The $M^X/G/1$ Queue with Single and Multiple Vacations under the LIFO Service Regime

Eitan Rosenberg Uri Yechiali

Department of Statistics and Operations Research
School of Mathematical Sciences
The Raymond and Beverly Sackler
Faculty of Exact Sciences
Tel Aviv University
Tel Aviv 69978, Israel

December 1992
Revised June 1993

Abstract

The $M^X/G/1$ queue without server vacations, with multiple vacations and with single vacations is studied under the LIFO service regime. For each model we derive explicit formulae for the Laplace-Stieltjes transform, mean and second moment of the waiting time W_{LIFO} of an arbitrary customer, and extend the range of Fuhrmann's general result, showing *directly* that for each case, $E[W^2_{\text{LIFO}}] = E[W^2_{\text{FIFO}}]/(1-\rho)$.

Keywords: Batch arrivals, $M^X/G/1$, LIFO, FIFO, vacation models, decomposition, second moments

1. Introduction.

We consider the $M^X/G/1$ queueing system where batches of customers arrive according to a Poisson process and the order in which batches are admitted to service is the Last In First Out (LIFO) regime. Service of individual customers is non-preemptive and within a batch customers are served according to their inner order. We study three models, not studied before under the LIFO regime:

(i) $M^X/G/1$ *without* server vacations; (ii) $M^X/G/1$ with *multiple* vacations; and (iii) $M^X/G/1$ with *single* vacations.

The *regular* $M/G/1$ queue with First In First Out (FIFO) service regime and with multiple and single vacations was studied by Levy and Yechiali [1975] who also were the first to indicate the decomposition phenomenon of waiting times for the $M/G/1$ process with multiple vacations. Doshi [1986] and others further studied decomposition properties and extended the results to more general systems. Scholl and Kleinrock [1983] investigated the multiple vacation $M/G/1$ queue and compared the moments of the waiting times under the FIFO, LIFO and ROS (Random Order of Service) regimes. Kella and Yechiali [1988] studied various $M/G/1$ priority queues with FIFO mechanism and derived the Laplace-Stieltjes transform (LST), mean and second moment of the waiting time of a class-k customer for the multiple and for the single vacations processes, both under the preemptive and non-preemptive priority disciplines.

The batch-arrival single-class (non-priority) $M^X/G/1$ queue without vacations and FIFO regime was analyzed by Burke [1975], and the corresponding queue with multiple vacations by Baba [1986]. A book by Chaudhry and Tempelton [1983] contains many results on bulk queues, and recently Takagi and Takahashi [1991] investigated *priority* queues with batch Poisson arrivals under the FIFO service regime, with multiple and with single vacations. None of the above works for batch-arrival queues attacks the LIFO service mechanism.

In this paper we investigate the $M^X/G/1$ queue with LIFO service discipline, concentrating on waiting times analysis. (The distribution of the number of customers present in the system is the *same* as the corresponding distribution in the $M^X/G/1$ queue under the FIFO regime). In section 2 we describe the details of the model. In sections 3, 4 and 5 we study, respectively, the cases with no vacations, with multiple vacations, and with single vacations. For each model we derive explicit formulae for the LST, and for the mean and second moment of the waiting time W_{LIFO} of an arbitrary customer in a batch, and compare these moments with the corresponding moments of the waiting time W_{FIFO} in an identical $M^X/G/1$ queue but with FIFO service regime. As expected, $E[W_{\text{LIFO}}] = E[W_{\text{FIFO}}]$ in all cases. However, considering the second moment, we show *explicitly* that in all models studied in this work, $E[W_{\text{FIFO}}^2] = (1-\rho)E[W_{\text{LIFO}}^2]$, thus extending the range of Fuhrmann's [1991] general result for regular $M/G/1$-type queues.

2. The Model.

We consider an $M^X/G/1$ queueing system were i.i.d random batches of customers arrive according to a Poisson process with rate λ, and the batch-size, X, has a probability mass function $P(X = n) = f_n$, with probability generating function (PGF) $F_X(z) \equiv \sum_{n=1}^{\infty} f_n z^n$. We let $f \equiv E[X]$, $f^{(2)} = E\big[X(X-1)\big]$, and $f^{(3)} = E\big[X(X-1)(X-2)\big]$, where $f^{(n)} \equiv \frac{d^n}{dz^n} F_X(z)\Big|_{z=1}$. Customers are served one at a time by a single server, and service times, S, of individual customers are i.i.d random variables with LST $\tilde{S}(\theta)$, mean $E[S]$, second and third moments $E[S^2]$ and $E[S^3]$, respectively. When a batch arrives and the server is busy, the residual service time of the customer being served, R_S, has a LST $\widetilde{R}_S(\theta) = \big[1 - \widetilde{S}(\theta)\big]/\big[\theta E[S]\big]$ with mean $E[R_S] = E[S^2]/\big[2E[S]\big]$. Batches are admitted to service following the LIFO regime. Within a batch, individual customers are served according to their inner order, and service is non-preemptive.

As indicated in the Introduction, we study in this work three models, all under the LIFO service discipline: (i) $M^X/G/1$ with no vacations, (ii) $M^X/G/1$ with multiple vacations, and (iii) $M^X/G/1$ with single vacations.

In the vacation models, as soon as the system becomes empty (following a busy period), the server takes a 'vacation', whose length U has LST $\widetilde{U}(\theta)$, mean, second and third moments $E[U]$, $E[U^2]$ and $E[U^3]$, respectively. The residual vacation time, R_U, has LST $\widetilde{R}_U(\theta) = \big[1 - \widetilde{U}(\theta)\big]/\big[\theta E[U]\big]$. Further vacations depend on whether it is a single, or multiple, vacation model.

Consider a busy period that starts upon an arrival of a random batch of size X to an *empty* system, and denote its length by B. The total service time of this batch, denoted by Y, is $Y = \sum_{i=1}^{X} S_i$, where S_i are i.i.d like S. Thus, $E[Y] = E[X]E[S] = fE[S]$, and the LST of Y is given by

$$\widetilde{Y}(\theta) = E\Big[e^{-\theta\sum_{i=1}^{X} S_i}\Big] = \sum_{n=1}^{\infty} f_n \big[\widetilde{S}(\theta)\big]^n = F_X\big(\widetilde{S}(\theta)\big)\ . \tag{1}$$

Differentiating, the second and third moments of Y are calculated: $E[Y^2] = f^{(2)}\big[E[S]\big]^2 + fE[S^2]$, and $E[Y^3] = f^{(3)}\big[E[S]\big]^3 + 3f^{(2)}E[S]E[S^2] + fE[S^3]$. Applying standard arguments, we readily derive the LST of the duration of a busy period B:

$$\widetilde{B}(\theta) = \widetilde{Y}\big(\theta + \lambda - \lambda\widetilde{B}(\theta)\big) = F_X\Big(\widetilde{S}\big(\theta + \lambda - \lambda\widetilde{B}(\theta)\big)\Big)\ , \tag{2}$$

so that $E[B] = E[Y]\big(1 + \lambda E[B]\big)$. Letting $\rho \equiv \lambda E[Y] = \lambda f E[S]$ we obtain

$$E[B] = \frac{E[Y]}{1-\rho} = \frac{fE[S]}{1 - \lambda f E[S]}\ . \tag{3}$$

Indeed, if we consider a batch as a single 'super' customer with mean service time $E[Y]$, then result (3) is just the mean duration of a busy period in a regular $M/G/1$ queue with 'super' customers and corresponding traffic intensity $\rho = \lambda E[Y]$. It follows, similarly to the regular $M/G/1$ queue, that a necessary and sufficient condition for steady state is $\rho < 1$.

The second and third moments of B are derived from (2) as

$$E[B^2] = \frac{E[Y^2]}{(1-\rho)^3}, \quad E[B^3] = \frac{3\lambda\big(E[Y^2]\big)^2}{(1-\rho)^5} + \frac{E[Y^3]}{(1-\rho)^4}\ . \tag{4}$$

3. The $M^X/G/1$ Queue with LIFO Service Regime.

In this section we analyze the $M^X/G/1$ queue with LIFO service regime and *no* vacations. We will derive the LST of W_{LIFO}, the waiting time of an arbitrary customer, and calculate its first two moments. We will then compare these moments with the mean and second moment of the waiting time, W_{FIFO}, in an identical $M^X/G/1$ system with FIFO service regime, and will show *explicitly* that $E[W_{\text{LIFO}}] = E[W_{\text{FIFO}}]$, while $E[W^2_{\text{LIFO}}] = E[W^2_{\text{FIFO}}]/(1-\rho)$.

Consider a random batch and select an arbitrary (test) customer C. The probability that C is in the n'th position in his batch is given by (see Burke [1975])

$$g_n = \frac{1}{f} \sum_{k=n}^{\infty} f_k \quad n = 1, 2, 3, \ldots \tag{5}$$

The PGF of $\{g_n\}$ is calculated as

$$\begin{aligned} G(z) \equiv \sum_{n=1}^{\infty} g_n z^n = \frac{1}{f} \sum_{k=1}^{\infty} f_k \sum_{n=1}^{k} z^n &= \frac{1}{f} \sum_{k=1}^{\infty} f_k \Big(\frac{z - z^{k+1}}{1 - z} \Big) \\ &= \frac{1}{f} \cdot \frac{z}{1 - z} \Big[1 - F_X(z) \Big] \ . \end{aligned} \tag{6}$$

Suppose C is the n'th in his batch, and denote the first $n-1$ customers in the batch by $C_1, C_2, \ldots, C_{n-1}$. When C_i starts service, and because of the LIFO regime, he generates a delay busy period (see e.g. Kella and Yechiali [1988]) of an $M^X/G/1$ queue, denoted by A, where the initial delay is customer's C_i service time, S. Therefore, the LST of A is given by

$$\widetilde{A}(\theta) = \widetilde{S}\big(\theta + \lambda - \lambda \widetilde{B}(\theta)\big) \tag{7}$$

so that $E[A] = E[S]/(1-\rho)$, and $E[A^2] = \lambda E[S] E[B^2] + \frac{E[S^2]}{(1-\rho)^2}$. Clearly, after service starts for the batch to which C belongs, C has to wait $(n-1)$ independent delay busy periods, all distributed like A. Thus, if the batch of C arrives to an *empty* system (which occurs with probability $1 - \rho$) and if C is the n'th in the batch, then his waiting time equals to $\sum_{i=1}^{n-1} A_i$, where A_i is distributed like A.

Suppose now that C arrives with his batch when the server is busy (this clearly occurs with probability ρ). The residual service time of the customer being served is R_S. During R_S the number of arriving batches is Poisson with rate λ. If there are m such batches, then because of the LIFO regime, C waits for his service to start a period of time equal to the sum

$$R_S + \sum_{j=1}^{m} B_j + \sum_{i=1}^{n-1} A_i, \quad \text{where } B_j \text{ are i.i.d like } B \ .$$

Thus,

$$E\left[e^{-\theta W_{\text{LIFO}}}\middle|\begin{matrix}C\text{'s batch arrives to a non-empty}\\ \text{system and } C \text{ is } n\text{'th in his batch}\end{matrix}\right]$$

$$= \int_0^\infty \sum_{m=0}^\infty e^{-\lambda t}\frac{(\lambda t)^m}{m!}\left[\widetilde{A}(\theta)\right]^{n-1} e^{-\theta t}\left[\widetilde{B}(\theta)\right]^m dR_S(t)$$

$$= \left[\widetilde{A}(\theta)\right]^{n-1}\widetilde{R}_S\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)\ .$$

It follows that the LST of the waiting time, W_{LIFO}, of an arbitrary customer is given by

$$E\Big\{e^{-\theta W_{\text{LIFO}}}\Big\} \equiv \widetilde{W}_{\text{LIFO}}(\theta) = (1-\rho)E\left[e^{-\theta W_{\text{LIFO}}}\middle|\begin{matrix}\text{the batch of } C \text{ arrives}\\ \text{to an empty system}\end{matrix}\right] +$$

$$+\ \rho E\left[e^{-\theta W_{\text{LIFO}}}\middle|\begin{matrix}\text{the batch of } C \text{ arrives}\\ \text{to a non-empty system}\end{matrix}\right]$$

$$= (1-\rho)\sum_{n=1}^\infty g_n\left[\widetilde{A}(\theta)\right]^{n-1} + \rho\sum_{n=1}^\infty g_n\left[\widetilde{A}(\theta)\right]^{n-1}\widehat{R}_S\big(\theta+\lambda-\lambda\widehat{B}(\theta)\big)\ .$$

That is,

$$\widetilde{W}_{\text{LIFO}}(\theta) = \frac{1-\rho}{\widetilde{A}(\theta)}G\big(\widetilde{A}(\theta)\big) + \frac{\rho}{\widetilde{A}(\theta)}\cdot G\big(\widetilde{A}(\theta)\big)\cdot\widetilde{R}_S\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)\ . \tag{8}$$

Now, using (6),

$$G\big(\widetilde{A}(\theta)\big) = \frac{1}{f}\cdot\frac{\widetilde{A}(\theta)}{1-\widetilde{A}(\theta)}\Big[1-F_X\big(\widetilde{A}(\theta)\big)\Big]\ . \tag{9}$$

Also, using (7),

$$\widetilde{R}_S\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big) = \frac{1-\widetilde{S}\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)}{\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)\cdot E[S]} = \frac{1-\widetilde{A}(\theta)}{\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)E[S]}\ .$$

Substituting in (8) and using $\rho = \lambda f E[S]$, we obtain

$$\widetilde{W}_{\text{LIFO}}(\theta) = \frac{1-\rho}{f}\cdot\frac{\Big[1-F_X\big(\widetilde{A}(\theta)\big)\Big]}{1-\widetilde{A}(\theta)} + \frac{\lambda\Big[1-F_X\big(\widetilde{A}(\theta)\big)\Big]}{\big[\theta+\lambda-\lambda\widetilde{B}(\theta)\big]}\ . \tag{10}$$

By taking derivatives, using L'Hospital rule, and after some tedious algebra, we derive

$$E[W_{\text{LIFO}}] = \frac{\lambda f E[S^2]}{2(1-\rho)} + \frac{f^{(2)}E[S]}{2f(1-\rho)}\ . \tag{11}$$

As expected, expression (11) for $E(W_{\text{LIFO}})$ *equals* the corresponding expression for $E(W_{\text{FIFO}})$ as given by Burke [2] for the $M^X/G/1$ with FIFO regime.

The calculation of $E(W^2_{\text{LIFO}})$ requires further endeavor, involving derivation of

$$E[A^3] = \lambda E[B^3]E[S] + \frac{3\lambda E[B^2]E[S^2]}{1-\rho} + \frac{E[S^3]}{(1-\rho)^3}$$

where $E[B^3]$ is given in (4).

With these results we finally obtain

$$E[W^2_{\text{LIFO}}] = \frac{\lambda f E[S^3]}{3(1-\rho)^2} + \frac{\lambda^2 f^{(2)}\left[E(S^2)\right]^2}{2(1-\rho)^3} + \frac{f^{(3)}\left[E[S]\right]^2}{3f(1-\rho)^2} + \frac{\lambda[f^{(2)}]^2\left[E[S]\right]^3 + (1+\rho)f^{(2)}E[S^2]}{2f(1-\rho)^3} \ . \tag{12}$$

Comparison with the $M^X/G/1$ Queue under the FIFO Regime.

The LST of the waiting time, W_{FIFO}, in an $M^{(X)}/G/1$ queue with FIFO regime is given by (see Baba [1986]) as

$$\widetilde{W}_{\text{FIFO}}(\theta) = \frac{(1-\rho)\theta\left[1 - F_X\left(\widetilde{S}(\theta)\right)\right]}{f\left[\theta - \lambda + \lambda F_X\left(\widetilde{S}(\theta)\right)\right]\left[1-\widetilde{S}(\theta)\right]} \ . \tag{13}$$

The mean is given by equation (11) and the second moment by

$$E[W^2_{\text{FIFO}}] = \frac{\lambda f E[S^3]}{3(1-\rho)} + \frac{\lambda^2 f^{(2)}\left[E[S^2]\right]^2}{2(1-\rho)^2} + \frac{f^{(3)}\left[E[S]\right]^2}{3f(1-\rho)} + \frac{\lambda[f^{(2)}]^2\left[E[S]\right]^3 + (1+\rho)f^{(2)}E[S^2]}{2f(1-\rho)^2} \ . \tag{14}$$

It readily follows that

$$E[W^2_{\text{LIFO}}] = \frac{E[W^2_{\text{FIFO}}]}{1-\rho} \ . \tag{15}$$

Relation (15) is well known for the *regular* $M/G/1$ queue and was extended by Fuhrmann [1991], using direct arguments, to $M/G/1$ queue with *exceptional* first service, to $M/G/1$ queue with (multiple) server vacation, to $M/G/1$ queue with static priorities, and to other $M/G/1$-type queues. Indeed, Fuhrmann's assumptions 1, 2 and 3 hold here too, and the value of the quantity Δ (see there) is given by $\Delta = E[A] = E[S]/[1-\rho]$.

We will show in the sequel that the *same* relation holds true for $M^X/G/1$ queues with single or with multiple vacations.

4. The $M^X/G/1$ Queue with LIFO Service Regime and Multiple Vacations.

In this section we analyze the system studied in section 3, but we let the server take a 'vacation' immediately at the end of a busy period. If the server returns from a vacation and the system is still empty (i.e. there were no arrivals during the vacation) he immediately takes another vacation, and continues in this manner until, upon return, he finds at least one customer waiting. An (extended) busy period starts right away. When this busy period ends, the process of multiple vacations repeats, and so on. (For more details on this process see Levy and Yechiali [1975]). The sequence of vacations $\{U_i\}$ are independent, all identically distributed like U. Consider the test customer C. When he arrives, there is either a customer being served or the server is on vacation. Baba [1986] has shown that the proportion of time that the server is on vacation in an $M^X/G/1$ queue with FIFO service regime is $(1-\rho)$. This holds true for the LIFO regime as well since the order of service does not affect the above proportion. Thus, letting W_{LIFO}^{MV} denote the waiting time of an arbitrary customer C when the multiple vacation (MV) procedure is applied and the service regime is LIFO, we have

$$\begin{aligned}
\widetilde{W}_{\text{LIFO}}^{MV}(\theta) &= (1-\rho)E\left[e^{\theta W}\middle|\begin{matrix}\text{the server is on vaca-}\\\text{tion when } C \text{ arrives}\end{matrix}\right] + \rho E\left[e^{-\theta W}\middle|\begin{matrix}\text{server is busy}\\\text{when } C \text{ arrives}\end{matrix}\right] \\
&= (1-\rho)\sum_{n=1}^{\infty} g_n \int_0^{\infty}\sum_{m=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^m}{m!}\left[\widetilde{A}(\theta)\right]^{n-1}e^{-\theta t}\left[\widetilde{B}(\theta)\right]^m dR_U(t) \\
&\quad + \rho\sum_{n=1}^{\infty} g_n \int_0^{\infty}\sum_{m=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^m}{m!}\left[\widetilde{A}(\theta)\right]^{n-1}e^{-\theta t}\left[\widetilde{B}(\theta)\right]^m dR_S(t) \\
&= (1-\rho)\sum_{n=1}^{\infty} g_n\left[\widetilde{A}(\theta)\right]^{(n-1)}\widetilde{R}_U\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big) + \rho\sum_{n=1}^{\infty} g_n\left[\widetilde{A}(\theta)\right]^{n-1}\widetilde{R}_S\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)
\end{aligned}$$

or,

$$\widetilde{W}_{\text{LIFO}}^{MV}(\theta) = (1-\rho)\frac{G\big(\widetilde{A}(\theta)\big)}{\widetilde{A}(\theta)}\widetilde{R}_U\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big) + \rho\frac{G\big(\widetilde{A}(\theta)\big)}{\widetilde{A}(\theta)}\widetilde{R}_S\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)\ . \quad (16)$$

Using (9) and $\widetilde{R}_S(\theta) = \big[1-\widetilde{S}(\theta)\big]/\big[\theta E[S]\big]$, $\widetilde{R}_U(\theta) = \big[1-\widetilde{U}(\theta)\big]/\big[\theta E[U]\big]$, we get

$$\widetilde{W}_{\text{LIFO}}^{MV}(\theta) = \frac{(1-\rho)\Big[1-F_X\big(\widetilde{A}(\theta)\big)\Big]\Big[1-\widetilde{U}\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)\Big]}{f\left[1-\widetilde{A}(\theta)\right]\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)E[U]} + \frac{\lambda\Big[1-F_X\big(\widetilde{A}(\theta)\big)\Big]}{\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big)}\ . \quad (17)$$

Comparing (17) to (10) (or (16) to (8)) we obtain, after differentiation,

$$E[W_{\mathrm{LIFO}}^{MV}] = E[W_{\mathrm{LIFO}}] + E[R_U] = \frac{\lambda f E[S^2]}{2(1-\rho)} + \frac{f^{(2)}E[S]}{2f(1-\rho)} + \frac{E[U^2]}{2E[U]} \ . \tag{18}$$

Observe that the mean waiting time in the $M^X/G/1$ queue with LIFO regime and multiple vacations is equal to the mean waiting time in the *same* system *without* vacations *plus* the mean residual time, $E[R_U]$, of a vacation. That is, regarding mean waiting times, we reveal here too a *decomposition phenomenon* for $M^X/G/1$-type queues with LIFO service regime.

Again, utilizing the calculations when deriving $E[W_{\mathrm{LIFO}}^2]$, we obtain, after lengthy algebra,

$$E\Big[(W_{\mathrm{LIFO}}^{MV})^2\Big] = \frac{\lambda f E[S^2]E[U^2]}{2(1-\rho)^2 E[U]} + \frac{f^{(2)}E[S]E[U^2]}{2f(1-\rho)^2 E[U]} + \frac{E[U^3]}{3(1-\rho)E[U]} + E[W_{\mathrm{LIFO}}^2] \tag{19}$$

where $E[W_{\mathrm{LIFO}}^2]$ is given by (12).

Using (11), result (19) can also be written as

$$E\Big[(W_{\mathrm{LIFO}}^{MV})^2\Big] = E[W_{\mathrm{LIFO}}]\frac{E[U^2]}{(1-\rho)E[U]} + \frac{E[U^3]}{3(1-\rho)E[U]} + E[W_{\mathrm{LIFO}}^2] \ . \tag{20}$$

Comparison with the $M^X/G/1$ Queue with Multiple Vacations under the FIFO Regime.

Baba [1986] derived the LST of W_{FIFO}^{MV}, the waiting time of an arbitrary (test) customer in the $M^X/G/1$ queue with FIFO service regime, and showed that

$$\begin{aligned} \widetilde{W}_{\mathrm{FIFO}}^{MV}(\theta) &= \frac{(1-\rho)\Big[1-F_X\big(\widetilde{S}(\theta)\big)\Big]\,[1-\widetilde{U}(\theta)]}{f\Big[\theta-\lambda+\lambda F_X\big(\widetilde{S}(\theta)\big)\Big]\,[1-\widetilde{S}(\theta)]\,E[U]} \\ &= \widetilde{W}_{\mathrm{FIFO}}(\theta)\cdot\frac{[1-\widetilde{U}(\theta)]}{\theta E[U]} = \widetilde{W}_{\mathrm{FIFO}}(\theta)\widetilde{R}_U(\theta) \ . \end{aligned} \tag{21}$$

That is, there exists a complete decomposition: $W_{\mathrm{FIFO}}^{MV} = W_{\mathrm{FIFO}} + R_U$.

He also calculated (see equation (25) there) the mean of W_{FIFO}^{MV} as

$$\begin{aligned} E[W_{\mathrm{FIFO}}^{MV}] &= \frac{\lambda f E[S^2]}{2(1-\rho)} + \frac{f^{(2)}E[S]}{2f(1-\rho)} + \frac{E[U^2]}{2E[U]} \\ &= E[W_{\mathrm{LIFO}}] + E[R_U] \ . \end{aligned} \tag{22}$$

As expected, equations (18) and (22) match, i.e. $E[W_{\text{LIFO}}^{MV}] = E[W_{\text{FIFO}}^{MV}]$. The second moment of W_{FIFO}^{MV} is calculated while using the decomposition result:

$$
\begin{aligned}
E\left[(W_{\text{FIFO}}^{MV})^2\right] &= 2E[W_{\text{FIFO}}]E[R_U] + E[R_U^2] + E[W_{\text{FIFO}}^2] \\
&= \frac{\lambda f E[S^2]E[U^2]}{2(1-\rho)E[U]} + \frac{f^{(2)}E[S]E[U^2]}{2f(1-\rho)E[U]} + \frac{E[U^3]}{3E[U]} + E[W_{\text{FIFO}}^2] \ ,
\end{aligned}
\tag{23}
$$

where $E[W_{\text{FIFO}}]$ is taken from (11), $E[R_U^2] = \frac{E[U^3]}{3E[U]}$ and $E[W_{\text{FIFO}}^2]$ is given by (14). Comparing (20) and (23), while using (15), it follows that in this case too,

$$
E\left[(W_{\text{FIFO}}^{MV})^2\right] = \frac{E\left[(W_{\text{FIFO}}^{MV})^2\right]}{1-\rho} \ . \tag{24}
$$

5. The $M^X/G/1$ Queue with Single Vacations under the LIFO Regime.

This section is devoted to the single-vacations process (see Levy and Yechiali [1975]) where the server takes only a single vacation at the end of a busy period. If, upon return from a vacation, there are customers present, their service starts, as in the previous model, with no delay, and the server serves exhaustively during the busy period until the system becomes empty, where he takes another vacation. However, if upon return from a vacation the server finds an empty system he waits idle for the first batch-arrival, where a busy period B starts. When B ends and the system is empty, the server takes another (single) vacation, etc.

To calculate the LST of the waiting time, W_{LIFO}^{SV}, of an arbitrary customer C we note that, in this model, when a batch arrives, the server can be found idle, busy or on vacation. We wish to calculate the probabilities of these three events.

Proposition. *The probability that the server is idle is given by*

$$
P_0 = \frac{(1-\rho)\widetilde{U}(\lambda)}{\widetilde{U}(\lambda) + \lambda E[U]} \ . \tag{25}
$$

Proof:

Let I denote the interarrival time between two batches. Clearly I is Exponential with LST $\widetilde{I}(\theta) = \lambda/(\lambda+\theta)$ and mean $1/\lambda$. Let T denote the cycle time, which is the time interval between two consecutive (single) vacations. Then, the LST of T is given by

$$\begin{aligned} E[e^{-\theta T}] = \widetilde{T}(\theta) &= \int_0^\infty e^{-\lambda t} E[e^{-\theta(t+I+B)}] dP(U \le t) \\ &+ \int_0^\infty \sum_{i=1}^\infty e^{-\lambda t} \frac{(\lambda t)^i}{i!} E\Big[e^{-\theta(t+\sum_{j=1}^{i} B_j)}\Big] dP(U \le t) \\ &= \Big(\frac{\lambda}{\lambda+\theta}\Big) \widetilde{B}(\theta)\widetilde{U}(\lambda+\theta) + \widetilde{U}\big(\theta+\lambda-\lambda\widetilde{B}(\theta)\big) - \widetilde{U}(\lambda+\theta) \ . \end{aligned} \tag{26}$$

(For a similar result, see equation (4) in Levy and Yechiali).

It follows from (26) that the mean cycle duration is

$$E[T] = \frac{1}{1-\rho}\Big[E[U] + \frac{\widetilde{U}(\lambda)}{\lambda}\Big] \ . \tag{27}$$

Clearly, a direct way to get $E[T]$ is to write

$$E[T] = E[U] + b_0\Big[\frac{1}{\lambda} + E[B]\Big] + \sum_{i=1}^\infty b_i i E[B] \tag{28}$$

where $b_i = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^i}{i!} dP(U \le t)$ is the probability that i batches arrive during a vacation period. As $\sum_{i=1}^\infty i b_i = \lambda E[U]$, $\frac{1}{\lambda} + E[B] = \frac{1}{\lambda(1-\rho)}$, and $b_0 = \widetilde{U}(\lambda)$, result (27) follows from (28).

Now, P_0 is the proportion of time that the server is idle, and hence, from (28)

$$P_0 = \frac{b_0/\lambda}{E[T]} = \frac{(1-\rho)\widetilde{U}(\lambda}{\lambda E[U] + \widetilde{U}(\lambda)} \ .$$

The probability that an arriving batch finds the server on vacation is $P_U = E[U]/E[T] = \frac{\lambda(1-\rho)E[U]}{\lambda E[U]+\widetilde{U}(\lambda)}$, and the probability that the server is busy is $1 - P_0 - E[U]/E[T] = \rho$, as expected. The LST of W_{LIFO}^{SV} is now derived as

$$\begin{aligned} \widetilde{W}_{\text{LIFO}}^{SV}(\theta) = P_0 \sum_{n=1}^\infty g_n \big[\widetilde{A}(\theta)\big]^{n-1} + \rho \sum_{n=1}^\infty g_n \big[\widetilde{A}(\theta)\big]^{n-1} \int_0^\infty \sum_{k=0}^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} e^{-\theta t} \big[\widetilde{B}(\theta)\big]^k \cdot dP(R_S \le t) \\ + P_U \sum_{n=1}^\infty g_n \big[A(\theta)\big]^{n-1} \int_0^\infty \sum_{m=0}^\infty e^{-\lambda t} \frac{(\lambda t)^m}{m!} e^{-\theta t} \big[\widetilde{B}(\theta)\big]^m dP(R_U \le t) \ . \end{aligned} \tag{29}$$

Using (9),

$$\widetilde{W}^{SV}_{\text{LIFO}}(\theta) = P_0 \frac{1 - F_X\big(\widetilde{A}(\theta)\big)}{f\big[1 - \widetilde{A}(\theta)\big]} + \rho \frac{1 - F_X\big(\widetilde{A}(\theta)\big)}{f\big[1 - \widetilde{A}(\theta)\big]} \widetilde{R}_S\big(\theta + \lambda - \lambda\widetilde{B}(\theta)\big) + P_U \frac{1 - F_X\big(\widetilde{A}(\theta)\big)}{f\big[1 - \widetilde{A}(\theta)\big]} \widetilde{R}_U\big(\theta + \lambda - \lambda\widetilde{B}(\theta)\big) \; . \tag{30}$$

Finally, substituting for $\widetilde{R}_S(\cdot)$ and $\widetilde{R}_U(\cdot)$ in (30), we obtain

$$\widetilde{W}^{SV}_{\text{LIFO}}(\theta) = P_0 \frac{\Big[1 - F_X\big(\widetilde{A}(\theta)\big)\Big]}{f\big[1 - \widetilde{A}(\theta)\big]} + \frac{\lambda\Big[1 - F_X\big(\widetilde{A}(\theta)\big)\Big]}{\big[\theta + \lambda - \lambda\widetilde{B}(\theta)\big]} + P_U \frac{\Big[1 - F_X\big(\widetilde{A}(\theta)\big)\Big]}{f\big[1 - \widetilde{A}(\theta)\big]} \cdot \frac{\Big[1 - \widetilde{U}\big(\theta + \lambda - \lambda\widetilde{B}(\theta)\big)\Big]}{\big[\theta + \lambda - \lambda\widetilde{B}(\theta)\big]E[U]} \; . \tag{31}$$

Note that equation (31) can be written as $\widetilde{W}^{SV}_{\text{LIFO}}(\theta) = \widetilde{W}^{MV}_{\text{LIFO}} + P_0 \frac{G\left(\widetilde{A}(\theta)\right)}{\widetilde{A}(\theta)} \Big[1 - \widetilde{R}_U\big(\theta + \lambda - \lambda\widetilde{B}(\theta)\big)\Big]$.

The mean and second moment of W^{SV}_{LIFO} are now calculated from (31) while using (11):

$$E[W^{SV}_{\text{LIFO}}] = E[W_{\text{LIFO}}] + \frac{P_U}{1-\rho} E[R_U] = \frac{\lambda f E[S^2]}{2(1-\rho)} + \frac{f^{(2)}E[S]}{2f(1-\rho)} + \frac{\lambda E[U^2]}{2\big[\lambda E[U] + \widetilde{U}(\lambda)\big]} \; . \tag{32}$$

Similarly to previous calculations, and letting $\Lambda \equiv \lambda E[U] + \widetilde{U}(\lambda)$, we obtain

$$E\Big[(W^{SV}_{\text{LIFO}})^2\Big] = E[W^2_{\text{LIFO}}] + \frac{\lambda f^{(2)}E[S]E[U^2]}{2f(1-\rho)^2\Lambda} + + \frac{\lambda^2 f E[S^2]E[U^2]}{2(1-\rho)^2\Lambda} + \frac{\lambda E[U^3]}{3(1-\rho)\Lambda} \; , \tag{33}$$

where $E[W^2_{\text{LIFO}}]$ is given in (12).

Comparison with the $M^X/G/1$ Queue with Single Vacations under the FIFO Regime.

This model was studied recently by Takagi and Takahashi [1991]. Applying the method of delay busy cycle analysis (as in Kella and Yechiali [1988]) they derived the LST and first two moments of the waiting times for the various classes in non-preemptive and preemptive resume priority queues.

From their equations (18), (20) and (38) we get

$$E[W^{SV}_{\text{FIFO}}] = \frac{\lambda E[U^2]}{2\Lambda} + \frac{\lambda f^{(2)} E[S]^2}{2(1-\rho)} + \frac{\lambda f E[S^2]}{2(1-\rho)} + \frac{f^{(2)} E[S]}{2f} \ . \tag{34}$$

Combining the 2nd and 4th terms in the RHS of (34) gives $\frac{f^{(2)}E[S]}{2f(1-\rho)}$, so that, as expected, (34) matches (32). That is, $E[W^{SV}_{\text{FIFO}}] = E[W^{SF}_{\text{LIFO}}]$.

Using equations (18) and (39) from Takagi and Takahashi, and completing the calculations of $E\left[(W^{SV}_{\text{FIFO}})^2\right]$, we finally obtain (the lengthy derivations are omitted), as in *all* previous models:

$$E\left[(W^{SV}_{\text{LIFO}})^2\right] = \frac{E\left[(W^{SV}_{\text{FIFO}})^2\right]}{1-\rho} \ . \tag{35}$$

References

[1] Y.Baba, "On the $M^X/G/1$ queue with vacation time", *Oper. Res. Letters* **5**, 93-98 (1986).

[2] P.J. Burke, "Delays in single-server queues with batch input", *Oper. Res.* **23**, 830-833 (1975).

[3] M.L. Chaudhry and J.G.C. Templeton, *A First Course in Bulk Queues*, John Wiley and Sons, New York, 1983.

[4] B.T. Doshi, "Queueing systems with vacations – a survey", *Queueing Systems* **1**, 29-66 (1986).

[5] S.W. Fuhrmann, "Second moment relationships for waiting times in queueing systems with Poisson input", *Queueing Systems* **8**, 397-406 (1991).

[6] O. Kella and U. Yechiali, "Priorities in $M/G/1$ queue with server vacations", *Naval Res. Logist.* **35** 23-34 (1988).

[7] Y. Levy and U. Yechiali, "Utilization of idle time in an $M/G/1$ queueing system", *Manag. Sci.* **22**, 202-211 (1975).

[8] M. Scholl and L. Kleinrock, "On the $M/G/1$ queue with rest periods and certain service – independent queueing disciplines", *Oper. Res.* **31**, 705-719 (1983).

[9] H. Takagi and Y. Takahashi, "Priority queues with batch Poisson arrivals", *Oper. Res. Letters* **10**, 225-232 (1991).