1}, i)]\cdot l\}.$$

If we define an *integer* $n_s$ such that

$$g-(c/\mu)(n_s+1)\geqq -l, \qquad g-(c/\mu)(n_s+2)<l, \tag{16}$$

then it is clear that the set

$$\{D_1^R(H_{m-1}, i)=1,\ i\leqq n_s;\ D_1^R(H_{m-1}, i)=0,\ i>n_s\} \tag{17}$$

is the desired one. That is, among all policies, the stationary nonrandomized policy is the optimal one. Note that, if $n_s$ is such that $g-(c/\mu)(n_s+1)>-l$, then the policy (17) is the *unique* optimal one and its interpretation is that an arrival joins the queue if and only if there are not more than $n_s$ customers ahead of him. That is, (17) defines a policy that is a deterministic control-limit rule.

If $g-(c/\mu)(n_s+1)=-l$, then any stationary policy such that

$$\{D_1^R(H_{m-1}, i)=1, i<n_s; 1\geqq D_1^R(H_{m-1}, n_s)\geqq 0; D_1^R(H_{m-1}, i)=0, i>n_s\}$$

is also optimal.

From (16) we have immediately

$$(\mu/c)(g-2c/\mu+l)<n_s\leqq(\mu/c)(g-c/\mu+l)<n_s+1, \tag{18}$$

and since, by assumption (*D*) above, every customer applies the $n_s$ policy, the process reduces to a $G/M/1/n_s$ process, whose imbedded Markov chain is given by:

$$\begin{aligned} P_{ij}&=q_{ij}(1), \qquad (i=0, 1, \cdots, n_s; j=0, 1, \cdots, n_s+1) \\ P_{n_s+1,j}&=P_{n_s,j}. \qquad (j\epsilon I_{n_s}) \end{aligned} \tag{19}$$

A brief observation reveals the interesting fact that the control-limit rule, as given by (16) or (18), is a function only of $g$, $c$, $l$, and $\mu$ and is *independent* of the arrival rate $\lambda$. This fact might lead one to suspect that, in the long run, a greater average reward per customer may be obtained by taking the arrival phenomena into consideration when looking for an optimal strategy than by ignoring them. These ideas are studied in the following section.

### *Social Optimization*

As noted above, one might ask himself whether the policy $n_s$, applied by every customer to optimize his individual expected net benefit, is also an optimal policy when the public or collective good is sought. In other words, suppose that the customers form a cooperative and their joint objective is to find a policy that will maximize the long-run average net benefit per customer (or, equivalently, per unit time) for all customers in the cooperative. Then, questions that obviously arise are as follows:

(i) Does an optimal policy exist?

(ii) If it does, is it a control-limit rule?

(iii) If the optimal rule is a control limit rule, is $n_s$ the control limit?

(iv) How does the average reward under a cooperative arrangement compare to the average reward under individualism?

We will show that, among all rules $R\epsilon C_S$, there exists a deterministic control-limit rule with *finite* control limit, denoted by $n_0$, that is optimal for social optimization. Moreover, it will be shown that, for fixed $g$, $c$, $\mu$, $l$

and $\lambda$, the optimal control limit $n_0$ is not necessarily the same as $n_s$, the optimal control limit for self-optimization, and, in fact, $n_0 \leqq n_s$.

We now show that, for our particular model and with respect to the average reward objective function $\phi_R$, the class $C_S$ is covered by the class $C_{CL}$. More precisely, let us define an equivalent relation as follows:

DEFINITION. Two policies $R$ and $R'$ are said to be $\phi$-equivalent if $\phi_R = \phi_{R'}$.

We then have the following:

LEMMA 4. *For every rule $R \epsilon C_S$, there exists a $\phi$-equivalent rule $R' \epsilon C_{CL}$.*

*Proof.* Let $\{d_i : 1 \geqq d_i \geqq 0\}$ be an arbitrary sequence of nonnegative numbers. Let $R \epsilon C_S$. As noted above, $R$ could be described completely by the set $\{D_i\}$ of joining probabilities. Hence, let $R = \{D_i : D_i = d_i,\ i = 0, 1, \cdots\}$. Let $J = \{j : d_j = 0\}$, that is, $J$ is the collection of all indices for which balking occurs with probability one. Let $k = \min\{j : j \epsilon J\}$, and let $r = \sup\{j : j \epsilon J\}$. If $J$ is empty, then $R \epsilon S$, and therefore $R \epsilon C_{CL}$. Suppose $J$ is not empty. If at time $\tau_0$ the system starts at state $\eta_0 \leqq r$, or whenever the system under $R$ is ergodic, then, any rule $R_k \epsilon C_{CL}$ such that $R_k = \{D_i : D_i = d_i,\ i \leqq k;\ D_i = 0,\ i > k\}$ is $\phi$-equivalent to $R$, since for both rules the effective state space is $I_k = \{0, 1, \cdots, k+1\}$. If the system under $R$ is not ergodic and $\eta_0 > r$, then there is a positive probability of the queue size increasing beyond all bounds, and thus any rule $R' \epsilon C_{CL}$ such that $R' = \{D_i : D_i > 0,\ i \leqq r;\ D_i = d_i,\ i > r\}$ is $\phi$-equivalent to $R$, since for both rules the average reward diverges to $-\infty$.

From Lemma 4 it follows that, as far as average reward is considered (and for stationary rules), it suffices to deal only with control-limit rules. Thus, in the rest of the section we will consider only this type of rule.

We recall that a customer who joins the queue when the system is in state $j$ spends an expected total time of $(j+1)/\mu$ in it. Thus, for every $m = 0, 1, \cdots$, we define the reward function $W_m = W(\eta_m, \Delta_m)$ as follows:

$$W_m = \begin{cases} g - (c/\mu)(\eta_m + 1), & \text{if} \quad \Delta_m = 1, \\ -l, & \text{if} \quad \Delta_m = 0. \end{cases}$$

From (14) it follows that, for $j \epsilon I$,

$$w_{j1} = g - (c/\mu)(j+1), \qquad w_{j0} = -l, \tag{20}$$

and from (15), confining our attention to rules $R \epsilon C_S$,

$$\begin{aligned} E_R W_m &= \textstyle\sum_j [w_{j1} P_R(\eta_m = j,\ \Delta_m = 1) + w_{j0} P_R(\eta_m = j,\ \Delta_m = 0)] \\ &= \textstyle\sum_j [w_{j1} P_R(\eta_m = j) \cdot P_R(\Delta_m = 1 | \eta_m = j) \\ &\quad + w_{j0} P_R(\eta_m = j) P_R(\Delta_m = 0 | \eta_m = j)] \\ &= \textstyle\sum_j [w_{j1} D_{j1} + w_{j0} D_{j0}) P_R(\eta_m = j)]. \end{aligned} \tag{21}$$

Using CHUNG's theorem (reference 2, pp. 85–87) and relations (20) and

(21), it follows that, for any $R=R_k \epsilon S_k$ or for any rule $R \epsilon S$ for which the system is ergodic, we can write

$$\phi_R = \sum_j \pi_j[[g-(c/\mu)(j+1)]D_j - l(1-D_j)] \\ = \sum_j \pi_j D_j[g-(c/\mu)(j+1)+l]-l, \tag{22}$$

where $\phi_R = \phi_R(i)$ for all $i \epsilon I$ (or $i \epsilon I_k$) independent of the initial state, and the $\{\pi_j\}$'s satisfy (6).

It is of interest to indicate that (22) could have been obtained directly by using results (11) for $I_k$ or the modified result for $I$ (*see* remark 1). Consider a rule with $D_i > 0$ for all $i \epsilon I = \{0, 1, \cdots\}$, such that the system is ergodic. Then the corresponding $\{\pi_i\}$'s are all positive and satisfy (6). The expected total time spent in the system by an arbitrary customer who joins the queue is $EW + 1/\mu$, and hence the expected net benefit of an arbitrary arrival is

$$\phi_R = (\sum_{i \epsilon I} D_i \pi_i)[g - c(EW+1/\mu)] - (1-\sum_{i \epsilon I} D_i \pi_i) l.$$

By using (11), (22) is readily obtained.

Thus, $\phi_R$ as given by (22) represents the collective objective function, i.e., the infinite-horizon average 'public good.'

Our objective now is to find a rule for social optimization, that is, our problem is to find a rule $R^*$ so as to maximize $\phi_R$ over all rules $R \epsilon C_S$ (practically, over all rules $R \epsilon C_{CL}$).

THEOREM 3. *For any finite state space,* $I_n = \{0, 1, \cdots, n+1\}$, *there exists a deterministic control-limit rule* $R \epsilon C_{DCL}$ *that maximizes* $\phi_R$.

*Proof.* It is well known[1,3] that, for a finite state space with a finite number of actions, there exists a nonrandomized rule $R^* \epsilon C_D$ that maximizes $\phi_R$. Let $R^* = \{D_i^*\}$, where $D_i^* = 0$ or $1$, $i \epsilon I$. Let $j = \min\{i : D_i^* = 0\}$. Clearly, $j \leqq n+1$. Thus, the process will consist of only $j+1$ recurrent states, as, once occupying any state below $j$, it will never get beyond $j$. The rule $R = \{D_i : D_i = 1,\ i < j;\ D_i = 0,\ i \geqq j\}$ will achieve the same long-run average reward as $R^*$, and hence is optimal. The fact that $R \epsilon C_{DCL}$ completes the proof.

The consequence of Theorem 3 is that, whenever $n < \infty$ is the capacity of the waiting room, then the optimal rule that maximizes $\phi_R$ can be found among the $n+1$ possible deterministic control-limit rules. More specifically, if $I = I_n = \{0, 1, \cdots, n+1\}$, then only rules $R_k^D = \{D_i : D_i = 1,\ i \leqq k;\ D_i = 0,\ i = k+1, \cdots, n+1\}$ for $k = 0, 1, \cdots, n$ need be considered. It follows that for $I_n$ finite our problem could be formulated as of finding $R_k^D$ so as to maximize $\phi_{R_k^D}$ over all $k \epsilon \{0, 1, \cdots, n\}$. Note that for fixed $k$ and rule $R_k^D$, the objective function as given by (22) is transformed into

$$\phi_{R_k^D} = \sum_{i=0}^{i=k} \pi_i(k)[g-(c/\mu)(i+1)+l]-l, \tag{23}$$

where the $\{\pi_i(k)\}$'s denote the steady-state probabilities of the process with rule $R_k^D$.

For the following two theorems, we restrict ourselves only to rules $R \epsilon C_{DCL}$. Our objective is to show that, although this class contains a denumerable number of policies, our search for the best rule of this class could be restricted to only a finite number of possible rules. We have the following:

THEOREM 4. *If $\lambda < \mu$, then there exists a rule $R_m^D \epsilon C_{DCL}$ such that, for all $R_n^D \epsilon C_{DCL}$, $\phi_{R_m^D} > \phi_{R_n^D}$ whenever $n > m$.*

*Proof.* We will show that there exists an $m$ such that, whenever $n \geqq m$, $\phi_{R_n^D} > \phi_{R_{n+1}^D}$. This would imply that $\phi_{R_m^D} > \phi_{R_n^D}$ for all $n > m$. Using (23) and (12), we obtain, after some algebraic manipulations,

$$\phi_{R_n^D} - \phi_{R_{n+1}^D} = -\pi_0(n+1)(c/\mu)[L(n)+1] - \pi_0(n+1)\pi_{n+1}(n)[g-(c/\mu)(n+2)+l] + (c/\mu)[1-\pi_{n+2}(n+1)].$$

Using Corollary 1, we write

$$\phi_{R_n^D} - \phi_{R_{n+1}^D} > -\pi_0(n+1)\pi_{n+1}(n)[g-(c/\mu)(n+2)+l] - (c/\mu)\pi_{n+2}(n+1).$$

Let $n \geqq n_s$. By (16) we have

$$g-(c/\mu)(n+2)+l = g-(c/\mu)(n_s+2)+l-(c/\mu)(n-n_s) < -(c/\mu)(n-n_s).$$

This relation, together with (12), implies

$$\phi_{R_n^D} - \phi_{R_{n+1}^D} > \pi_0(n+1)\pi_{n+1}(n)(c/\mu)(n-n_s) - (c/\mu)\pi_{n+2}(n+1) = (c/\mu)\pi_{n+1}(n)[\pi_0(n+1)(n-n_s+1)-1].$$

Thus, a sufficient condition for $\phi_{R_n^D} > \phi_{R_{n+1}^D}$ is that

$$\pi_0(n+1) > 1/(n-n_s+1). \qquad (24)$$

Since $\lambda < \mu$ the (infinite) $GI/M/1$ queue is ergodic, and from KENDALL[12] it is well known that $\pi_0(\infty) = 1 - Z_0$, where $Z_0$ is the unique solution $A(Z_0) = Z_0 (0 < Z_0 < 1)$ for $A(Z) = \sum_{k=0}^{k=\infty} a_k Z^k$. By Lemma 3 it follows that $\pi_0(n) \geqq 1 - Z_0$ for every $n$, and therefore, for (24) to hold, it is sufficient to choose $n$ so large that $1 - Z_0 \geqq 1/(n-n_s+1)$. By letting

$$m \geqq n_s + Z_0/(1-Z_0), \qquad (25)$$

the proof is complete.

THEOREM 5. *For any $\lambda$ and $\mu$, there exists a rule $R_{n_0}^D$ with finite $n_0$ such that, for all $R \epsilon C_{DCL}$, $\phi_{R_{n_0}^D} = \sup_n \phi_{R_n^D}$.*

*Proof.* Suppose $\lambda \geqq \mu$. Then the (infinite) $GI/M/1$ queue is not ergodic and $L(n)\to\infty$ as $n\to\infty$. Thus, $\phi_{R_n D}\to-\infty$ as $n\to\infty$. Since, for every finite $n$, $\phi_{R_n D}$ is finite, and since $\phi_{R_n D}\leqq g-c/\mu$ for every $n$, it follows that there exists a finite number $n_0$ such that $\phi_{R^D_{n_0}}=\sup_n \phi_{R_n D}$. If $\lambda<\mu$ then, by Theorem 4 and letting $m$ be the largest integer not greater than $n_s+Z_0/(1-Z_0)+1$, we can restrict ourselves to the finite set $I'=\{0, 1, \cdots, m\}$, which obviously contains an $n_0$ such that $\phi_{R^D_{n_0}}=\max_{n\epsilon I'} \phi_{R_n D}$.

We can now show that, among all stationary policies $R\epsilon C_S$, the optimal one is indeed a finite deterministic control-limit rule; that is:

THEOREM 6. *There exists a finite number $n_0$ such that $\phi_{R^D_{n_0}}=\sup_{R\epsilon C_S}\phi_R$.*

*Proof.* From Lemma 4 it follows that only control-limit rules need to be considered. If the control-limit rule generates a finite effective state space, then our assertion follows immediately from Theorem 3. So, it remains to consider only infinite control-limit rules. Let $R$ be arbitrary such that $R=\{D_i:D_i>0, i=0, 1, \cdots\}$. If the process generated by $R$ is not ergodic, then, by the discussion of the preceding theorem, any finite control-limit rule $R_k$ would be better. Since a rule $R_k$ generates an actual finite state space, the existence of $n_0$ follows from Theorem 3. Thus, we may suppose that $R$ generates an ergodic process. We define the sequence of control-limit rules $\{R_k, k=0, 1, \cdots\}$ such that $R_k=\{D_i:D_i>0, i\leqq k, D_i>0, i>k\}$. Clearly, $R_k\to R$ as $k\to\infty$. We define also the sequence of average rewards $\{\phi_{R_k}, k=0, 1, \cdots\}$. Since the process is ergodic, $\phi_R$ exists and $\phi_{R_k}\to\phi_R$ as $k\to\infty$. From Theorem 3 it follows that, for each $R_k$, there exists a deterministic control-limit rule $R^D_{n(k)}$ such that $\phi_{R^D_{n(k)}}\geqq\phi_{R_k}$ and $n(k)\leqq k$. From Theorem 5, it follows that $\phi_{R^D_{n_0}}\geqq\phi_{R^D_{n(k)}}\geqq\phi_{R_k}$ for all $k=0, 1, \cdots$, and hence, $\phi_{R^D_{n_0}}\geqq\lim_{k\to\infty}\phi_{R_k}=\phi_R$.

We can now find a stronger upper bound than (25) on $n_0$, the optimal (deterministic) control limit for social optimization. Eventually we will show that $n_0\leqq n_s$. For this purpose we use HOWARD's algorithm.[10] Consider a finite state space $I_M=\{0, 1, 2, \cdots, M\}$, where $M$ is a finite bound on $n_0$ such that

$$M\geqq\min\{n:\pi_0(n+1)\geqq 1/(n-n_s+1), n\geqq n_s\}.$$

Thus, $n_0\epsilon\{0, 1, \cdots, M\}$. We recall that we need to consider only deterministic control-limit rules, and therefore we start Howard's procedure by letting $k=1$ whenever $\eta_r=i\leqq n_s$ and $k=0$ whenever $\eta_r=i$, $n_s<i\leqq M$. That is, we start with a deterministic control-limit rule $R^D_{n_s}=\{D_i:D_i=1, i\leqq n_s; D_i=0, i>n_s\}$. Using (20), the 'value determination operation' is now to find $\phi, v_0, v_1, \cdots, v_M$ (where one of the $\{v_i\}$'s is arbitrarily determined) that satisfy

$$\begin{aligned}
\phi+v_0&=g-c/\mu+\textstyle\sum_{j=0}^{j=1} q_{0j}(1)v_j,\\
&\vdots\\
\phi+v_i&=g-(c/\mu)(i+1)+\textstyle\sum_{j=0}^{i+1} q_{ij}(1)v_j,\ i<n_s,\\
&\vdots\\
\phi+v_{n_s}&=g-(c/\mu)(n_s+1)+\textstyle\sum_{j=0}^{n_s+1} q_{n_sj}(1)v_j,\\
\phi+v_{n_s+1}&=-l+\textstyle\sum_{j=0}^{n_s+1} q_{n_s+1,j}(0)v_j,\\
&\vdots\\
\phi+v_M&=-l+\textstyle\sum_{j=0}^{j=M} q_{M,j}(0)v_j.
\end{aligned} \tag{26}$$

We need the following:

LEMMA 5. *For any solution of* (26) *we have*

$$v_0\geqq v_1\geqq \cdots \geqq v_{n_s}\geqq v_{n_s+1}\geqq \cdots \geqq v_M.$$

*Proof*. First, we establish that $v_{n_s}\geqq v_{n_s+1}$. Since $q_{ij}(1)=q_{i+1,j}(0)$ for all $i$, $j$, then, by subtracting in (26) the $(n_s+2)$nd row from the $(n_s+1)$st, and using (16), we obtain

$$\begin{aligned}
v_{n_s}-v_{n_s+1}&=g-(c/\mu)(n_s+1)+l+\textstyle\sum_{j=0}^{n_s+1} [q_{n_sj}(1)-q_{n_s+1,j}(0)]v_j\\
&=g-(c/\mu)(n_s+1)+l\geqq 0,
\end{aligned}$$

where equality holds if and only if $g-(c/\mu)(n_s+1)=-l$.

We will now use backward induction to show that $v_0\geqq v_1\geqq \cdots \geqq v_{n_s}\geqq v_{n_s+1}$. Thus, we assume $v_1\geqq v_2\geqq \cdots \geqq v_{n_s}$ and show that $v_0\geqq v_1$. By subtracting the second row of (26) from its first, we have:

$$\begin{aligned}
v_0-v_1&=c/\mu+a_1(v_0-v_1)+a_0(v_1-v_2)\\
&=[1/(1-a_1)][c/\mu+a_0(v_1-v_2)]\geqq 0.
\end{aligned}$$

Next, we use forward induction to obtain that $v_{n_s}\geqq v_{n_s+1}\geqq \cdots \geqq v_M$. We assume that $v_{n_s+1}\geqq \cdots \geqq v_{M-1}$ and show that $v_{M-1}\geqq v_M$. Using (26) we have

$$v_{M-1}-v_M=[1/(1-a_0)][\textstyle\sum_{j=0}^{M-2} a_{M-1-j}(v_j-v_{j+1})]\geqq 0.$$

We can now prove the following:

THEOREM 7. $n_0\leqq n_s$.

*Proof*. Since only deterministic control-limit rules need to be considered, it suffices to show that $k=1$ for $i>n_s$ is never an improvement on (26). (*See* the 'policy-improvement-routine' in reference 10). To show this, we define, for every $i$,

$$g_i(k)=w_{ik}+\textstyle\sum_j q_{ij}(k)v_j,\qquad k=0,\ 1.$$

Thus, it suffices to show that, for every $i>n_s$, $g_i(0)>g_i(1)$. But, for every $i$,

$$g_i(1)-g_i(0)=g-(c/\mu)(i+1)+l-a_i(v_0-v_1) \\ -a_{i-1}(v_1-v_2)-\cdots-a_0(v_i-v_{i+1}).$$

By using Lemma 5 and the fact that for every $i>n_s$, $g-(c/\mu)(i+1)+l<0$, it follows that $g_i(0)>g_i(1)$ for all $i>n_s$. This shows that $\phi_{R_{n_s}^D}>\phi_{R_n{}^D}$ for $n>n_s$, which completes the proof.

From calculations made in reference 16 for the $M/M/1$ queue, it is evident that, in general, $n_0<n_s$, and only because of the integer properties of $n_s$ and $n_0$ we sometimes have $n_0=n_s$. The interpretation of this fact and Theorem 7 is that, for the system studied, exercising narrow self-interest by all customers seldom optimizes public good.

### *Formulation as a Linear Program*

We have just shown that $n_0\leqq n_s$. Thus, following MANNE,[15] WOLFE AND DANTZIG,[20] DERMAN,[3] and others, the problem of finding $n_0$ can be formulated as a linear program.

By Theorem 7, our objective is to find a rule $R^*\epsilon C_S$ so as to

$$\text{maximize}_{R\epsilon C_S}\left\{\sum_{i=0}^{i=n_s}\pi_i(R)\cdot D_{i}[g-(c/\mu)(i+1)+l]-l=\phi_R\right\} \tag{27}$$

subject to (6) with $I_{n_s}$ replacing $I$.

Clearly, (27) is equivalent to

$$\text{maximize}_{R\epsilon C_S}\left\{\sum_{i=0}^{i=n_s}\pi_i(R)D_{i}[g-(c/\mu)(i+1)+l]\right\}. \tag{28}$$

To formulate this as a linear program, we consider the variables $\{x_{ik}\}$ as follows: Let

$$x_{ik}=\pi_i(R)D_{ik}. \qquad (i=0,1,\cdots,n_s+1;\ k=0,1)$$

Obviously, we have $x_{i1}=\pi_i(R)(1-D_{i0})=\pi_i(R)-x_{i0}$. Substituting in (28), we get the following linear program:

$$\text{maximize } \sum_{i=0}^{i=n_s} x_{i1}[g-(c/\mu)(i+1)+l] \text{ subject to}$$

$$x_{jk}\geqq 0, \qquad (j=0,1,\cdots,n_s+1;\ k=0,1)$$

$$\sum_{k=0}^{k=1}x_{jk}-\sum_{i=0}^{i=n_s+1}\sum_{k=0}^{k=1}x_{ik}q_{ij}(k)=0, \qquad (j=0,1,\cdots,n_s+1)$$

$$\sum_{j=0}^{j=n_s+1}\sum_{k=0}^{k=1}x_{jk}=1.$$

Once the $\{x_{ik}\}$'s are determined, the $\{\pi_i(R)\}$'s and the $\{D_{ik}\}$'s can be obtained from:

$$\pi_i(R)=\sum_{k=0}^{k=1}x_{ik};$$

$$D_{ik}=x_{ik}/\pi_i(R)=x_{ik}/(x_{i1}+x_{i0}), \quad \text{if} \quad \pi_i(R)>0;$$
$$D_{ik}=\text{arbitrary}, \quad \text{if} \quad \pi_i(R)=0.$$

References 3 and 20 show that the optimal rule is such that $D_{ik}=0$ or 1. Furthermore, reference 20 shows that there are at most $n_s+2$ variables $x_{jk}$ that are positive. Clearly, if $x_{j1}=0$ (that is, $D_j=0$), then $\pi_{j+1}(R)=0$. Hence, $n_0$ is found by

$$n_0=\max\{i:x_{i1}>0\}. \tag{29}$$

We may summarize this result by saying that, among all stationary policies, the optimal rule for social optimization is: *join if and only if the observed queue size is not greater than* $n_0$ *as given by* (29).

## OPTIMAL STATION-TOLL CHARGES

IN THIS SECTION, we relax the assumption that $\theta$, the service charge, is fixed. We will treat $\theta$ as a controllable parameter and determine its optimal values in various circumstances. First, we analyze the situation where the agency that operates the service station is governed by the customers themselves, and thus, both the policy of balking or joining and the level of the service toll $\theta$ are solely decided by the customers. Next, we treat the case where the toll-collecting agency is a profit-making organization, completely divorced from the individual or collective economic interests of the customers. In this case, the agency will seek to impose a toll $\theta_r$ designed to maximize its own revenue rather than to optimize the whole system. We show how $\theta_r$ is determined for the two distinguished possibilities: (i) The customers employ an individual policy $n_s$. (ii) The customers employ the 'collective' policy $n_0$. We observe that, for both cases, the fact that the optimal (deterministic control limit) rule $n(\theta)$ is completely known to the service station for any toll charge $\theta$ is analogous to the monopoly model of price theory.

### *Over-all Optimization*

In the preceding sections, it was assumed that $G$, $c$, $\lambda$, $\mu$, and $\theta$ were fixed. We have seen that the arrival rate $\lambda$ could be changed into an effective arrival rate by applying a control-limit rule—$n_s$ or $n_0$—in order to optimize the individual or the collective objective function. In many queuing models, the attempt to improve the *over-all* performance of the system is made by proposing a modification in the service process itself, or by assigning some priorities in order to minimize costs. Another approach might be to examine how the join-or-balk decisions of the customers —and thus the queue size—are affected by changes in $\theta$, the service toll.

It was shown that for any $\theta \geqq 0$, $n_0(\theta) \leqq n_s(\theta)$, where $n_0(\theta)$ and $n_s(\theta)$

are obtained from (29) and (18) respectively. As noted, the fact that, except for particular values of the parameters, we would have $n_0 < n_s$ points out that consideration of narrow self-interest does not ordinarily lead to over-all optimality. That is, a situation may occur in which—if the customers behave according to a criterion of narrow self-interest—the facilities of the system will be over-congested. However, if the service station is operated by some nonprofit organization dedicated to a more global concept of optimization, or whenever the customers themselves collectively govern the operations of the station (through an appointed agency, say), then an ameliorated state of affairs can be achieved. More specifically, we view the problem as of how to cause an individual customer to employ the $n_0$ strategy rather than the $n_s$. This last statement follows from the observation that, for any strategy $R_n^D$, the average reward per customer is given by $\phi_{R_n^D}$ and thus, by definition of $n_0$, $\phi_{R_{n_0}^D} \geqq \phi_{R_{n_s}^D}$.

If the individual customers are not likely to be persuaded by argument alone, then two distinct ways might be used in order to reduce the queue size. One way is an administrative measure that will limit the capacity of the system to the $n_0$ level. Since $n_0 \leqq n_s$, the only obstacle for individual customers to join the queue will be the limited capacity, and thus an over-all optimization will be achieved.

Perhaps of more interest is the situation where the toll charge is used as a device for controlling the queue size. Without loss of generality, we may assume that the initial charge is $\theta = 0$ (i.e., $g = G$) and our objective is to find $\theta_0$ so as to achieve

$$n_s(\theta_0) = n_0(0), \tag{30}$$

where, clearly, $n_s(0) \geqq n_0(0)$.

When requiring (30), we implicitly assume that the additional toll revenue—at the level of $[1 - \pi_{n+1}(n)]\theta_0$ per arriving customer (when applying some rule $R_n^D$)—is redistributed among the participants, either directly or by some other means like service improvement, dividends from a cooperative company, etc. Thus, for an over-all optimization, the public objective is to find an $n^*$ so as to

$$\begin{aligned} \text{maximize}_n \ \{ \textstyle\sum_{i=0}^{i=n} \pi_i(n) [G - \theta_0 - (c/\mu)(i+1) + l] \\ - l + [1 - \pi_{n+1}(n)]\theta_0 \}, \end{aligned} \tag{31}$$

which is equivalent to

$$\text{maximize}_n \ \textstyle\sum_{i=0}^{i=n} \pi_i(n) [G - (c/\mu)(i+1) + l]. \tag{32}$$

Clearly, $n^*$ obtained from (32) is such that $n^* = n_0(0)$, which motivates (30). Note that another implicit assumption is that the station's operating costs are independent of the particular level of the control-limit rule.

If, however, there are some additional costs involved in the collecting of the extra service charge, and/or the redistributed amount is discounted, then, still, an over-all optimization can be achieved, although the computations will not be so easy as in (30). Suppose that the (present) value of the money returned to the customers is at the rate of $(1-\alpha)[1-\pi_{n+1}(n)]\theta_0$ per arrival, where $0 \leqq \alpha \leqq 1$. Then (31) is equivalent to finding $n^*$ so as to

$$\text{maximize}_n \sum_{i=0}^{i=n} \pi_i(n)[G-\alpha\theta_0-(c/\mu)(i+1)+l].$$

Clearly, $n^* = n_0(\alpha\theta_0)$, from which it follows that $\theta_0$ is such that $n_s(\theta_0) = n_0(\alpha\theta_0)$.

It just remains to find the actual value, or, rather, the range, of $\theta_0$. Using (18), we have

$$(\mu/c)(G-\theta_0-2c/\mu+l) < n_s(\theta_0) = n_0(\alpha\theta_0) \leqq (\mu/c)(G-\theta_0-c/\mu+l),$$

from which we get

$$G-(c/\mu)[n_0(\alpha\theta_0)+2]+l < \theta_0 \leqq G-(c/\mu)[n_0(\alpha\theta_0)+1]+l.$$

When $\alpha=0$, $\theta_0$ could easily be obtained by calculating $n_0(0)$ and having

$$G-(c/\mu)[n_0(0)+2]+l < \theta_0 \leqq G-(c/\mu)[n_0(0)+1]+l.$$

That is, any value of $\theta_0$ in the above interval will compel the selfish customers to employ the $n_s(\theta_0)$ policy and thus cause—willingly or not—an over-all optimization.

### *Station Optimization*

Suppose now that the toll-collecting agency is a profit-making organization that is completely divorced from the individual or collective economic interests of the customers. In this case, the agency will seek to impose a toll $\theta_r$ designed to maximize its own revenue rather than to optimize the whole system. We assume that the agency's objective is to maximize the expected average collected toll per unit time, or, equivalently, per arriving customer. (Once again, we assume that either the station's operating costs are negligible, or independent of the policy used by the customers.) However, the agency realizes that for any service charge $\theta$ the customers, trying to achieve the best for themselves, employ some control-limit rule $n=n(\theta)$ [either $n_s(\theta)$ or $n_0(\theta)$], and thus the state space $I$ consists of finite number of states with $I=\{0, 1, \cdots, n, n+1\}$. The probability transition matrix for this situation is given now by (19), with $n=n(\theta)$ replacing $n_s$.

If $X_m$ denotes the revenue of the service station from the $m$th arrival, then

$$X_m = \begin{cases} \theta, & \text{if } \Delta_m = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since, for the deterministic control-limit rule $n(\theta)$, $\Delta_m = 1$ if the system is in state $i \leqq n(\theta)$ and $\Delta_m = 0$ whenever $i \geqq n(\theta)+1$, then the station's objective is to find $\theta_r$ so as to

$$\text{maximize}_\theta\ [1-\pi_{n+1}(n)]\cdot\theta = \psi_n(\theta), \tag{33}$$

subject to the usual stationary equations (6), where $n = n(\theta)$.

The expression for $\psi_n(\theta)$ exhibits the conflict the agency is confronted with. Since both $n_s(\theta)$ and $n_0(\theta)$ are nonincreasing functions of $\theta$, then, roughly speaking, the use of a larger toll $\theta$ implies a smaller control limit $n(\theta)$, which, in turn, implies a smaller value of $[1-\pi_{n+1}(n)]$.

We consider two cases:

*Case* 1: $n_s$. In this case, we assume that the customers are not organized as a cooperative, but, rather, each individual is doing his best for himself; that is, for any service charge $\theta$, there is an $n_s(\theta)$, given by (16), that is the control-limit balking rule used by all the customers. It is seen that $n_s(\theta)$ is a step function of $\theta$, continuous from the left, with jump points at values $\theta_n = G-(c/\mu)(n+1)+l$ for $n = 0, 1, \cdots$. We assert that the agency is interested just in the set of jump points lying in the interval $(0, G-c/\mu+l]$. This is the result of the fact that, if $\theta_r \leqq 0$, then $\psi_n(\theta) \leqq 0$ for every rule $n_s$ chosen by the customers according to self optimization, whereas, for $\theta > G-c/\mu+l$, $n_s(\theta) \leqq (\mu/c)(G-\theta-c/\mu+l) < 0$, which implies that none of the customers joins the queue, and hence, once again, $\psi_n(\theta) = 0$. We let $N$ be an integer defined by the bracket function as follows: $N = [(\mu/c)(G-c/\mu+l)]$. Then we have to consider only the $N+1$ values of $\theta_n$ for $n = 0, 1, \cdots, N$. Thus, we see that $n_s(\theta)$ is constant over intervals of length $c/\mu$.

The graph of $n_s(\theta)$ is given in Fig. 1.

Since, for any $\theta \epsilon (\theta_{n+1}, \theta_n]$, the control limit is $n_s(\theta) = n$, then, from the station's point of view, it is profitable to choose the largest possible value of $\theta$ without changing the policy of the customers. Thus, for $\theta_{n+1} < \theta \leqq \theta_n$, $\theta$ should be fixed such that $\theta = \theta_n$. Hence (33) is equivalent to finding $n_r$ so as to

$$\text{maximize}_{0 \leqq n \leqq N}\ \{[1-\pi_{n+1}(n)]\theta_n = [1-\pi_{n+1}(n)][G-(c/\mu)(n+1)+l]\},$$

which, in turn, is equivalent to

$$\min_{0 \leqq n \leqq N}\ \{(c/\mu)n + \pi_{n+1}(n)\theta_n\}. \tag{34}$$

Since $(c/\mu)n$ is an increasing function of $n$ and $\pi_{n+1}(n)\theta_n$ is a decreasing function of $n$, it is clear that the optimal value $n_r$ is obtained from

$$n_r = \min\{n \epsilon I : \pi_{n+1}(n)\theta_n - \pi_{n+2}(n+1)\ \theta_{n+1} < c/\mu\}, \tag{35}$$

from which we have, finally,

$$\theta_r = G-(c/\mu)(n_r+1)+l. \tag{36}$$

It is of interest to note that an analogy can be drawn to an inventory model with lost sales and no set-up costs. Expression (34) is equivalent to $\min_n \{cn+\mu\pi_{n+1}(n)\theta_n\}$. Thus $c$, the waiting cost per customer per unit time, is analogous to the holding cost of inventory per item per unit time; $n$, which is the maximal joining number of the customers, and hence serves as the capacity of the waiting room, is analogous to the lot size; $\mu$ represents the demand rate; $\theta_n$ is the cost incurred when there is a shortage, and

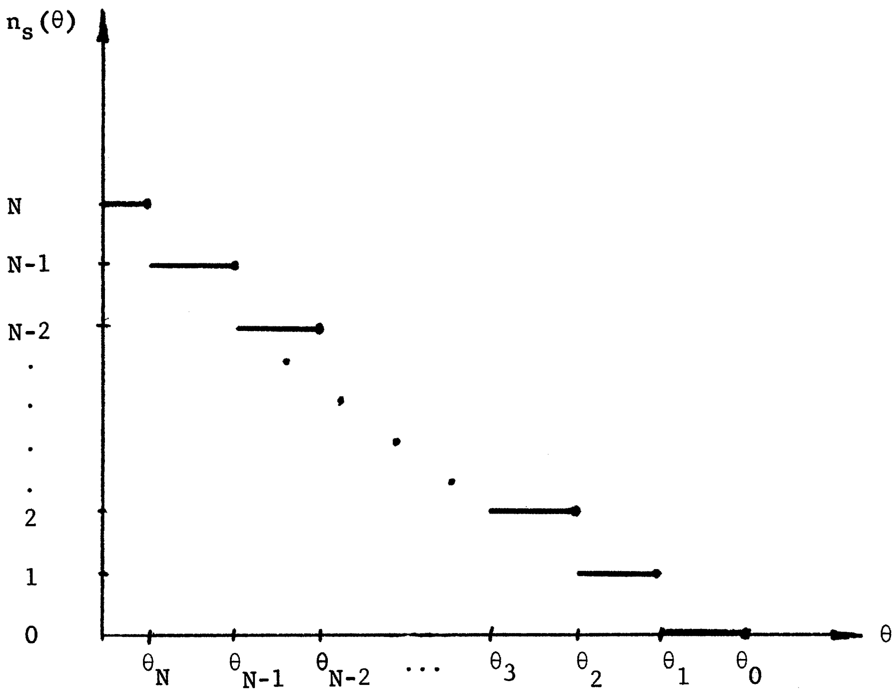


**Fig. 1.** The graph of $n_s(\theta)$.

$\pi_{n+1}(n)$, the probability of balking, is equivalent to the probability of a lost sale.

*Case* 2: $n_0$. In this case, we assume that the customers are organized and try to achieve the 'best' for their entire community. Thus, for every service charge $\theta$, they employ a control limit rule $n_0(\theta)$ that maximizes the average net benefit per customer.

The agency's objective now is to find $\theta_r^*$ so as to

$$\max_\theta \{[1-\pi_{n_0+1}(n_0)]\theta\},$$

where, for any $\theta$, $n_0$ is determined by the solution of the problem

$$\max_n \left\{ \sum_{i=0}^{i=n} \pi_i(n)[G-\theta-(c/\mu)(i+1)+l] \right\}, \tag{37}$$

with $\{\pi_i(n)\}$'s obtained from the probability transition matrix of the pure $GI/M/1/n$ queuing process (or, equivalently from the solution of the linear programming problem of Section 3). Since $n_0(\theta) \leqq n_s(\theta)$ then, as in case 1, the search for the optimal value $\theta_r^*$ can be restricted to the interval $(0, G-c/\mu+l]$. Moreover, there can be at most $N+1$ distinct values of $n_0(\theta)$.

Before showing how $\theta_r^*$ can be found, we emphasize the conflict of interests between the customers and the agency as expressed by their corresponding objective functions. A simple modification shows that (37) is equivalent to

$$\min_n \{[1-\pi_{n+1}(n)]\theta+[G-(c/\mu)(n+2)+l]\pi_{n+1}(n)+(c/\mu)L(n)\}.$$

Notice that $[1-\pi_{n+1}(n)]\theta$, which is to be maximized by the agency, is one of the terms to be minimized by the customers.

We now give a straightforward, one-pass algorithm to find $\theta_r^*$.

*Step 1.* For every $\theta_k=G-(c/\mu)(k+1)+l$, $k=0, 1, \cdots, N$, find the corresponding $n_0(\theta_k)$ by solving a linear program, say. By Theorem 7, $n_0(\theta_k)$ has the property that $n_0(\theta_k) \leqq n_s(\theta_k)=k$. Hence, $n_0(\theta_0) \leqq n_0(\theta_1) \leqq \cdots \leqq n_0(\theta_N)$. Denote by $n_1, n_2, \cdots, n_m$, with $n_1<n_2<\cdots<n_m$, the $m$ distinct values of the $\{n_0(\theta_k)\}$'s. Clearly $m \leqq N+1$.

*Step 2.* Let $S_i=\{\theta_k : n_0(\theta_k)=n_i\}$, $i=1, 2, \cdots, m$. For every $S_i$ let $\hat{\theta}_i=\max\{\theta_k : \theta_k \epsilon S_i\}$. Since $\{\theta_k \epsilon S_i, \theta_l \epsilon S_{i+1}\}$ implies $\theta_k>\theta_l$, then $\hat{\theta}_i>\hat{\theta}_{i+1}$ for $i=1, \cdots, m-1$.

*Step 3.* Find $\max_{1\leqq i \leqq m}\{[1-\pi_{n_i+1}(n_i)]\hat{\theta}_i\}$ and denote by $j$ the index for which this maximum is achieved. Let $t$ be such that $\theta_t=\max\{\theta_k : \theta_k \epsilon S_j\}$, that is, $\theta_t=\hat{\theta}_j$. If $t=0$, then $\theta_r^*=\theta_0$. If $t\neq 0$, go to step 4.

*Step 4.* The optimal value $\theta_r^*$ is now in the interval $[\theta_t, \theta_{t-1})$. Thus we are looking for $\theta \epsilon [\theta_t, \theta_{t-1})$ such that

$$\begin{aligned}&[1-\pi_{n_j+1}(n_j)]\theta+[G-(c/\mu)(n_j+2)+l]\pi_{n_j+1}(n_j)+(c/\mu)L(n_j)\\ &=[1-\pi_{n_j}(n_j-1)]\theta+[G-(c/\mu)(n_j+1)+l]\pi_{n_j}(n_j-1)+(c/\mu)L(n_j-1),\end{aligned}$$

from which we have, finally,

$$\theta_r^*=(c/\mu)[L(n_j-1)-L(n_j)+\pi_{n_j+1}(n_j)]/[\pi_{n_j}(n_j-1)-\pi_{n_j+1}(n_j)]+\theta_{n_j}, \tag{38}$$

where, from step 1, $\theta_{n_j}=G-(c/\mu)(n_j+1)+l$.

## CONCLUSION

THE SITUATION here may, perhaps, be better understood by drawing an equivalence between our queuing model and the general economic model of monopoly. If the service agency (the monopolist) realizes that the customers are selfish, that is, their type of policy is the $n_s$ one, then their demand curve, represented by Fig. 1 is completely known, and $\theta_r$, as given by (36) and (35), is the toll that maximizes $\psi_n(\theta)$. If, however, the customers are organized, then a graph similar to Fig. 1 could be obtained—again with the interpretation of a demand curve—and $\theta_r^*$, as given by (38), is the desired service charge. In fact, this $\theta_r^*$ dictates the actual value of the customers' control limit $n_0(\theta_r^*)$, since any deviation by them from this value would have the effect of reducing their net benefit.

## ACKNOWLEDGMENTS

THIS PAPER is based on a part of an Eng. Sc.D. dissertation submitted to the department of Mathematical Methods and Operations Research at Columbia University, New York. The author is greatly indebted to PETER J. KOLESAR and CYRUS DERMAN for their advice, criticism, and many valuable suggestions.

## REFERENCES

1. D. BLACKWELL, "Discrete Dynamic Programming," *Ann. Math. Stat.* **33,** 719–726 (1962).
2. K. L. CHUNG, *Markov Chains with Stationary Transition Probabilities*, Springer, Berlin, 1960.
3. C. DERMAN, "On Sequential Decisions and Markov Chains," *Management Sci.* **9,** 16–24 (1962).
4. ———, "On Optimal Replacement Rules when Changes of States Are Markovian," Chap. 9 in *Mathematical Optimization Techniques*, R. BELLMAN (ed.), The Rand Corp., 1963.
5. ———, "Denumerable State Markovian Decision Processes—Average Cost Criteria," *Ann. Math. Stat.* **37,** 1545–1554 (1966).
6. ———, "A Markovian Decision Process—Average Cost Criterion," *Lectures in Applied Mathematics*, Vol. 12, Part 2 (pp. 139–148), American Mathematical Society, Providence, 1968.
7. W. FELLER, *An Introduction to Probability Theory and Its Applications*, Vol. 1, 2nd edition, Wiley, New York, 1964.
8. P. D. FINCH, "Balking in the Queueing System *GI/M/1*," *Acta Math, Acad. Sci. Hungr.* **10,** 241–247 (1959).
9. T. HOMMA, "On the Many Server Queueing Process with Particular Type of Queue Discipline," pp. 91–101, *Rep. Stat. Appl. Res. JUSE*, Vol. 4, No. 3, 1956.
10. R. A. HOWARD, *Dynamic Programming and Markov Processes*, Wiley, New York, 1960.
11. S. KARLIN, *A First Course in Stochastic Processes*, Academic Press, New York, 1966.
12. D. G. KENDALL, "Stochastic Processes Occurring in the Theory of Queues and

Their Analysis by the Method of the Imbedded Markov Chain," *Ann. Math. Stat.* **24,** 338–354 (1953).
13. W. A. LEEMAN, "The Reduction of Queues through the Use of Price," *Opns. Res.* **12,** 783–785 (1964).
14. ———, "Comments on Preceding Note," *Opns. Res.* **13,** 680–681 (1965).
15. A. S. MANNE, "Linear Programming and Sequential Decisions," *Management Sci.* **6,** 259–267 (1960).
16. P. NAOR, "On the Regulation of Queue Size by Levying Tolls," *Econometrics* **37,** 15–24 (1969).
17. T. L. SAATY, "The Burdens of Queuing Charges—Comment on a Letter by Leeman," *Opns. Res.* **13,** 679–680 (1965).
18. L. TAKÁCS, "On a Combined Waiting Time and Loss Problem Concerning Telephone Traffic," *Ann. Univ. Sci. Budapest, Sectio Math.*, **1,** 73–82 (1958).
19. ———, *Introduction to the Theory of Queues,* Oxford University Press, New York, 1962.
20. P. WOLFE AND G. B. DANTZIG, "Linear Programming in a Markov Chain," *Opns. Res.* **10,** 702–710 (1962).
21. U. YECHIALI, "On Optimal Balking Rules and Toll Charges in the $GI/M/1$ Queuing Process," Technical Report No. 47, Operations Research Group, Columbia University, New York, 1969.