A Note on an $M/G/1$ Queue with a Waiting Server, Timer and Vacations

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Abstract

We analyze a generalized protocol of an $M/G/1$ queue with server vacations where after returning from a vacation to an empty system, the server (as in many real-life situations and common also to human behaviour) activates a Timer and waits dormant. If an arrival occurs before the Timer expires, a busy period starts immediately. If the Timer is shorter than the inter-arrival time, the server does not wait any more and leaves for a new vacation, and so on. We derive transforms and performance measures of the system's key variables and show how the general results reduce to their two extreme cases: (i) zero-length Timer yields the multiple vacation model and (ii) infinite-length Timer yields the single vacation case.

1 Introduction

Two important extensions of the classical $M/G/1$ queue, which have been studied extensively in the literature, are the multiple and single vacation models (see Levy and Yechiali [1975], Takagi [1991] and references there). Under the multiple vacation protocol, whenever the server returns from a vacation and finds an empty queue, he immediately leaves for another vacation; while under the single vacation regime, if the server finds an empty

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system after a vacation, he waits dormant until the first arrival thereafter occurs, serves a full busy period, and only then takes another vacation. These two protocols represent two extremes: in the multiple vacation scheme the server *never* waits in front of an empty queue, whereas in the single vacation case the server *always* waits for an arriving customer (job) when he finds an empty queue upon returning from a vacation.

Yet, in many and various situations (common also to human behaviour) the server, finding an empty queue upon returning from a vacation, may wait idle in front of the queue for a *pre-specified* (random or constant) duration of time (which we call a 'Timer'), hoping for an arrival during that 'waiting' period. Only if the Timer expires before an arrival occurs, the server immediately takes another vacation.

Such a protocol *generalizes* the two different vacation models into a unified one: if the Timer's duration is set to zero, the multiple vacation regime is obtained, while if the Timer's duration is set to infinity, the result is the single vacation model.

In this note we analyze the above generalized protocol. We derive transforms and performance measures of key variables as follows: probability generating function (PGF) and mean of the number of jobs present in the system; Laplace Stieltjes transforms (LST) and means of the variables (i) waiting time of an arbitrary customer, (ii) duration of a busy period, (iii) length of a 'Vacation Period' (i.e. the continuous period of time the server is not working during a cycle), (iv) length of a cycle, and (v) non-productive idle time during a cycle. For those variables and their transforms we show how the results of the generalized model reduce to their corresponding results for the two extreme cases: the multiple and the single vacation regimes. Section 2 presents the model description. Sections 3 and 4 are devoted to the analysis of the number of customers and of the waiting time, respectively. Section 5 discusses various busy and non-busy periods.

2 The Model

We consider an $M/G/1$ queue where customers arrive according to a Poisson process $\{A(t), t \geq 0\}$ with intensity λ, each requiring a service time B with distribution $B(\cdot)$, mean b, and LST $B^*(\cdot)$. A similar notational convention is used for other random variables introduced in the sequel. The traffic intensity is denoted by $\rho = \lambda b$. At the termination of a busy period, when the queue becomes empty, the server takes a (random) vacation U. At the end of a vacation U the server returns to the (main) queue. If, upon return, the number of waiting jobs in the queue (denoted by Y) is greater than 0, the server serves exhaustively (that is, he stays Y regular $M/G/1$-type busy periods) and then takes a new vacation U. However, if $Y = 0$, the server activates a (random) Timer T and waits. If an arrival occurs before T expires, the arriving customer is immediately taken into service and the server is kept busy a full ($M/G/1$-type) busy period, at the end of which he takes another vacation U. If no arrival occurs during the Timer's duration (i.e. the Timer is shorter than the inter-arrival time), the server does not wait any more and leaves for a vacation U. We invoke the usual independence assumptions between arrivals, service times, vacation lengths and Timer durations. The distributions of U and T are assumed to be general.

3 Number of Customers

3.1 Law of Motion

In this section we shall derive the steady state distribution of the number of customers in the $M/G/1$ queue with vacations and a timer. Just as in the ordinary $M/G/1$ queue, it is easily verified that this steady state distribution coincides (if it exists) with the steady state distribution of the number of customers immediately after the departure of a customer. Let us first concentrate on the latter quantity. Let $X_n :=$ number of customers left behind by the nth departing customer. Then the law of motion of the system's state X is given as follows:

If $X_n \geq 1$, then

$$X_{n+1} = X_n - 1 + A(B), \tag{1}$$

where $A(t)$:= number of Poisson arrivals during a time interval of length t. (We use B, rather than B_n, as a generic random variable.)

If $X_n = 0$, then

$$X_{n+1} = \begin{cases} A(U)|_{A(U)\geq 1} - 1 + A(B), & \text{w.p.} \quad \frac{1-U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)}, \\ A(B), & \text{w.p.} \quad \frac{U^*(\lambda)(1-T^*(\lambda))}{1-U^*(\lambda)T^*(\lambda)}, \end{cases} \tag{2}$$

where $T^*(\lambda) = P(\text{no arrivals in } T)$, $U^*(\lambda) = P(\text{no arrivals in } U)$.

The explanation of (2) is as follows: when the server takes a vacation, the probability of no arrivals during U is $\int_0^\infty e^{-\lambda u} \mathrm{d}P(U \leq u) = U^*(\lambda)$. Then, upon finding an empty system, the server activates a Timer T. The probability of no arrivals during T is $T^*(\lambda)$, and the server takes another vacation, etc. This combined process repeats itself $k \geq 0$ times with probability $[U^*(\lambda)T^*(\lambda)]^k$, until, after k repetitions, there is an arrival during U. This last event occurs with probability $1 - U^*(\lambda)$ and the server then finds $A(U)|_{A(U)\geq 1}$ waiting customers. Thus, the next departure will leave behind $A(U)|_{A(U)\geq 1} - 1 + A(B)$ waiting customers with probability $(1-U^*(\lambda))\sum_{k=0}^{\infty}[U^*(\lambda)T^*(\lambda)]^k = \frac{1-U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)}$. The other possibility is that after k repeated pairs of U and T without arrivals, there will be another vacation with no arrivals, but with an arrival during the following T. This occurs with probability $U^*(\lambda)(1-T^*(\lambda))\sum_{k=0}^{\infty}[U^*(\lambda)T^*(\lambda)]^k = \frac{U^*(\lambda)(1-T^*(\lambda))}{1-U^*(\lambda)T^*(\lambda)}$. Then, when the job arriving within T departs, it leaves $A(B)$ waiting jobs behind.

3.2 Generating Function

Let $\hat{X}(z) := E[z^X]$ denote the PGF of a random variable (r.v.) X.

Using (1) and (2) we have

$$\begin{aligned} E[z^{X_{n+1}}] &= E[z^{X_n - 1 + A(B)} | X_n \geq 1] \cdot P(X_n \geq 1) \\ &\quad + \Bigg[E[z^{(A(U)|_{A(U)\geq 1} - 1 + A(B))}] \cdot \frac{1 - U^*(\lambda)}{1 - U^*(\lambda)T^*(\lambda)} \\ &\qquad + E[z^{A(B)}] \cdot \frac{U^*(\lambda)(1 - T^*(\lambda))}{1 - U^*(\lambda)T^*(\lambda)} \Bigg] \cdot P(X_n = 0). \end{aligned}$$

We consider the system in steady state. It can easily be shown (e.g. by relating the system to those with single and multiple vacations) that $\rho < 1$ is a necessary and sufficient condition for the existence of a steady state solution.

Set $p_0 := P(X_n = 0)$, $\delta := \lambda(1 - z)$, $\hat{X}(z) := \lim_{n\to\infty} E[z^{X_n}]$. Then, with $E[z^{A(B)}] = B^*(\delta)$,

$$\begin{aligned} \hat{X}(z) = z^{-1}B^*(\delta)\big(\hat{X}(z) - p_0\big) + \Bigg[z^{-1}B^*(\delta)\, \frac{U^*(\delta) - U^*(\lambda)}{1 - U^*(\lambda)}\, \frac{1 - U^*(\lambda)}{1 - U^*(\lambda)T^*(\lambda)} \\ + B^*(\delta)\, \frac{U^*(\lambda)(1 - T^*(\lambda))}{1 - U^*(\lambda)T^*(\lambda)} \Bigg] p_0. \end{aligned} \tag{3}$$

Result (3) follows since

$$E[z^{X_n} \mid X_n \geq 1] P(X_n \geq 1) = \sum_{k=1}^{\infty} z^k \frac{P(X_n = k)}{P(X_n \geq 1)} P(X_n \geq 1) = \hat{X}(z) - p_0$$

and ($I_{(\cdot)}$ denoting an indicator function)

$$E[z^{A(U)} \mid A(U) \geq 1] = \frac{E[z^{A(U)}] - E[z^{A(U)} I_{(A(U)=0)}]}{P(A(U) \geq 1)} = \frac{U^*(\delta) - U^*(\lambda)}{1 - U^*(\lambda)}. \tag{4}$$

Rearranging terms we obtain

$$\hat{X}(z) = p_0 \frac{B^*(\delta)\Big[U^*(\lambda)\big(T^*(\lambda) - 1\big)(1 - z) + U^*(\delta) - 1\Big]}{\big[z - B^*(\delta)\big]\,\big[1 - U^*(\lambda)T^*(\lambda)\big]}.$$

To calculate p_0 we use $\hat{X}(1) = 1$, which leads to

$$1 = \frac{p_0}{\big[1 - U^*(\lambda)T^*(\lambda)\big]} \cdot \lim_{z\to 1} \left[\frac{U^*(\lambda)(T^*(\lambda)-1)(1-z) + U^*(\delta) - 1}{z - B^*(\delta)} \right].$$

Applying L'Hospital's rule we obtain (with $\rho = \lambda b$),

$$p_0 = (1-\rho)\left[\frac{1 - U^*(\lambda)T^*(\lambda)}{U^*(\lambda)\big[1 - T^*(\lambda)\big] + \lambda E[U]} \right].$$

Finally,

$$\begin{aligned} \hat{X}(z) &= \left[\frac{(1-\rho)B^*(\delta)(1-z)}{B^*(\delta) - z} \right] \left[\frac{U^*(\lambda)\big[1 - T^*(\lambda)\big] + \big(1 - U^*(\delta)\big)/(1-z)}{U^*(\lambda)\big[1 - T^*(\lambda)\big] + \lambda E[U]} \right] \\ &= \hat{X}(z)\big|_{M/G/1} \cdot \xi(z), \end{aligned} \tag{5}$$

where $\hat{X}(z)\big|_{M/G/1} = (1-\rho)(1-z)B^*(\delta)/(B^*(\delta) - z)$ is the PGF of the number of customers in a stationary $M/G/1$ queue with arrival rate λ and service times B, and $\xi(z)$ is the PGF of an independent random variable, XI, as will become apparent in the sequel.

The case of *multiple* vacations (MV) is obtained from (5) by setting $T = 0$, implying $T^*(\lambda) = 1$. Thus,

$$\begin{aligned} \hat{X}(z)\big|_{MV} &= \frac{(1-\rho)B^*(\delta)}{B^*(\delta) - z} \left[\frac{1 - U^*(\delta)}{\lambda E[U]} \right] \\ &= \hat{X}(z)\big|_{M/G/1} \cdot \frac{1 - U^*(\delta)}{(1-z)\lambda E[U]}, \end{aligned}$$

which coincides with Takagi [1991], equation (2.12c).

The case of *single* vacation (SV) is obtained from (5) by setting $T = \infty$, implying $T^*(\lambda) = 0$. Thus,

$$\hat{X}(z)\big|_{SV} = \hat{X}(z)\big|_{M/G/1} \frac{U^*(\lambda) + \frac{1 - U^*(\delta)}{1-z}}{U^*(\lambda) + \lambda E[U]}. \tag{6}$$

Equation (6) is identical with Takagi [1991], equation (2.23).

3.3 An Alternative Derivation

As indicated in the beginning of this section, the steady state distribution of X is also the steady state distribution of the number of customers in the system at an arbitrary epoch. The PGF of the latter quantity, also denoted by X, may also be obtained by using the queue-length decomposition result of Fuhrmann and Cooper [1985] (which holds for our model) stating that

$$X = X_{M/G/1} + XI, \tag{7}$$

where $XI :=$ number of customers at an arbitrary non-serving epoch, while $X_{M/G/1}$ and XI are independent. Borst and Boxma [1997] showed that

$$E[z^{XI}] = \frac{\hat{X}_S(z) - \hat{X}_E(z)}{(1-z)\big(E[X_E] - E[X_S]\big)}, \tag{8}$$

where X_S (X_E) denotes the number of customers at the start (end) of a non-serving period.

In our case, $X_S = 0$, while (see (2))

$$X_E = \begin{cases} 1 & \text{w.p.} \;\; \frac{U^*(\lambda)[1-T^*(\lambda)]}{1-U^*(\lambda)T^*(\lambda)}, \\ A(U)\big|_{A(U)\geq 1} & \text{w.p.} \;\; \frac{1-U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)}. \end{cases}$$

Thus, $\hat{X}_S(z) = 1$, while (see (4))

$$\begin{aligned} \hat{X}_E(z) &= z\frac{U^*(\lambda)\big[1-T^*(\lambda)\big]}{1-U^*(\lambda)T^*(\lambda)} + E\big[z^{A(U)|_{A(U)\geq 1}}\big]\frac{1-U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)} \\ &= \frac{zU^*(\lambda)\big[1-T^*(\lambda)\big] + U^*(\delta) - U^*(\lambda)}{1-U^*(\lambda)T^*(\lambda)}. \end{aligned} \tag{9}$$

Equation (9) yields

$$E[X_E] = \frac{U^*(\lambda)\big[1-T^*(\lambda)\big] + \lambda E[U]}{1-U^*(\lambda)T^*(\lambda)}. \tag{10}$$

Combining equations (7),(8),(9) and (10) leads to the PGF of X:

$$\hat{X}(z) = \hat{X}(z)\big|_{M/G/1} \cdot \frac{U^*(\lambda)\big[1-T^*(\lambda)\big] + \big(1-U^*(\delta)\big)/(1-z)}{U^*(\lambda)\big[1-T^*(\lambda)\big] + \lambda E[U]},$$

which is equation (5) above.

3.4 Mean Queue Size

The mean queue size is obtained by differentiating equation (5) , and is given by

$$E[X] = E[X_{M/G/1}] + \frac{\lambda^2 E[U^2]}{2} \cdot \frac{1}{\gamma}, \tag{11}$$

where $\gamma = \lambda E[U] + U^*(\lambda)(1 - T^*(\lambda))$, and $E[X_{M/G/1}] = \frac{\lambda^2 E[B^2]}{2(1-\rho)} + \rho$ is the mean number of customers in a stationary $M/G/1$ system.

4 Waiting Time

Let $W :=$ waiting time (excluding service) of an arbitrary customer. Then, since

$$\hat{X}(z) = W^*(\delta) B^*(\delta) \, ,$$

we have

$$\begin{aligned} W^*(s) &= \frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)} \cdot \frac{\lambda\big[1 - U^*(s)\big] + sU^*(\lambda)\big[1 - T^*(\lambda)\big]}{s\big(U^*(\lambda)\big[1 - T^*(\lambda)\big] + \lambda E[U]\big)} \qquad (12) \\ &= W^*_{M/G/1}(s) \cdot \chi(s), \end{aligned}$$

where $\chi(s)$ is the LST of a random variable independent of $W_{M/G/1}$.

For the MV case we obtain (as in Levy and Yechiali [1975], equation (36), and Takagi [1992], equation (2.13)),

$$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)} \cdot \frac{1 - U^*(s)}{sE[U]} = W^*_{M/G/1}(s) \cdot R^*_U(s),$$

where R_U is the residual life (forward recurrence time) of the vacation length U.

For the SV case, equation (12) reduces to

$$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)} \cdot \frac{\lambda\big[1 - U^*(s)\big] + sU^*(\lambda)}{s\big(U^*(\lambda) + \lambda E[U]\big)},$$

as in Levy and Yechiali [1975], equation (22).

4.1 Mean Waiting Time

Differentiating equation (12) results in

$$E[W] = E[W_{M/G/1}] + \frac{\lambda E[U^2]}{2} \cdot \frac{1}{\gamma}, \tag{13}$$

where $E[W_{M/G/1}] = \frac{\lambda E[B^2]}{2(1-\rho)}$ and γ is defined below (11).

For the MV case, equation (13) reduces to

$$E[W_{MV}] = E[W_{M/G/1}] + \frac{E[U^2]}{2E[U]},$$

as in Levy and Yechiali [1975], equation (38).

For the SV case, equation (13) results in

$$E[W_{SV}] = E[W_{M/G/1}] + \frac{E[U^2]}{2\big(E[U] + \frac{U^*[\lambda]}{\lambda}\big)}.$$

Clearly, $E[W_{SV}] \leq E[W] \leq E[W_{MV}]$.

5 Busy and Non-Busy Periods

5.1 Busy Period

Let $\theta :=$ length of a busy period, i.e. the time interval extending from the moment the server starts working after a vacation (and possibly activating a Timer) until it leaves again for a new vacation. Following the law of motion (2), θ starts with either $A := A(U)\big|_{A(U)\geq 1}$ jobs (with probability $\alpha = (1 - U^*(\lambda))/(1 - U^*(\lambda)T^*(\lambda))$), or with a single job (with probability $1 - \alpha$). Thus,

$$\theta^*(s) = E\big[e^{-s\theta}\big] = E\big[e^{-s\sum_{i=1}^{A}\theta_i}\big] \cdot \alpha + E\big[e^{-s\theta_1}\big](1-\alpha),$$

where each θ_i is distributed like a regular $M/G/1$-type busy period with LST satisfying $\theta_1^*(s) = B^*\big(s + \lambda(1 - \theta_1^*(s))\big)$ and mean $E[\theta_1] = b/(1-\rho)$.

Now,

$$E\big[e^{-s\sum_{i=1}^{A}\theta_i}\big] = E_A\big[(\theta_1^*(s))^A\big] = \hat{A}\big(\theta_1^*(s)\big)\ ,$$

where (see (4)),

$$\hat{A}(z) = \frac{U^*\big(\lambda(1-z)\big) - U^*(\lambda)}{1 - U^*(\lambda)} \ .$$

Clearly, $E[A] = \frac{\lambda E[U]}{1-U^*(\lambda)}$.

Combining the above we obtain

$$\theta^*(s) = \frac{U^*\big(\lambda(1-\theta_1^*(s))\big) - U^*(\lambda)}{1 - U^*(\lambda)T^*(\lambda)} + \theta_1^*(s)\frac{U^*(\lambda)\big(1-T^*(\lambda)\big)}{1 - U^*(\lambda)T^*(\lambda)}. \tag{14}$$

It follows that

$$\begin{aligned} E[\theta] &= E[A] \cdot E[\theta_1] \cdot \alpha + E[\theta_1](1-\alpha) \\ &= E[\theta_1]\left[\frac{\lambda E[U] + U^*(\lambda)\big(1-T^*(\lambda)\big)}{1 - U^*(\lambda)\cdot T^*(\lambda)}\right]. \end{aligned} \tag{15}$$

For the MV case, where $T = 0$ and $T^*(\lambda) = 1$, equation (14) reduces to

$$\theta^*(s) = \frac{U^*\big(\lambda(1-\theta_1^*(s))\big) - U^*(\lambda)}{1 - U^*(\lambda)}, \tag{16}$$

leading to

$$E[\theta] = E[\theta_1] \cdot \frac{\lambda E[U]}{1 - U^*(\lambda)} = \frac{\rho E[U]}{(1-\rho)\big[1 - U^*(\lambda)\big]}. \tag{17}$$

Equations (16) and (17) coincide with Levy and Yechiali [1975], equations (24) and (26), respectively.

For the SV case, where $T = \infty$ and $T^*(\lambda) = 0$, equation (14) becomes

$$\theta^*(s) = U^*\Big(\lambda\big(1-\theta_1^*(s)\big)\Big) - U^*(\lambda) + \theta_1^*(s)U^*(\lambda),$$

leading to

$$E[\theta] = E[\theta_1]\big(\lambda E[U] + U^*(\lambda)\big).$$

The above coincide, respectively, with equations (2.30a) and (2.30b) of Takagi [1991].

5.2 Vacation Period

Let I be the Exponential(λ) inter-arrival time, with LST $I^*(s) = \frac{\lambda}{s+\lambda}$. Denote by V_P the *vacation period*, i.e. the time interval beginning at the end of an active busy period and extending to the start of the next busy period. Let $\{U_i, i = 1, 2, 3, ...\}$ ($\{T_i, i = 1, 2, 3, ...\}$) be a sequence of i.i.d. random variables having LST $U^*(\cdot)$ ($T^*(\cdot)$).

For particular realizations $U_1, U_2, \ldots, U_{k+1}, T_1, T_2, \ldots, T_k$, and I we have

$$V_P = \begin{cases} (U_1+T_1)+\cdots+(U_k+T_k)+U_{k+1} & \text{w.p.} \quad e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)} \\ & \qquad\qquad \times(1-e^{-\lambda U_{k+1}}) \\ (U_1+T_1)+\cdots+(U_k+T_k)+U_{k+1}+I & \text{w.p.} \quad e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)} \\ & \qquad\qquad \times e^{-\lambda U_{k+1}}\overline{F}_{T_{k+1}}(I) \end{cases}$$

where $\overline{F}_{T_{k+1}}(I) = P[T_{k+1} > I]$. Hence

$$\begin{aligned} V_P^*(s) &= \sum_{k=0}^{\infty} E\Big[e^{-s\sum_{i=1}^{k}(U_i+T_i)}e^{-sU_{k+1}}e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}(1-e^{-\lambda U_{k+1}})\Big] \\ &\quad + \sum_{k=0}^{\infty} E\Big[e^{-s\sum_{i=1}^{k}(U_i+T_i)}e^{-sU_{k+1}}e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}e^{-\lambda U_{k+1}}\Big]E\big[e^{-sI}\overline{F}_{T_{k+1}}(I)\big] \\ &= \Big[\big[U^*(s)-U^*(s+\lambda)\big] + U^*(s+\lambda)\int_{I=0}^{\infty}\lambda e^{-(\lambda+s)I}\overline{F}_T(I)dI\Big] \\ &\qquad\qquad \times\sum_{k=0}^{\infty}\big[U^*(s+\lambda)T^*(s+\lambda)\big]^k. \end{aligned}$$

Thus,

$$V_P^*(s) = \frac{U^*(s) - U^*(s+\lambda) + U^*(s+\lambda)\frac{\lambda}{\lambda+s}[1-T^*(s+\lambda)]}{1-U^*(s+\lambda)T^*(s+\lambda)}. \tag{18}$$

Differentiating (18) yields

$$E[V_P] = \frac{E[U] + \frac{U^*(\lambda)}{\lambda}[1-T^*(\lambda)]}{1-U^*(\lambda)T^*(\lambda)} = \frac{\gamma}{\lambda\big[1-U^*(\lambda)T^*(\lambda)\big]}. \tag{19}$$

For the MV case (with $T = 0$, $T^*(\cdot) = 1$), equation (18) reduces to

$$V_P^*(s) = \frac{U^*(s) - U^*(s+\lambda)}{1-U^*(s+\lambda)},$$

which coincides with Takagi [1991] equation (2.18a), while equation (19) reduces to

$$E[V_P] = \frac{E[U]}{1 - U^*(\lambda)}. \tag{20}$$

Equation (20) is identical to equation (25) of Levy and Yechiali [1975] and equation (2.18b) of Takagi [1991].

For the SV case (with $T = \infty$, $T^*(\cdot) = 0$), equations (18) and (19) yield, respectively,

$$V_P^*(s) = U(s) - U^*(s+\lambda) + U^*(s+\lambda)I^*(s),$$

and

$$E[V_P] = E[U] + \frac{U^*(\lambda)}{\lambda}. \tag{21}$$

5.3 Cycle Time

Let $C := V_P + \theta$ denote the cycle time. Then, for given $U_1, T_1, U_2, T_2, \dots, U_k$, T_k, U_{k+1} and I, and with $A := A(U_{k+1})|_{A(U_{k+1})\geq 1}$ we have

$$C = \begin{cases} \sum_{i=1}^{k}(U_i + T_i) + U_{k+1} + \sum_{j=1}^{A}\theta_j & \text{w.p.} \quad e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}(1 - e^{-\lambda U_{k+1}}), \\ \sum_{i=1}^{k}(U_i + T_i) + U_{k+1} + I + \theta_1 & \text{w.p.} \ e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}e^{-\lambda U_{k+1}}\overline{F}_{T_{k+1}}(I), \end{cases}$$

where, as before, θ_1 is the busy period of a regular $M/G/1$ queue with arrival rate λ and service times B, and $\overline{F}_{T_{k+1}}(I) = P(T_{k+1} > I)$. Then, the LST of C is given by

$$
\begin{aligned}
C^*(s) &= \sum_{k=0}^{\infty} E\Big[e^{-s\sum_{i=1}^{k}(U_i+T_i)}e^{-sU_{k+1}} \cdot e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}(1-e^{-\lambda U_{k+1}}) \cdot e^{-s\sum_{j=1}^{A}\theta_j}\Big] \\
&\quad + \sum_{k=0}^{\infty} E\Big[e^{-s\sum_{i=1}^{k}(U_i+T_i)}e^{-sU_{k+1}}e^{-sI} \cdot e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}e^{-\lambda U_{k+1}}\overline{F}_{T_{k+1}}(I)e^{-s\theta_1}\Big] \\
&= \sum_{k=0}^{\infty}\big[U^*(s+\lambda)T^*(s+\lambda)\big]^k \\
&\qquad \times E\Bigg[e^{-sU_{k+1}}(1-e^{-\lambda U_{k+1}}) \cdot \Big(\frac{e^{-\lambda U_{k+1}(1-\theta_1^*(s))} - e^{-\lambda U_{k+1}}}{1-e^{-\lambda U_{k+1}}}\Big)\Bigg] \\
&\quad + \sum_{k=0}^{\infty}\big[U^*(s+\lambda)T^*(s+\lambda)\big]^k U^*(s+\lambda)\frac{\lambda}{\lambda+s}\big[1-T^*(s+\lambda)\big]\theta_1^*(s)\ .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&C^*(s) \\
&= \frac{U^*\big[s+\lambda(1-\theta_1^*(s))\big] - U^*(s+\lambda) + U^*(s+\lambda)\frac{\lambda}{\lambda+s}\big[1-T^*(s+\lambda)\big]\theta_1^*(s)}{1-U^*(s+\lambda)T^*(s+\lambda)}.
\end{aligned}
\tag{22}
$$

By differentiation,

$$
E[C] = \frac{E[U] + \frac{U^*(\lambda)}{\lambda}\big[1-T^*(\lambda)\big]}{(1-\rho)\big[1-U^*(\lambda)T^*(\lambda)\big]}. \tag{23}
$$

Note: It readily follows from (15) and (23) that the fraction of time the server is busy is, as expected,

$$
\frac{E[\theta]}{E[C]} = E[\theta_1]\lambda(1-\rho) = \rho.
$$

For the MV case, equation (22) reduces to

$$
C_{MV}^*(s) = \frac{U^*\big[s+\lambda(1-\theta_1^*(s))\big] - U^*(s+\lambda)}{1-U^*(s+\lambda)},
$$

while equation (23) reduces to

$$
E[C_{MV}] = \frac{E[U]}{(1-\rho)[1-U^*(\lambda)]}. \tag{24}
$$

Equation (24) coincides with Levy and Yechiali [1975], equation (27).

For the SV case, equation (22) reduces to

$$C^*_{SV}(s) = U^*\Big[s + \lambda\big(1 - \theta^*_1(s)\big)\Big] - U^*(s+\lambda) + U^*(s+\lambda)\frac{\lambda}{\lambda+s}\theta^*_1(s),$$

which coincides with Levy and Yechiali [1975], equation (4), while equation (23) reduces to

$$E[C_{SV}] = \frac{1}{1-\rho}\left[E[U] + \frac{U^*(\lambda)}{\lambda}\right],$$

which is identical with Levy and Yechiali [1975], equation (6).

5.4 Sum of U Durations within a Vacation Period

Let H be the sum of the durations of the vacations within a vacation period V_P. Then, similar to the derivation of $V^*_P(\cdot)$ we write

$$\begin{aligned} H^*(s) &= \sum_{k=0}^{\infty} E\Big[e^{-s\sum_{i=1}^{k}U_i} \cdot e^{-sU_{k+1}} \cdot e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}\big(1 - e^{-\lambda U_{k+1}}\big)\Big] \\ &\quad + \sum_{k=0}^{\infty} E\big[e^{-s\sum_{i=1}^{k}U_i} \cdot e^{-sU_{k+1}} \cdot e^{-\lambda\sum_{i=1}^{k}(U_i+T_i)}e^{-\lambda U_{k+1}}\big]\big(1 - T^*(\lambda)\big) \\ &= \big[U^*(s) - U^*(s+\lambda)\big]\sum_{k=0}^{\infty}\big[U^*(s+\lambda)T^*(\lambda)\big]^k \\ &\quad + U^*(s+\lambda)\big[1 - T^*(\lambda)\big]\sum_{k=0}^{\infty}\big[U^*(s+\lambda)T^*(\lambda)\big]^k. \end{aligned}$$

Thus,

$$H^*(s) = \frac{U^*(s) - U^*(s+\lambda)T^*(\lambda)}{1 - U^*(s+\lambda)T^*(\lambda)}.$$

This implies that

$$E[H] = \frac{E[U]}{1 - U^*(\lambda)T^*(\lambda)}. \tag{25}$$

Indeed, the server continues taking vacations U in a Bernoulli fashion with probability $U^*(\lambda)T^*(\lambda)$.

5.5 Non-Productive Idle Time

The total time within a cycle in which the server is non-productive, i.e. waiting idle for either an arrival or the expiration of the Timer, is given by $NP := V_p - H$.

Similarly to the derivations of V_P and H we write

$$\begin{aligned} NP^*(s) &= \sum_{k=0}^{\infty} E\Big[e^{-s\sum_{i=1}^{k} T_i} e^{-\lambda \sum_{i=1}^{k}(U_i+T_i)}(1-e^{-\lambda U_{k+1}})\Big] \\ &\quad + \sum_{k=0}^{\infty} E\big[e^{-s\sum_{i=1}^{k} T_i} e^{-\lambda \sum_{i=1}^{k}(U_i+T_i)} e^{-\lambda U_{k+1}}\big] \frac{\lambda}{s+\lambda}\big(1-T^*(s+\lambda)\big) \\ &= \frac{1-U^*(\lambda)\frac{s}{s+\lambda} - U^*(\lambda)\frac{\lambda}{s+\lambda}T^*(s+\lambda)}{1-T^*(s+\lambda)U^*(\lambda)}. \end{aligned}$$

By differentiation we get

$$E[NP] = \frac{\frac{U^*(\lambda)}{\lambda}(1-T^*(\lambda))}{1-T^*(\lambda)U^*(\lambda)}. \tag{26}$$

Equation (26) can also be obtained from (21) and (25), since $NP = V_P - H$.

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