Burst Arrival Queues with Server Vacations and Random Timers

Merav Shomrony and Uri Yechiali

Department of Statistics and Operations Research

School of Mathematical Sciences

Raymond and Beverly Sackler Faculty of Exact Sciences

Tel Aviv University, Tel Aviv 69978, Israel

Abstract

We derive performance measures for burst arrival (e.g. messages of variable length packets) $M^K / G / 1$ queues with server vacations, controlled by the, so called, Randomly Timed Gated (RTG) protocol, operating as follows : Whenever the server returns from a (general-type) vacation and initiates a busy period, a Timer with random duration T is activated. If the server empties the queue before time T, he leaves for another vacation. Otherwise (i.e. if there are still customers (packets) in the system when the timer expires), two versions of terminating the busy period, each leading to a different model, are studied : (i). The server completes service (e.g. transmission) only to the customer being served at time T and leaves. (ii). The server leaves immediately. We derive both state-dependent and steady-state performance measures as a function of the (vacation-type dependent variable) number of customers present at the initiation of a busy period. When the vacation policy is specified (i.e. Multiple or Single), we obtain explicit formulas for the various performance measures, derive the distribution and mean of the waiting and sojourn times of a customer, and compare between the two versions.

Analysis of the conditions for stability concludes the paper.

Keywords: Batch arrivals, Vacations, Random Timer

1. Model and Notation

We study a burst arrival (such as messages comprised of variable length packets) $M^K / G / 1$ queues with general server vacations controlled by the Randomly Timed Gated (RTG) protocol (Eliazar and Yechiali [1998]), operating as follows : Whenever the server returns from a (general-type) vacation and initiates a busy period, a Timer with random duration T is activated. If the server empties the queue before time T, he leaves for another vacation. Otherwise (i.e. if there are still customers (e.g. packets) in the system when the timer expires), two versions of terminating the busy period, each leading to a different model, are studied :

Version 1 : The server completes service (e.g. transmission) only to the customer (packet) being served at time T and leaves. At the next busy period the service starts with the next customer in queue.

Version 2 : The server leaves immediately for a vacation. The service of the preempted customer will have to be repeated (resampled).

Bulk queues have been studied extensively in the literature and a wealth of results is contained in books by Cohen [1982] , Chaudhry and Templeton [1983] , Medhi [1984] and Takagi [1991], as well as in various papers such as Baba [1986], and Rosenberg and Yechiali [1993]. In modern communication networks, batch arrivals models may be used to represent bursty traffic (such as variable-size messages), while the RTG regime may be considered as either a time-limit on the duration of each busy period (e.g. transmission duration), or as a failing process that affects the operation of the system.

The regular $M^K / G / 1$ queueing system is characterized by a Poisson arrival process (with rate λ) of i.i.d random batches of customers (e.g. packets), where each batch (message) size, K, has a probability mass function $P(K = n) = f_n$, ($n \geq 1$). Customers are served one by one by a single server and the service times of individual customers ,V , are i.i.d continuous random variables with density $f_V(\cdot)$.

Batches (messages) are admitted to service according to the First Come First Served (FCFS) order. Within a batch, customers (packets) are served (transmitted) according to their inner order. However, under the RTG protocol, the length of each busy period is governed by a random - duration Timer T, as described above. T is distributed Exponentially with density $f_T(t) = \mu e^{-\mu t}$. At the termination of a busy period, the server leaves for a random vacation of duration U. Distinct vacations are i.i.d continuous random variables having density $f_U(\cdot)$. We denote by A(t) the number of batches arriving in (0,t] and assume that the Poisson arrival process $\{ A(t) \}_{t \geq 0}$, the sequence of service times , the Timer T and the sequence of vacations are mutually independent.

We define the following :

Busy period : the time interval from the moment when the server starts serving the queue, after returning from a vacation, until he leaves the system for another vacation.

Cycle : the time interval between two consecutive moments in which the server 'enters' the system.

If, upon returning from a vacation, the system is not empty, the server starts immediately a new busy period. Otherwise, if the system is empty, we analyze in the sequel two policies, known as: 1. Multiple vacation and 2. Single vacation (see Levy and Yechiali [1975] , Kella and Yechiali [1988] , Takagi [1991]) .

We also use the following notation :

$\hat{M}(z) = E[z^M]$: probability generating function (PGF) of a discrete random variable M, where $|z| \leq 1$.

$\tilde{M}(w) = E[e^{-wM}]$: Laplace-Stieltjes transform (LST) of a non negative random variable M,

where $Re(w) \geq 0$.

B_r : length of a busy period initiated by r waiting customers ($r \geq 0$)

Y_r : queue size at the end of a busy period initiated by r waiting customers ($r \geq 0$)

N_r : number of customers whose service has been fully completed during a busy period initiated by r waiting customers ($r \geq 0$)

For Version 2 , we also define B_r^{eff} and Y_r^{eff} as follows : B_r^{eff} is the effective part of B_r , i.e. the time interval from the beginning of B_r until the last full service completion in B_r. Y_r^{eff} is the queue size at the end of B_r^{eff} .

The structure of the paper and the main results are the following :

In section 2 we develop the joint distribution of the state-dependent pair (B_r , Y_r) for each of the two versions. In section 3 we analyze the system in steady state and obtain the following performance measures : (1) LST and mean of the length of a busy period , B. (2) PGF and mean of the number of customers , Y, left behind at the end of a busy period . (3) Mean number of customers , N, served during a busy period . (4) Mean length of a cycle , C. (5) P_{busy} , the probability that the server is busy, and P_{eff} , the proportion of time that he is working effectively. (6) PGF and mean of L , the queue size at service completion instants.

Each of these measures is expressed as a function of the variable X, number of customers present at the beginning of a busy period. This variable depends heavily on the vacation policy.

In particular , in section 4, we consider the Multiple and Single vacations policies and find the distribution of X , from which we derive explicit expressions for the X -dependent general measures obtained earlier in section 3. Mean values of certain measures are compared between the two versions. In section 5 we develop an expression for the queue size at an arbitrary moment. Note that, since the arrivals are in batches, PASTA is not valid here, and the above expression differs from L. In section 6 we derive the LST and mean of the waiting time of a customer, as well as the LST and mean of its sojourn time in the system, for each version and vacation type. We conclude by deriving necessary and sufficient conditions for stability for both versions.

2. The joint distribution of (B_r , Y_r)

In this section we develop the joint distribution of the random pair (B_r , Y_r), r = 1,2,3,... for each of the two versions. (Note that $B_0 = Y_0 = N_0 = 0$). We will later use these results to obtain the joint distribution and performance measures for the system in steady state.

Define the joint t:

$$\begin{aligned} \Phi_r(w,z) &= E\,[\ e^{-wB_r}\ z^{Y_r}\ ] \\ \Phi_r^{eff}(w,z) &= E\,[\ e^{-wB_r^{eff}}\ z^{Y_r^{eff}}\ ] \end{aligned} \quad , \quad \text{Re}(w) \geq 0,\ |z| \leq 1 \qquad (2.1)$$

By observing the process at the first service completion within B_r , we can analyze Versions 1 and 2 simultaneously.

Indeed, if $T > V_1$, at time V_1 the process regenerates itself with $r-1+\sum_{i=1}^{A(V_1)} K_i$ waiting customers, where $\{ K_i \}_{i=1}^{\infty}$ are i.i.d , all distributed like K .

Therefore, for $r \geq 1$:

(i) if $T > V_1$, then for both versions :

$$(B_r , Y_r) \overset{d}{=} \left(V_1 + B_{r-1+\sum_{i=1}^{A(V_1)} K_i} \ , \ Y_{r-1+\sum_{i=1}^{A(V_1)} K_i} \right)$$

where the equality holds also for (B_r^{eff} , Y_r^{eff}).

(ii) if $T \le V_1$, then, for Version 1 : $(B_r, Y_r) \overset{d}{=} (V_1, \; r-1+\sum_{i=1}^{A(V_1)} K_i)$

for Version 2 : $(B_r, Y_r) \overset{d}{=} (T, \; r+\sum_{i=1}^{A(T)} K_i)$

$(B_r^{eff}, Y_r^{eff}) \overset{d}{=} (0, \; r)$

Derivation of $\Phi_r(w,z) = E[\, e^{-wB_r} \; z^{Y_r} \,]$

We write

$$E[\, e^{-wB_r} \; z^{Y_r}] = E[\, e^{-wB_r} \; z^{Y_r} \mid T \le V_1 \,] \cdot P(T \le V_1) + E[\, e^{-wB_r} \; z^{Y_r} \mid T > V_1 \,] \cdot P(T > V_1) \tag{2.2}$$

For the two versions, suppressing the index from V_1, and setting $R = r-1+\sum_{i=1}^{A(v)} K_i$, we get

$$E[e^{-wB_r} \; z^{Y_r} \mid T > V_1] \cdot P(T > V_1) = \int_{v=0}^{\infty}\int_{t=v}^{\infty} E[e^{-wB_r} z^{Y_r} \mid T=t, V=v] \cdot f_T(t)dt \cdot f_V(v)dv$$

$$= \int_{v=0}^{\infty}\int_{t=v}^{\infty} e^{-wv} \cdot E[\, e^{-wB_R} \cdot z^{Y_R} \,] \cdot f_T(t)dt \cdot f_V(v)dv$$

$$= \int_{v=0}^{\infty}\int_{t=v}^{\infty} e^{-wv} \cdot \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} E[e^{-w B_{r-1+k}} \cdot z^{Y_{r-1+k}}] \cdot P(\sum_{i=1}^{j} K_i = k) \cdot P(A(v)=j) \cdot f_T(t)dt \cdot f_V(v)dv$$

$$= \int_{v=0}^{\infty} e^{-(\lambda+w+\mu)v} \cdot \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} \Phi_{r-1+k}(w,z) \cdot P(\sum_{i=1}^{j} K_i = k) \cdot \frac{(\lambda v)^j}{j!} \cdot f_V(v)dv$$

$$= \sum_{k=0}^{\infty} \Phi_{r-1+k}(w,z) \cdot \sum_{j=0}^{k} P(\sum_{i=1}^{j} K_i = k) \cdot E[\, e^{-(\lambda+w+\mu)V} \cdot \frac{(\lambda V)^j}{j!} \,]$$

From the above we obtain

$$E[\, e^{-wB_r} \; z^{Y_r} \mid T > V_1 \,] \cdot P(T > V_1) = \sum_{k=0}^{\infty} \Phi_{r-1+k}(w,z) \cdot a_k(w) \tag{2.3}$$

where

$$a_k(w) \equiv \sum_{j=0}^{k} P(\sum_{i=1}^{j} K_i = k) \cdot E[\, e^{-(\lambda+w+\mu)V} \cdot \frac{(\lambda V)^j}{j!} \,] \tag{2.4}$$

Again, the last equality stands also for $E[\, e^{-wB_r^{eff}} \; z^{Y_r^{eff}} \mid T > V_1 \,] \cdot P(T > V_1)$ with

Φ_r^{eff} replacing Φ_r .

Now, when $T \le V_1$, the derivation is carried out for each version separately.

Version 1

$$E[\, e^{-wB_r} z^{Y_r} \mid T \le V_1 \,] \cdot P(\, T \le V_1 \,) \;=\; \int_{v=0}^{\infty} \int_{t=0}^{v} E[e^{-wB_r} z^{Y_r} \mid T = t, V = v] \cdot f_T(t)dt \cdot f_V(v)dv$$

$$= \int_{v=0}^{\infty} \int_{t=0}^{v} e^{-wv} \cdot E[\, z^{\,r-1+\sum_{i=1}^{A(v)} K_i} \,] \cdot f_T(t)\, dt \cdot f_V(v)\, dv$$

$$= z^{r-1} \cdot \int_{v=0}^{\infty} (1 - e^{-\mu v}) \cdot e^{-wv} \cdot \sum_{j=0}^{\infty} P(\, A(v) = j) \cdot E[z^{\sum_{i=1}^{j} K_i}] \cdot f_V(v)dv$$

$$= z^{r-1} \cdot \int_{v=0}^{\infty} (1 - e^{-\mu v}) \cdot e^{-wv} \cdot \sum_{j=0}^{\infty} \frac{e^{-\lambda v} \cdot (\lambda v)^j}{j!} \cdot [\, \hat{K}(z) \,]^j \cdot f_V(v)\, dv$$

$$= z^{r-1} \cdot \int_{v=0}^{\infty} (1 - e^{-\mu v}) \cdot e^{-(w+\lambda)v} \cdot e^{\lambda v \hat{K}(z)} \cdot f_V(v)dv$$

$$= z^{r-1} \cdot \left[\tilde{V}(w + \lambda\,(1 - \hat{K}(z))) - \tilde{V}(\, w + \mu + \lambda\,(1 - \hat{K}(z))) \right] \tag{2.5}$$

Version 2

$$E[\, e^{-wB_r} z^{Y_r} \mid T \le V_1 \,] \cdot P(\, T \le V_1 \,) \;=\; \int_{v=0}^{\infty} \int_{t=0}^{v} E[e^{-wB_r} z^{Y_r} \mid T = t, V = v] \cdot f_T(t)dt \cdot f_V(v)dv$$

$$= \int_{v=0}^{\infty} \int_{t=0}^{v} e^{-wt} \cdot z^r \cdot E[\, z^{\sum_{i=1}^{A(t)} K_i} \,] \cdot \mu\, e^{-\mu t} dt \cdot f_V(v)dv$$

$$= z^r \cdot \int_{v=0}^{\infty} \int_{t=0}^{v} e^{-(\lambda + w + \mu)t} \cdot \mu\, e^{\lambda t \hat{K}(z)} dt \cdot f_V(v)\, dv$$

$$= z^r \cdot \int_{v=0}^{\infty} \frac{\mu}{w + \mu + \lambda(1 - \hat{K}(z))} \cdot (\, 1 - e^{-(\, w + \mu + \lambda(1 - \hat{K}(z)))v} \,) \cdot f_V(v)\, dv$$

$$= \frac{\mu\, z^r \cdot [\, 1 - \tilde{V}(\, w + \mu + \lambda(1 - \hat{K}(z)))\,]}{w + \mu + \lambda(1 - \hat{K}(z))} \tag{2.6}$$

Also,

$$E[\, e^{-wB_r^{eff}} z^{Y_r^{eff}} \mid T \le V_1 \,] \cdot P(\, T \le V_1 \,) \;=\; \int_{v=0}^{\infty} \int_{t=0}^{v} z^r \cdot \mu\, e^{-\mu t} dt \cdot f_V(v)dv$$

$$= \; z^r \cdot \int_{v=0}^{\infty} (1 - e^{-\mu v}) \cdot f_V(v)\, dv \;=\; z^r \cdot (1 - \tilde{V}(\mu)) \tag{2.7}$$

To conclude the calculation we substitute results (2.2) through (2.7) in (2.1) and obtain infinite sets of equations :

$$\begin{bmatrix} \Phi_0(w,z) = 1 \\ \Phi_r(w,z) = \sum_{k=0}^{\infty} \Phi_{r-1+k}(w,z) \cdot a_k(w) + c\,(w,z) \cdot z^r, \quad r > 0 \end{bmatrix} \tag{2.8}$$

$$\begin{bmatrix} \Phi_0{}^{eff}(w,z) = 1 \\ \Phi_r{}^{eff}(w,z) = \sum_{k=0}^{\infty} \Phi_{r-1+k}{}^{eff}(w,z) \cdot a_k(w) + c^{eff}(w,z) \cdot z^r, \quad r > 0 \end{bmatrix}$$

where $a_k(w)$ is given by (2.4) and

$$c\,(w,z) = \begin{cases} z^{-1} \cdot \left[\tilde{V}(w+\lambda(1-\hat{K}(z))) - \tilde{V}(\,w+\mu+\lambda(1-\hat{K}(z)))\, \right] & Version\ 1 \\ \dfrac{\mu \cdot \left[\, 1-\tilde{V}(w+\mu+\lambda(1-\hat{K}(z)))\right]}{w+\mu+\lambda(1-\hat{K}(z))} & Version\ 2 \end{cases}$$

$$c^{eff}(w,z) = 1 - \tilde{V}(\mu)$$

<u>Solution of the set (2.8)</u>

It has been shown by Eliazar and Yechiali [1998] that the set of equations of the form (2.8) admits a unique solution. In reference [13] we show (and it can be also be verified by substitution) that the solution of (2.8) is

$$\Phi_r(w,z) = \varphi\,(w,z) \cdot z^r + (1-\varphi(w,z)) \cdot [\beta(w,z)]^r \tag{2.9}$$

where

$$\varphi(w,z) = \frac{c(w,z) \cdot z}{z - \tilde{V}(w+\mu+\lambda(1-\hat{K}(z)))} \tag{2.10}$$

and

$$\beta(w,z) = \beta(w) = \tilde{V}(\,w+\mu+\lambda(1-\tilde{\theta}_K(w+\mu)))\,.$$

$\tilde{\theta}_K(w)$ is the LST of the length of a busy period in $M^K / G / 1$ model, initiated by an arrival of a batch into an empty system, and it is known (see Rosenberg and Yechiali [1993]) that :

$$\tilde{\theta}_K(w) = \hat{K}(\tilde{V}(w+\lambda(\,1-\tilde{\theta}_K(w)\,))\,) \tag{2.11}$$

By substituting c(w,z) in (2.10) and setting $\sigma \equiv w+\mu+\lambda(1-\hat{K}(z))$ we get

<u>Version 1</u>

$$\varphi\,(w,z) = \frac{\tilde{V}(\sigma-\mu) - \tilde{V}(\sigma)}{z - \tilde{V}(\sigma)} \tag{2.12}$$

Version 2

$$\varphi(w,z) = \frac{z\mu \cdot (1-\tilde{V}(\sigma))}{\sigma \cdot (z-\tilde{V}(\sigma))}$$

$$\varphi^{eff}(w,z) = \frac{z \cdot (1-\tilde{V}(\mu))}{z-\tilde{V}(\sigma)} \tag{2.13}$$

3. System in steady state

In this section we derive various performance measures for the system in steady state (conditions are discussed in section 7). We define the following :

C - Cycle length.

B - Length of a busy period during a cycle (B may be 0, depending on the particular vacation regime applied in the system).

N - Number of customers successfully served during a busy period.

X - Queue size at the start of a busy period.

Y - Queue size at the end of a busy period.

Now, from (2.9), the joint transform of (B,Y) is given by

$$\begin{aligned}\Phi(w,z) &= E\,[e^{-wB}z^{Y}] = E_X\Big[\,E\,\big[\,e^{-wB}\,z^{Y}\,\big|\,X\,\big]\Big] = E_X\big[\Phi_X(w,z)\big]\\ &= \varphi(w,z)\cdot E[z^X] + (1-\varphi(w,z))\cdot E\Big[[\tilde{V}(w+\mu+\lambda(1-\tilde{\theta}_K(w+\mu)))]^X\Big]\\ &= \varphi(w,z)\cdot\hat{X}(z) + (1-\varphi(w,z))\cdot\hat{X}(\tilde{V}(w+\mu+\lambda(1-\tilde{\theta}_K(w+\mu))))\end{aligned} \tag{3.1}$$

Equation (3.1) implies that the joint distribution of (B,Y) , as well as the marginal distributions of B and Y, are explicitly given in terms of $\hat{X}(\cdot)$, the PGF of X (which, by itself, depends heavily on the type of vacation policy being employed and on the busy period termination version).

3.1 Performance Measures

Set $\delta \equiv \tilde{V}(\mu+\lambda(1-\tilde{\theta}_K(\mu)))$, $\alpha \equiv \lambda(1-\hat{K}(z))$. Then from (3.1) :

$$\tilde{B}(w) = \Phi(w,1) = \varphi(w,1) + (1-\varphi(w,1))\cdot\hat{X}(\tilde{V}(w+\mu+\lambda(1-\tilde{\theta}_K(w+\mu)))) \tag{3.2}$$

$$\hat{Y}(z) = \Phi(0,z) = \varphi(0,z)\cdot\hat{X}(z) + (1-\varphi(0,z))\cdot\hat{X}(\delta) \tag{3.3}$$

Version 1

By taking derivatives of (3.2) and (3.3), while using (2.12) we obtain

$$E[B] = \frac{E[V]\cdot(1-\hat{X}(\delta))}{1-\tilde{V}(\mu)} \tag{3.4}$$

$$E[Y] = E[X] - \frac{(1-\lambda E[V]E[K])\cdot(1-\hat{X}(\delta))}{1-\tilde{V}(\mu)} \tag{3.5}$$

<u>Version 2</u> In a similar manner, by using (2.13),

$$E[B] = \frac{1}{\mu}\cdot(1-\hat{X}(\delta)) \tag{3.6}$$

$$E[Y] = E[X] + \left[1-\hat{X}(\delta)\right]\cdot\left[\frac{\lambda E[K]\cdot(1-\tilde{V}(\mu)) - \mu\tilde{V}(\mu)}{\mu(1-\tilde{V}(\mu))}\right] \tag{3.7}$$

$$E[B^{eff}] = \frac{\tilde{V}'(\mu)\cdot[\hat{X}(\delta)-1]}{1-\tilde{V}(\mu)} \tag{3.8}$$

$$E[Y^{eff}] = E[X] + \frac{[\lambda E[K]\cdot\tilde{V}'(\mu) + \tilde{V}(\mu)]\cdot[\hat{X}(\delta)-1]}{1-\tilde{V}(\mu)} \tag{3.9}$$

where $\tilde{V}'(\mu) = \frac{d}{dw}\tilde{V}(w)\Big|_{w=\mu}$.

We now calculate E[N], E[C], P_{busy} and P_{eff} for both versions.

<u>Version 1</u>

E[N] is calculated by using Wald's identity and (3.4). Clearly, $B = \sum_{i=1}^{N} V_i$, leading to

$$E[N] = \frac{E[B]}{E[V]} = \frac{1-\hat{X}(\delta)}{1-\tilde{V}(\mu)} \tag{3.10}$$

Since in steady state E[A(C)] = E[N] we get $\lambda E[K]E[C] = E[N]$. That is,

$$E[C] = \frac{E[N]}{\lambda E[K]} = \frac{1-\hat{X}(\delta)}{\lambda E[K]\cdot(1-\tilde{V}(\mu))} \tag{3.11}$$

Define P_{busy} as the proportion of time that the server is busy . Then,

$$P_{busy} = \frac{E[B]}{E[C]} = \lambda E[V]E[K] \equiv \rho \tag{3.12}$$

Clearly, in Version 1, the rate of work flowing into the system, ρ , must be equal to the fraction of time the server is busy.

<u>Version 2</u>

Define S, the duration of a successful service attempt, as

$$S \sim V \,\big|\, V < T. \quad Then, \quad E[S] = -\frac{\tilde{V}'(\mu)}{\tilde{V}(\mu)}$$

Now, $B^{eff} = \sum_{i=1}^{N} S_i$, where S_i are i.i.d as S.

Hence, by using (3.8)

$$E[N] \;=\; \frac{E[B^{eff}\,]}{E[S]} \;=\; \frac{\tilde{V}(\mu)\;\cdot\;[\,1-\hat{X}(\delta)\,]}{1-\tilde{V}(\mu)} \tag{3.13}$$

$$E[C] \;=\; \frac{E[N]}{\lambda E[K]} \;=\; \frac{\tilde{V}(\mu)\;\cdot[\,1-\hat{X}(\delta)]}{\lambda E[K]\cdot(1-\tilde{V}(\mu))} \tag{3.14}$$

$$P_{busy} \;=\; \frac{E[B]}{E[C]} \;=\; \frac{\lambda E[K]\cdot(1-\tilde{V}(\mu))}{\mu\tilde{V}(\mu)} \tag{3.15}$$

For Version 2 we also define P_{eff} as the proportion of time the server is working 'effectively'.

i.e. $P_{eff} \;=\; \dfrac{E[B^{eff}\,]}{E[C]}$. Using (3.8) and (3.14) we have

$$P_{eff} \;=\; -\frac{\tilde{V}\,'(\mu)}{\tilde{V}(\mu)}\cdot\lambda E[K] = \lambda E[K]E[S] \tag{3.16}$$

Indeed, the mean number of individual arrivals is $\lambda E[K]$, each being effectively served exactly once, requiring E[S] units of time.

3.2 Queue size at service completion instants

We now present results regarding $\hat{L}(z)$, the PGF of the steady state queue size at service completion instants. The derivations are omitted and may be found in reference [13].

<u>Version 1</u>

$$\hat{L}(z) = (\,1-\rho\,)\;\cdot\frac{\tilde{V}(\lambda(1-\hat{K}(z)))}{z-\tilde{V}(\lambda(1-\hat{K}(z)))}\;\cdot\;\frac{\hat{X}(z)-\hat{Y}(z)}{E[X]-E[Y]} \tag{3.17}$$

Note that, for the regular M^K / G / 1 queue, Cohen ([1982] Eq. (2.10) , p. 386), has shown that the PGF of the number of customers left behind in the system by a departing customer is given by

$$\hat{L}_{M^K/G/1}(z) \;=\; (1-\rho)\cdot\frac{\tilde{V}(\lambda(1-\hat{K}(z)))}{z-\tilde{V}(\lambda(1-\hat{K}(z)))}\cdot\frac{\hat{K}(z)-1}{E[K]} \tag{3.18}$$

It follows that for the RTG regime

$$\hat{L}(z) = \hat{L}_{M^K/G/1}(z)\cdot\frac{E[K]}{\hat{K}(z)-1}\cdot\frac{\hat{X}(z)-\hat{Y}(z)}{E[X]-E[Y]} \tag{3.19}$$

If K≡1, Eq. (3.19) coincides with the Fuhrmann - Cooper decomposition [1985] (see also Borst [1995]).

In order to obtain expressions for $\hat{L}(z)$ in terms of $\hat{X}(\cdot)$ alone, we use (3.3), (2.12), (3.5) and get

$$\hat{L}(z) = (1-\tilde{V}(\mu))\cdot\frac{\tilde{V}(\alpha)}{[z-\tilde{V}(\alpha+\mu)]}\cdot\frac{[\hat{X}(z)-\hat{X}(\delta)]}{[1-\hat{X}(\delta)]} \qquad (3.20)$$

Differentiating,

$$E[L] = \rho + \frac{E[X]}{1-\hat{X}(\delta)} - \frac{1-P_{eff}\cdot\tilde{V}(\mu)}{1-\tilde{V}(\mu)} \qquad (3.21)$$

Version 2

$$\hat{L}(z) = \frac{1-\tilde{V}(\mu)}{\tilde{V}(\mu)}\cdot\frac{\tilde{V}(\alpha+\mu)}{[z-\tilde{V}(\alpha+\mu)]}\cdot\frac{[\hat{X}(z)-\hat{X}(\delta)]}{[1-\hat{X}(\delta)]} \qquad (3.22)$$

$$E[L] = \frac{E[X]}{1-\hat{X}(\delta)} - \frac{1-P_{eff}}{1-\tilde{V}(\mu)} \qquad (3.23)$$

To summarize, all performance measures derived in this section are given in terms of the PGF and mean of X, the queue size at the beginning of a busy period. However, as mentioned, X itself depends heavily on the characteristics of the vacation policy.

We therefore turn to study two common vacation regimes, namely, the Multiple and Single vacation policies.

4. Multiple and Single Vacation Policies

When the vacation policy is specified, it becomes possible to obtain explicit expressions for the various performance measures derived previously. In what follows we consider the Multiple and Single vacation policies (see Levy and Yechiali, [1975], Takagi [1991]).

4.1 Multiple Vacation

If, upon returning from a vacation, the system is not empty, the server starts immediately a busy period. Otherwise, if the system is empty, we say that the length of the initiated busy period is $B_0 = 0$ and the server leaves immediately for another random vacation. All vacations are i.i.d, distributed as a random variable U, and are independent of the other underlying processes.

The server continues in this manner until, upon return, he finds at least one batch waiting and starts a nongenerate busy period.

For both versions, $Y + \sum_{i=1}^{A(U)} K_i = X$. Hence $\left(recall \;\; that \;\; \alpha = \lambda(1-\hat{K}(z)) \right)$,

$$\hat{X}(z) \;=\; E[z^{Y+\sum_{i=1}^{A(U)} K_i}] \;=\; \hat{Y}(z)\cdot\tilde{U}(\lambda(1-\hat{K}(z))) \;=\; \hat{Y}(z)\cdot\tilde{U}(\alpha) \tag{4.1}$$

Substituting $\hat{Y}(z)$ from (3.3) in (4.1) yields

$$\hat{X}(z) = \left[\, \varphi(0,z)\cdot\hat{X}(z) + (1-\varphi(0,z))\cdot\hat{X}(\delta) \,\right]\cdot\tilde{U}(\alpha) \quad , \; \textit{implying}$$

$$\hat{X}(z) = \frac{(1-\varphi(0,z))\cdot\hat{X}(\delta)\cdot\tilde{U}(\alpha)}{1-\varphi(0,z)\cdot\tilde{U}(\alpha)} \tag{4.2}$$

Since in both versions $\varphi(0,1)=1$, by using L'hospital's rule in (4.2) , we obtain

$$1 = \hat{X}(1) \;=\; \frac{\hat{X}(\delta)\cdot\left(-\left.\frac{\partial\varphi(0,z)}{\partial z}\right|_{z=1}\right)}{-\left.\frac{\partial\varphi(0,z)}{\partial z}\right|_{z=1} \;-\; \lambda E[U]E[K]} \tag{4.3}$$

4.1.1 Version 1

$$\left.\frac{\partial\varphi(0,z)}{\partial z}\right|_{z=1} \;=\; \frac{\rho-1}{1-\tilde{V}(\mu)} \quad . \qquad \textit{Thus,} \;\; \textit{from} \;\; (4.3)$$

$$\hat{X}(\delta) \;=\; 1 \;-\; \frac{\lambda E[U]E[K]\cdot(1-\tilde{V}(\mu))}{1-\rho} \tag{4.4}$$

Substituting (4.4) and (2.12) in (4.2) leads to

$$\hat{X}(z) = \frac{\left[z-\tilde{V}(\alpha)\right]\cdot\left[1-\frac{\lambda E[U]E[K]\cdot(1-\tilde{V}(\mu))}{1-\rho}\right]\cdot\tilde{U}(\alpha)}{z-\tilde{V}(\alpha+\mu)-\left[\tilde{V}(\alpha)-\tilde{V}(\alpha+\mu)\right]\cdot\tilde{U}(\alpha)} \tag{4.5}$$

Moments of X (in particular E[X]) can now be derived by differentiating $\hat{X}(z)$ in (4.5).

<u>Performance Measures</u>

Using equations (4.4) and (4.5), we derive explicit expressions for various performance measures.

From (3.4), (3.5), (3.10) and (3.11) we get

$$E[B] \;=\; E[V] \;\cdot\; \frac{\lambda E[U]E[K]\cdot(1-\tilde{V}(\mu))}{(1-\rho)\cdot(1-\tilde{V}(\mu))} \;=\; \frac{\rho E[U]}{1-\rho} \tag{4.6}$$

$$E[Y] \;=\; E[X]-\lambda E[U]E[K] \tag{4.7}$$

$$E[N] \;=\; \frac{\lambda E[U]E[K]}{1-\rho} \tag{4.8}$$

$$\mathrm{E[C]} \;=\; \frac{\mathrm{E[U]}}{1-\rho} \tag{4.9}$$

Clearly, E[C] can also be obtained by setting E[C] = E[B] + E[U].

Note that for Version 1 with Multiple vacations, E[B] , E[N] and E[C] do not depend on the Timer T and have the same values as in the regular (no timer) Multiple vacations $M^K / G / 1$ queue.

This follows since, under Version 1, there is no loss of effective work.

Calculation of $\hat{L}(z)$

Using equations (4.2), (4.4) and (3.20) we obtain

$$\hat{L}(z) = \frac{\tilde{V}(\alpha)}{\lambda E[U]E[K]} \cdot \frac{\left[1-\rho-\lambda E[U]E[K]\cdot(1-\tilde{V}(\mu))\right]\cdot\left[\tilde{U}(\alpha)-1\right]}{\left[z-\tilde{V}(\alpha+\mu)-\left(\tilde{V}(\alpha)-\tilde{V}(\alpha+\mu)\right)\cdot\tilde{U}(\alpha)\right]} \tag{4.10}$$

4.1.2 Version 2

Similarly to the derivation for Version 1, one gets

$$\hat{X}(\delta) = 1 - \frac{\lambda E[U]E[K]\cdot\mu(1-\tilde{V}(\mu))}{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))} \tag{4.11}$$

$$\hat{X}(z) = \frac{\left[\dfrac{(\mu+\alpha)\cdot(z-\tilde{V}(\mu+\alpha))-z\mu(1-\tilde{V}(\mu+\alpha))}{(\mu+\alpha)\cdot(z-\tilde{V}(\mu+\alpha))}\right]\cdot\left[1-\dfrac{\lambda E[U]E[K]\cdot\mu(1-\tilde{V}(\mu))}{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))}\right]\cdot\tilde{U}(\alpha)}{1-\left[\dfrac{z\mu(1-\tilde{V}(\mu+\alpha))}{(\mu+\alpha)\cdot(z-\tilde{V}(\mu+\alpha))}\right]\cdot\tilde{U}(\alpha)} \tag{4.12}$$

Performance Measures

By using (4.11) and (4.12) in (3.6),(3.7),(3.8),(3.9),(3.13) and (3.14), we get

$$E[B] = \frac{\lambda E[U]E[K]\cdot(1-\tilde{V}(\mu))}{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))} \tag{4.13}$$

$$E[Y] = E[X] - \lambda E[U]E[K] \tag{4.14}$$

$$E[B^{eff}] = \frac{-\lambda E[U]E[K]\cdot\mu\tilde{V}'(\mu)}{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))} \tag{4.15}$$

$$E[Y^{eff}] = E[X] - \frac{\lambda E[U]E[K]\cdot\mu\cdot\left(\lambda E[K]\cdot\tilde{V}'(\mu)+\tilde{V}(\mu)\right)}{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))} \tag{4.16}$$

$$E[N] = \frac{\mu\tilde{V}(\mu)\cdot\lambda E[U]E[K]}{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))} \tag{4.17}$$

$$E[C] = \frac{\mu\tilde{V}(\mu)\cdot E[U]}{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))} \tag{4.18}$$

It should be emphasized that, in distinct to Version 1, all performance measures for Version 2 do depend on the parameter μ of the Timer.

Calculation of $\hat{L}(z)$

Using equations (4.2) and (4.11) and inserting the result for $\frac{\hat{X}(z)-\hat{X}(\delta)}{1-\hat{X}(\delta)}$ in (3.22), yield

$$\hat{L}(z) = \left[\frac{\tilde{V}(\alpha+\mu)}{\mu\tilde{V}(\mu)\cdot\lambda E[U]E[K]}\right]\cdot\frac{\left[\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))\cdot(1+\mu E[U])\right]\cdot\left[\tilde{U}(\alpha)-1\right]}{\left[(\alpha+\mu)\cdot(z-\tilde{V}(\alpha+\mu))-z\mu(1-\tilde{V}(\alpha+\mu))\cdot\tilde{U}(\alpha)\right]} \tag{4.19}$$

4.1.3 A Limiting case : the Exhaustive Regime

The regular Exhaustive regime becomes a limiting case of the general RTG regime when $\mu \to 0$.

That is, the server leaves for a vacation if and only if the system is empty. This clearly holds for both versions since, without a Timer, both coincide. We have, for both versions,

$$\hat{X}(z) \xrightarrow{\mu\to 0} \tilde{U}(\lambda(1-\hat{K}(z))) \qquad E[X] \xrightarrow{\mu\to 0} \lambda E[U]E[K]$$

$$\tilde{B}(w) \xrightarrow{\mu\to 0} \tilde{U}(\lambda(1-\tilde{\theta}_K(w))) \qquad E[B] \xrightarrow{\mu\to 0} \frac{\rho E[U]}{1-\rho}$$

$$\hat{L}(z) \xrightarrow{\mu\to 0} \frac{(1-\rho)\cdot\tilde{V}(\alpha)}{\tilde{V}(\alpha)-z}\cdot\frac{(1-\tilde{U}(\alpha))}{\lambda E[U]E[K]} = \hat{L}_{M^K/G/1}(z)\cdot\frac{(1-\tilde{U}(\alpha))}{\lambda E[U]\cdot(1-\hat{K}(z))} \tag{4.20}$$

Where $\hat{L}_{M^K/G/1}(z)$ is the PGF of the queue size at service completions in $M^K / G / 1$ queue with no vacations (see (3.18)). For K=1, result (4.20) reduces to Takagi's result ([1991] , vol 1 , Eq. (2.12c) , p. 122) for simple M / G / 1 queue with Multiple vacations.

Equation (4.20) exhibits a decomposition phenomenon. Let R_U be the 'remaining time' of a random variable U. It is well known that the LST of R_U is $\tilde{R}_U(w)=\frac{1-\tilde{U}(w)}{wE[U]}$.

The PGF of the total number of arrivals during R_U is given by :

$$\tilde{R}_U(\lambda(1-\hat{K}(z))) = \frac{1-\tilde{U}(\lambda(1-\hat{K}(z)))}{\lambda(1-\hat{K}(z))\cdot E[U]} \tag{4.21}$$

Thus, when $\mu\to 0$, (4.20) implies that $L = L_{M^K/G/1} + AR$, where AR is the total number of individual customers arriving during R_U.

By using L'hospital's rule, we get from (4.20)

$$E[L] \xrightarrow{\mu\to 0} \rho + \frac{\lambda^2\cdot E^2[K]\cdot E[V^2]}{2\cdot(1-\rho)} + \frac{E[K(K-1)]}{2E[K]\cdot(1-\rho)} + \frac{\lambda E[K]\cdot E[U^2]}{2E[U]} \tag{4.22}$$

Again, by setting K = 1 , equation (4.22) reduces to the corresponding result for M / G / 1 queue (see Levy and Yechiali [1975]).

Finally, when $\mu \to 0$, (4.17) and (4.18) coincide with (4.8) and (4.9) respectively :

$$E[N] \xrightarrow{\mu \to 0} \frac{\lambda E[U]E[K]}{1-\rho} \qquad , \qquad E[C] \xrightarrow{\mu \to 0} \frac{E[U]}{1-\rho}$$

4.1.4 Comparison between the two Versions

Comparing equations (4.6) and (4.13), it follows that

$$E[\,B \mid version1\,] \le E[\,B \mid version2\,] \quad \Leftrightarrow \quad \mu E[V]\cdot\tilde{V}(\mu) \le 1-\tilde{V}(\mu)$$

For V having the Gamma distribution with scale parameter γ and shape parameter α (such that $E[V] = \alpha/\gamma$) this conditon reduces to $1+\mu\dfrac{\alpha}{\gamma} \le \left(1+\dfrac{\mu}{\gamma}\right)^{\alpha}$.

This implies that, for $\alpha = 1$ (i.e. V Exponential with mean γ^{-1}), equality holds, i.e. the mean length of a busy period is <u>equal</u> in both versions. However, for $\alpha > 1$ (e.g. family of Erlang distributions with $\alpha \equiv 2,3,\ldots$), $\mu E[V]\cdot\tilde{V}(\mu) < 1-\tilde{V}(\mu)$ (which is also the case for V Deterministic with E[V]= γ^{-1}). That is $E[\,B \mid version1\,] < E[\,B \mid version2\,]$.

On the other hand, if $\alpha < 1$, the inequality <u>reverses</u> its orientation. Similar conclusions hold for E[N] and E[C].

Note that, although the actual value of a busy period under each version does depend on the vacation variable U, their <u>ratio</u> depends only on the distribution of the service time V and the Timer parameter μ.

4.2 Single vacation

If the server returnes from a vacation and finds no customers waiting (X = 0), he waits idle for the first batch to arrive, upon which he starts a busy period. Otherwise (X ≥ 1), he immediately starts a busy period.

The computation of $\Phi_r(w,z) = E\left[\, e^{-wB_r} z^{Y_r} \,\right]$ in section 2 remains the same, since the length of a busy period initiated by r customers (B_r) and the number of customers at the end of this busy period (Y_r) do not depend on the type of vacation. The only difference is that in a Single vacation, r > 0. Nevertheless, we set here the same starting condition, $\Phi_0(w,z) = 1$, for r = 0.

The steady state equation (3.1) for $\Phi(w,z) = E_x[\Phi_X(w,z)]$ is valid here too, and so are the measures in section 3 for the variables B , Y , N , C and L (as functions of X).

For both versions :

$$X = \begin{cases} Y + \sum_{i=1}^{A(U)} K_i & \text{if} \quad Y + \sum_{i=1}^{A(U)} K_i > 0 \\ K & \text{if} \quad Y + \sum_{i=1}^{A(U)} K_i = 0 \end{cases} \tag{4.23}$$

Thus ,

$$\hat{X}(z) = E\left[z^X \,\middle|\, Y + \sum_{i=1}^{A(U)} K_i > 0 \right] \cdot P(Y + \sum_{i=1}^{A(U)} K_i > 0) + E\left[z^X \,\middle|\, Y + \sum_{i=1}^{A(U)} K_i = 0 \right] \cdot P(Y + \sum_{i=1}^{A(U)} K_i = 0) \tag{4.24}$$

Using (3.3), (2.12) and (2.13), one gets (after some calculations),

<u>Version 1</u> $$\hat{X}(z) = \tilde{U}(\lambda) \cdot \frac{\tilde{V}(\lambda) \cdot \hat{X}(\delta)}{\tilde{V}(\lambda+\mu)} \cdot \left(\hat{K}(z) - 1\right) + \hat{Y}(z) \cdot \tilde{U}(\alpha) \tag{4.25}$$

<u>Version 2</u> $$\hat{X}(z) = \tilde{U}(\lambda) \cdot \hat{X}(\delta) \cdot \left(\hat{K}(z) - 1\right) + \hat{Y}(z) \cdot \tilde{U}(\alpha) \tag{4.26}$$

4.2.1 Version 1

Substituting $\hat{Y}(z)$ from (3.3) in (4.26) and rearranging , yield

$$\hat{X}(z) = \hat{X}(\delta) \cdot \frac{\dfrac{\tilde{U}(\lambda) \cdot \tilde{V}(\lambda)}{\tilde{V}(\lambda+\mu)} \cdot \left(\hat{K}(z) - 1\right) + \left(1 - \varphi(0,z)\right) \cdot \tilde{U}(\alpha)}{1 - \varphi(0,z) \cdot \tilde{U}(\alpha)} \tag{4.27}$$

where, in a similar way to the Multiple vacation case, we derive,

$$\hat{X}(\delta) = \frac{1 - \rho - \lambda E[U] E[K] \cdot (1 - \tilde{V}(\mu))}{\dfrac{\tilde{V}(\lambda) \cdot \tilde{U}(\lambda) \cdot E[K] \cdot (1 - \tilde{V}(\mu))}{\tilde{V}(\lambda+\mu)} + (1 - \rho)} \tag{4.28}$$

<u>Performance measures</u>

Inserting $\hat{X}(z)$ *and* $\hat{X}(\delta)$ suitably in the relevant equations of section 3, we obtain explicit expressions for E[B], E[N] and E[C].

Note that E[C] can also be computed through

$$E[C] = E[B] + E[U] + \frac{1}{\lambda} \cdot P(\sum_{i=1}^{A(U)} K_i = 0) \cdot P(Y = 0) = E[B] + E[U] + \frac{1}{\lambda} \cdot \tilde{U}(\lambda) \cdot \frac{\tilde{V}(\lambda) \cdot \hat{X}(\delta)}{\tilde{V}(\lambda+\mu)}$$

This follows since the mean idle time (while waiting for the first group to arrive) is $\frac{1}{\lambda}$.

Calculation of $\hat{L}(z)$

Using equations (4.27) and (4.28) and inserting the result for $\dfrac{\hat{X}(z)-\hat{X}(\delta)}{1-\hat{X}(\delta)}$ in (3.20) lead to

$$\hat{L}(z)=\frac{\dfrac{\tilde{V}(\alpha)}{z-\tilde{V}(\alpha+\mu)}\cdot\left[\dfrac{\tilde{U}(\lambda)\cdot\tilde{V}(\lambda)}{\tilde{V}(\lambda+\mu)}\cdot\left(\hat{K}(z)-1\right)+\tilde{U}(\alpha)-1\right]\cdot\left[1-\rho-\lambda E[U]E[K]\cdot(1-\tilde{V}(\mu))\right]}{\left[1-\dfrac{\tilde{V}(\alpha)-\tilde{V}(\alpha+\mu)}{z-\tilde{V}(\alpha+\mu)}\cdot\tilde{U}(\alpha)\right]\cdot\left[\dfrac{\tilde{U}(\lambda)\cdot\tilde{V}(\lambda)\cdot E[K]}{\tilde{V}(\lambda+\mu)}+\lambda E[U]E[K]\right]}$$

(4.29)

4.2.2 Version 2

Similarly to Version 1 ,

$$\hat{X}(z)\;=\;\hat{X}(\delta)\;\cdot\;\frac{\tilde{U}(\lambda)\cdot\left(\hat{K}(z)-1\right)+\left(1-\varphi(0,z)\right)\cdot\tilde{U}(\alpha)}{1-\;\varphi(0,z)\cdot\tilde{U}(\alpha)}\tag{4.30}$$

where,

$$\hat{X}(\delta)\;=\;\frac{\dfrac{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))}{\mu(1-\tilde{V}(\mu))}\;-\;\lambda E[U]E[K]}{\tilde{U}(\lambda)\cdot E[K]\;+\;\dfrac{\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))}{\mu(1-\tilde{V}(\mu))}}\tag{4.31}$$

<u>Performance measures</u>

Again, explicit formulas for E[B], E[N] and E[C] are obtained by inserting the above results for $\hat{X}(z)$ and $\hat{X}(\delta)$ in the appropriate equations of section 3.

Calculation of $\hat{L}(z)$

Using equations (4.30) and (4.31) while inserting the resulting expression for $\dfrac{\hat{X}(z)-\hat{X}(\delta)}{1-\hat{X}(\delta)}$

in (3.22) lead to

(4.32)

$$\hat{L}(z)\;=\;\frac{\tilde{V}(\alpha+\mu)}{\tilde{V}(\mu)\cdot(z-\tilde{V}(\alpha+\mu))}\cdot\frac{\left[\mu\tilde{V}(\mu)-\lambda E[K]\cdot(1-\tilde{V}(\mu))\cdot(1+\mu E[U])\right]\cdot\left[\tilde{U}(\lambda)\cdot\left(\hat{K}(z)-1\right)+\tilde{U}(\alpha)-\right.}{\left[1-\dfrac{z\mu(1-\tilde{V}(\alpha+\mu))\cdot\tilde{U}(\alpha)}{(\alpha+\mu)\cdot(z-\tilde{V}(\alpha+\mu))}\right]\cdot\left[\left(\tilde{U}(\lambda)+\lambda E[U]\right)\cdot\mu E[K]\right]}$$

4.2.3 A Limiting case

When $\mu\to 0$, the results are readily reduced to those for the $M^K/G/1$ system with Single vacations (no Timer).

For both versions,

$$\hat{X}(z) \xrightarrow{\mu\to 0} \tilde{U}(\lambda)\cdot(\hat{K}(z)-1)+\ \tilde{U}(\lambda(1-\hat{K}(z)))$$

$$E[X] \xrightarrow{\mu\to 0} \left(\tilde{U}(\lambda)+\lambda E[U]\right)\cdot E[K]$$

$$\tilde{B}(w) \xrightarrow{\mu\to 0} \tilde{U}(\lambda)\cdot\left(\tilde{\theta}_K(w)-1\right) + \tilde{U}(\lambda(1-\tilde{\theta}_K(w)))$$

$$E[B] \xrightarrow{\mu\to 0} \frac{E[V]E[K]\cdot\left(\tilde{U}(\lambda)+\lambda E[U]\right)}{1-\rho} = \frac{E[V]}{1-\rho}\cdot\lim_{\mu\to 0} E[X]$$

$$\hat{L}(z) \xrightarrow{\mu\to 0} \frac{(1-\rho)\cdot\tilde{V}(\alpha)}{z-\tilde{V}(\alpha)}\cdot\frac{\left[\tilde{U}(\lambda)\cdot\left(\hat{K}(z)-1\right)+\tilde{U}(\alpha)-1\right]}{\left(\tilde{U}(\lambda)+\lambda E[U]\right)\cdot E[K]} \tag{4.33}$$

By using L'hospital's rule,

$$E[L] \xrightarrow{\mu\to 0} \rho + \frac{\lambda^2\cdot E^2[K]\cdot E[V^2]}{2\cdot(1-\rho)} + \frac{\lambda^2\cdot E[K]\cdot E[U^2]}{2\cdot(\ \tilde{U}(\lambda)+\lambda E[U]\ )} + \frac{E[K(K-1)]}{2E[K]\cdot(1-\rho)} \tag{4.34}$$

Again, for K=1, results (4.34) and (4.33) reduce respectively to those of Levy and Yechiali ([1975], p. 206) and Takagi ([1991], Eq (2.23), p. 125).

$$E[N] \xrightarrow{\mu\to 0} \frac{\left(\lambda E[U]+\tilde{U}(\lambda)\right)\cdot E[K]}{1-\rho} = \frac{1}{1-\rho}\cdot\lim_{\mu\to 0} E[X]$$

$$E[C] \xrightarrow{\mu\to 0} \frac{\frac{1}{\lambda}\tilde{U}(\lambda)+E[U]}{1-\rho} = \frac{1}{\rho}\cdot\lim_{\mu\to 0} E[B]$$

5. Queue size at an arbitrary moment

In this section we derive the PGF of the queue size at an arbitrary moment for the two versions, each under both the Multiple and Single vacation policies (Note that $\hat{L}(z)$ is the PGF of the queue size at service completion instants).

First we define the PGFs of the following variables :

PGF Variable

F(z) - queue size at an *arbitrary* moment.

V(z) - queue size at an *arbitrary* moment during a vacation.

S(z) - queue size at an *arbitrary* moment during a service time.

$g_1(z)$ - number of customers arriving during R_V , the 'remaining' part of a service time.

$g_2(z)$ - number of customers arriving during R_U , the 'remaining' part of a vacation time.

We also define the probabilities :

P_1 = P(the server is serving)

P_2 = P(the server is on a vacation)

5.1 Multiple Vacation

For both versions the following relations hold :

$$F(z) = S(z) \cdot P_1 + V(z) \cdot P_2 \tag{5.1}$$

$$V(z) \cdot g_2(z) = \hat{X}(z) \tag{5.2}$$

$$S(z) \cdot g_1(z) \cdot \frac{1}{z} = \hat{L}(z) \tag{5.3}$$

Substituting (5.2) and (5.3) in (5.1) yields

$$F(z) = P_1 \cdot \frac{z \cdot \hat{L}(z)}{g_1(z)} + P_2 \cdot \frac{\hat{X}(z)}{g_2(z)} \tag{5.4}$$

Where, (see (4.21)),

$$g_1(z) = \frac{1 - \tilde{V}(\lambda(1 - \hat{K}(z)))}{\lambda(1 - \hat{K}(z)) \cdot E[V]} \qquad g_2(z) = \frac{1 - \tilde{U}(\lambda(1 - \hat{K}(z)))}{\lambda(1 - \hat{K}(z)) \cdot E[U]}$$

Now, for <u>Version 1</u> ,

From (3.12),

$P_1 = P_{busy} = \rho$, $P_2 = 1 - P_1 = 1 - \rho$, while $\hat{L}(z)$ *and* $\hat{X}(z)$ are taken from (4.10) and (4.5), respectively.

For <u>Version 2</u> ,

From (3.15),

$P_1 = P_{busy} = \dfrac{\lambda E[K] \cdot (1 - \tilde{V}(\mu))}{\mu \tilde{V}(\mu)}$, $P_2 = 1 - P_1$, while $\hat{L}(z)$ *and* $\hat{X}(z)$ are taken from (4.19) and (4.12), respectively.

5.2 Single vacation

We add the following definitions :

h(z) - PGF of the number of customers arriving during a vacation.

I(z) - PGF of the queue size at an *arbitrary* moment during <u>‘idle time’</u>.

(the server enters the ‘idle time’ period at the end of a vacation if and only if at this instant the system is empty).

P_3 = P(the server is idle)

$\overline{V}(z)$ - PGF of the queue size at the end of a vacation.

For both versions, the following relations hold,

$$F(z) = S(z) \cdot P_1 + V(z) \cdot P_2 + I(z) \cdot P_3 \tag{5.5}$$

(Clearly, I(z)=1).

$$\overline{V}(z) = \hat{Y}(z) \cdot h(z) \tag{5.6}$$

$$V(z) \cdot g_2(z) = \overline{V}(z) \tag{5.7}$$

Equation (5.3) holds here, as well.

Substituting (5.3) and (5.7) in(5.5) and using (5.6) yields

$$F(z) = P_1 \cdot \frac{z \cdot \hat{L}(z)}{g_1(z)} + P_2 \cdot \frac{\hat{Y}(z) \cdot h(z)}{g_2(z)} + P_3 \tag{5.8}$$

Where $h(z) = \tilde{U}(\lambda(1-\hat{K}(z)))$.

Now, for both versions,

$$P_1 = \frac{E[B]}{E[C]} \left(= \rho \ for\ version 1 \right) \quad ; \quad P_2 = \frac{E[U]}{E[C]} \quad ; \quad P_3 = \frac{E[I]}{E[C]}$$

while $\hat{L}(z)$ *and* $\hat{Y}(z)$ are taken, respectively (with the relevant φ and $\hat{X}(\cdot)$), from (4.29) and (3.3) for Version 1, and from (4.32) and (3.3) for Version 2.

6. Waiting and Sojourn times

In this section we derive explicit formulas for the LSTs and means of the waiting and sojourn times of an arbitrary customer for both versions and for both vacation policies. The results are summerized in tables 1 and 2 in the sequel.

The sojourn time of an arbitrary customer is composed of three components :

W^1 - time from arrival of the customer's group (batch) until the group starts service.

Clearly, W^1 depends on the vacations policy.

W^2 - time from the instant that the group starts service until the customer's service is initiated.

W^3 - time from the beginning of service of the customer until the customer leaves the system.

The waiting (queueing) time of a customer before first service initiation is $W = W^1 + W^2$.

A customer's sojourn time is $W + W^3$, and since W and W^3 are independent, its LST is the product $\tilde{W}(s) \cdot \tilde{W}^3(s)$. We now calculate those distributions.

6.1 The Distribution of W^2

Define W_k^2 to be W^2 for the k[th] customer in a group.

The following recursive relations hold :

$$
\begin{array}{lllll}
\textit{Version} & 1 & 2 & & \\
W_k^2 = & \begin{cases} V_1 + W_{k-1}^2 & V_1 + W_{k-1}^2 \quad , \quad V_1 < T \\ \\ V_1 + U + W_{k-1}^2 & T + U + W_k^2 \quad , \quad V_1 \ge T \end{cases} & & k \ge 2 & (6.1)
\end{array}
$$

$$\mathrm{W}_1^2 = 0$$

In order to calculate the LST of W_k^2 we write

$$\widetilde{W}_k^2(s) = E[e^{-sW_k^2}] = E[e^{-sW_k^2} \mid T > V_1] \cdot P(T > V_1) + E[e^{-sW_k^2} \mid T \le V_1] \cdot P(T \le V_1) \quad (6.2)$$

When $T > V_1$ we have, for <u>both versions</u> (where $V_1 \sim V$),

$$E[e^{-sW_k^2} \mid T > V_1] \cdot P(T > V_1) = \int_{v=0}^{\infty} \int_{t=v}^{\infty} E\left[e^{-s(v+W_{k-1}^2)}\right] \cdot f_T(t)dt \cdot f_V(v)dv$$

$$= \int_{v=0}^{\infty} e^{-sv} \cdot e^{-\mu v} \cdot \widetilde{W}_{k-1}^2(s) \cdot f_V(v)dv = \widetilde{V}(\mu+s) \cdot \widetilde{W}_{k-1}^2(s) \quad (6.3)$$

When $T \le V_1$ we have,

<u>Version 1</u>

$$E[e^{-sW_k^2} \mid T \le V_1] \cdot P(T \le V_1) = \int_{v=0}^{\infty} \int_{t=0}^{v} E[e^{-s(v+U+W_{k-1}^2)}] \cdot f_T(t)dt \cdot f_V(v)dv$$

$$= \int_{v=0}^{\infty} (1-e^{-\mu v}) \cdot e^{-sv} \cdot f_V(v)dv \cdot \widetilde{W}_{k-1}^2(s) \cdot \widetilde{U}(s) = \widetilde{W}_{k-1}^2(s) \cdot \widetilde{U}(s) \cdot (\widetilde{V}(s) - \widetilde{V}(\mu+s)) \quad (6.4)$$

<u>Version 2</u>

$$E[e^{-sW_k^2} \mid T \le V_1] \cdot P(T \le V_1) = \int_{v=0}^{\infty} \int_{t=0}^{v} e^{-st} \cdot \widetilde{U}(s) \cdot \widetilde{W}_k^2(s) \cdot \mu e^{-\mu t} dt \cdot f_V(v)dv$$

$$= \frac{\mu}{\mu+s} \cdot \int_{v=0}^{\infty} (1-e^{-(\mu+s)v}) \cdot f_V(v)dv \cdot \widetilde{W}_k^2(s) \cdot \widetilde{U}(s) = \frac{\mu}{\mu+s} \cdot \widetilde{W}_k^2(s) \cdot \widetilde{U}(s) \cdot [1 - \widetilde{V}(\mu+s)] \quad (6.5)$$

By inserting (6.3) and (6.4) or (6.5) into (6.2) for each version, respectively, we get

<u>Version 1</u>

$$
\begin{aligned}
\widetilde{W}_k^2(s) &= \widetilde{V}(\mu+s) \cdot \widetilde{W}_{k-1}^2(s) + \widetilde{W}_{k-1}^2(s) \cdot \widetilde{U}(s) \cdot [\widetilde{V}(s) - \widetilde{V}(\mu+s)] \\
&= \widetilde{W}_{k-1}^2(s) \cdot \left[\widetilde{V}(\mu+s) + \widetilde{U}(s) \cdot [\widetilde{V}(s) - \widetilde{V}(\mu+s)]\right] \\
&= \widetilde{W}_{k-2}^2(s) \cdot \left[\widetilde{V}(\mu+s) + \widetilde{U}(s) \cdot [\widetilde{V}(s) - \widetilde{V}(\mu+s)]\right]^2 = \cdots = \\
&= \widetilde{W}_1^2(s) \cdot \left[\widetilde{V}(\mu+s) + \widetilde{U}(s) \cdot [\widetilde{V}(s) - \widetilde{V}(\mu+s)]\right]^{k-1}
\end{aligned}
$$

Hence, since $W_1^2 = 0$,

$$\tilde{W}_k^2(s) = \left[\tilde{V}(\mu+s) + \tilde{U}(s)\cdot[\,\tilde{V}(s) - \tilde{V}(\mu+s)\,]\right]^{k-1} \qquad , \; k \ge 1 \tag{6.6}$$

By differentiation,

$$E[W_k^2] = (k-1)\cdot\left[E[V] \; + (1-\tilde{V}(\mu))\cdot E[U]\right] \tag{6.7}$$

<u>Version 2</u>

$$\tilde{W}_k^2(s) = \tilde{V}(\mu+s)\cdot \tilde{W}_{k-1}^2(s) + \frac{\mu}{\mu+s}\cdot \tilde{W}_k^2(s)\cdot \tilde{U}(s)\cdot[1-\tilde{V}(\mu+s)]$$

$$= \tilde{W}_{k-1}^2(s) \;\cdot\; \frac{\tilde{V}(\mu+s)}{1-\frac{\mu}{\mu+s}\cdot\tilde{U}(s)\cdot[\,1-\tilde{V}(\mu+s)\,]}$$

Hence,

$$\tilde{W}_k^2(s) = \left[\frac{\tilde{V}(\mu+s)}{1-\frac{\mu}{\mu+s}\cdot\tilde{U}(s)\cdot[\,1-\tilde{V}(\mu+s)]}\right]^{k-1} \qquad , \quad k \ge 1 \tag{6.8}$$

This implies,

$$E[W_k^2] \;=\; (k-1)\cdot\frac{1-\tilde{V}(\mu)}{\tilde{V}(\mu)}\cdot\left(\tfrac{1}{\mu} + E[U]\right) \tag{6.9}$$

Define ,

P_k = P{ a customer is in the k^th position in his group }

f_j = P{ size of a group is j } = P(K = j)

It is known, (see Burke [1975] , Rosenberg and Yechiali [1993]) that

$$P_k = \frac{1}{E[K]}\cdot\sum_{j=k}^{\infty} P(K=j) = \frac{1}{E[K]}\cdot\sum_{j=k}^{\infty} f_j$$

Since, with probability P_k , $W^2 = W_k^2$, We have,

$$\tilde{W}^2(s) = E[\,e^{-sW^2}\,] = \sum_{k=1}^{\infty} P_k\cdot E[\,e^{-sW_k^2}\,] = \sum_{k=1}^{\infty} P_k\cdot \tilde{W}_k^2(s) = \frac{1}{E[K]}\cdot\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} f_j\cdot\tilde{W}_k^2(s)$$

By substituting (6.6) and (6.8) in the above, we get :

<u>Version 1</u>

$$\tilde{W}^2(s) = \frac{1}{E[K]}\cdot\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} f_j\cdot\left[\tilde{V}(\mu+s) + \tilde{U}(s)\cdot[\tilde{V}(s) - \tilde{V}(\mu+s)]\right]^{k-1}$$

$$= \frac{1}{E[K]}\cdot\sum_{j=1}^{\infty} f_j\cdot\sum_{k=1}^{j}\left[\,\tilde{V}(\mu+s) + \tilde{U}(s)\cdot[\,\tilde{V}(s) - \tilde{V}(\mu+s)]\right]^{k-1}$$

$$= \frac{1}{E[K]} \cdot \sum_{j=1}^{\infty} f_j \cdot \left[\frac{1 - \left[\tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s)] \right]^{j}}{1 - \left[\tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s)] \right]} \right]$$

Thus,

$$\tilde{W}^2(s) = \frac{1 - \hat{K}\left(\tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s)\,] \right)}{E[K] \cdot \left(1 - \left[\tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s)\,] \right] \right)} \tag{6.10}$$

Now,

$$E[W^2] = \sum_{k=1}^{\infty} P_k \cdot E[W_k^2] = \left[E[V] + (1 - \tilde{V}(\mu)) \cdot E[U] \right] \cdot \frac{E[K(K-1)]}{2E[K]} \tag{6.11}$$

Note that $E[V] + (1 - \tilde{V}(\mu)) \cdot E[U]$ (see also below) is the mean time interval between two consecutive services in a group, while $\frac{E[K(K-1)]}{2E[K]}$ is the equivalent of mean 'remaining' time for the group size variable , K.

<u>Version 2</u>

In a similar manner,

$$\tilde{W}^2(s) = \frac{1 - \hat{K}\left[\dfrac{\tilde{V}(\mu+s)}{1 - \frac{\mu}{\mu+s} \cdot \tilde{U}(s) \cdot [\, 1 - \tilde{V}(\mu+s)\,]} \right]}{E[K] \cdot \left[1 - \dfrac{\tilde{V}(\mu+s)}{1 - \frac{\mu}{\mu+s} \cdot \tilde{U}(s) \cdot [\, 1 - \tilde{V}(\mu+s)\,]} \right]} \tag{6.12}$$

$$E[W^2] = \frac{1 - \tilde{V}(\mu)}{\tilde{V}(\mu)} \cdot \left(\tfrac{1}{\mu} + E[U] \right) \cdot \frac{E[K(K-1)]}{2E[K]} \tag{6.13}$$

It is important to indicate that, for each version , $\tilde{W}^2(s)$ is the same for both vacation regimes.

By taking $\mu \to 0$ in equations (6.10) and (6.12) we get

$$\tilde{W}^2(s) = \frac{1 - \hat{K}\left(\tilde{V}(s) \right)}{E[K] \cdot \left(1 - \tilde{V}(s) \right)}$$

which is the same result as in Baba [1986] for the regular $M^K / G / 1$ queue.

6.2 The Distribution of W^1

Each group may be considered as a 'super' customer, with service time being the time interval extending from the instant at which the service of the group's first customer starts, through the instant at which its last customer leaves the system. This service time will be noted as V^g.

We therefore have a M / G / 1 queue with vacations (Multiple or Single) and service times distributed as V^g. The waiting time of the super customer in such a system is identical to W^1.

First we compute $\tilde{V}^g(s)$, the LST of V^g.

Define $V^g{}_m$ to be V^g for a group of size m.

We have the following recursive relation :

$$
\begin{array}{lll}
 & \textit{Version} \quad 1 & 2 \\
V^g{}_m = & \begin{cases} V_1 + V^g{}_{m-1} \\ \\ V_1 + U + V^g{}_{m-1} \end{cases} & \begin{array}{ll} V_1 + V^g{}_{m-1} , & V_1 < T \\ & \\ T + U + V^g{}_m , & V_1 \geq T \end{array} \qquad m \geq 1
\end{array}
$$

($V^g{}_0 = 0$)

This last relation is identical to the expression for W_k^2 in (6.1) , where $V^g{}_m$ here replaces W_k^2 there (except for m=1). Hence, the solution is :

<u>Version 1</u>

$$\tilde{V}^g{}_m(s) = \tilde{V}^g{}_{m-1}(s) \cdot \left[\tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s)] \right]$$

$$= \ \ldots = \tilde{V}^g{}_0(s) \cdot \left[\tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s)] \right]^m$$

Thus,

$$\tilde{V}^g{}_m(s) = \left[\tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s)] \right]^m \qquad , \ m \geq 1 \qquad (6.14)$$

<u>Version 2</u>

$$\tilde{V}^g{}_m(s) \ = \ \left[\frac{\tilde{V}(\mu+s)}{1 - \frac{\mu}{\mu+s} \cdot \tilde{U}(s) \cdot [\, 1 - \tilde{V}(\mu+s)]} \right]^m \qquad , \ m \geq 1 \qquad (6.15)$$

For both versions, for m ≥ 1, $V^g = V^g{}_m$ *with probability* f_m. Hence,

$$\tilde{V}^g(s) = \sum_{m=1}^{\infty} f_m \cdot \tilde{V}^g{}_m(s)$$

By inserting (6.14) and (6.15) in the above , we obtain :

<u>Version 1</u>

$$\tilde{V}^g(s) = \sum_{m=1}^{\infty} f_m \cdot \left[\ \tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s)] \ \right]^m$$

$$= \ \hat{K}\left(\tilde{V}(\mu+s) + \tilde{U}(s) \cdot [\, \tilde{V}(s) - \tilde{V}(\mu+s) \ ] \right) \qquad (6.16)$$

and by differentiation,

$$E[V^g] = E[K]\cdot\left[E[V] + (1-\tilde{V}(\mu))\cdot E[U]\right] \tag{6.17}$$

Indeed, the mean time interval between the start of two consecutive services in a group is given by :

$$E[\,V|\;V \le T]\cdot P(V \le T) + \left(E[\,V|\;V > T] + E[U]\right)\cdot P(V > T) = E[V] + (1-\tilde{V}(\mu))\cdot E[U]$$

Since the mean size of a batch is E[K] , result (6.17) follows.

Version 2

$$\tilde{V}^g(s) = \sum_{m=1}^{\infty} f_m \cdot \left[\frac{\tilde{V}(\mu+s)}{1-\frac{\mu}{\mu+s}\cdot\tilde{U}(s)\cdot[\,1-\tilde{V}(\mu+s)\,]}\right]^m = \hat{K}\left(\frac{\tilde{V}(\mu+s)}{1-\frac{\mu}{\mu+s}\cdot\tilde{U}(s)\cdot[\,1-\tilde{V}(\mu+s)\,]}\right) \tag{6.18}$$

and

$$E[V^g] \;=\; \frac{E[K]}{\tilde{V}(\mu)}\cdot(1-\tilde{V}(\mu))\cdot(\tfrac{1}{\mu} + E[U]) \tag{6.19}$$

The LST for the waiting time of the super customer is given for the Multiple vacation and the Single vacation by Equations (36) and (22), respectively, of Levy and Yechiali [1975] as follows :

<u>For Multiple vacation</u> :
$$\tilde{W}^1(s) = \left[\frac{1-\lambda E[V^g]}{\lambda E[U]}\right]\cdot\left[\frac{1-\tilde{U}(s)}{\tilde{V}^g(s)-(1-\frac{s}{\lambda})}\right] \tag{6.20}$$

Note that if we substitute for V^g and take $\mu \to 0$ in (6.20), we get, for the two versions,

$E[V^g] = E[K]\cdot E[V]$, $\tilde{V}^g(s) = \hat{K}(\tilde{V}(s))$, which leads to

$$\tilde{W}^1(s) = \left[\frac{1-\lambda E[K]E[V]}{\lambda E[U]}\right]\cdot\left[\frac{1-\tilde{U}(s)}{\hat{K}(\tilde{V}(s))-(1-\frac{s}{\lambda})}\right].$$

This result is the same as in Baba [1986] , for the regular $M^K / G / 1$ model.

<u>For Single vacation</u> :
$$\tilde{W}^1(s) = \left[\frac{1-\lambda E[V^g]}{\tilde{U}(\lambda)+\lambda E[U]}\right]\cdot\left[\frac{1-\tilde{U}(s)+\frac{\tilde{U}(\lambda)}{\lambda}\cdot s}{\tilde{V}^g(s)-(1-\frac{s}{\lambda})}\right] \tag{6.21}$$

The LST of the waiting time of a customer is $\tilde{W}(s) = \tilde{W}^1(s)\cdot\tilde{W}^2(s)$.

Table 1 below gives the prescription of how to calculate $\tilde{W}(s)$ for each of the four cases.

Case	Product of equations	$\tilde{V}^g(s)$ is given by	$E[V^g]$ is given by
Multiple, Version 1	(6.10)*(6.20)	(6.16)	(6.17)
Multiple, Version 2	(6.12)*(6.20)	(6.18)	(6.19)

Single, Version 1	(6.10)*(6.21)	(6.16)	(6.17)
Single, Version 2	(6.12)*(6.21)	(6.18)	(6.19)

Table 1 : Calculation of $\tilde{W}(s) = \tilde{W}^1(s) \cdot \tilde{W}^2(s)$

6.3 The Distribution of W^3

Version 1

In this version W^3 = V since there are no service preemptions.

Version 2

$$W^3 = \begin{cases} V & , \quad V < T \\ T + U + W^3 & , \quad V \geq T \end{cases} \tag{6.22}$$

$$E[e^{-sW^3} \mid T > V] \cdot P(T > V) = \int_{v=0}^{\infty} \int_{t=v}^{\infty} e^{-sv} \cdot f_T(t)\, dt \cdot f_V(v)\, dv$$

$$= \int_{v=0}^{\infty} e^{-sv} \cdot e^{-\mu v} \cdot f_V(v)\, dv = \tilde{V}(\mu + s) \tag{6.23}$$

$$E[e^{-sW^3} \mid T \leq V] \cdot P(T \leq V) = \int_{v=0}^{\infty} \int_{t=0}^{v} E[e^{-s(t+U+W^3)}] \cdot f_T(t)\, dt \cdot f_V(v)\, dv$$

$$= \frac{\mu}{\mu + s} \cdot \int_{v=0}^{\infty} (1 - e^{-(\mu+s)v}) \cdot f_V(v)\, dv \cdot \tilde{W}^3(s) \cdot \tilde{U}(s) = \frac{\mu}{\mu + s} \cdot \tilde{W}^3(s) \cdot \tilde{U}(s) \cdot [1 - \tilde{V}(\mu + s)] \tag{6.24}$$

Combining (6.23) and (6.24) we get

$$\tilde{W}^3(s) = \tilde{V}(\mu + s) + \frac{\mu}{\mu + s} \cdot \tilde{W}^3(s) \cdot \tilde{U}(s) \cdot [1 - \tilde{V}(\mu + s)], \quad \textit{which leads to}$$

$$\tilde{W}^3(s) = \frac{\tilde{V}(\mu + s)}{1 - \frac{\mu}{\mu+s} \cdot \tilde{U}(s) \cdot [1 - \tilde{V}(\mu + s)]} \tag{6.25}$$

By differentiating (6.25) we get

$$E[W^3] = \frac{1}{\tilde{V}(\mu)} \cdot \left[1 - \tilde{V}(\mu)\right] \cdot \left(\tfrac{1}{\mu} + E[U] \right) \tag{6.26}$$

Clearly, (6.26) can be obtained directly by taking expectations on (6.22), or by setting k=2 in (6.9). This follows since W^3 in this version is the time interval between the start of two consecutive services in a group, so that W^3 is distributed as W_2^2 . Also , by comparing (6.26) with (6.19) , we see that $E[V^g] = E[K] \cdot E[W^3]$.

We further note that W^2 , as well as W^3 , has the same distribution in the the two vacation models (Multiple and Single). This follows since during such a period (W^2 or W^3) the system is not empty and the two regimes are indistinguishable. Note that V^g , the service time of a group, is also the same in both vacation models.

What distinguishes between the waiting times in the two models is W^1 , which depends on the vacation regime.

The results regarding $E[W^1]$, $E[W^2]$ and $E[W^3]$ are summerized in Table 2 below.

	Multiple vacation		Single vacation	
	Version 1	Version 2	Version 1	Version 2
$E[W^1]$	$\frac{\lambda\, E[(V^g)^2]}{2(1-\lambda E[V^g])}+\frac{\lambda E[U^2]}{2E[U]}$		$\frac{\lambda\, E[(V^g)^2]}{2(1-\lambda E[V^g])}+\frac{\lambda E[U^2]}{2(\tilde{U}(\lambda)+\lambda E[U])}$	
$E[W^2]$	$\eta\cdot\Omega$	$\xi\cdot\Omega$	$\eta\cdot\Omega$	$\xi\cdot\Omega$
$E[W^3]$	E[V]	ξ	E[V]	ξ
$\Omega=\frac{E[K(K-1)]}{2E[K]}$, $\xi=\frac{1-\tilde{V}(\mu)}{\tilde{V}(\mu)}\cdot\left(\frac{1}{\mu}+E[U]\right)$, $\eta=E[V]+(1-\tilde{V}(\mu))\cdot E[U]$				

Table 2: Mean Waiting Times

($E[V^g]$ is given for the two versions respectively, by equations (6.17) and (6.19), while $E[(V^g)^2]$ can be calculated by differentiating (6.16) and (6.18)).

A quick calculation reveals that, for sufficiently large E[U],

$E[W^2 | version1] < E[W^2 | version2]$ and $E[W^3 | version1] < E[W^3 | version2]$.

7. Stability Conditions

In this section we derive stability conditions for both versions. We first use conditions applicable only to work - conserving systems (Version1) and then use the notion of super customer to derive conditions for both versions.

Conditions for Version 1

In this case we can use Fricker and Jaibi's (FJ) result [1994] regarding necessary and sufficient condition for stability. Note that our model is a polling system with a single queue and batch arrivals. The switchover times in FJ are taken here to be the vacation times, and the service policies are work conserving, where for Single vacation model, the switching time includes the length of time in which the server waits for the first group to arrive after a vacation (whenever the system is empty).

Thus, the necessary and sufficient condition for stability is :

$$\rho \; + \; \frac{\lambda \, E[K]}{E[G]} \cdot E[D] \; < \; 1 \tag{7.1}$$

Where $\rho \; = \; \lambda E[K]E[V]$ is the traffic load of the system, λE[K] is the arrival rate to the queue, G is the number of customers served during an imaginary busy period which is initiated by infinite number of waiting customers, and D is the total switchover time between queues (vacations). In our case,

$G \sim Geometric(\varphi)$ where $\varphi \; = \; P(T \le V) \; = \; 1 - \tilde{V}(\mu)$. Hence,

$$E[G] \; = \; \frac{1}{\varphi} \; = \; \frac{1}{1 - \tilde{V}(\mu)} \tag{7.2}$$

E[G] can also be obtained by using the performance measure E[N] from equation (3.10) :

$$E[G] \; = \; \lim_{X \to \infty} E[N] \; = \; \lim_{X \to \infty} \frac{1 - \hat{X}(\delta)}{1 - \tilde{V}(\mu)} \; = \; \frac{1}{1 - \tilde{V}(\mu)} \quad ,$$

since $\lim_{X \to \infty} \hat{X}(\delta) = 0 \;\; for \;\; 0 < \delta < 1$ where δ is defined in Section 3.1.

We now compute E[D] seperately for the Multiple and Single vacation regimes.

$Multiple \;\; vacation \quad : \qquad D = U$

$$Single \;\; vacation \quad : \quad E[D] \; = \; E[U] + \tfrac{1}{\lambda} \cdot P(\, no \; arrivals \; during \; U \,) \cdot P(Y = 0) \tag{7.3}$$

$$= \; E[U] + \tfrac{1}{\lambda} \cdot \tilde{U}(\lambda) \cdot \frac{\tilde{V}(\lambda) \cdot \hat{X}(\delta)}{\tilde{V}(\lambda + \mu)} \qquad (\text{ Since } \; P(Y = 0) \; = \hat{Y}(0) \;)$$

By substituting (7.2) and (7.3) in (7.1) we get the conditions :

<u>Multiple vacation</u>

$$\begin{aligned} &\rho \; + \; \lambda E[K] \cdot (1 - \tilde{V}(\mu)) \cdot E[U] \; < \; 1 \\ &That \;\; is, \qquad \lambda \, E[K] \cdot \Big(\, E[V] + (1 - \tilde{V}(\mu)) \cdot E[U] \Big) < 1 \end{aligned} \tag{7.4}$$

<u>Single vacation</u>

$$\rho \; + \; \lambda E[K] \cdot (1 - \tilde{V}(\mu)) \cdot \left(E[U] + \tfrac{1}{\lambda} \cdot \tilde{U}(\lambda) \cdot \frac{\tilde{V}(\lambda) \cdot \hat{X}(\delta)}{\tilde{V}(\lambda + \mu)} \right) \; < \; 1$$

Substituting $\hat{X}(\delta)$ from (4.28) in the above equation we get,

$$1 \ + \ \frac{(1-\rho)\cdot\tilde{V}(\lambda+\mu)\cdot\left(\rho+\lambda E[K]E[U]\cdot(1-\tilde{V}(\mu))-1\right)}{\tilde{V}(\lambda)\cdot\tilde{U}(\lambda)\cdot E[K]\cdot(1-\tilde{V}(\mu))+(1-\rho)\cdot\tilde{V}(\lambda+\mu)} \ < \ 1$$

Since $\rho < 1$, the last condition reduces to

$$\rho \ + \ \lambda E[K]E[U]\cdot(1-\tilde{V}(\mu))-1 \ < \ 0$$

$$\textit{That is,} \qquad \lambda\, E[K]\cdot\left(E[V]+(1-\tilde{V}(\mu))\cdot E[U]\right) \ < \ 1 \tag{7.5}$$

Comparing (7.5) to (7.4), we see that the stability condition is the same for the two vacation regimes.

Another aproach applicable for both versions is the following,

Conditions for Version 1 and Version 2

Since the service policy in Version 2 is not work conserving, one has to use a different approach (which can be applied to both versions) in order to get the stability conditions.

As mentioned previously in this section, both versions can be looked upon as variations of the standard $M / G / 1$ queue, where each group is considered to be a super customer which loads the system with an actual service time of length V^g. The mean of V^g is given in (6.17) and (6.19), for Version 1 and Version 2, respectively

A necessary and sufficient condition is $\lambda E[V^g] < 1$. That is,

Version 1

$$\lambda\, E[K]\cdot\left(E[V]+(1-\tilde{V}(\mu))\cdot E[U]\right) \ < \ 1 \tag{7.6}$$

Clearly, this condition is identical with (7.4).

Version 2

$$\frac{\lambda\, E[K]}{\tilde{V}(\mu)}\cdot(1-\tilde{V}(\mu))\cdot\left(\tfrac{1}{\mu}+E[U]\ \right) \ < \ 1 \tag{7.7}$$

As mentioned, V^g is independent of the vacation type and therefore results (7.6) and (7.7) hold for both vacation regimes.

References

[1] Y. Baba, “On the $M^X / G / 1$ Queue with Vacation Time”, Operations Research Letters 5, 93-98 (1986).

[2] S.C. Borst, “Polling Systems with Multiple Coupled Servers”, Queueing Systems 20, 369-393 (1995).

[3] P.J. Burke, “Delays in Single-Server Queues with Batch Input”, Operations Research 23, 830-833 (1975).

[4] M.H. Chaudhry and J.G.C Templeton, “A First Course in Bulk Queues”, Wiley (1983).

[5] J.W. Cohen, “The Single Server Queue” , North Holland (1982).

[6] I. Eliazar and U. Yechiali, “Randomly Timed Gated Queueing Systems”, To appear in SIAM J. on Applied

Mathematics (1998).

[7] C. Fricker and M.R. Jaibi, “Monotonicity and Stability of Periodic Polling Models”, Queueing Systems 15, 211-238 (1994).

[8] S.W. Fhurmann and R.B. Cooper, “Stochastic Decomposition in the M / G / 1 Queue with Generalized Vacations”, Operations Research 33, 1117-1129 (1985).

[9] O. Kella and U. Yechiali, “Priorities in M / G / 1 Queue with Server Vacations”, Naval Research Logistics Vol 35, No. 1, 23-34 (1988).

[10] Y. Levy and U. Yechiali, “Utilization of Idle Time in an M / G / 1 Queue with Server Vacations”, Management Science 22, 202-211 (1975).

[11] J. Medhi, “Recent Developments in Bulk Queueing Models”, Wiley (1984).

[12] E. Rosenberg and U. Yechiali, “The M^X / G / 1 Queue with Single and Multiple Vacations under the LIFO Service Regime”, Oprations Research Letters 14, 171-179 (1993).

[13] M. Shomrony and U. Yechiali, “Batch arrival Queues with Server Vacations and Busy Periods Controlled by a Random Timer”, Technical Report, Dept. of Statistics and OR, Tel-Aviv University (1998).

[14] H. Takagi, “Queueing Analysis”, Vol 1, North Holland (1991).