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A retrial system with two input streams and two orbit queues

Konstantin Avrachenkov · Philippe Nain · Uri Yechiali

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Abstract Two independent Poisson streams of jobs flow into a single-server service system having a limited common buffer that can hold at most one job. If a type-i job ($i = 1, 2$) finds the server busy, it is blocked and routed to a separate type-i retrial (orbit) queue that attempts to re-dispatch its jobs at its specific Poisson rate. This creates a system with three dependent queues. Such a queueing system serves as a model for two competing job streams in a carrier sensing multiple access system. We study the queueing system using multi-dimensional probability generating functions, and derive its necessary and sufficient stability conditions while solving a Riemann–Hilbert boundary value problem. Various performance measures are calculated and numerical results are presented. In particular, numerical results demonstrate that the proposed multiple access system with two types of jobs and constant retrial rates provides incentives for the users to respect their contracts.

Keywords Retrial queues · Constant retrial rate · Riemann–Hilbert boundary value problem · Carrier sensing multiple access system

Mathematics Subject Classification (2000) 60K25 · 30E25 · 35Q15

K. Avrachenkov (✉) · P. Nain
Inria Sophia Antipolis, Sophia Antipolis Cedex, France
e-mail: K.Avrachenkov@sophia.inria.fr

P. Nain
e-mail: Philippe.Nain@inria.fr

U. Yechiali
Tel Aviv University, Tel Aviv, Israel
e-mail: uriy@post.tau.ac.il

1 Introduction

We investigate a single-server system with two independent exogenous Poisson streams flowing into a common buffer that can hold at most one job. If a type-i job finds the server busy, it is routed to a separate retrial (orbit) queue from which jobs are re-transmitted at an exponential rate. The rates of retransmissions may be different from the rates of the original input streams.

Such a queueing system serves as a model for two competing job streams in a carrier sensing multiple access system, where the jobs, after a failed attempt to network access, wait in an orbit queue [32,33]. An example of carrier sensing multiple access system is a local area computer network (LAN) with bus architecture. The two types of customers can be interpreted as customers with different priority requirements. The constant retrial rate helps to stabilize and control the multiple access system [9]. Jobs with higher retrial rate can be viewed as higher priority jobs. The setting with two levels of priority can be applied to train or vehicular onboard networks, where high priority jobs correspond to critical system control signals and low priority jobs correspond to onboard passenger Internet access traffic.

Queues with blocking and with retrials have been studied extensively in the literature (e.g., [1–8,13–15,19,20,35] and references therein). The important features of the retrial system under consideration are two-class setting and constant retrial rate. While there have been studies of multi-class retrial systems and systems with constant retrial rate, to the best of our knowledge, performance evaluation of the two-class retrial system with a constant retrial rate is carried out in the present paper for the first time. Specifically, the retrial queueing systems with a constant retrial rate and a single type of jobs were considered in [4–7,14,15,20]. A two-class retrial system with a single-server, no waiting room, batch arrivals and classical retrial scheme (when each orbit job retries individually after a random time exponentially distributed with a fixed parameter) was introduced and analyzed by Kulkarni [26]. Then, Falin [18] extended the analysis of the model in [26] to the multi-class setting with arbitrary number of classes. We note that in both [18,26] the authors have only derived expressions for the first two moments. In [25] Grishechkin has established equivalence between the multi-class batch arrival retrial queues with classical retrial policy and branching processes with immigration. However, no closed-form Laplace transforms of the waiting times have been provided.

In [30] a non-preemptive priority mechanism was added to the model of [18,26]. In [28] Langaris and Dimitriou have considered a multi-class retrial system where retrial classes are associated with different phases of service. In both [28,30] only the mean number of jobs in each retrial class was calculated. In our model we consider only the single-server case with no waiting room for primary customers. If one needs to study the case of several servers and/or waiting room for primary customers, we feel that a feasible approach would be to use the matrix-analytic and quasi-Toeplitz Markov chain methods as this has been done for single-class retrial systems in [17,29]. Of course, that approach would not provide explicit analytic formulae but will likely result in efficient numerical algorithms.

We formulate our retrial system as a three-dimensional Markovian queueing network, derive its necessary and sufficient stability conditions and calculate the

Probability Generating Function (PGF) of the joint orbits and main buffer occupation. Recently, these stability conditions have been shown by simulations to hold for a more general system with generally distributed service times [8]. Our general approach for calculating the PGF of the joint buffer occupation is to reduce the problem to a Riemann–Hilbert boundary value problem. To the best of our knowledge, this is the first application of the Riemann–Hilbert boundary value technique to the analysis of retrial queueing systems. For a single-class retrial queue Artalejo and Gomez-Corral in [2] obtained the limiting distribution of the system state via the solution of a Fredholm integral equation. The technique of reducing the solution of certain two-dimensional functional equations for generating functions to the solution of a boundary value problem (typically Rieman–Hilbert or Dirichlet problem)—whose solution is known in closed-form—is due to Fayolle and Iasnogorodski [21]. For a short primer on Riemann–Hilbert boundary value technique specialized to our problem and related work see Sect. 4.1.

Let us outline the structure of the paper. After the Introduction in Sect. 2 we present the model and derive balance equations and generating functions. Necessary stability conditions are obtained in Sect. 3. Using the technique developed by Fayolle and Iasnogoroski [21], in Sect. 4 we show that these generating functions are obtained, in closed-form, via the solution of a Riemann–Hilbert boundary value problem. In order for the paper to be self-contained, we have added a special Sect. 4.1 describing the main steps of the Riemann–Hilbert boundary value technique. In particular, this technique allows us to show that the necessary stability conditions found in Sect. 3 are also sufficient. Performance measures are calculated in Sect. 5, and numerical results are presented in Sect. 6. Our numerical results demonstrate that the proposed multiple access system with two types of jobs and constant retrial rates provides incentives for the users to respect the contracts.

2 Model, balance equations, and generating functions

Two independent Poisson streams of jobs, S_1 and S_2, flow into a single-server service system. The service system can hold *at most* one job. The arrival rate of stream S_i is λ_i, $i = 1, 2$, with $\lambda := \lambda_1 + \lambda_2$. The required service time of each job is independent of its type and is exponentially distributed with mean $1/\mu$. If an arriving type-i job finds the (main) server busy, it is routed to a dedicated retrial (orbit) queue that operates as an $\cdot/M/1/\infty$ queue. That is, blocked jobs of type i form a type-i single-server orbit queue that attempts to retransmit jobs (if any) to the main service system at a Poisson rate of μ_i, $i = 1, 2$. Let $L(t)$ denote the number of jobs in the main queue. $L(t)$ assumes the values of 0 or 1. Let $Q_i(t)$ be the number of jobs in orbit queue i, $i = 1, 2$. The transition-rate diagram of the system is depicted in Fig. 1. The Markov process $\{(Q_1(t), Q_2(t), L(t)) : t \in [0, \infty)\}$ is irreducible on the state-space $\{0, 1, \ldots\} \times \{0, 1, \ldots\} \times \{0, 1\}$.

Consider the system in steady-state, where we define by (Q_1, Q_2, L) the stationary version of the Markov chain $\{(Q_1(t), Q_2(t), L(t)) : t \in [0, \infty)\}$. Later on we establish necessary and sufficient stability conditions. Define the set of stationary probabilities $\{P_{mn}(k)\}$ as follows:

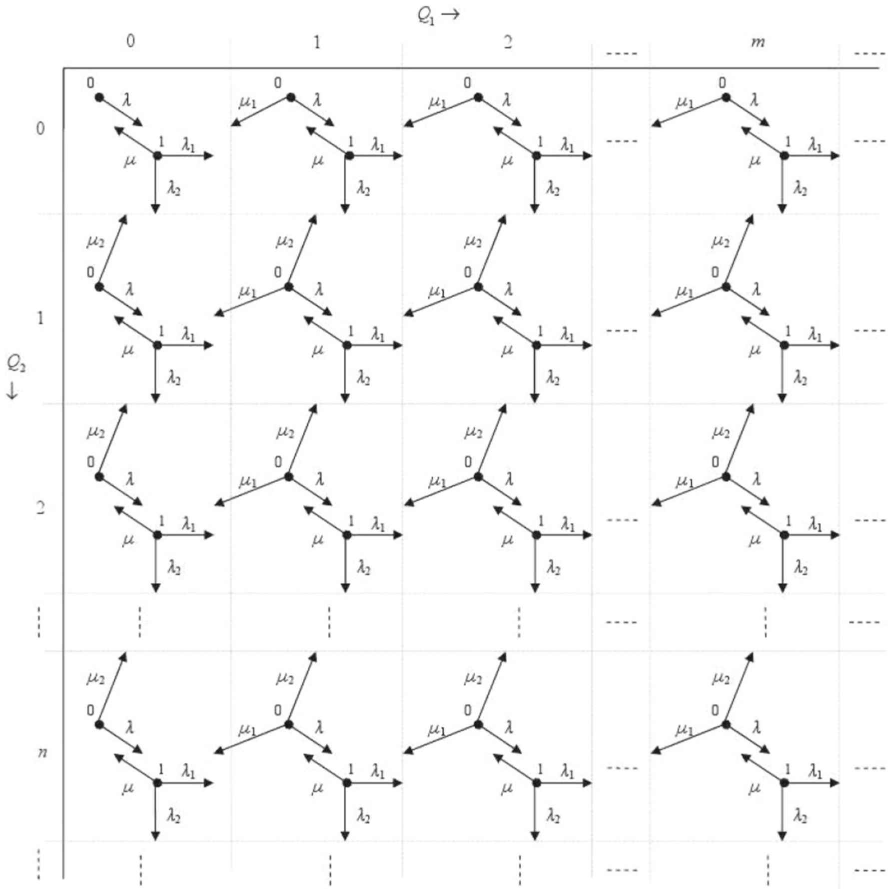

Fig. 1 Transition-rate diagram. The numbers 0 or 1 appearing next to each node indicate whether $L = 0$ or $L = 1$, respectively

$$P_{mn}(k) = \lim_{t\to\infty} P(Q_1(t)=m, Q_2(t)=n, L(t) = k) = P(Q_1 = m, Q_2 = n, L = k),$$

for $m, n = 0, 1, \ldots$ and $k = 0, 1$, when these limits exist. Define the marginal probabilities

$$P_{m\bullet}(k) = \sum_{n=0}^{\infty} P_{mn}(k) = P(Q_1 = m, L = k), \quad m = 0, 1, 2, \ldots \quad k = 0, 1,$$

and

$$P_{\bullet n}(k) = \sum_{m=0}^{\infty} P_{mn}(k) = P(Q_2 = n, L = k), \quad n = 0, 1, 2, \ldots \quad k = 0, 1.$$

Let us write the balance equations. If $Q_2 = 0$, we have

(a) for $Q_1 = 0$ and $k = 0$,

$$\lambda P_{00}(0) = \mu P_{00}(1), \tag{1}$$

(b) for $Q_1 = m \geq 1$ and $k = 0$,

$$(\lambda + \mu_1) P_{m0}(0) = \mu P_{m0}(1), \tag{2}$$

(c) for $Q_1 = 0$ and $k = 1$,

$$(\lambda + \mu) P_{00}(1) = \lambda P_{00}(0) + \mu_1 P_{10}(0) + \mu_2 P_{01}(0), \tag{3}$$

(d) for $Q_1 = m \geq 1$ and $k = 1$,

$$(\lambda + \mu) P_{m0}(1) = \lambda P_{m0}(0) + \mu_1 P_{m+1,0}(0) + \mu_2 P_{m1}(0) + \lambda_1 P_{m-1,0}(1). \tag{4}$$

If $Q_2 = n, n \geq 1$, we have

(e) for $Q_1 = 0$ and $k = 0$,

$$(\lambda + \mu_2) P_{0n}(0) = \mu P_{0n}(1), \tag{5}$$

(f) for $Q_1 = m \geq 1$ and $k = 0$,

$$(\lambda + \mu_1 + \mu_2) P_{mn}(0) = \mu P_{mn}(1), \tag{6}$$

(g) for $Q_1 = 0$ and $k = 1$,

$$(\lambda + \mu) P_{0n}(1) = \lambda P_{0n}(0) + \mu_1 P_{1n}(0) + \mu_2 P_{o,n+1}(0) + \lambda_2 P_{0,n-1}(1), \tag{7}$$

(h) for $Q_1 = m \geq 1$ and $k = 1$,

$$\begin{aligned}(\lambda + \mu) P_{mn}(1) = \lambda P_{mn}(0) + \mu_1 P_{m+1,n}(0) + \mu_2 P_{m,n+1}(0) \\ + \lambda_1 P_{m-1,n}(1) + \lambda_2 P_{m,n-1}(1).\end{aligned} \tag{8}$$

The PGF of the stationary version of the Markov process $\{(Q_1(t), Q_2(t), L(t)) : t \in [0, \infty)\}$ is given by

$$H(x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{1} P_{mn}(k) x^m y^n z^k. \tag{9}$$

Let us also define the following (partial) PGFs

$$G_n^{(k)}(x) = \sum_{m=0}^{\infty} P_{mn}(k) x^m, \quad k = 0, 1, \quad n = 0, 1, \ldots$$

and

$$H^{(k)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{mn}(k) x^m y^n = \sum_{n=0}^{\infty} G_n^{(k)}(x) y^n, \quad k = 0, 1. \tag{10}$$

Note that

$$H(x, y, z) = H^{(0)}(x, y) + z H^{(1)}(x, y), \quad |x| \leq 1, |y| \leq 1. \tag{11}$$

Then, for $n = 0$ and $k = 0$, multiplying each equation from (1) and (2) by x^m, respectively, and summing over m results in

$$\lambda \sum_{m=0}^{\infty} P_{m0}(0) x^m + \mu_1 \sum_{m=1}^{\infty} P_{m0}(0) x^m = \mu \sum_{m=0}^{\infty} P_{m0}(1) x^m,$$

or

$$(\lambda + \mu_1) G_0^{(0)}(x) - \mu_1 P_{00}(0) = \mu G_0^{(1)}(x). \tag{12}$$

Similarly, for $n = 0$ and $k = 1$, using Eqs. (3) and (4) leads to

$$(\lambda + \mu) G_0^{(1)} = \lambda G_0^{(0)} + \mu_1 \sum_{m=0}^{\infty} P_{m+1,0}(0) x^m + \mu_2 G_1^{(0)}(x) + \lambda_1 \sum_{m=1}^{\infty} P_{m-1,0}(1) x^m.$$

That is,

$$(\lambda + \mu) G_0^{(1)}(x) = \lambda G_0^{(0)}(x) + \frac{\mu_1}{x} \left(G_0^{(0)}(x) - P_{00}(0) \right) + \mu_2 G_1^{(0)}(x) + \lambda_1 x G_0^{(1)}(x).$$

Multiplying by x and arranging terms, we obtain

$$-(\lambda x + \mu_1) G_0^{(0)}(x) + (\lambda_1 (1 - x) + \lambda_2 + \mu) x G_0^{(1)}(x) - \mu_2 x G_1^{(0)}(x) = -\mu_1 P_{00}(0). \tag{13}$$

Using Eqs. (5) and (6) for $n \geq 1$ and $k = 0$ results in

$$(\lambda + \mu_2) G_n^{(0)}(x) + \mu_1 \left(G_n^{(0)}(x) - P_{0n}(0) \right) = \mu G_n^{(1)}(x),$$

or

$$(\lambda + \mu_1 + \mu_2) G_n^{(0)}(x) - \mu G_n^{(1)}(x) = \mu_1 P_{0n}(0). \tag{14}$$

Similarly, for $n \geq 1$ and $k = 1$, Eqs. (7) and (8) lead to

$$(\lambda + \mu)G_n^{(1)}(x) = \lambda G_n^{(0)}(x) + \frac{\mu_1}{x}\left(G_n^{(0)}(x) - P_{0n}(0)\right) + \mu_2 G_{n+1}^{(0)}(x)$$
$$+\lambda_1 x G_n^{(1)}(x) + \lambda_2 G_{n-1}^{(1)}(x),$$

or

$$-(\lambda x + \mu_1)G_n^{(0)}(x) + (\lambda_1(1-x) + \lambda_2 + \mu)xG_n^{(1)}(x) - \mu_2 x G_{n+1}^{(0)}(x)$$
$$-\lambda_2 x G_{n-1}^{(1)}(x) = -\mu_1 P_{0n}(0). \tag{15}$$

Using Eqs. (12) and (14), multiplying respectively by y^n and summing over n, we obtain

$$(\lambda + \mu_1)H^{(0)}(x, y) + \mu_2\left(H^{(0)}(x, y) - G_0^{(0)}(x)\right) - \mu H^{(1)}(x, y) = \mu_1 H^{(0)}(0, y). \tag{16}$$

Similarly, using Eqs. (13) and (15), we obtain

$$-(\lambda x + \mu_1)H^{(0)}(x, y) + (\lambda_1(1-x) + \lambda_2 + \mu)xH^{(1)}(x, y)$$
$$-\frac{\mu_2 x}{y}\left(H^{(0)}(x, y) - G_0^{(0)}(x)\right) - \lambda_2 xyH^{(1)}(x, y) = -\mu_1 H^{(0)}(0, y). \tag{17}$$

Noting that $G_0^{(0)}(x) = H^{(0)}(x, 0)$ and denoting $\alpha := \lambda + \mu_1 + \mu_2$, we can rewrite Eqs. (16) and (17) as

$$\alpha H^{(0)}(x, y) - \mu H^{(1)}(x, y) = \mu_2 H^{(0)}(x, 0) + \mu_1 H^{(0)}(0, y), \tag{18}$$

$$(\lambda xy + \mu_1 y + \mu_2 x)H^{(0)}(x, y) - (\lambda_1(1-x) + \lambda_2(1-y) + \mu)xyH^{(1)}(x, y)$$
$$= \mu_2 x H^{(0)}(x, 0) + \mu_1 y H^{(0)}(0, y), \tag{19}$$

or, equivalently, in a matrix form

$$\mathbf{C}(x, y)\mathbf{H}(x, y) = \mathbf{g}(x, y), \tag{20}$$

where

$$\mathbf{C}(x, y) = \begin{bmatrix} \alpha & -\mu \\ \lambda xy + \mu_1 y + \mu_2 x & -(\lambda_1(1-x) + \lambda_2(1-y) + \mu)xy \end{bmatrix},$$

$$\mathbf{H}(x, y) = \begin{bmatrix} H^{(0)}(x, y) \\ H^{(1)}(x, y) \end{bmatrix},$$

$$\mathbf{g}(x, y) = \begin{bmatrix} \mu_2 H^{(0)}(x, 0) + \mu_1 H^{(0)}(0, y) \\ \mu_2 x H^{(0)}(x, 0) + \mu_1 y H^{(0)}(0, y) \end{bmatrix}.$$

Now, if we calculate $H^{(0)}(x, 0)$ and $H^{(0)}(0, y)$, the two-dimensional PGF $\mathbf{H}(x, y)$ is immediately obtained from Eq. (20). This calculation will be performed in Sect. 4.

3 Necessary stability conditions

We first establish necessary conditions, which will also turn out to be sufficient. The proposition below already shows that $\lambda/\mu \leq 1$ and $(\lambda/\mu)(1+\lambda_i/\mu_i) \leq 1$ for $i = 1, 2$ are necessary conditions for the existence of a steady-state.

Proposition 1

$$H^{(1)}(1, 1) = P(L = 1) = \frac{\lambda}{\mu} \tag{21}$$

and

$$H^{(0)}(0, 1) = P(Q_1 = 0, L = 0) = 1 - \frac{\lambda}{\mu}\left(1 + \frac{\lambda_1}{\mu_1}\right) \tag{22}$$

$$H^{(0)}(1, 0) = P(Q_2 = 0, L = 0) = 1 - \frac{\lambda}{\mu}\left(1 + \frac{\lambda_2}{\mu_2}\right). \tag{23}$$

Note that the condition (i) $\lambda/\mu \leq 1$ is a consequence of conditions (ii) $(\lambda/\mu)(1 + \lambda_i/\mu_i) \leq 1$ for $i = 1, 2$, so that in the following we will not consider condition (i) but only conditions (ii).

Proof For each $m = 0, 1, 2, \ldots$ we consider a vertical "cut" (see Fig. 1) between the column representing the states $\{Q_1 = m, L = 1\}$ and the column representing the states $\{Q_1 = m + 1, L = 0\}$. According to the local balance equation approach [12], we can write the balance of rates between the states from the left of the cut and the states from the right of the cut. Namely, we have

$$\lambda_1 P_{m\bullet}(1) = \mu_1 P_{m+1\bullet}(0), \quad m = 0, 1, 2, \ldots. \tag{24}$$

Summing (24) over all m results in

$$\lambda_1 H^{(1)}(1, 1) = \mu_1(1 - H^{(1)}(1, 1) - P_{0\bullet}(0)). \tag{25}$$

Clearly, $P(L = k) = \sum_{m=0}^{\infty} P_{m\bullet}(k) = H^{(k)}(1, 1), k = 0, 1$.

From (25) we readily get

$$1 - P_{0\bullet}(0) = \frac{\lambda_1 + \mu_1}{\mu_1} H^{(1)}(1, 1). \tag{26}$$

Since $P_{0\bullet}(0) = H^{(0)}(0, 1)$, we can write (26) as

$$1 - H^{(0)}(0, 1) = \frac{\lambda_1 + \mu_1}{\mu_1} H^{(1)}(1, 1), \tag{27}$$

and, by symmetry,

$$1 - H^{(0)}(1,0) = \frac{\lambda_2 + \mu_2}{\mu_2} H^{(1)}(1,1). \tag{28}$$

Substituting (27) and (28) in Eq. (18), with $x = y = 1$, yields

$$H^{(1)}(1,1) = P(L = 1) = \frac{\lambda}{\mu}.$$

Now, from (27) and (28), respectively, we obtain

$$H^{(0)}(0,1) = P(Q_1 = 0, L = 0) = 1 - \frac{\lambda}{\mu}\left(\frac{\lambda_1 + \mu_1}{\mu_1}\right) \tag{29}$$

and

$$H^{(0)}(1,0) = P(Q_2 = 0, L = 0) = 1 - \frac{\lambda}{\mu}\left(\frac{\lambda_2 + \mu_2}{\mu_2}\right), \tag{30}$$

which completes the proof. □

The next result shows that the system cannot be stable if either $(\lambda/\mu)(1+\lambda_1/\mu_1) = 1$ or $(\lambda/\mu)(1 + \lambda_2/\mu_2) = 1$.

Proposition 2 *If either* $(\lambda/\mu)(1 + \lambda_1/\mu_1) = 1$ *or* $(\lambda/\mu)(1 + \lambda_2/\mu_2) = 1$ *then* $P_{m,n}(0) = P_{m,n}(1) = 0$ *for all* $m, n = 0, 1, \ldots$ *or, equivalently, both queues* Q_1 *and* Q_2 *are unbounded with probability one.*

Proof Assume, for instance, that $(\lambda/\mu)(1 + \lambda_2/\mu_2) = 1$ so that $H^{(0)}(1,0) = 0$ from (23). Since $H^{(0)}(1,0) = \sum_{m\geq 0} P_{m,0}(0)$ (see (10)), the condition $H^{(0)}(1,0) = 0$ implies that

$$P_{m,0}(0) = 0 \quad \text{for } m = 0, 1, \ldots, \tag{31}$$

so that from (1) to (2)

$$P_{m,0}(1) = 0 \quad \text{for } m = 0, 1, \ldots . \tag{32}$$

We now use an induction argument to prove that

$$P_{m,n}(0) = 0 \quad \text{for } m, n = 0, 1, \ldots . \tag{33}$$

We have already shown in (31) that (33) is true for $n = 0$. Assume that (33) is true for $n = 0, 1, \ldots, k$ and let us show that it is still true for $n = k + 1$.

From (6) and the induction hypothesis we get that $P_{m,k}(0) = P_{m,k}(1) = 0$ for $m = 1, 2, \ldots$. The latter equality implies, using (8), that $P_{m,k+1}(0) = 0$. This shows that (33) holds for $m = 0, 1, \dots$ and $n = k+1$, and completes the induction argument, proving that (33) is true.

We have therefore proved that $P_{m,n}(0) = 0$ for all $m, n = 0, 1, \dots$. Let us prove that $P_{m,n}(1) = 0$ for all $m, n = 0, 1, \dots$. The latter is true for $m, n = 1, 2, \dots$ thanks to (6). It is also true for $n = 0, m = 0, 1, \dots$ from (32). It remains to investigate the case where $m = 0$ and $n = 0, 1, \dots$. By (5) and (33) we get that $P_{0,n}(1) = 0$ for $n = 1, 2, \dots$, whereas we have already noticed that $P_{0,0}(1) = 0$.

In summary, $P_{m,n}(0) = P_{m,n}(1) = 0$ for all $m, n = 0, 1, \dots$, so that

$$P(Q_1 = m, Q_2 = n) = P_{m,n}(0) + P_{m,n}(1)$$

for all $m, n = 0, 1, \dots$, which completes the proof. □

We conclude from Propositions (1) and (2) that conditions

$$\left(\frac{\lambda}{\mu}\right)\left(1 + \frac{\lambda_1}{\mu_1}\right) < 1 \quad \text{and} \quad \left(\frac{\lambda}{\mu}\right)\left(1 + \frac{\lambda_2}{\mu_2}\right) < 1 \tag{34}$$

are necessary for the system to be stable.

We will show in Sect. 4 that under conditions (34) the matrix equation (20) has a unique solution $\mathbf{H}(x, y) = (H^{(0)}(x, y), H^{(1)}(x, y))$ which is analytic for $|x| < 1, |y| < 1$ and continuous for $|x| \le 1, |y| \le 1$. As a result, conditions (34) will turn out to be the necessary and sufficient conditions for the system stability.

Before ending this section, let us give an intuitive explanation of the stability conditions. In a stable system, λ/μ is the fraction of time the server in the main queue is busy. Thus, this is also the proportion of jobs sent to the orbit queues. Therefore, the maximal rates at which jobs flow into orbit queue 1 and into orbit queue 2 are $(\lambda_1 + \mu_1)\lambda/\mu$ and $(\lambda_2 + \mu_2)\lambda/\mu$, respectively. Each of these rates must be smaller than the corresponding maximal service rate, μ_1 or μ_2, respectively.

4 Derivation of $H^{(0)}(x, 0)$ and $H^{(1)}(0, y)$

Throughout we assume that the necessary stability conditions found in (34) hold. Our analysis below will formally show that these conditions are also sufficient for the stability of the system.

Some additional notation: $C_a = \{z \in \mathbb{C} : |z| = a\}$ $(a > 0)$ denotes the circle centered in 0 of radius a and $C_a^+ = \{z \in \mathbb{C} : |z| < a\}$ denotes the interior of C_a, with $\mathbb{C}$ denoting the complex plane.

Let $\hat{\lambda}_i := \alpha\lambda_i$, $\hat{\mu}_i := \mu\mu_i$ for $i = 1, 2$ and $\hat{\lambda} := \hat{\lambda}_1 + \hat{\lambda}_2$, so that $\hat{\lambda} = \alpha\lambda$.

Convention: Lemma 1 in the Appendix says that either $\alpha\lambda_1 < \mu\mu_1$ or $\alpha\lambda_2 < \mu\mu_2$ should hold under the enforced necessary conditions (34) for stability. Without loss of generality, we will assume throughout the paper that $\alpha\lambda_1 < \mu\mu_1$ or, equivalently, that

$$\hat{\lambda}_1 < \hat{\mu}_1. \tag{35}$$

From Eqs. (18)–(19) we readily derive the two-dimensional functional equation

$$R(x, y)H^{(0)}(x, y) = A(x, y)H^{(0)}(x, 0) + B(x, y)H^{(0)}(0, y), \quad |x| \leq 1, |y| \leq 1, \tag{36}$$

with

$$R(x, y) := \hat{\lambda}_1(1-x)xy + \hat{\lambda}_2(1-y)xy - \hat{\mu}_1(1-x)y - \hat{\mu}_2(1-y)x \tag{37}$$
$$A(x, y) := ((1-y)(\lambda_2 y - \mu) + \lambda_1(1-x)y)\mu_2 x \tag{38}$$
$$B(x, y) := ((1-x)(\lambda_1 x - \mu) + \lambda_2(1-y)x)\mu_1 y. \tag{39}$$

For further use note that

$$R(x, y) = \frac{\alpha}{\mu_2}A(x, y) + \lambda\mu(1-y)x + \mu\mu_1(x-y) \tag{40}$$
$$R(x, y) = \frac{\alpha}{\mu_1}B(x, y) + \lambda\mu(1-x)y + \mu\mu_2(y-x). \tag{41}$$

The so-called kernel $R(x, y)$ of the functional equation (36) is the same as the kernel in [21, Eq. (1.3)] upon replacing λ_i and μ_i in [21] by $\hat{\lambda}_i$ and $\hat{\mu}_i$, respectively, for $i = 1, 2$. All results in [21] which we will use to solve (36) are collected in Proposition 3 below. Note, however, that the r.h.s. of (36) is different from the r.h.s of Eq. (1.3) in [21], thereby ruling out a direct application of the results in [21] to solve (36).

Once $H^{(0)}(x, y)$ is known for all $|x| \leq 1$ and $|y| \leq 1$ then $H^{(1)}(x, y)$ can be found from (18). In the following we will therefore only focus on the calculation of $H^{(0)}(x, y)$ or, equivalently from (36), on the calculation of $H^{(0)}(x, 0)$ and $H^{(0)}(0, y)$ for all $|x| \leq 1$ and $|y| \leq 1$.

4.1 Overview of the approach used to solve the functional equation (36)

To help the reader navigating this technical section, we sketch the method that we will use to solve (36). It is due to Fayolle and Ianogorodski [21].

It starts with the observation that the r.h.s. of (36) vanishes whenever $R(x, y) = 0$ provided that $H^{(0)}(x, y)$ is finite. More precisely, the equation $R(x, y) = 0$ has one root $y = h(x)$ which is analytic in the whole complex plane $\mathbb{C}$ cut along two real-line segments $[y_1, y_2]$ and $[y_3, y_4]$ such that $0 < y_1 < y_2 < 1 < y_3 < y_4$ (see Proposition 3). Hence,

$$A(x, h(x))H^{(0)}(x, 0) + B(x, h(x))H^{(0)}(0, h(x)) = 0 \tag{42}$$

as long as $H^{(0)}(x, h(x))$ is finite. It will turn out that when x describes the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$, $h(x)$ describes the real-line segment $[y_1, y_2]$. Dividing (42) by $B(x, h(x))$ for $x \in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$ (we will prove that this division is allowed) and multiplying both sides of the resulting equation by the complex number i, yields

$$i\frac{A(x,h(x))}{B(x,h(x)}H^{(0)}(x,0)=-iH^{(0)}(0,h(x)),\quad \forall x\in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}. \tag{43}$$

Since the r.h.s. of (43) is an imaginary complex number whenever $x \in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$, taking the real part in both sides of (43) gives

$$\Re\left(i\frac{A(x,h(x))}{B(x,h(x))}H^{(0)}(x,0)\right)=0,\quad \forall x\in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}. \tag{44}$$

Define $U(x)=A(x,h(x))/(B(x,h(x))(x-x_0)^r)$ and $\tilde{H}(x)=H(x,0)(x-x_0)^r$ where constants $x_0>1$ and $r\in\{0,1\}$ explicitly depend on the model parameters [see (63) and (64)]. In this notation, (44) becomes

$$\Re\left(iU(x)\tilde{H}(x)\right)=0,\quad \forall x\in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}. \tag{45}$$

We will show that $U(x)$ does not vanish on the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$ and that the unknown function $\tilde{H}(x)$ is analytic inside the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$ and is continuous on the circle (by definition of $\tilde{H}(x)$ this property is true when $\hat{\mu}_1/\hat{\lambda}_1\le 1$ and it will have to be established for $\hat{\mu}_1/\hat{\lambda}_1>1$). As a result, the problem of finding $\tilde{H}(x)$ (or equivalently $H^{(0)}(x,0)$) is reduced to what is known as a Riemann–Hilbert boundary value problem, namely, finding an analytic function inside the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$, and satisfying on $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$ a boundary condition of the form (45) (see [24, Chap. 2], [31, pp. 99-107]). This problem has $\chi+1$ linearly independent solutions, where χ is the so-called index of the problem, defined as the variation of the argument of the function $U(z)$ when z describes the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$ in the positive direction [see (68)]. If $\chi=0$, which will turn out to be the case for our problem under the necessary stability conditions (34), then $H^{(0)}(x,0)$ is uniquely and explicitly defined for all $|x|\le\sqrt{\hat{\mu}_1/\hat{\lambda}_1}$. From the latter result we will easily derive $H^{(0)}(x,0)$ for all $|x|\le 1$ when $\hat{\mu}_1/\hat{\lambda}_1<1$. The explicit form taken by $H^{(0)}(x,0)$ is given in (69)–(71) (see also [21]).

Once $H^{(0)}(x,0)$ is known for all $|x|\le 1$, we will easily identify $H^{(0)}(0,y)$ for $|y|\le 1$ [see (72)].

This technique of reducing the solution of certain two-dimensional functional equations (Eq. (36) in our case) to the solution of a boundary value problem (typically Rieman-Hilbert or Dirichlet problem)—whose solution is known in closed-form—is due to Fayolle and Iasnogorodski [21]. In [21] (see also [22] that generalizes the work in [21]) the unknown function is the generating function of a two-dimensional stationary Markov chain describing the joint queue-length in a two-queue system. Cohen and Boxma [16] extended the work in [21,22] to two-dimensional stationary Markov chains taking real values, typically representing the joint waiting time or the joint unfinished work in a variety of two-queue systems. Other related papers include [10,11,23,32] (non-exhaustive list).

4.2 Zeros of $R(x, y)$ and their properties

For y fixed, $R(x, y)$ vanishes at

$$x(y) = \frac{-b(y) \pm \sqrt{c(y)}}{2\hat{\lambda}_1 y}, \tag{46}$$

where

$$b(y) := \hat{\lambda}_2 y^2 - (\hat{\mu}_1 + \hat{\mu}_2 + \hat{\lambda})y + \hat{\mu}_2, \tag{47}$$

$$c(y) := b_-(y)b_+(y), \tag{48}$$

with

$$b_-(y) := b(y) - 2y\sqrt{\hat{\lambda}_1\hat{\mu}_1}, \quad b_+(y) := b(y) + 2y\sqrt{\hat{\lambda}_1\hat{\mu}_1}. \tag{49}$$

We have

$$b_-(y) = \hat{\lambda}_2(y - y_1)(y - y_4), \quad b_+(y) = \hat{\lambda}_2(y - y_2)(y - y_3), \tag{50}$$

with

$$y_1 = \frac{\xi_1 - \sqrt{\xi_1^2 - 4\hat{\lambda}_2\hat{\mu}_2}}{2\hat{\lambda}_2}, \quad y_2 = \frac{\xi_2 - \sqrt{\xi_2^2 - 4\hat{\lambda}_2\hat{\mu}_2}}{2\hat{\lambda}_2}, \tag{51}$$

$$y_3 = \frac{\xi_2 + \sqrt{\xi_2^2 - 4\hat{\lambda}_2\hat{\mu}_2}}{2\hat{\lambda}_2}, \quad y_4 = \frac{\xi_1 + \sqrt{\xi_1^2 - 4\hat{\lambda}_2\hat{\mu}_2}}{2\hat{\lambda}_2}, \tag{52}$$

$$\xi_1 = \hat{\mu}_1 + \hat{\mu}_2 + \hat{\lambda} + 2\sqrt{\hat{\lambda}_1\hat{\mu}_1}, \quad \xi_2 = \hat{\mu}_1 + \hat{\mu}_2 + \hat{\lambda} - 2\sqrt{\hat{\lambda}_1\hat{\mu}_1}. \tag{53}$$

$y_1, \ldots, y_4$ are the branch points of $x(y)$ (since $c(y_i) = 0$ for $i = 1, \ldots, 4$). It is easily seen that (Hint: $y_2 < 1$ and $y_3 > 1$, both from Convention (35))

$$0 < y_1 < y_2 < 1 < y_3 < y_4. \tag{54}$$

Remark 1 The algebraic function $x(y)$ has two algebraic branches, denoted by $k(y)$ and $k^\sigma(y)$, related via the relation $k(y)k^\sigma(y) = \hat{\mu}_1/\hat{\lambda}_1$. When $y \in (y_1, y_2) \cup (y_3, y_4)$, $k(y)$ and $k^\sigma(y)$ are complex conjugate numbers (since $c(y) < 0$ for those values of y), with $k(y_i) = k^\sigma(y_i)$ for $i = 1, \ldots, 4$. In particular, $|k(y)| = \sqrt{k(y)k^\sigma(y)} = \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ for $y \in [y_1, y_2] \cup [y_3, y_4]$, thereby showing that for $y \in [y_1, y_2]$ (resp. $y \in [y_3, y_4]$) $k(y)$ and $k^\sigma(y)$ lie on the circle centered in 0 with radius $\sqrt{\hat{\mu}_1/\hat{\lambda}_1}$.

When x is fixed similar results hold. We will denote by

$$y(x) = \frac{-e(x) \pm \sqrt{d(x)}}{2\hat{\lambda}_2 x} \tag{55}$$

the algebraic function solution of $R(x, y) = 0$ for x fixed, where $e(x) := \hat{\lambda}_1 x^2 - (\hat{\mu}_1 + \hat{\mu}_2 + \hat{\lambda})x + \hat{\mu}_1$ and $d(x) := e_-(x)e_+(x)$, with

$$e_-(x) := e(x) - 2x\sqrt{\hat{\lambda}_2\hat{\mu}_2}, \quad e_+(x) := e(x) + 2x\sqrt{\hat{\lambda}_2\hat{\mu}_2}.$$

We denote by $x_i, i = 1, \dots, 4$ the four branch points of $y(x)$, namely, the zeros of $d(x)$; they are obtained by interchanging indices 1 and 2 in (51)–(53).

We have

$$e_-(x) = \hat{\lambda}_1(x - x_1)(x - x_4), \quad e_+(x) = \hat{\lambda}_1(x - x_2)(x - x_3), \tag{56}$$

where

$$0 < x_1 < x_2 \leq 1 < x_3 < x_4, \tag{57}$$

with $x_2 = 1$ iff $\hat{\lambda}_2 = \hat{\mu}_2$.

The following results, found in [21, Lemmas 2.2, 2.3, 3.1], hold and will extensively be used in the next subsection:

Proposition 3 *For y fixed, the equation* $R(x, y) = 0$ *has one root* $x(y) = k(y)$ *which is analytic in the whole complex plane* $\mathbb{C}$ *cut along the real-line segments* $[y_1, y_2]$ *and* $[y_3, y_4]$. *Moreover*[1]

- (a1) $|k(y)| \leq 1$ *if* $|y| = 1$. *More precisely,* $|k(y)| < 1$ *if* $|y| = 1$ *with* $y \neq 1$, *and* $k(1) = \min(1, \hat{\mu}_1/\hat{\lambda}_1) = 1$ *under Convention* (35).
- (b1) $|k(y)| \leq \sqrt{\frac{\hat{\mu}_1}{\hat{\lambda}_1}}$ *for all* $y \in \mathbb{C}$;
- (c1) *when* y *sweeps twice* $[y_1, y_2]$, $k(y)$ *describes a circle centered in* 0 *with radius* $\sqrt{\frac{\hat{\mu}_1}{\hat{\lambda}_1}}$, *so that* $|k(y)| = \sqrt{\frac{\hat{\mu}_1}{\hat{\lambda}_1}}$ *for* $y \in [y_1, y_2]$.

Similarly, for x fixed, the equation $R(x, y) = 0$ *has one root* $y(x) = h(x)$ *which is analytic in* $\mathbb{C} - [x_1, x_2] - [x_3, x_4]$, *and*

- (a2) $|h(x)| < 1$ *if* $|x| = 1, x \neq 1$, *and* $h(1) = \min(1, \hat{\mu}_2/\hat{\lambda}_2) \leq 1$.
- (b2) $|h(x)| \leq \sqrt{\frac{\hat{\mu}_2}{\hat{\lambda}_2}}$ *for all* $x \in \mathbb{C}$;
- (c2) $|h(x))| = \sqrt{\frac{\hat{\mu}_2}{\hat{\lambda}_2}}$ *if* $x \in [x_1, x_2]$

Moreover,

- (d1) $h(k(y)) = y$ *for* $y \in [y_1, y_2]$ *and* $k(h(x)) = x$ *for* $x \in [x_1, x_2]$.

[1] Apply Rouché's theorem to $R(x, y)$ to get (a1), and the "maximum modulus principal" to the analytic function $k(y)$ in $\mathbb{C} - [y_1, y_2] - [y_3, y_4]$ to get (b1). (c1) follows from Remark 1.

(d2) $h\left(\sqrt{\hat{\mu}_1/\hat{\lambda}_1}\right) = y_2$ *and* $h\left(-\sqrt{\hat{\mu}_1/\hat{\lambda}_1}\right) = y_1$.

(d3) $k\left(\sqrt{\hat{\mu}_2/\hat{\lambda}_2}\right) = x_2$ *and* $k\left(-\sqrt{\hat{\mu}_2/\hat{\lambda}_2}\right) = x_1$.

Last

(e) $|h(x)| \leq 1$ *for* $1 \leq |x| \leq \sqrt{\frac{\hat{\mu}_1}{\hat{\lambda}_1}}$ *(recall that* $\hat{\lambda}_1 < \hat{\mu}_1$*).*

4.3 A boundary value problem and its solution

In reference to the program set in Sect. 4.1, we are now in a position to set a boundary value problem that is satisfied by the unknown function $H^{(0)}(x, 0)$.

We know that $R(k(y), y) = 0$ by definition of $k(y)$. On the other hand, $H^{(0)}(x, y)$ is well-defined for all $(x, y) = (k(y), y)$ with $|y| = 1$, since (i) $H^{(0)}(x, y)$ is well-defined for $|x| \leq 1$, $|y| \leq 1$, (ii) $k(y)$ is continuous for $|y| = 1$ (from Proposition 3 we know that $k(y)$ is analytic in $\mathbb{C} - [y_1, y_2]$ and we know that $0 < y_1 < y_2 < 1$ so that $k(y)$ is continuous for $|y| = 1$), (iii) $|k(y)| \leq 1$ for $|y| = 1$ (cf. Proposition 3-(a1)). Therefore, the l.h.s. of (36) must vanish for all pairs $(k(y), y)$ such that $|y| = 1$, which yields

$$A(k(y), y)H^{(0)}(k(y), 0) = -B(k(y), y)H^{(0)}(0, y), \quad \forall |y| = 1. \tag{58}$$

The r.h.s. of (58) is analytic for $|y| \leq 1$ with $y \notin [y_1, y_2]$ and continuous for $|y| \leq 1$, so that the r.h.s. of (58) can be analytically continued up to the interval $[y_1, y_2]$.

This gives

$$A(k(y), y)H^{(0)}(k(y), 0) = -B(k(y), y)H^{(0)}(0, y), \quad \forall y \in [y_1, y_2]. \tag{59}$$

It is shown in Lemma 2 that $B(k(y), y) \neq 0$ for $y \in [y_1, y_2]$. We may therefore divide both sides of (59) by $B(k(y), y)$ to get

$$\frac{A(k(y), y)}{B(k(y), y)}H^{(0)}(k(y), 0) = -H^{(0)}(0, y), \quad \forall y \in [y_1, y_2]. \tag{60}$$

Take $y \in [y_1, y_2]$ and let $k(y) = x$. We know by Proposition 3-(c1) that $x \in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$; furthermore, we know by Proposition 3-(d1) that $h(k(y)) = y$ so that $y = h(x)$. We may therefore rewrite (60) as

$$\frac{A(x, h(x))}{B(x, h(x))}H^{(0)}(x, 0) = -H^{(0)}(0, h(x)), \quad \forall x \in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}. \tag{61}$$

By multiplying both sides of (61) by the imaginary complex number i and by noting that $H^{(0)}(0, h(x))$ is a real number when $x \in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$ since $h(x) \in [y_1, y_2]$, we get

$$\Re\left(i\frac{A(x, h(x))}{B(x, h(x))}H^{(0)}(x, 0)\right) = 0, \quad \forall x \in C_{\sqrt{\frac{\hat{\mu}_1}{\hat{\lambda}_1}}}. \tag{62}$$

Equation (62) would define a Riemann–Hilbert boundary value problem for the function $H^{(0)}(x,0)$ if the following two conditions were satisfied (see Sect. 4.1): (a) $H^{(0)}(x,0)$ is analytic for $|x| < \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ (observe that this function is initially only analytic for $|x| < 1$) and (b) $A(x,h(x))/B(x,h(x))$ does not vanish on the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$.

Let us see if one can prove that conditions (a) and (b) above hold.

It is shown in Lemma 3 that $h(x)$ is analytic for $1 < |x| < \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ and continuous for $1 \le |x| \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$; furthermore $|h(x)| \le 1$ for $1 \le |x| \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ by Proposition 3-(e). These two properties together imply that $H^{(0)}(0,h(x))$ is analytic for $1 < |x| < \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ and continuous for $1 \le |x| \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$, which allows us to conclude from (61) and from the principle of analytic continuation that $A(x,h(x))H^{(0)}(x,0)/B(x,h(x))$, the l.h.s. of (61), inherits these two properties.

Define $v(x) := A(x,h(x))H^{(0)}(x,0)/B(x,h(x))$. We have just shown that $v(x)$ is analytic for $1 < |x| < \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ and continuous for $1 \le |x| \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$. If $A(x,h(x))$ did not vanish for $1 \le |x| \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ then we could conclude that property (b) above is satisfied. But we have shown in Lemma 4 that, depending on the model parameters (see below), $A(x,h(x))$ has at most one zero $x = x_0$ in the region $D_x := \{x \in \mathbb{C} : 1 \le |x| \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}\}$.

In order to state a more precise result, define the constants x_0 and $r \in \{0,1\}$ as follows

$$x_0 = \frac{-(\lambda+\mu_1-\mu)\lambda\mu_1 + \sqrt{((\lambda+\mu_1-\mu)\lambda\mu_1)^2 + 4\lambda\lambda_1(\lambda+\mu_1)\mu\mu_1^2}}{2\lambda\lambda_1(\lambda+\mu_1)}, \tag{63}$$

$$r = \begin{cases} 1, & \text{if } x_0 \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1} \text{ and } \frac{(\lambda+\mu_1)x_0}{\lambda x_0+\mu_1} \le \sqrt{\hat{\mu}_2/\hat{\lambda}_2}, \\ 0, & \text{otherwise.} \end{cases} \tag{64}$$

If $r = 1$ then $A(x,h(x))$ has a unique zero in D_x given by $x = x_0$, with multiplicity one, whereas if $r = 0$ then $A(x,h(x))$ does not vanish in D_x (see Lemma 4). Introduce

$$U(x) := \frac{A(x,h(x))}{B(x,h(x))(x-x_0)^r} \quad \text{and} \quad \tilde{H}(x) := H^{(0)}(x,0)(x-x_0)^r. \tag{65}$$

By construction

$$\frac{A(x,h(x))}{B(x,h(x))} H^{(0)}(x,0) = U(x)\tilde{H}(x) \tag{66}$$

so that, from (62),

$$\Re\left(i\,U(x)\tilde{H}(x)\right) = 0, \quad \forall x \in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}. \tag{67}$$

Furthermore, still by construction, $U(x)$ does not vanish on the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$ and we have shown that $\tilde{H}(x)$ is analytic inside the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$. In other words, $\tilde{H}(x)$ satisfies a Riemann–Hilbert problem with the boundary condition (67), whose solution is given below.

Define

$$\chi := -\frac{1}{\pi}[\arg U(x)]_{x\in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}} \tag{68}$$

the so-called index of the Riemann–Hilbert problem, where $[\arg \alpha(z)]_{z\in C}$ denotes the variation of the argument of the function $\alpha(z)$ when z moves on a closed curved C in the positive direction, provided that $\alpha(z) \neq 0$ for $z \in C$).

The Riemann–Hilbert problem has $\chi + 1$ independent solutions [31, p. 104]. It is shown in Lemma 5 that, as expected, $\chi = 0$ under conditions (34), thereby showing that the solution of the Riemann–Hilbert problem (67) is unique under conditions (34).

With $\chi = 0$ the solution of the Riemann–Hilbert problem is (we have retuned to the sought function $H^{(0)}(x, 0)$, which is the function of interest to us)

$$H^{(0)}(x,0) = D(x - x_0)^{-r} \exp\left(\frac{1}{2\pi i}\int_{|z|=\sqrt{\hat{\mu}_1/\hat{\lambda}_1}} \frac{\log(J(z))}{z-x}\mathrm{d}z\right), \quad \forall\, |x| < \sqrt{\hat{\mu}_1/\hat{\lambda}_1}, \tag{69}$$

where D is a constant (to be determined) and (with $\overline{z}$ the complex conjugate of $z \in \mathbb{C}$)

$$J(z) = -\frac{\overline{iU(z)}}{iU(z)}.$$

We are left with calculating the constant D in (69). Setting $x = 1$ in (69) gives

$$D = (1 - x_0)^r \left(1 - \frac{\lambda}{\mu}\left(1 + \frac{\lambda_2}{\mu_2}\right)\right) \exp\left(-\frac{1}{2\pi i}\int_{|z|=\sqrt{\hat{\mu}_1/\hat{\lambda}_1}} \frac{\log(J(z))}{z-1}\mathrm{d}z\right) \tag{70}$$

by using the value of $H^{(0)}(1, 0)$ found in (23). We may therefore rewrite (69) as

$$\begin{aligned} H^{(0)}(x,0) &= \left(\frac{1-x_0}{x-x_0}\right)^r \left(1 - \frac{\lambda}{\mu}\left(1 + \frac{\lambda_2}{\mu_2}\right)\right) \\ &\quad \times \exp\left(\frac{1}{2\pi i}\int_{|z|=\sqrt{\hat{\mu}_1/\hat{\lambda}_1}} \frac{\log(J(z))(x-1)}{(z-x)(z-1)}\mathrm{d}z\right) \end{aligned} \tag{71}$$

for all $|x| < \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$. The expression (71) allows us to show that the necessary stability conditions (34) are also sufficient.

Proposition 4 *The Markovian retrial queueing system with two classes of jobs and constant retrial rates is positive recurrent if and only if the conditions* (34) *are satisfied.*

Proof The "only if" part has been proven in Sect. 3. Since by Lemma 5 $\chi = 0$ under conditions (34), there exists a unique normalized invariant measure. In addition, the Markov process $\{(Q_1(t), Q_2(t), L(t)) : t \in [0, \infty)\}$ is irreducible and non-explosive (all transition rates are bounded). Thus, using Theorem 3.18 from [27], we conclude that the Markov process is positive recurrent. □

We also need to calculate the other boundary function $H^{(0)}(0, y)$ for $|y| \leq 1$. For $|y| = 1$, $H^{(0)}(0, y)$ is given in (58). For $|y| < 1$, $H^{(0)}(0, y)$ is obtained from (58) and Cauchy's formula, which gives

$$H^{(0)}(0, y) = \frac{1}{2\pi i} \int_{|t|=1} \frac{V(t)}{t - y} dt, \quad |y| < 1, \tag{72}$$

where

$$V(t) := -\frac{A(k(t), t)}{B(k(t), t)} H^{(0)}(k(t), 0), \quad |t| = 1, \tag{73}$$

does not vanish for all $|t| = 1$, as shown in Lemma 6.

Introducing (71) and (72) into (20) uniquely determines the functions $H^{(0)}(x, y)$ and $H^{(1)}(x, y)$ for all $|x| \leq 1, |y| \leq 1$.

5 Performance measures

Later on in this section we shall need the derivatives $\frac{\mathrm{d}}{\mathrm{d}x} H^{(0)}(x, 0)|_{x=1}$ and $\frac{\mathrm{d}}{\mathrm{d}y} H^{(0)}(0, y)|_{y=1}$.

Differentiating (71) w.r.t x gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} H^{(0)}(x, 0) &= \left(\frac{1 - x_0}{x - x_0}\right)^r \left(1 - \frac{\lambda}{\mu}\left(1 + \frac{\lambda_2}{\mu_2}\right)\right) \\
&\quad \times \exp\left(\frac{1}{2\pi i} \int_{|z|=\sqrt{\hat{\mu}_1/\hat{\lambda}_1}} \frac{\log(J(z))(x-1)}{(z-x)(z-1)} \mathrm{d}z\right) \\
&\quad \times \left(\frac{-r}{x - x_0} + \frac{1}{2\pi i} \int_{|z|=\sqrt{\hat{\mu}_1/\hat{\lambda}_1}} \frac{\log(J(z)}{(z-x)^2} \mathrm{d}z\right) \\
&= H^0(x, 0) \left(\frac{-r}{x - x_0} + \frac{1}{2\pi i} \int_{|z|=\sqrt{\hat{\mu}_1/\hat{\lambda}_1}} \frac{\log(J(z)}{(z-x)^2} \mathrm{d}z\right).
\end{aligned} \tag{74}$$

Letting $x = 1$ in (74) and using (23) yields

$$\frac{\mathrm{d}}{\mathrm{d}x} H^{(0)}(x, 0)|_{x=1} = \left(1 - \frac{\lambda}{\mu}\left(1 + \frac{\lambda_2}{\mu_2}\right)\right) \left(\frac{r}{x_0 - 1} + \frac{1}{2\pi i} \int_{|z|=\sqrt{\hat{\mu}_1/\hat{\lambda}_1}} \frac{\log(J(z)}{(z-1)^2} \mathrm{d}z\right). \tag{75}$$

The derivative $\frac{\mathrm{d}}{\mathrm{d}y}H^{(0)}(0, y)|_{y=1}$ is obtained from (58). By Lemma 6, we have

$$\frac{\mathrm{d}}{\mathrm{d}y}H^{(0)}(0, y)|_{y=1} = -\lim_{y\to 1}\frac{A(k(y), y)}{B(k(y), y)}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x, 0)|_{x=1}\, k'(1) - \lim_{y\to 1}\frac{\mathrm{d}}{\mathrm{d}y}\frac{A(k(y), y)}{B(k(y), y)}H^{(0)}(1, 0), \tag{76}$$

where $\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x, 0)|_{x=1}$ and $H^{(0)}(1, 0)$ are given in (75) and (23), respectively. The limits in the above expression can be calculated by L'Hôpital's rule. Lengthy but easy algebra gives

$$\lim_{y\to 1}\frac{A(k(y), y)}{B(k(y), y)} = \frac{(\lambda_2 - \mu + \lambda_1 k'(1))\mu_2}{(\lambda_2 + (\lambda_1 - \mu)k'(1))\mu_1}$$

and

$$\lim_{y\to 1}\frac{\mathrm{d}}{\mathrm{d}y}\frac{A(k(y), y)}{B(k(y), y)} = -\frac{(-\lambda_2+(-\lambda_1 + \mu)k'(1)+(\lambda_2 - \mu)k'(1)^2 + \lambda_1 k'(1)^3 + (\mu - \lambda_1 - \lambda_2)k''(1))\mu\mu_2}{(\lambda_2 + (\lambda_1 - \mu)k'(1))\mu_1},$$

where

$$k'(1) = \frac{\hat{\lambda}_2 - \hat{\mu}_2}{\hat{\mu}_1 - \hat{\lambda}_1},$$

and

$$k''(1) = 2\frac{(\hat{\mu}_1 + \hat{\mu}_2 - 2(\hat{\lambda}_1 + \hat{\lambda}_2))\hat{\mu}_1\hat{\mu}_2 + \hat{\lambda}_1^2\hat{\mu}_2 + \hat{\lambda}_2^2\hat{\mu}_1}{(\hat{\mu}_1 - \hat{\lambda}_1)^3}.$$

We are now in a position to calculate some important performance measures.

By setting $x = 0$ in Eq. (71), we immediately obtain the probability of empty system

$$P(Q_1 = 0, Q_2 = 0, L = 0) = \left(\frac{x_0 - 1}{x_0}\right)^r \left(1 - \frac{\lambda}{\mu}\left(1 + \frac{\lambda_2}{\mu_2}\right)\right) \times \exp\left(\frac{1}{2\pi i}\int_{|z|=\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}\frac{\log(J(z))}{z(1-z)}\mathrm{d}z\right) \tag{77}$$

Next, we calculate the expected orbit queue lengths. For the first queue, we have

$$E[Q_1] = \sum_{m=1}^{\infty} m\left(\sum_{n=0}^{\infty}P_{mn}(0) + \sum_{n=0}^{\infty}P_{mn}(1)\right) = \frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x, 1)|_{x=1} + \frac{\mathrm{d}}{\mathrm{d}x}H^{(1)}(x, 1)|_{x=1}. \tag{78}$$

Thus, we need to calculate $\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,1)|_{x=1}$ and $\frac{\mathrm{d}}{\mathrm{d}x}H^{(1)}(x,1)|_{x=1}$. From (36) we have

$$H^{(0)}(x,y)=\frac{A(x,y)}{R(x,y)}H^{(0)}(x,0)+\frac{B(x,y)}{R(x,y)}H^{(0)}(0,y). \tag{79}$$

Using (37)–(39) and setting $y=1$ in (79), yields

$$H^{(0)}(x,1)=\frac{\lambda_1\mu_2 x}{\alpha\lambda_1 x-\mu\mu_1}H^{(0)}(x,0)+\frac{(\lambda_1 x-\mu)\mu_1}{\alpha\lambda_1 x-\mu\mu_1}H^{(0)}(0,1).$$

Next, by differentiating the above relation with respect to x we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,1)=&-\frac{\lambda_1\mu_2\mu\mu_1}{(\alpha\lambda_1 x-\mu\mu_1)^2}H^{(0)}(x,0)+\frac{\lambda_1\mu_2 x}{\alpha\lambda_1 x-\mu\mu_1}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,0)\\&+\frac{\lambda_1\mu_1\mu(\alpha-\mu_1)}{(\alpha\lambda_1 x-\mu\mu_1)^2}H^{(0)}(0,1).\end{aligned}$$

Setting $x=1$ in the above, yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,1)|_{x=1}=&\frac{\lambda_1\mu_1\mu}{(\mu\mu_1-\alpha\lambda_1)^2}\Big((\alpha-\mu_1)H^{(0)}(0,1)-\mu_2H^{(0)}(1,0)\Big)\\&-\frac{\lambda_1\mu_2}{\mu\mu_1-\alpha\lambda_1}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,0)|_{x=1},\end{aligned} \tag{80}$$

where $H^{(0)}(0,1)$, $H^{(0)}(1,0)$ and $\mathrm{d}H^{(0)}(x,0)/\mathrm{d}x|_{x=1}$ are given in (22), (23) and (75), respectively.

It remains to find $\mathrm{d}H^{(1)}(x,1)/\mathrm{d}x|_{x=1}$. Differentiating (18) with respect to x and setting $x=y=1$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}H^{(1)}(x,1)|_{x=1}&=\frac{\alpha}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,1)|_{x=1}-\frac{\mu_2}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,0)|_{x=1}\\&=\frac{\alpha\lambda_1\mu_1}{(\mu\mu_1-\alpha\lambda_1)^2}\Big((\alpha-\mu_1)H^{(0)}(0,1)-\mu_2H^{(0)}(1,0)\Big)\\&\quad-\frac{\mu_1\mu_2}{\mu\mu_1-\alpha\lambda_1}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,0)|_{x=1},\end{aligned} \tag{81}$$

by using (80).

By combining (78), (80) and (81) we finally obtain

$$\begin{aligned}E[Q_1]=&\frac{(\alpha+\mu)\lambda_1\mu_1}{(\mu\mu_1-\alpha\lambda_1)^2}\Big((\alpha-\mu_1)H^{(0)}(0,1)-\mu_2H^{(0)}(1,0)\Big)\\&-\frac{\mu_2(\lambda_1+\mu_1)}{\mu\mu_1-\alpha\lambda_1}\frac{\mathrm{d}}{\mathrm{d}x}H^{(0)}(x,0)|_{x=1},\end{aligned} \tag{82}$$

where $H^{(0)}(0, 1)$, $H^{(0)}(1, 0)$ and $\mathrm{d}H^{(0)}(x, 0)/\mathrm{d}x|_{x=1}$ are given in (22), (23) and (75), respectively.

Similarly, the expected queue length for the second orbit is given by

$$\begin{aligned} E[Q_2] &= \frac{\mathrm{d}}{\mathrm{d}y} H^{(0)}(1, y)|_{y=1} + \frac{\mathrm{d}}{\mathrm{d}y} H^{(1)}(1, y)|_{y=1} \\ &= \frac{(\alpha + \mu)\lambda_2\mu_2}{(\mu\mu_2 - \alpha\lambda_2)^2} \left((\alpha - \mu_2) H^{(0)}(1, 0) - \mu_1 H^{(0)}(0, 1)\right) \\ &\quad - \frac{\mu_1(\lambda_2 + \mu_2)}{\mu\mu_2 - \alpha\lambda_2} \frac{\mathrm{d}}{\mathrm{d}y} H^{(0)}(0, y)|_{y=1}, \end{aligned} \tag{83}$$

where $\mathrm{d}H^{(0)}(0, y)/\mathrm{d}y|_{y=1}$ is given in (76).

Finally, we recall that [see (21)]

$$E[L] = P(L = 1) = \frac{\lambda}{\mu}.$$

6 Numerical examples

To gain more insights into the performance of the system, let us consider numerical examples. First, we set $\mu_1 = \mu_2 = 2$, $\mu = 4$, $\lambda_1 = 0.1$ and vary λ_2 in the interval [0.2; 1.9]. In Fig. 2 we plot the probability of an empty system $P(Q_1 = 0, Q_2 = 0, L = 0)$ calculated by (77) as a function of λ_2. We also plot $H^{(0)}(1, 0)$, see formula (23), which corresponds, if λ_1 is small, to the probability of empty system with one type of jobs and a single orbit queue. Now if we change the value of λ_1 from 0.1 to 1.0, we observe that the value of $P(Q_1 = 0, Q_2 = 0, L = 0)$ deviates significantly from $H^{(0)}(1, 0)$.

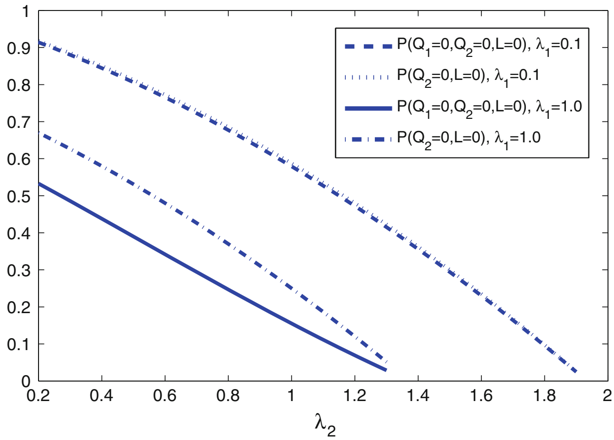

Fig. 2 Probability of an empty system ($\mu = 4$, $\mu_1 = \mu_2 = 2$)

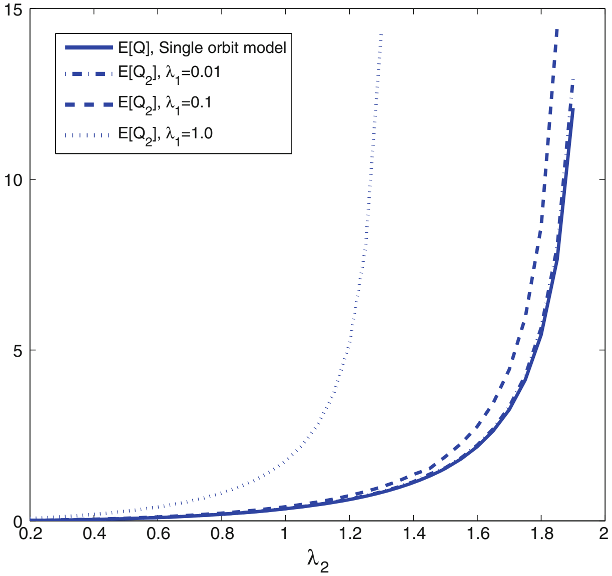

Fig. 3 The expected orbit queue size, $E[Q_2]$ ($\mu = 4$, $\mu_1 = \mu_2 = 2$)

Let us explain how we evaluate the contour integrals like the one in formula (77). We evaluate such integrals by numerical integration as in [34] using the trapezium method. Specifically, we first change the variable $z = \sqrt{\hat{\mu_1}/\hat{\lambda_1}} \exp(i\varphi)$, $\varphi \in [0, 2\pi)$, changing the differential as $dz = \sqrt{\hat{\mu_1}/\hat{\lambda_1}} i \exp(i\varphi) d\varphi$. Then, we have divided the interval $[0, 2\pi)$ into K equal parts and apply the trapezium numerical integration method evaluating the integrand at the points $\varphi_k = 2\pi k/K$ with $k = 0, 1, \ldots, K-1$. We have chosen $K = 30000$. This should guarantee a good accuracy, since a good accuracy has been reported for similar integrals in [34] for $K = 250$.

Keeping $\mu_1 = \mu_2 = 2$, $\mu = 4$, in Fig. 3 we plot the expected queue length of the second orbit $E[Q_2]$ calculated by (83) as a function of λ_2 for $\lambda_1 = 0.01; 0.1; 1.0$. We also plot the expected queue length of the orbit queue for the single orbit retrial system [5], which is given by

$$E[Q] = \frac{\lambda_2^2(\lambda_2 + \mu + \mu_2)}{\mu(\mu\mu_2 - \lambda_2^2 - \lambda_2\mu_2)}.$$

Again, as expected, when λ_1 goes to zero, $E[Q_2]$ approaches the expected queue length of the orbit queue in the single orbit retrial system.

Next, we investigate how the retrial rates affect the system performance. Let us fix $\lambda_1 = \lambda_2 = 1.2$, $\mu = 4$, $\mu_1 = 2$ and we vary μ_2 in the interval [2.0; 2.15]. With such parameter setting, the system is not too far from the stability boundary. We plot in Fig. 4 the expected lengths of the orbit queues, $E[Q_1]$ and $E[Q_2]$, as functions of μ_2. We can see that if the jobs of type 2 retry at a bit faster rate than the jobs of type 1, they can gain significantly in terms of the waiting time. Specifically, an increase of

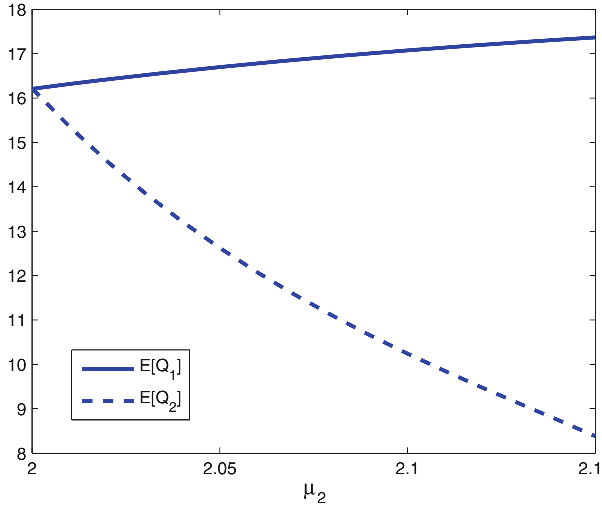

Fig. 4 The expected queue lengths of the orbit queues as functions of μ_2 ($\lambda_1 = \lambda_2 = 1.2$, $\mu = 4$, $\mu_1 = 2$)

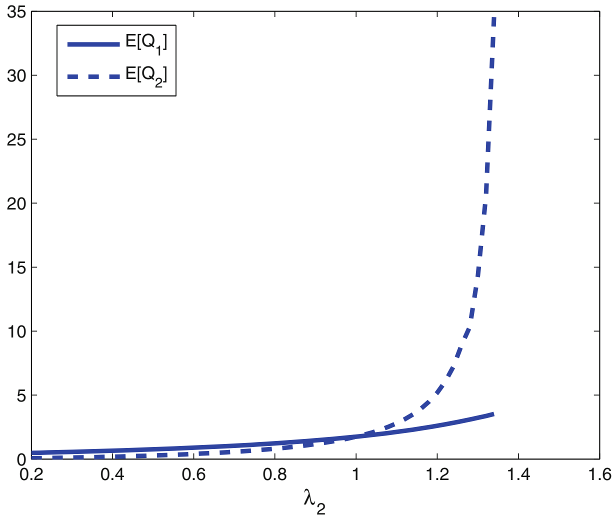

Fig. 5 The expected queue lengths of the orbit queues as functions of λ_2 ($\lambda_1 = 1$, $\mu_1 = \mu_2 = 2$, $\mu = 4$)

less than 10 % of the retrial rate of jobs of type 2 helps them to reduce the expected orbit queue length by 50 %. Clearly, if there is no cost for retrials, it is beneficial for the jobs to increase their retrial rate. However, there are good reasons to keep the control of the retrial rates in the hand of the system administrator and not to set them too high. As was just mentioned, the first reason is the possible cost for retrials. The second reason is the creation of incentives to respect the contract. To illustrate this point, we fix $\lambda_1 = 1$, $\mu_1 = \mu_2 = 2$, $\mu = 4$, and vary λ_2 in the interval [0.2; 1.34]. In Fig. 5, we plot the expected queue lengths of the orbit queues. We see that if the jobs of type 2 increase their input rate beyond their fair share, they will be severely penalized in terms of the expected delay, whereas the increase of the input rate of jobs of type 2 does not inflict any significant damage to the jobs of type 1.

Acknowledgments We thank anonymous reviewers for their comments which helped us to significantly improve the presentation of the results. We would also like to thank Efrat Perel for helping us in drawing the figure of the transition-rate diagram.

Appendix

Lemma 1 *Conditions* (34) *imply that either* $\alpha\lambda_1 < \mu\mu_1$ *or* $\alpha\lambda_2 < \mu\mu_2$.

Proof Assume that $\alpha\lambda_1 \geq \mu\mu_1$ and $\alpha\lambda_2 \geq \mu\mu_2$

Multiplying the first inequality in (34) by $\mu\mu_1$ and using the definition of λ and α gives

$$(\lambda_1 + \lambda_2)(\lambda_1 + \mu_1) < \mu\mu_1 \leq \alpha\lambda_1 = (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\lambda_1$$

which is true if and only if (a) $\lambda_2\mu_1 < \lambda_1\mu_2$.

Multiplying now the second inequality in (34) by $\mu\mu_2$ gives

$$(\lambda_1 + \lambda_2)(\lambda_2 + \mu_2) < \mu\mu_2 \leq \alpha\lambda_2 = (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\lambda_2$$

which is true if and only if (b) $\lambda_1\mu_2 < \lambda_2\mu_1$.

Since inequalities (a) and (b) cannot be true simultaneously we conclude that either $\alpha\lambda_1 < \mu\mu_1$ or $\alpha\lambda_2 < \mu\mu_2$, which concludes the proof. □

Lemma 2 *Under conditions* (34)*,* (i) $A(k(y), y) \neq 0$ *and* (ii) $B(k(y), y) \neq 0$ *for* $y \in [y_1, y_2]$.

Equivalently, (iii) $A(x, h(x)) \neq 0$ *and* (iv) $B(x, h(x)) \neq 0$ *for* $x \in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$.

Proof From (39) and (41) we see that $R(x, y)$ and $B(x, y)$ vanish simultaneously if and only if

$$(1 - x)(\lambda_1 x - \mu) + \lambda_2(1 - y)x = 0$$
$$\lambda(1 - x)y + \mu_2(y - x) = 0.$$

The second equation gives $x = (\lambda + \mu_2)y/((\lambda y + \mu_2))$. Plugging this value of x into the first equation yields (Hint: use $\lambda = \lambda_1 + \lambda_2$)

$$P_1(y) := (1 - y)Q_1(y) = 0$$

with $Q_1(y) := \lambda\lambda_2(\lambda + \mu_2)y^2 + (\lambda + \mu_2 - \mu)\lambda\mu_2 y - \mu\mu_2^2 = 0$.

From $\lim_{y\to\pm\infty} Q_1(y) = +\infty$ and $Q_1(0) = -\mu\mu_2^2$ we conclude that the polynomial $Q_1(y)$ has two real roots, $y_- < 0 < y_+$ and that $Q_1(y) < 0$ for $0 \leq y < y_+$. Since

$$Q_1(1) = \left(\frac{\lambda + \mu_2}{\mu\mu_2}\right)\left(\frac{\lambda}{\mu}\left(1 + \frac{\lambda_2}{\mu_2}\right) - 1\right) < 0, \quad (84)$$

where the latter inequality holds under conditions (34), we conclude that $Q_1(y) < 0$ for $y \in [0, 1]$, which in turn implies that $P_1(y) < 0$ for $y \in [0, 1)$. The latter completes the proof of (ii) since $[y_1, y_2] \subset [0, 1)$ [see (54)].

The proof of (i) is the same as the proof of (ii) up to interchanging incides 1 and 2.

Eqns (iii) and (iv) both follow from the fact that $k([y_1, y_2]) = C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$ (cf. Proposition 3-(11)) and the relation $h(k(y)) = y$ for $y \in [y_1, y_2]$ (cf. Proposition 3-(d1)). □

Lemma 3 *Under Convention* (35)*,* $h(x)$ *is analytic for* $1 < |x| < \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ *and continuous for* $1 \le |x| \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$.

Proof We already know by Proposition 3 that $h(x)$ is analytic for $x \in \mathbb{C} - [x_1, x_2] - [x_3, x_4]$ where $x_2 \le 1 < x_3$. It is therefore enough to show that $\sqrt{\hat{\mu}_1/\hat{\lambda}_1} < x_3$ or, equivalently from (56) that $e_+\left(\sqrt{\hat{\mu}_1/\hat{\lambda}_1}\right) < 0$. Easy algebra shows that $e_+\left(\sqrt{\hat{\mu}_1/\hat{\lambda}_1}\right) = -\sqrt{\hat{\mu}_1/\hat{\lambda}_1}\left(\left(\sqrt{\hat{\lambda}_1} - \sqrt{\hat{\mu}_1}\right)^2 + \left(\sqrt{\hat{\lambda}_2} + \sqrt{\hat{\mu}_2}\right)^2\right) < 0$, which concludes the proof. □

Lemma 4 *Assume that conditions* (34) *hold. Define*

$$x_0 := \frac{-(\lambda + \mu_1 - \mu)\lambda\mu_1 + \sqrt{((\lambda + \mu_1 - \mu)\lambda\mu_1)^2 + 4\lambda\lambda_1(\lambda + \mu_1)\mu\mu_1^2}}{2\lambda\lambda_1(\lambda + \mu_1)} > 1$$

If $x_0 \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ *and if* $(\lambda+\mu_1)x_0/(\lambda x_0+\mu_1) \le \sqrt{\hat{\mu}_2/\hat{\lambda}_2}$ *then* $A(x, h(x))$ *has exactly one zero* $x = x_0$ *in the region* $D_x := \left\{x \in \mathbb{C} : 1 < |x| \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}\right\}$ *and this zero has multiplicity one. Otherwise* $A(x, h(x))$ *has no zero in* D_x.

Proof From (38) and (40) we see that $R(x, y)$ and $A(x, y)$ vanish simultaneously if and only if

$$(1 - y)(\lambda_2 y - \mu) + \lambda_1(1 - x)y = 0 \tag{85}$$

$$\lambda(1 - y)x + \mu_1(x - y) = 0. \tag{86}$$

Eq. (86) gives

$$y = \frac{(\lambda + \mu_1)x}{\lambda x + \mu_1}. \tag{87}$$

Plugging this value of y into (85) yields

$$\frac{1 - x}{(\lambda x + \mu_1)^2}\, Q_2(x) = 0$$

with $Q_2(x) := \lambda\lambda_1(\lambda + \mu_1)x^2 + (\lambda + \mu_1 - \mu)\lambda\mu_1 x - \mu\mu_1^2$.

Therefore, $A(x, h(x))$ will vanish in the region D_x if and only if the polynomial $Q_2(x)$ vanishes in D_x. From $\lim_{x\to\pm\infty} Q_2(x) = +\infty$, $Q_2(0) = \mu\mu_1^2 < 0$ and

$$Q_2(1) = \mu\mu_1(\lambda + \mu_1)\left(\frac{\lambda}{\mu} + \frac{\lambda\lambda_1}{\mu\mu_1} - 1\right) < 0 \tag{88}$$

we conclude that $Q_2(x)$ has always two real zeros with opposite sign.

Let us first focus on the negative zero of $Q_2(x)$, denoted by x_-. Let us show that x_- cannot belong to the region D_x or, equivalently, that x_- cannot satisfies the inequalities $-\sqrt{\hat{\mu}_1/\hat{\lambda}_1} \le x_- < -1$. Assume that $-\sqrt{\hat{\mu}_1/\hat{\lambda}_1} \le x_- < -1$ and that $A(x_-, h(x_-)) = 0$. By (40) the latter equality implies (Hint: $R(x_-, h(x_-)) = 0$ by definition of $h(x)$)

$$\lambda\mu(1 - h(x_-))x_- + \mu\mu_1(x_- - h(x_-)) = 0. \tag{89}$$

Since $-1 \le h(x_-) \le 1$ from Proposition 3-(e), we observe that $(1 - h(x_-))x_- \le 0$ and $(x_- - h(x_-)) < 0$ so that the l.h.s. of (89) cannot be equal to zero. Therefore, $A(x), h(x))$ does not vanish at $x = x_-$.

We now focus on the positive zero of $Q_2(x)$, denoted by x_0. Note that $x_0 > 1$ from (88). If $x_0 > \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ then clearly $A(x, h(x))$ has no zero in $(1, \sqrt{\hat{\mu}_1/\hat{\lambda}_1}]$. If $1 < x_0 \le \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ then $A(x, h(x))$ as a unique zero in $(1, \sqrt{\hat{\mu}_1/\hat{\lambda}_1}]$, given by $x = x_0$, *provided* that [see (87)] $h(x_0) = (\lambda + \mu_1)x_0/(\lambda x_0 + \mu_1) \le \sqrt{\hat{\mu}_2/\hat{\lambda}_2}$ since we know from Proposition 3-(b2) that the branch $h(x)$ is such that $|h(x)| \le \sqrt{\hat{\mu}_2/\hat{\lambda}_2}$ for all $x \in \mathbb{C}$; if $(\lambda + \mu_1)x_0/(\lambda x_0 + \mu_1) > \sqrt{\hat{\mu}_2/\hat{\lambda}_2}$ then $A(x, h(x))$ does not vanish in $(1, \sqrt{\hat{\mu}_1/\hat{\lambda}_1}]$.

We are left with proving that when $A(x, h(x))$ vanishes at $x = x_0$ then this zero has multiplicity one. From now on we assume that $A(x_0, h(x_0)) = 0$.

From the definition of $h(x)$ and (40) we get

$$0 = R(x, h(x)) = \frac{\alpha}{\mu_2}A(x, h(x)) + \mu[\lambda(1 - h(x))x + \mu_1(x - h(x))].$$

Differentiating this equation w.r.t. x gives

$$0 = \frac{\alpha}{\mu_2}\frac{dA(x, h(x))}{\mathrm{d}x} + \mu[-\lambda h'(x)x + \lambda(1 - h(x)) + \mu_1(1 - h'(x))]. \tag{90}$$

Assume that $dA(x, h(x))/\mathrm{d}x = 0$ at point $x = x_0$, namely, assume that $A(x, h(x))$ has a zero of multiplicity at least two at $x = x_0$. From (90) this implies

$$-\lambda h'(x_0)x_0 + \lambda(1 - h(x_0)) + \mu_1(1 - h'(x_0) = 0$$

that is

$$h'(x_0) = \mu_1 \frac{\lambda + \mu_1}{(\lambda x_0 + \mu_1)^2} \tag{91}$$

with $h(x_0) = (\lambda + \mu_1)x_0/(\lambda x_0 + \mu_1)$ [see (87)].

On the other hand, letting $(x, y) = (x, h(x))$ in (38) yields

$$A(x, h(x)) = ((1 - h(x))(\lambda_2 h(x) - \mu) + \lambda_1(1 - x)h(x))\mu_2 x. \tag{92}$$

Differentiating $A(x, h(x)$ wrt x in (92) and letting $x = x_0$ gives

$$\begin{aligned}
\frac{\mathrm{d}A(x, h(x))}{\mathrm{d}x}|_{x=x_0} &= [-h'(x_0)(\lambda h(x_0) - \mu) + \lambda_2(1 - h(x_0))h'(x_0) \\
&\quad -\lambda_1 h(x_0) + \lambda_1(1 - x_0)h'(x_0)]\mu_2 x_0 \\
&\quad +\frac{\mu_2}{x_0} A(x_0, h(x_0)) \\
&= [h'(x_0)(-2\lambda_2 h(x_0) + \lambda_2 + \mu + \lambda_1(1 - x_0)) - \lambda_1 h(x_0)]\mu_2 x_0 \\
&\quad +\frac{\mu_2}{x_0} A(x_0, h(x_0)) \\
&= [h'(x_0)(-2\lambda_2 h(x_0) + \lambda_2 + \mu + \lambda_1(1 - x_0)) - \lambda_1 h(x_0)]\mu_2 x_0
\end{aligned}$$

since $A(x_0, h(x_0)) = 0$. Therefore, $dA(x, h(x))/\mathrm{d}x = 0$ at point $x = x_0$ iff (note that $x_0 \neq 0$)

$$h'(x_0)(-2\lambda_2 h(x_0) + \lambda_2 + \mu + \lambda_1(1 - x_0)) - \lambda_1 h(x_0) = 0.$$

Since $-2\lambda_2 h(x_0) + \lambda_2 + \mu + \lambda_1(1 - x_0) < 0$ because $x_0 > 1$, we get

$$h'(x_0) = \frac{\lambda_1 h(x_0)}{-2\lambda_2 h(x_0) + \lambda_2 + \mu + \lambda_1(1 - x_0)}$$

with [see (87)] $h(x_0) = (\lambda + \mu_1)x_0/(\lambda x_0 + \mu_1)$, so that $h'(x_0) < 0$. However, $h'(x_0) > 0$ in (91). This yields a contradiction, thereby implying that $dA(x, h(x))/\mathrm{d}x$ does not vanish at point $x = x_0$ when $A(x, h(x))$ does or, equivalently, that x_0 is a zero of multiplicity one. □

Lemma 5 *Under conditions* (34) *and Convention* (35) *the index* χ *of the Riemann–Hilbert problem (the index is defined in* (68)*) is equal to zero.*

Proof Recall the definition of $U(x)$ in (65). First, by studying $U(\sqrt{\hat{\mu}_1/\hat{\lambda}_1}e^{i\theta})$ for $\theta \in [0, 2\pi)$ it is easily seen that $U(x)$ describes a closed (and simple) contour when x describes the circle $C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$; moreover, for $x \in C_{\sqrt{\hat{\mu}_1/\hat{\lambda}_1}}$, $U(x)$ takes only real values when $x \in \{-\sqrt{\hat{\mu}_1/\hat{\lambda}_1}, \sqrt{\hat{\mu}_1/\hat{\lambda}_1}\}$.

As a result, we will show that $\chi = 0$ if we show that

$$U\left(-\sqrt{\hat{\mu}_1/\hat{\lambda}_1}\right) \times U\left(\sqrt{\hat{\mu}_1/\hat{\lambda}_1}\right) > 0, \tag{93}$$

since (93) will imply that the contour defined by $\{U(x) : |x| = \sqrt{\hat{\mu}_1/\hat{\lambda}_1}\}$ does not contain the point $x = 0$ in its interior, so that by definition of the index, $\chi = 0$.

We have from (40)–(41) (Hint: $R(x, h(x)) = 0$ by definition of $h(x)$))

$$A(x, h(x)) = -\frac{\mu\mu_2}{\alpha}(\lambda(1 - h(x))x + \mu_1(x - h(x)) \tag{94}$$

$$B(x, h(x)) = -\frac{\mu\mu_1}{\alpha}(\lambda(1 - x)h(x) + \mu_2(h(x) - x)). \tag{95}$$

Define $x_- := -\sqrt{\hat{\mu}_1/\hat{\lambda}_1}$ and $x_+ := \sqrt{\hat{\mu}_1/\hat{\lambda}_1}$.

By Convention (35) we know that $x_- < -1$ and $x_+ > 1$. Also note that $h(x_-) = y_1 < 1$ and $h(x_+) = y_2 < 1$ from Proposition 3-(d2) and (54). With this, it it is easily seen from (94)–(95) that

$$A(x_-, h(x_-)) > 0 \quad \text{and} \quad A(x_+, h(x_+)) < 0$$

and

$$B(x_-, h(x_-)) < 0 \quad \text{and} \quad B(x_+, h(x_+)) > 0$$

so that

$$A(x_-, h(x_-))/B(x_-, h(x_-)) < 0 \quad \text{and} \quad (A(x_+, h(x_+))/B(x_+, h(x_+)) < 0.$$

and, therefore,

$$A(x_-, h(x_-))/B(x_-, h(x_-))\ A(x_+, h(x_+))/B(x_+, h(x_+)) > 0. \tag{96}$$

The above shows that (93) is true if $r = 0$ in the definition of $U(x)$ since in this case $U(x) = A(x, h(x))/B(x, h(x))$.

Assume that $r = 1$ in the definition of $U(x)$ with $x_0 < x_+$ and $(\lambda + \mu_1)x_0/(\lambda x_0 + \mu_1) \le \sqrt{\hat{\mu}_2/\hat{\lambda}_2}$. Since $(x - x_0) < 0$ for $x = x_-$ and $(x - x_0) > 0$ for $x = x_+$ we conclude from (96) that $U(x_-) > 0$ and $U(x_+) > 0$, thereby showing that (93) is also true in this case.

It remains to investigate the case when $r = 1$ with $x_0 = x_+$ and $(\lambda + \mu_1)x_0/(\lambda x_0 + \mu_1) \le \sqrt{\hat{\mu}_2/\hat{\lambda}_2}$. Clearly, $U(x_-) > 0$ since, from (96), $A(x_-, h(x_-))/B(x_-, h(x_-)) < 0$ and $(x_- - x_0) < 0$ because $x_- < -1$.

Let us focus on the sign of $U(x_+)$. We know that the mapping $x \to U(x)$ is continuous for $|x| \le x_+$ and that $U(x_+) \ne 0$ when $x_+ = x_0$. Since we have shown that $U(x_+) > 0$ when $x_0 < x_+$ and $(\lambda+\mu_1)x_0/(\lambda x_0+\mu_1) \leq \sqrt{\hat{\mu}_2/\hat{\lambda}_2}$, we deduce, by continuity, that necessarily $U(x_+) > 0$ when $x_+ = x_0$ and $(\lambda + \mu_1)x_0/(\lambda x_0 + \mu_1) \leq \sqrt{\hat{\mu}_2/\hat{\lambda}_2}$, which concludes the proof. □

Lemma 6 *Under condition* (34) *and Convention* (35)*,* $B(k(y), y) \neq 0$ *for* $|y| = 1$, $y \neq 1$*. Also,* $B(k(y), y)$ *has a zero at* $y = 1$*, with multiplicity one.*

Proof Fix $|y| = 1$, $y \neq 1$. We know from Proposition 3-(a1) that $|k(y)| < 1$.

From (41) and the fact that $R(k(y), y) = 0$ by definition of $k(y)$, we see that $B(k(y), y) = 0$ is equivalent to

$$0 = \lambda(1 - k(y))y + \mu_2(y - k(y)) = (\lambda(1 - k(y)) + \mu_2)y - \mu_2 k(y)$$

that is,

$$\lambda(1 - k(y) + \mu_2)y = \mu_2 k(y).$$

Taking the absolute value in both sides of the above equation yields

$$|\lambda(1 - k(y) + \mu_2)| = |\lambda(1 - k(y) + \mu_2)y| = |\mu_2 k(y)| < \mu_2. \tag{97}$$

But $|\lambda(1 - k(y)) + \mu_2)| > \mu_2$ which contradicts (97). Hence, $B(k(y), y) \neq 0$ for $|y| = 1$, $y \neq 1$.

Since $k(1) = 1$, we see that $B(k(1), 1) = B(1, 1) = 0$ from the definition of $B(x, y)$. Let us show that the multiplicity of this zero is one. This amounts to showing that $\mathrm{d}B(k(y), y)/\mathrm{d}y$ does not vanish at $y = 1$.

Differentiating $B(k(y), y)$ w.r.t. y in (41) (Hint: $R(k(y), y) = 0$) and setting $y = 1$, gives

$$\frac{\mathrm{d}B(k(y), y)}{\mathrm{d}y}|_{y=1} = \frac{\mu\mu_1}{\alpha}((\lambda + \mu_2)k'(1) - \mu_2). \tag{98}$$

Let us calculate $k'(1)$, the derivative of $k(y)$ at $y = 1$. To this end, let us use (37) to differentiate $R(k(y), y)$ (which is equal to zero) w.r.t. y, which gives

$$0 = \frac{\mathrm{d}R(k(y), y)}{\mathrm{d}y}|_{y=1} = (\mu\mu_1 - \alpha\lambda_1)k'(1) + \mu\mu_2 - \alpha\lambda_2 \tag{99}$$

so that $k'(1) = (\alpha\lambda_2 - \mu\mu_2)/(\mu\mu_1 - \alpha\lambda_1)$ (note that $\mu\mu_1 - \alpha\lambda_1 \neq 0$ from Convention (35), which shows that $k'(1)$ is well defined). Plugging this value of $k'(1)$ into (98) gives

$$\begin{aligned}\frac{dB(k(y),y)}{dy}\Big|_{y=1} &= \frac{\mu\mu_1}{\alpha(\mu\mu_1-\alpha\lambda_1)}\left((\alpha\lambda_2-\mu\mu_2)(\lambda+\mu_2)-\mu_2(\mu\mu_1-\alpha\lambda_1)\right)\\ &= \frac{\mu\mu_1}{\alpha(\mu\mu_1-\alpha\lambda_1)}\alpha(\lambda\lambda_2+\lambda\mu_2-\mu\mu_2)\\ &= \frac{\mu\mu_1}{\mu\mu_1-\alpha\lambda_1}\mu\mu_2\left(\frac{\lambda\lambda_2}{\mu\mu_2}+\frac{\lambda}{\mu}-1\right)<0\end{aligned}$$

under the conditions in (34) (to establish the 2nd equality we have used the definitions of α and λ). This proves that $dB(k(y), y)/dy|_{y=1} \neq 0$ and completes the proof. □

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