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EXPLICIT SOLUTIONS FOR CONTINUOUS-TIME QBD PROCESSES BY USING RELATIONS BETWEEN MATRIX GEOMETRIC ANALYSIS AND THE PROBABILITY GENERATING FUNCTIONS METHOD

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Two main methods are used to solve continuous-time quasi birth-and-death processes: matrix geometric (MG) and probability generating functions (PGFs). MG requires a numerical solution (via successive substitutions) of a matrix quadratic equation $A_0 + RA_1 + R^2A_2 = 0$. PGFs involve a row vector $\vec{G}(z)$ of unknown generating functions satisfying $H(z)\vec{G}(z)^{\mathrm{T}} = \vec{b}(z)^{\mathrm{T}}$, where the row vector $\vec{b}(z)$ contains unknown "boundary" probabilities calculated as functions of roots of the matrix $H(z)$. We show that: (a) $H(z)$ and $\vec{b}(z)$ can be explicitly expressed in terms of the triple A_0, A_1, and A_2; (b) when each matrix of the triple is lower (or upper) triangular, then (i) R can be explicitly expressed in terms of roots of $\det[H(z)]$; and (ii) the stability condition is readily extracted.

Keywords: continuous-time QBD processes, probability generating functions, matrix geometric, calculation of the rate matrix R

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1. INTRODUCTION

Continuous-time quasi birth-and-death (CTQBD) processes have been studied extensively in the literature and applied to solve a large number of problems (see e.g., books by Neuts [13] and Latouche and Ramaswami [11]). A CTQBD process is a two-dimensional continuous-time Markov process where one (bounded) dimension represents the so-called "phases" of the process, and the other (unbounded) dimension represents the so-called "levels." In this work we consider a CTQBD process with $n+1$ phases, denoted by the index $j = 0, 1, 2, \ldots, n$, and with infinite number of levels, denoted by the index $i = 0, 1, 2, \ldots$. For such a process let (L_1, L_2) denote the two-dimensional system state, where L_1 represents the levels and L_2 the phases.

Let $p_{i,j} \equiv P(L_1 = i, L_2 = j)$ denote the steady-state joint probabilities of the system states.

Two solution methods are often used to analyze and solve such problems: (i) via probability generating functions (PGFs) and (ii) via matrix geometric (MG) analysis. These methods are summarized below.

1.1 The PGF method

For this method one first defines $n+1$ partial PGFs, one for each phase, as follows:

$$G_j(z) = \sum_{i=0}^{\infty} p_{i,j} z^i, \quad j = 0, 1, 2, \ldots, n; \ |z| \leq 1. \tag{1}$$

Then, by formulating explicit balance equations for each state, multiplying every equation with index (i, j) by z^i, and then summing over all i for each j separately, one obtains (after some algebra) a finite set of $n+1$ linear equations for the $n+1$ unknown PGFs, which can be expressed as

$$H(z)\vec{G}(z)^{\mathrm{T}} = \vec{b}(z)^{\mathrm{T}}, \tag{2}$$

where $H(z)$ is an $(n+1)$-dimensional square matrix based on the system's parameters; $\vec{G}(z) = (G_0(z), G_1(z), \ldots, G_n(z))$ is a row vector of the unknown PGFs; and $\vec{b}(z) = (b_0(z), b_1(z), \ldots, b_n(z))$ is the right-hand side row vector of Eq. (2).

To obtain each $G_j(z)$ one uses Cramer's rule. That is,

$$G_j(z) = \frac{\det[H_j(z)]}{\det[H(z)]}, \quad j = 0, 1, 2, \ldots, n, \tag{3}$$

where $H_j(z)$ is a matrix obtained from $H(z)$ by replacing its j-th column with the right-hand side vector $\vec{b}(z)^{\mathrm{T}}$. However, the vector $\vec{b}(z)$ contains a finite number of unknown probabilities, usually called "boundary" probabilities. To obtain the latter probabilities one calculates numerically the roots of $\det[H(z)]$, and then argues that since $G_j(z)$ is an analytic function for $|z| \leq 1$, every root of $\det[H(z)]$ within $|z| \leq 1$ is also a root of $\det[H_j(z)]$. Let $\{z_v\}$ be the set of roots of $\det[H(z)]$. By using the appropriate root for each $\det[H_j(z)]$, $j = 0, 1, 2, \ldots, n$, one obtains a linear set of equations $\det[H_j(z_v)] = 0$, $j = 0, 1, 2, \ldots, n$, where the unknowns are the boundary probabilities. Using the required number of roots within the open interval $(0, 1)$, together with (if needed) direct equations from the set of balance equations, one obtains a unique linear set of independent equations in the above probabilities, which is solved numerically, so that the generating functions can be obtained (see e.g., [12, 15]). Consequently, by using the PGFs, the steady state probabilities and the system's various performance measures are derived (see e.g., [1, 2, 9, 14, 16, 17, 18, 19]).

1.2 The MG method

For this method of solution, let $\vec{p}_i = (p_{i,0}, p_{i,1}, \ldots, p_{i,n})$ denote the probability row vector of the states of level $i = 0, 1, 2, \ldots$, and let $\vec{p} = (\vec{p}_0, \vec{p}_1, \vec{p}_2, \ldots)$ denote the row vector of all system's probabilities. Then, by formulating accordingly an infinitesimal generator matrix Q, the system's set of balance equations and its solution for the probability vector $\vec{p}$ is given by

$$\vec{p}Q = \vec{0}, \ \vec{p} \cdot \vec{e} = 1, \tag{4}$$

where $\vec{0}$ is a row vector with all its elements equal to 0, and $\vec{e}$ is a column vector of ones.

A general form of the matrix Q looks like

$$Q = \begin{pmatrix} B_{0,0} & B_{0,1} & \cdots & B_{0,m} & 0 & 0 & 0 & 0 & \cdots \\ B_{1,0} & B_{1,1} & \cdots & B_{1,m} & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\ B_{m,0} & B_{m,1} & \cdots & B_{m,m} & A_0 & 0 & 0 & 0 & \cdots \\ B_{m+1,0} & B_{m+1,1} & \cdots & B_{m+1,m} & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & A_2 & A_1 & A_0 & \\ 0 & 0 & \cdots & 0 & 0 & 0 & A_2 & A_1 & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & A_2 & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}, \qquad \mathbf{(5)}$$

where the initial $m+1$ block-columns of matrices $B_{i,j}$ deal with the first $m+1$ levels. Each matrix $B_{i,j}$, $i = 0, 1, 2, \ldots, m+1$, $j = 0, 1, 2, \ldots, m$, may be a matrix of zeros, one of the triple matrices (A_0, A_1, A_2), or some other matrix. From block-column $m+1$ and on, the triple of $(n+1)$-dimensional square matrices A_0, A_1, and A_2 repeats itself indefinitely in each block-column. At column $j = m+1$ the triple begins from block-row $i = m$ with a diagonal down shift of one block from row to row.

Following Neuts [13] the steady state probability vectors can be calculated by

$$\vec{p}_i = \vec{p}_m R^{i-m}, \quad i = m, m+1, m+2, \ldots, \qquad \mathbf{(6)}$$

where using $\vec{p}_{m+1} = \vec{p}_m R$, the vector $\vec{p}_m$ (as well as $\vec{p}_0, \vec{p}_1, \ldots, \vec{p}_{m-1}$) is calculated by the set of equations

$$\sum_{i=0}^{m+1} \vec{p}_i B_{i,j} = \vec{0}, \quad j = 0, 1, 2, \ldots, m,$$

$$\left(\sum_{i=0}^{m-1} \vec{p}_i + \vec{p}_m [I - R]^{-1} \right) \vec{e} = 1,$$

and the so-called "rate matrix" R is a square matrix which is a minimal non-negative solution (see [13, Chap. 3, p. 82]) of the matrix quadratic equation

$$A_0 + RA_1 + R^2 A_2 = 0. \qquad \mathbf{(7)}$$

In most cases the matrix R is numerically calculated via successive substitutions. Several iterative algorithmic methods have been developed for this purpose (see e.g., [11, Chap. 8, 4, 3]), although there are structured special cases (see Section 1.3) that allow for an efficient solution of the non-linear matrix equation. Finally, by using the matrix R and the probability vectors $\{\vec{p}_i\}$ one can calculate appropriate performance measures of the system.

1.3 Contribution

As indicated, both the PGF and the MG methods require numerical calculations: the first requires calculations of the roots of $\det[H(z)]$, while the latter requires calculation of the rate matrix R, where both calculations are based on the *same* system's parameters. Several relationships between the methods have been established and are reported in the above-mentioned books. Additional results have been derived recently (see e.g., [16, 14, 18, 9, 2]). We also refer the reader to a recent paper by Kapodistria and Palmowski [10] and references therein, in which MG approach for solving certain random walk processes is discussed where both the phase and the level dimensions are unbounded.

In this paper, we add a few relationships between the above two solution methods. Our contribution is four-fold:

- Derivation of closed-form expressions for $H(z)$ and $\vec{b}(z)$ of the PGF method in terms of the triple matrices A_0, A_1, and A_2 used to calculate the rate matrix R.
- Showing that when each of the matrices A_0, A_1, and A_2 is lower triangular, or when all three are upper triangular, the diagonal elements of R are expressed as functions of the *roots* of $\det[H(z)]$.
- Obtaining directly calculated (*finite* sums) expressions of the entries of R in cases where the matrices A_0, A_1, and A_2 are each lower triangular, or when all three are upper triangular.
- Deriving readily extracted stability condition when the above triple is lower (upper) triangular.

We note that Latouche and Ramaswami [11, Chap. 8, p. 197], expressed the matrix R in terms of the matrices A_0 and A_1 in the case where $A_2 = \vec{v} \cdot \vec{\alpha}$, where $\vec{v}$ is a column vector and $\vec{\alpha}$ is a row vector. Similarly, R can be expressed in terms of A_2 and A_1 when A_0 is a product of a column vector and a row vector. Further to that result, Van Leeuwaarden and Winands [20] showed that for a specific class of QBD processes the rate matrix R can be determined while based on probabilistic arguments "by monitoring the QBD process from the time it leaves a certain level until it returns to that same level for the first time." Van Houdt and Van Leeuwaarden [21] considered discrete time M/G/1-type and tree-like QBD Markov chains where A_0, A_1, and A_2 are triangular and derived directly calculated expressions (in the form of *infinite* series) for a matrix G, which is the solution of the matrix equation $G = \sum_{k=0}^{2} A_k G^k$, stating that a similar approach can be taken on the form $R = \sum_{k=0}^{2} R^k A_k$. In an earlier work [22] it is shown that the infinite series can be written in terms of hypergeometric functions. Very recently Hanukov et al. [5–7] considered queueing-inventory problems with "preliminary services" and derived explicit expressions for the entries of $R = [r^{v,t}]$, where some coefficients of $r^{v,t}$ are Catalan numbers. Phung-Duc [19] derived explicit expressions of R for an M/M/c/Setup queue and presented a computational complexity comparison between the generating function approach and the matrix analytic method.

For sake of clarity of exposition, we summarize in Section 1.4 the notations used in this paper.

1.4 Notation

(L_1, L_2)—The two-dimensional system state, where L_1 represents the levels and L_2 the phases.

$p_{i,j} \equiv P(L_1 = i, L_2 = j)$—The steady-state joint probabilities of the system states. $i = 0, 1, 2, \ldots,\ j = 0, 1, 2, \ldots, n$.

$G_j(z)$—Partial PGF for phase $j = 0, 1, 2, \ldots, n$.

$H(z) \equiv [h^{v,t}(z)]$—An $(n+1)$-dimensional square matrix based on the system's parameters.

$\vec{b}(z)$—The right-hand side row vector of Eq. (2).

Q—An infinitesimal generator matrix.

$\vec{p}_i = (p_{i,0}, p_{i,1}, \ldots, p_{i,n})$—The probability row vector of the states of level $i = 0, 1, 2, \ldots$.

$\vec{p} = (\vec{p}_0, \vec{p}_1, \vec{p}_2, \ldots)$—The row vector of all system's probabilities.

$\vec{e}$—A column vector of ones.

$\vec{0}$—A row vector with all its elements equal to 0.

$B_{i,j}$—Initial sub-matrices of Q, $i = 0, 1, 2, \ldots, m+1$, $j = 0, 1, 2, \ldots, m$.

$A_0 \equiv [a_0^{v,t}]$, $A_1 \equiv [a_1^{v,t}]$, and $A_2 \equiv [a_2^{v,t}]$—A triple of sub-matrices of Q repeating itself indefinitely in each block-column, starting from block-column $m+1$.

$R \equiv [r^{v,t}]$—The square "rate" matrix satisfying the matrix quadratic equation $A_0 + RA_1 + R^2 A_2 = 0$.

$r^{2,v,t}$—Entries of R^2.

z_v—A root of $\det[H(z)]$.

$A \equiv [a^{v,t}] = A_0 + A_1 + A_2$.

$\vec{\pi} = (\pi_0, \pi_1, \pi_2 \ldots \pi_n)$—The unique solution of $\vec{\pi} A = \vec{0}$, $\vec{\pi}\vec{e} = 1$.

2. CLOSED FORM EXPRESSIONS FOR $H(z)$ AND $\vec{b}(z)$ IN TERMS OF A_0, A_1, AND A_2

In this section, we derive closed form expressions for $H(z)$ and $\vec{b}(z)$, so that the set of linear equations given by Eq. (2) can be obtained with no need in specifically formulating the balance equations and constructing the equations for the PGFs, as described in Section 1.1.

We now rewrite the PGF row vector as

$$\vec{G}(z) = \sum_{i=0}^{\infty} \vec{p}_i z^i \tag{8}$$

and use it to derive closed form expressions for $H(z)$ and $\vec{b}(z)$ in terms of the triple (A_0, A_1, A_2), as stated in Theorem 1:

Theorem 1:

$$H(z) = (z^2 A_0 + z A_1 + A_2)^T$$

and

$$\vec{b}(z) = \sum_{j=0}^{m-1} z^{j+2} \vec{p}_j A_0 + \sum_{j=0}^{m} z^{j+1} \vec{p}_j A_1 + \sum_{j=0}^{m+1} z^j \vec{p}_j A_2 - \sum_{j=0}^{m} \sum_{i=0}^{m+1} z^{j+1} \vec{p}_i B_{i,j}.$$

Proof: Consider the matrix Q in its form (5). By Eq. (4) we have

$$\sum_{i=0}^{m+1} \vec{p}_i B_{i,j} = \vec{0}, \quad j = 0, 1, \ldots, m, \tag{9}$$

$$\vec{p}_{j-1} A_0 + \vec{p}_j A_1 + \vec{p}_{j+1} A_2 = \vec{0}, \quad j = m+1, m+2, \ldots, \infty. \tag{10}$$

By multiplying each equation by z^j and summing over j we get

$$\sum_{j=0}^{m} \sum_{i=0}^{m+1} z^j \vec{p}_i B_{i,j} + \sum_{j=m+1}^{\infty} z^j \vec{p}_{j-1} A_0 + \sum_{j=m+1}^{\infty} z^j \vec{p}_j A_1 + \sum_{j=m+1}^{\infty} z^j \vec{p}_{j+1} A_2 = \vec{0}$$

or

$$\sum_{j=0}^{m}\sum_{i=0}^{m+1} z^j \vec{p}_i B_{i,j} + \left(z\sum_{j=0}^{\infty} z^j \vec{p}_j A_0 - z\sum_{j=0}^{m-1} z^j \vec{p}_j A_0 \right) + \left(\sum_{j=0}^{\infty} z^j \vec{p}_j A_1 - \sum_{j=0}^{m} z^j \vec{p}_j A_1 \right)$$
$$+ \left(z^{-1}\sum_{j=0}^{\infty} z^j \vec{p}_j A_2 - z^{-1}\sum_{j=0}^{m+1} z^j \vec{p}_j A_2 \right) = \vec{0}.$$

Multiplying by z and substituting Eq. (8) leads to

$$z^2\vec{G}(z)A_0 + z\vec{G}(z)A_1 + \vec{G}(z)A_2 = z^2\sum_{j=0}^{m-1} z^j \vec{p}_j A_0 + z\sum_{j=0}^{m} z^j \vec{p}_j A_1$$
$$+ \sum_{j=0}^{m+1} z^j \vec{p}_j A_2 - \sum_{j=0}^{m}\sum_{i=0}^{m+1} z^{j+1} \vec{p}_i B_{i,j},$$

or

$$\vec{G}(z)(z^2A_0 + zA_1 + A_2) = \sum_{j=0}^{m-1} z^{j+2} \vec{p}_j A_0 + \sum_{j=0}^{m} z^{j+1} \vec{p}_j A_1$$
$$+ \sum_{j=0}^{m+1} z^j \vec{p}_j A_2 - \sum_{j=0}^{m}\sum_{i=0}^{m+1} z^{j+1} \vec{p}_i B_{i,j}, \tag{11}$$

which completes the proof (see Eq. (2)). ∎

Note that a similar result for the matrix $H(z)$ as given in the first part of Theorem 1, has been established in Altman et al. [1] for a specific model.

3. CONNECTIONS BETWEEN $H(z)$ AND R AND ITS DIRECT CALCULATION

In this section, we derive explicit relations between the matrix $H(z)$ and the entries of the matrix R for cases where each matrix A_0, A_1, and A_2 is lower triangular or when all three are upper triangular.

Denote the elements of the various matrices as follows:

$$R \equiv [r^{v,t}],\ R^2 \equiv [r^{2,v,t}],\ A_0 \equiv [a_0^{v,t}],\ A_1 \equiv [a_1^{v,t}], A_2 \equiv [a_2^{v,t}],\ H(z) \equiv [h^{v,t}(z)].$$

Lemma 1: *If the matrices A_0, A_1, and A_2 are each lower triangular, namely $a_0^{v,t} = 0$, $a_1^{v,t} = 0$, and $a_2^{v,t} = 0$ for $v < t$, then*

(i) $h^{v,t}(z) = 0$ *for $v > t$, i.e., $H(z)$ is upper triangular.*
(ii) $r^{v,t} = 0$ *for $v < t$, i.e., R is lower triangular.*
(iii) $r^{2,v,t} = 0$ *for $v < t$, i.e., R^2 is also lower triangular.*
(iv) $r^{2,v,v} = (r^{v,v})^2$, $\forall v$

Proof: (i) Follows immediately from Theorem 1.

(ii) According to Neuts [13, Chap. 1, p. 8] and Haviv [8, Chap. 12, p. 202], starting from state (i, v) for any $i > m$, $r^{v,t}$ is equal to the expected number of visits in state $(i+1, t)$ before the process first re-enters level i. Since, for $v < t$, $a_0^{v,t} = 0$, $a_1^{v,t} = 0$, and $a_2^{v,t} = 0$, there is no feasible way to visit the state $(i+1, t)$ before re-entering level i, implying that $r^{v,t} = 0$ for $v < t$.

(iii) The entries of R^2 are given by $r^{2,v,t} = \sum_{\tau=0}^{n} r^{v,\tau} r^{\tau,t}$. By (ii), $r^{\tau,t} = 0$ for $\tau < t$ and $r^{v,\tau} = 0$ for $v < \tau$. Since $v < t$, we get $r^{2,v,t} = \sum_{\tau=0}^{n} r^{v,\tau} r^{\tau,t} = 0$.

(iv) The entries of the main diagonal of R^2 are given by $r^{2,v,v} = \sum_{\tau=0}^{n} r^{v,\tau} r^{\tau,v}$. By (ii), $r^{\tau,v} = 0$ for $\tau < v$ and $r^{v,\tau} = 0$ for $v < \tau$. Thus, $r^{2,v,v} = \sum_{\tau=0}^{n} r^{v,\tau} r^{\tau,t} = (r^{v,v})^2$. ■

Consider now $H(z)$. By (i) of the above Lemma, $h^{v,t}(z) = 0$ for $v < t$. Thus, the determinant of $H(z)$ is equal to the product of its main diagonal entries. That is, $\det[H(z)] = \prod_{\forall v} h^{v,v}(z)$. Hence, calculating the roots of this determinant, namely finding the roots of the polynomial equation $\prod_{\forall v} h^{v,v}(z) = 0$, translates to a set of $n+1$ equations $h^{v,v}(z) = 0$, $v = 0, 1, 2, \ldots, n$, which, by Theorem 1, results in $z^2 a_0^{v,v} + z a_1^{v,v} + a_2^{v,v} = 0$, $\forall v$.

Let z_v be a root of $\det[H(z)]$. Then, since $a_1^{v,v} < 0$ and $|a_1^{v,v}| \geq a_0^{v,v} + a_2^{v,v}$, the non-negative roots of $\det[H(z)]$ (that will be used in the sequel) are given by

$$z_v = \begin{cases} \dfrac{-a_1^{v,v} - \sqrt{(a_1^{v,v})^2 - 4a_0^{v,v}\, a_2^{v,v}}}{2a_0^{v,v}}, & a_0^{v,v} > 0,\ a_2^{v,v} > 0 \\ \dfrac{-a_1^{v,v}}{a_0^{v,v}}, & a_0^{v,v} > 0,\ a_2^{v,v} = 0 \\ \dfrac{-a_2^{v,v}}{a_1^{v,v}}, & a_0^{v,v} = 0 \end{cases}, \ \forall v. \tag{12}$$

We are ready now to establish the connections between $H(z)$ and R.

THEOREM 2: *If $a_0^{v,t} = 0$, $a_1^{v,t} = 0$, and $a_2^{v,t} = 0$ for $v < t$, then*

$$r^{v,v} = \begin{cases} z_v \dfrac{a_0^{v,v}}{a_2^{v,v}}, & a_0^{v,v} > 0,\ a_2^{v,v} > 0 \\ z_v = 0, & a_0^{v,v} = a_2^{v,v} = 0 \\ \dfrac{1}{z_v}, & a_0^{v,v} > 0, a_2^{v,v} = 0 \end{cases}, \ \forall v \tag{13}$$

PROOF: By Eq. (7) and the Lemma, $r^{v,v}$ is calculated by the following set of equations: $a_0^{v,v} + r^{v,v} a_1^{v,v} + (r^{v,v})^2 a_2^{v,v} = 0$, $\forall v$. Since the minimal nonnegative solution of R must be taken [13], $r^{v,v}$ is given by

$$r^{v,v} = \begin{cases} \dfrac{-a_1^{v,v} - \sqrt{(a_1^{v,v})^2 - 4a_0^{v,v}\, a_2^{v,v}}}{2a_2^{v,v}}, & a_2^{v,v} > 0,\ a_0^{v,v} > 0 \\ 0, & a_2^{v,v} > 0,\ a_0^{v,v} = 0 \\ \dfrac{-a_0^{v,v}}{a_1^{v,v}}, & a_2^{v,v} = 0 \end{cases}, \ \forall v. \tag{14}$$

Combining (12) and (14) leads to (13), which completes the proof. ■

We further note that the first-type relation of Theorem 2 (Eq. (13)) was revealed in Perel and Yechiali [16] when analyzing the so-called "Israeli queue." The first- and third-type relations were indicated in Paz and Yechiali [14] when analyzing an M/M/1 queue in random environment with disasters. The third-type was also pointed at in Armony et al. [2] when studying a multi-server queueing system with cross-selling. (see further discussion of those papers in Section 4).

The Lemma above establishes that $r^{v,t}=0$ for $v<t$, while Theorem 2 gives the values of $r^{v,v}$ for all v. In order to complete the calculation of all entries of R, it remains to obtain the values of $r^{v,t}$ for $v>t$. The following theorem shows how the latter values can be calculated directly, thus concluding an explicit derivation of all entries of the rate matrix R, which eliminates the need to use numerical successive substitutions when calculating R.

THEOREM 3: *If $a_0^{v,t}=0,\ a_1^{v,t}=0,$ and $a_2^{v,t}=0$ for $v<t$, then*

$$r^{v,t}=-\frac{a_0^{v,t}+\sum_{k=t+1}^{v} r^{v,k}a_1^{k,t}+\sum_{\tau=t+1}^{v-1} r^{v,\tau}r^{\tau,t}a_2^{t,t}+\sum_{k=t+1}^{v}\sum_{\tau=k}^{v} r^{v,\tau}r^{\tau,k}a_2^{k,t}}{a_1^{t,t}+a_2^{t,t}(r^{t,t}+r^{v,v})}\ \text{for } v>t$$

PROOF: By Eq. (7), $r^{v,t}$ is calculated by the following equation: $a_0^{v,t}+\sum_{k=0}^{n} r^{v,k}a_1^{k,t}+\sum_{k=0}^{n} r^{2,v,k}a_2^{k,t}=0$, where $r^{2,v,k}=\sum_{\tau=0}^{n} r^{v,\tau}r^{\tau,k}$. By the Theorem's condition $a_1^{k,t}=0$ and $a_2^{k,t}=0$ for $k<t$. By the Lemma, $r^{v,k}=0$ for $k>v$, $r^{v,\tau}=0$ for $\tau>v$ and $r^{\tau,k}=0$ for $k>\tau$. Thus, the above equation reduces to $a_0^{v,t}+\sum_{k=t}^{v} r^{v,k}a_1^{k,t}+\sum_{k=t}^{v}\sum_{\tau=k}^{v} r^{v,\tau}r^{\tau,k}a_2^{k,t}=0$, which can be written as follows: $a_0^{v,t}+\left(r^{v,t}a_1^{t,t}+\sum_{k=t+1}^{v} r^{v,k}a_1^{k,t}\right)+\left(r^{v,t}r^{t,t}a_2^{t,t}+\sum_{\tau=t+1}^{v-1} r^{v,\tau}r^{\tau,t}a_2^{t,t}+\sum_{k=t+1}^{v}\sum_{\tau=k}^{v} r^{v,\tau}r^{\tau,k}a_2^{k,t}+r^{v,v}r^{v,t}a_2^{t,t}\right)=0$. Eliminating $r^{v,t}$ completes the proof. ■

Note the order of calculation in Theorem 3: the main diagonal is calculated first; then the one bellow it, and so on, until reaching the last element in the bottom left corner of the matrix R.

We also note that a non-successive substitution procedure to calculate the entries of the matrix G is given in [21] where its entries are expressed as infinite series.

When each of the matrices A_0, A_1, and A_2 is lower triangular, the stability condition for the QBD process is readily obtained, *with no need for any calculation.*

THEOREM 4 (Stability Condition): *If $a_0^{v,t}=0,\ a_1^{v,t}=0,$ and $a_2^{v,t}=0$ for $v<t$, then the stability condition is readily given by $a_0^{0,0}<a_2^{0,0}$.*

PROOF: Let $A\equiv[a^{v,t}]=A_0+A_1+A_2$ and let the row vector $\vec{\pi}=(\pi_0,\pi_1,\pi_2\ldots\pi_n)$ be the unique solution of $\vec{\pi}A=\vec{0}$, $\vec{\pi}\vec{e}=1$. That is,

$$\sum_{v=0}^{n}\pi_v a^{v,t}=0,\quad t=1,2,3,\ldots,n, \tag{15}$$

$$\sum_{v=0}^{n}\pi_v=1. \tag{16}$$

The general stability condition [13, Chap. 3, p. 83] is given by

$$\vec{\pi}\,A_0\,\vec{e}<\vec{\pi}\,A_2\,\vec{e}. \tag{17}$$

We first prove, by induction, that $\pi_v=0$ for all $v=1,2,3,\ldots,n$.

(i) Starting from the last equation of (15) (i.e., $t = n$) we have $\sum_{v=0}^{n} \pi_v a^{v,n} = 0$. Since A is lower triangular (i.e., $a^{v,n} = 0$ for all $v = 0, 1, 2, \ldots, n-1$), the latter equation reduces to $\pi_n a^{n,n} = 0$, implying that $\pi_n = 0$.

(ii) We now assume that $\pi_v = 0$ for $v = n, n-1, \ldots, t+2, t+1$ and show that $\pi_t = 0$ for every $t \geq 1$. Since A is lower triangular, $a^{v,t} = 0$ for all $v = 0, 1, 2, \ldots, t-1$. By the induction assumption, $\pi_v = 0$ for all $v = t+1, t+2, \ldots, n$. Thus, Eq. (15) reduces to $\pi_t a^{t,t} = 0$, implying that $\pi_t = 0$, as claimed. By substituting $\pi_t = 0$, for all $t = 1, 2, 3, \ldots, n$ in Eq. (16), we get $\pi_0 = 1$. Finally, by substituting $\vec{\pi} = (1, 0, 0, \ldots, 0)$ in Eq. (17) the stability condition reduces to $a_0^{0,0} < a_2^{0,0}$. ■

This result can be explained intuitively as follows: the matrix A_0 represents the process level's forward direction, whereas A_2 represents the backward direction. Since the matrices A_0, A_1, and A_2 are lower triangular, the process reaches phase 0 with probability 1, and within this phase, stability requires that the forward rate ($a_0^{0,0}$) is smaller than the backward rate ($a_2^{0,0}$).

Note: All entries in the zero-th row in A are zeros. Since there is a unique solution, then $a^{t,t} < 0$ for $t = 1, 2, 3, \ldots, n$. This follows since the sum of entries of each row in A is zero, and $a^{v,t} \geq 0$ for $v \neq t$, with at least one $a^{v,t} > 0$.

The case where each of the three matrices A_0, A_1, and A_2 is upper-triangular is treated in the following theorem.

THEOREM 5: *Suppose that $a_0^{v,t} = 0$, $a_1^{v,t} = 0$, and $a_2^{v,t} = 0$ for $v > t$. Then, the Lemma holds for $v > t$, as well as Eqs. (13) and (14) of Theorem 2. The stability condition is given by $a_0^{n,n} < a_2^{n,n}$. The entries $r^{v,t}$ for $v < t$ are given by*

$$r^{v,t} = -\frac{a_0^{v,t} + \sum_{k=v}^{t-1} r^{v,k} a_1^{k,t} + \sum_{\tau=v+1}^{t-1} r^{v,\tau} r^{\tau,t} a_2^{t,t} + \sum_{k=v}^{t-1} \sum_{\tau=v}^{k} r^{v,\tau} r^{\tau,k} a_2^{k,t}}{a_1^{t,t} + a_2^{t,t}(r^{t,t} + r^{v,v})}.$$

PROOF: The proof follows by similar arguments leading to the Lemma and to Theorem 2, where the expression of $r^{v,t}$ for $v < t$ is derived similarly to Theorem 3, with the required change of the summation indices.

There are cases where the matrices A_0, A_1, and A_2 are only partial lower or upper triangular. Such cases are treated in the following two corollaries. ■

COROLLARY 1: *Consider the triple A_0, A_1, and A_2, each of which not necessarily lower triangular. Suppose that, from some column k and up, all entries above the main diagonal of each matrix are zeros, that is, $a_0^{v,t} = 0$, $a_1^{v,t} = 0$, and $a_2^{v,t} = 0$ for $v < t$, where $t \geq k$. Then,*

(I) *similarly to the Lemma,*

(a) $h^{v,t}(z) = 0$ *for* $t = k, k+1, \ldots, n$; $v = 0, 1, \ldots, t-1$,

(b) $r^{v,t} = 0$ *for* $t = k, k+1, \ldots, n$; $v = 0, 1, \ldots, t-1$,

(c) $r^{2,v,t} = 0$ *for* $t = k, k+1, \ldots, n$; $v = 0, 1, \ldots, t-1$.

(II) *Similarly to Theorem 2,*

$$r^{v,v} = \begin{cases} z_v \dfrac{a_0^{v,v}}{a_2^{v,v}}, & a_0^{v,v} > 0,\ a_2^{v,v} > 0 \\ z_v = 0, & a_0^{v,v} = a_2^{v,v} = 0 \\ \dfrac{1}{z_v}, & a_0^{v,v} > 0, a_2^{v,v} = 0 \end{cases}, \quad v = k, k+1, \ldots, n.$$

(III) *Similarly to Theorem* 3,

$$r^{v,t} = -\frac{a_0^{v,t} + \sum_{k=t+1}^{v} r^{v,k} a_1^{k,t} + \sum_{\tau=t+1}^{v-1} r^{v,\tau} r^{\tau,t} a_2^{t,t} + \sum_{k=t+1}^{v} \sum_{\tau=k}^{v} r^{v,\tau} r^{\tau,k} a_2^{k,t}}{a_1^{t,t} + a_2^{t,t}(r^{t,t} + r^{v,v})}$$

for $t = k, k+1, \ldots, n;\ v = t+1, t+2, \ldots, n.$

Corollary 2: *Consider the triple* A_0, A_1, *and* A_2. *Suppose that, for each matrix, from some column* k *and below, all entries below the main diagonal are zeros, that is* $a_0^{v,t} = 0$, $a_1^{v,t} = 0$, *and* $a_2^{v,t} = 0$ *for* $v > t$, *where* $t \leq k$. *Then,*

(i) *part* (I) *from Corollary 1 holds for* $t = 0, 1, \ldots, k;\ v = t+1, t+2, \ldots, n$,

(ii) *Part* (II) *from Corollary 1 holds for* $v = 0, 1, \ldots, k$, *and*

(iii) $r^{v,t} = -\frac{a_0^{v,t} + \sum_{k=v}^{t-1} r^{v,k} a_1^{k,t} + \sum_{\tau=v+1}^{t-1} r^{v,\tau} r^{\tau,t} a_2^{t,t} + \sum_{k=v}^{t-1} \sum_{\tau=v}^{k} r^{v,\tau} r^{\tau,k} a_2^{k,t}}{a_1^{t,t} + a_2^{t,t}(r^{t,t} + r^{v,v})}$ *for* $t = 0, 1, \ldots, k;$ $v = 0, 1, \ldots, t-1.$

We refer the reader to [5] *where a system with such structure is analyzed along with numerical calculations.*

4. EXAMPLES

In this section, we present a few examples of interesting problems analyzed in the literature that are formulated as QBD processes with lower (upper) triangular matrices A_0, A_1, and A_2, and show how the solutions can be obtained, and how the stability condition is derived, by applying our closed-form expressions.

4.1 Example 1: A queueing-inventory system with preliminary services

In Hanukov et al. [6], the following QBD process is introduced. Consider a single-server system with a Poisson arrival rate λ and exponentially-distributed service time with mean $1/\mu$. The service can be split into two consecutive stages. The first, denoted “preliminary service” (PS), can be performed in the absence of customers, and its outcome can be stored until a full service is requested. The second stage, denoted “complementary service” (CS), requires the actual presence of a customer to be completed. When the system is empty, the server produces an inventory of PSs at a Poisson rate α. The aim of this policy is to utilize the server’s idle time in order to reduce customers sojourn times. The inventory-size of PSs is limited to n units, and when the number of stored PSs reaches n, the server stops producing PSs and stays idle. If a customer arrives at the front of the queue and a PS is available, the server immediately starts rendering the CS for that customer. Otherwise, the customer receives a “full service” (FS). The CS time is assumed to be exponentially distributed with mean $1/\beta$, where $\beta > \mu$. Such systems are common in the fast food industry.

Let L and S be the number of customers and the number of PSs in the system in steady state, respectively. A transition rate diagram of the two-dimensional process (L, S) is depicted in Hanukov et al. [6]. The infinitesimal generator matrix Q is given by

$$Q = \begin{pmatrix} B & A_0 & 0 & 0 \cdots \\ A_2 & A_1 & A_0 & 0 \cdots \\ 0 & A_2 & A_1 & A_0 \\ \vdots & & \ddots & \ddots \ddots \end{pmatrix},$$

where the matrices B, A_0, A_1, and A_2, each of order $(n+1)\times(n+1)$, are given by

$$B=\begin{pmatrix}-(\alpha+\lambda) & \alpha & 0 & \cdots & 0 & 0\\ 0 & -(\alpha+\lambda) & \alpha & & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots\\ 0 & 0 & 0 & & -(\alpha+\lambda) & \alpha\\ 0 & 0 & 0 & \cdots & 0 & -\lambda\end{pmatrix},\quad A_0=\begin{pmatrix}\lambda & 0 & \cdots & 0 & 0\\ 0 & \lambda & & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & & \lambda & 0\\ 0 & 0 & \cdots & 0 & \lambda\end{pmatrix}$$

$$A_1=\begin{pmatrix}-(\mu+\lambda) & 0 & \cdots & 0 & 0\\ 0 & -(\beta+\lambda) & & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & & -(\beta+\lambda) & 0\\ 0 & 0 & \cdots & 0 & -(\beta+\lambda)\end{pmatrix},\quad A_2=\begin{pmatrix}\mu & 0 & \cdots & 0 & 0\\ \beta & 0 & \cdots & 0 & 0\\ 0 & \beta & \ddots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \beta & 0\end{pmatrix}.$$

Clearly, each matrix A_0, A_1, and A_2 is lower triangular.
By Theorem 1, $H(z)$ is readily given by

$$H(z)=\begin{pmatrix}\lambda z^2-(\mu+\lambda)z+\mu & \beta & 0 & \cdots & 0 & 0\\ 0 & \lambda z^2-(\beta+\lambda)z & \beta & & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots\\ 0 & 0 & 0 & & \lambda z^2-(\beta+\lambda)z & \beta\\ 0 & 0 & 0 & \cdots & 0 & \lambda z^2-(\beta+\lambda)z\end{pmatrix}.$$

Since, in this example, $m=0$, $B_{0,0}=B$, and $B_{1,0}=A_2$, the expression of $\vec{b}(z)$ is reduced to

$$\begin{aligned}\vec{b}(z)&=z\vec{p}_0A_1+\sum_{j=0}^{1}z^j\vec{p}_jA_2-\sum_{i=0}^{1}z\vec{p}_iB_{i,0}=z\vec{p}_0A_1+\vec{p}_0A_2+z\vec{p}_1A_2-z\vec{p}_0B-z\vec{p}_1A_2\\ &=\vec{p}_0(zA_1+A_2-zB)\end{aligned}$$

Thus, $\vec{b}(z)$ is calculated directly by substituting $\vec{p}_0$, A_1, A_2, and B in the latter expression:

$$\vec{b}(z)^{\mathrm{T}}=\begin{pmatrix}(\alpha z-\mu(z-1))p_{0,0}+\beta p_{0,1}\\ -\alpha zp_{0,0}-(\beta z-\alpha z)p_{0,1}+\beta p_{0,2}\\ -\alpha zp_{0,1}-(\beta z-\alpha z)p_{0,2}+\beta p_{0,3}\\ \vdots\\ -\alpha zp_{0,n-2}-(\beta z-\alpha z)p_{0,n-1}+\beta p_{0,n}\\ -\alpha zp_{0,n-1}-\beta zp_{0,n}\end{pmatrix}.$$

The roots of $\det[H(z)]$ are derived by using Eq. (12). Since $a_0^{0,0}>0$ and $a_2^{0,0}>0$, the first term in Eq. (12) leads to $z_0=1$. As, $a_0^{v,v}>0$ and $a_2^{v,v}=0$ for $v=1,2,\ldots,n$, then

$z_v = (\beta+\lambda)/\lambda$ for $v = 1, 2, \ldots, n$ by using the second term in Eq. (12). The matrix R is calculated as follows: clearly, $r^{v,t} = 0$ for $v < t$. The entries $(r^{v,v})$ of the main diagonal of R can be calculated by Eq. (13) or by Eq. (14) and are given by

$$r^{v,v} = \begin{cases} \lambda/\mu & v = 0 \\ \lambda/(\beta+\lambda) & v \geq 1 \end{cases}.$$

Other entries of R can be calculated by the expression in Theorem 3. In the current example $a_0^{v,t} = 0$ for $v > t$, $a_1^{k,t} = 0$ for $k > t$, $a_2^{t,t} \neq 0$ only for $t = 0$ and $a_2^{k,t} \neq 0$ only for $k = t+1$. Thus, the expression in Theorem 3 reduces to

$$\begin{aligned} r^{v,0} &= -\frac{\sum_{\tau=t+1}^{v-1} r^{v,\tau} r^{\tau,0} a_2^{0,0} + \sum_{\tau=1}^{v} r^{v,\tau} r^{\tau,1} a_2^{1,0}}{a_1^{0,0} + a_2^{0,0}(r^{0,0} + r^{v,v})} \\ &= \frac{\mu \sum_{\tau=1}^{v-1} r^{v,\tau} r^{\tau,0} + \beta \sum_{\tau=1}^{v} r^{v,\tau} r^{\tau,1}}{\mu\beta/(\beta+\lambda)} \text{ for } v = 1, 2, \ldots, n, \\ r^{v,t} &= -\frac{\sum_{\tau=t+1}^{v} r^{v,\tau} r^{\tau,t+1} a_2^{t+1,t}}{a_1^{t,t}} = \frac{\beta \sum_{\tau=t+1}^{v} r^{v,\tau} r^{\tau,t+1}}{\beta+\lambda} \text{ for } v > t \geq 1. \end{aligned}$$

Finally, by Theorem 4, the stability condition $a_0^{0,0} < a_2^{0,0}$ is translated into the simple expression $\lambda < \mu$.

4.2 Example 2: The Israeli Queue with priorities

Consider the so-called "Israeli Queue" (see [17]) where the waiting line is composed of N different groups, with corresponding N "leaders," each standing in front of its group. A newly arriving customer joins group i with probability $q_i \geq 0$ where $\sum_{i=1}^{N} q_i = 1$. Each group is served in one batch and the service time is independent of the group's size. The next group to be served is the one that its leader is the one that has been waiting for the longest time. As an example, one may think of a traffic light where the next green light is given to the direction where the first car has been waiting for the longest time. Consider now the following QBD process introduced in Perel and Yechiali [16]. A single server attends two classes of customers: VIP (class 1, high priority) and regular (class 2, low priority). The VIP customers form a classical infinite-buffer M/M/1 queue, while the customers of class 2 form the so-called Israeli Queue with batch service and at most N groups. The arrival stream of VIP (of regular) customers follows a homogeneous Poisson process with rate λ_1 (λ_2), while service time is exponentially distributed with rate μ_1 (μ_2).

Let L_1 and L_2 be the total number of VIP and of regular customers in the system, respectively. A transition rate diagram of the two-dimensional process (L_1, L_2) is depicted in Perel and Yechiali [16]. The infinitesimal generator matrix Q is given by

$$Q = \begin{pmatrix} B & A_0 & 0 & 0 \cdots \\ A_2 & A_1 & A_0 & 0 \cdots \\ 0 & A_2 & A_1 & A_0 \\ \vdots & & \ddots & \ddots \ddots \end{pmatrix},$$

where the matrices B, A_0, A_1, and A_2, each of order $(n+1)\times(n+1)$, are given by

$$B = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_2 & 0 & \cdots & \cdots & 0 & 0 \\ \mu_2 & -(\lambda_1+\lambda_2(1-p)+\mu_2) & \lambda_2(1-p) & 0 & & 0 & 0 \\ 0 & \mu_2 & -(\lambda_1+\lambda_2(1-p)^2+\mu_2) & \lambda_2(1-p)^2 & & & \\ \vdots & \ddots & & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & & 0 & & -(\lambda_1+\lambda_2(1-p)^{N-1}+\mu_2) & \lambda_2(1-p)^{N-1} \\ 0 & 0 & & 0 & \cdots & \mu_2 & -(\lambda_1+\mu_2) \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -(\lambda_1+\lambda_2+\mu_1) & \lambda_2 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -(\lambda_1+\lambda_2(1-p)+\mu_1) & \lambda_2(1-p) & 0 & & 0 & 0 \\ 0 & 0 & -(\lambda_1+\lambda_2(1-p)^2+\mu_1) & \lambda_2(1-p)^2 & & & \\ \vdots & \ddots & & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & & 0 & & -(\lambda_1+\lambda_2(1-p)^{N-1}+\mu_1) & \lambda_2(1-p)^{N-1} \\ 0 & 0 & & 0 & \cdots & 0 & -(\lambda_1+\mu_1) \end{pmatrix},$$

$$A_0 = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \lambda_1 & 0 \\ 0 & 0 & \cdots & 0 & \lambda_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 & 0 \\ 0 & \mu_1 & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \mu_1 & 0 \\ 0 & 0 & \cdots & 0 & \mu_1 \end{pmatrix}.$$

As can be seen, the matrices A_0, A_1, and A_2 are upper triangular, and so the solution of this QBD process can be easily calculated by our expressions. For example, the stability condition $a_0^{n,n} < a_2^{n,n}$ given in Theorem 5 readily results in $\lambda_1 < \mu_1$.

4.3 Example 3: Multi-server queueing systems with cross-selling

The following cross-selling QBD process was introduced in Armony et al. [2]. Consider a multi-server queueing system with N parallel servers and unlimited waiting room, to which customers arrive according to a Poisson process with rate λ. A customer service has two potential phases. Phase 1 is experienced by every customer with exponential duration having mean $1/\mu$. After a completion of phase 1 service a customer is identified as a so-called “cross-selling” candidate with probability p, or the customer completes service and leaves the system with the complementary probability $q = 1 - p$. If the customer is a cross-selling candidate and the system manager decides to go ahead and discuss a cross-selling deal with the customer, phase 2 of the service begins, having exponential duration with mean $1/\xi$.

Let L be the total number of customers in the system and L_2 be the number of customers in cross-selling. A transition rate diagram of the two-dimensional process (L, L_2) is depicted in Armony et al. [2]. The infinitesimal generator matrix Q is given by

$$Q = \begin{pmatrix}
B_1^{(0)} & B_0^{(0)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
B_2^{(1)} & B_1^{(1)} & B_0^{(1)} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & B_2^{(2)} & B_1^{(2)} & B_0^{(2)} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & B_2^{(N)} & B_1^{(N)} & B_0^{(N)} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & B_2^{(N+1)} & B_1^{(N+1)} & B_0^{(N+1)} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & B_2^{(T)} & B_1^{(T)} & B_0^{(T)} & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & A_2 & A_1 & A_0 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & A_2 & A_1 & A_0 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

where the matrices A_0, A_1, and A_2, each of order $(n+1) \times (n+1)$, are given by

$$A_0 = \begin{pmatrix}
\lambda & 0 & \cdots & 0 & 0 \\
0 & \lambda & & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & & \lambda & 0 \\
0 & 0 & \cdots & 0 & \lambda
\end{pmatrix}, \quad
A_2 = \begin{pmatrix}
N\mu & 0 & 0 & \cdots & 0 & 0 \\
\xi & (N-1)\mu & 0 & \cdots & 0 & 0 \\
0 & 2\xi & (N-2)\mu & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \mu & 0 \\
0 & 0 & 0 & \cdots & N\xi & 0
\end{pmatrix},$$

$$A_1 = \begin{pmatrix}
-(\lambda + N\mu) & 0 & \cdots & 0 & 0 \\
0 & -(\lambda + (N-1)\mu + \xi) & & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & & -(\lambda + \mu + (N-1)\xi) & 0 \\
0 & 0 & \cdots & 0 & -(\lambda + N\xi)
\end{pmatrix}.$$

Evidently, each of the triple is lower triangular, leading to stability condition $\lambda < N\mu$.

4.4 Example 4: An M/M/1 queue in random environment with disasters

Paz and Yechiali [14] analyzed the following model. Consider a M/M/1 type queue operating in a special "random environment" as follows: when in operative phase $i \geq 1$ the system acts as an $\mathrm{M}(\lambda_i)/\mathrm{M}(\mu_i)/1$ queue, with Poisson arrival rate λ_i and service rate μ_i. The duration of time the system resides in phase i is an exponentially distributed random variable with mean $1/\eta_i$, $i = 1, 2, \ldots, n$. Furthermore, when in operative phase $i \geq 1$, the system suffers occasionally a disastrous failure (catastrophe), causing it to move to a "failure" phase, denoted by $i = 0$. A disaster causes all present customers to be cleared out of the system. When in the failure phase $i = 0$, the system undergoes a repair process, having exponentially distributed duration with mean $1/\eta_0$.

Let U be the phase in which the system operates and X denote the number of customers in the system. A transition rate diagram of the two-dimensional process (U, X) is depicted in Paz and Yechiali [14]. The infinitesimal generator matrix Q is given by

$$Q = \begin{pmatrix} B + A_2 + A_1 & A_0 & 0 & 0 & \cdots \\ B + A_2 & A_1 & A_0 & 0 & \cdots \\ B & A_2 & A_1 & A_0 & \\ B & 0 & A_2 & A_1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix},$$

where the matrices A_0, A_1, and A_2, each of order $(n+1) \times (n+1)$, are given by

$$A_0 = \begin{pmatrix} \lambda_0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu_1 & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \mu_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \mu_n \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -(\lambda_0 + \eta_0) & q_1\eta_0 & \cdots & q_{n-1}\eta_0 & q_n\eta_0 \\ 0 & -(\lambda_1 + \eta_1 + \mu_1) & & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & -(\lambda_{n-1} + \eta_{n-1} + \mu_{n-1}) & 0 \\ 0 & 0 & \cdots & 0 & -(\lambda_n + \eta_n + \mu_n) \end{pmatrix}.$$

A_0, A_1, and A_2 are each upper triangular so, by Theorem 5, the stability condition is $\lambda_n < \mu_n$.

5. CONCLUSIONS

In this work, further and explicit relationships are derived between the two mostly used methods to solve CTQBD problems. One method is by constructing a set of linear equations, expressed as $H(z)\vec{G}(z)^{\mathrm{T}} = \vec{b}(z)^{\mathrm{T}}$, where $H(z)$ is a matrix containing the system's parameters, $\vec{G}(z)$ is a row vector of PGFs, while $\vec{b}(z)$ is a row vector containing unknown "boundary probabilities." The derivation of the latter probabilities relies on numerical calculations of the *roots* of $\det[H(z)]$. The other method is based on a MG analysis requiring numerical calculations of the so-called "rate matrix" R, which is obtained by solving a quadratic matrix equation for R. Some specific relations between the roots of $\det[H(z)]$ and the entries of R have been established recently for specific problems (see e.g., [14, 16, 17, 18, 5]). The current work continues in this direction. Specifically, it is shown that the matrix $H(z)$ and the vector $\vec{b}(z)$ can be expressed directly in terms of the matrix-triple (A_0, A_1, A_2) of the MG's CTQBD formulation, thus eliminating the lengthy derivation of the balance equations when using the PGFs method. Furthermore, in cases where each of the matrices A_0, A_1, and A_2 is lower triangular or when all three matrices are upper triangular, it is shown that (i) the entries of the rate matrix R can be calculated directly with no need to apply successive substitution algorithms, thus reducing the calculation effort considerably; (ii) explicit formulas for the diagonal entries of the matrix R in terms of the roots of $\det[H(z)]$ are obtained; and (iii) the system's stability condition is readily derived.

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