Polling Systems with Positive and Negative Customers

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Abstract

We study Polling Systems (POLS) with both positive and negative customers, a direction not yet pursued in the literature on POLS. We analyze the Gated, Exhaustive and Globally-Gated service regimes and derive expressions for the Probability Generating Functions ($PGFs$) of the system's states and for the Laplace-Stieltjes Transforms of the waiting times. Since the transforms depend on an unknown PGF of a key variable, we bring into use Interpolation Theory, never before exploited in the analyses of POLS, and obtain (approximated) explicit results for the $PGFs$ and means of the variables involved. Numerical examples are presented, demonstrating the use of the solution procedure.

Key words: Polling, negative customers, gated, exhaustive, globally gated, Interpolation theory.

1. Introduction

We study single-server polling systems where, in addition to the regular arrival stream of positive customers, there is a flow of 'negative' arrivals. A negative customer does not receive service and has the effect of removing a (positive) customer from the queue. If it arrives when the queue is empty, it has no effect on the system and it is lost.

Networks with positive and negative arrivals, called G-networks, were inspired by neural networks that communicate via 'impulse' signals. A positive customer is interpreted as 'excitation' while a negative one as 'inhibition'. The goal of this work is to study and analyze polling systems with *both* positive and negative arrivals, a direction not yet pursued in the literature on polling systems.

Polling systems consisting of N queues (channels), receiving independent streams of positive arrivals and served by a single server which incurs switch-over periods when moving from one channel to another, have been widely studied in the literature and used as a central model for the analysis of a large variety of applications in the areas of computer networks, telecommunication systems, manufacturing, traffic control, etc. What makes these systems even more complex is the polling table according to which the single server visits the various queues. In most studies, however, the visit order is assumed to be cyclic (Round Robin).

Common service policies, which control the amount of service given to each queue during a server's visit, are the Exhaustive, Gated (locally and globally) and Limited regimes. Under the Exhaustive regime, at each visit to a queue, the server continues serving until the queue becomes empty, and only then the server is allowed to move further. Under the (locally) Gated regime, the customers that are served during the current visit in a certain queue are only those present when the server starts visiting (polls) the queue. Customers arriving while the queue is attended by the server will be served only during the next server's visit. Under the Globally Gated regime only customers present at the *start of the cycle* are served during the current visit, while under the K_i-Limited service discipline only a limited number of jobs (at most K_i) are served at each server's visit to queue i.

Most of the results on polling systems until mid 80s are presented in the excellent manuscript by Takagi [1986]. An update is given in Takagi [1990]. The Globally Gated regime was introduced by Boxma, Levy and Yechiali [1992] and further studied by Boxma, Weststrate and Yechiali [1993]. Polling systems with breakdowns are treated in Kofman and Yechiali [1996, 1997] while polling systems involving both breakdowns and repairs were analyzed in Nakdimon and Yechiali [2003]. The randomly timed gated regime has been studied by Eliazar and Yechiali [1998_a, 1998_b]. Pseudo conservation laws were established by Boxma and Groenndijk [1987], while optimal visit-orders of the server are discussed in Browne and Yechiali [1989] and in Yechiali [1993]. G-networks have been introduced by Gelenbe [1991_a, 1991_b] and extended to networks with resets in Gelenbe and Fourneau [2002].

As indicated above, our aim is to extend the analysis spectrum of polling systems to include both positive and negative customers so as to construct a framework for G- type polling networks. Our analysis method is based on the 'laws of motion' which connect the occupancy variables X_i^j s (number of customers present in queue j at polling instant of queue i). The traditional method of analysis utilizes joint multi-variable probability generating functions ($PGFs$) of the system state to derive a set of recursive equations for the means (and higher moments) of the variables X_i^j s. However, in the models presented here this method does not yield a complete solution. We thus bring into use (a version of) the "interpolation theory" numerical method. This approach is new and has not been used before for the analysis of polling systems.

In this work we analyze the Gated, Exhaustive and Globally-Gated service regimes. The organization of the paper is the following. In chapter 2 we present the basic model, set the notation and derive some preliminary results which are used in the sequel. In chapter 3 we study the Gated service discipline under the following assumptions: (i) Positive customers and their service times generate, in each queue, a $M/G/1$- type queue. (ii) negative customers arrive continuously and independently at all queues. Their arrival process is independent of the positive arrival stream. A negative arrival to a queue attended by the server, 'kills' the customer being served. The server then takes the next customer in line. Arrival of a negative customer at a queue not attended by the server, kills the first customer in line, if any. If the queue is empty, the negative customer has no effect on the system. We derive the multi-dimensional Probability Generating Function (PGF) of the system state (a vector of queue sizes) at polling instants, and calculate the mean queue sizes at those moments. We also calculate the PGF of the queue size in each station at an arbitrary moment of time, and derive the Laplace Stieltjes Transform (LST) of the waiting time of a positive customer in each queue. The explicit derivation of the PGF of X_i^i, the queue size in queue i at polling instant, employs techniques used in interpolation theory. Furthermore, the analysis requires the use of *transient* solutions for the $M/M/1$ queue, involving Bessel functions. Numerical examples are presented, demonstrating the calculation method. Chapter 4 deals with the Exhaustive service discipline. We consider the same scenario as for the Gated regime and derive the system's various $PGFs$, $LSTs$ and means. Again, transient solutions for the $M/M/1$ queue are used, interpolation theory is employed and a numerical example is presented. Chapter 5 studies the Globally-Gated regime.

2. The Basic Model

2.1 Model description

We consider a polling system comprised of N channels (stations, queues), labeled $1, 2, \ldots, N$, with *both* positive (regular) and negative customers. Positive customers (messages, jobs) arrive at channel i according to a Poisson process with rate λ_i, while negative customers arrive at channel i, independently of the positive customers, according to a Poisson process with rate γ_i.

A single server renders service to the entire network by moving from one channel to another, in a cyclic order, i.e., visiting the queues in the order $1, 2, \ldots, N-1, N, 1, 2, \ldots$. The server resides at a queue for a length of time determined by the service discipline and then moves (switches over) to the next channel. The switch-over duration from channel i to the next is a random variable D_i, independent of the other processes. Flows of positive and negative customers are independent of the server's position.

Each positive job in channel i is characterized by an independent random service requirement B_i.

A negative customer arriving to a channel being attended by the server causes the positive customer in service to leave the system **immediately**. That is, the service of the latter is not completed. The negative customer has no other effect on the system. If a negative customer arrives to a queue not being attended by the server, it removes the first positive customer in the queue (if any) from the system. If a negative customer arrives to an empty queue, it has no effect.

2.2 Notation and definitions

$\widehat{X}(z) = E[z^X]$ denotes the probability generating function (PGF) of a discrete random variable X.

$\widetilde{Y}(s) = E[e^{-sY}]$ denotes the Laplace Stieltjes transform (LST) of a non-negative continuous random variable Y.

$X \sim Y$ denotes two random variables having the same probability distribution function.

X_i^j Number of jobs present in queue j at polling instant of queue i. $f_i(j) = E[X_i^j]$.

T_i Inter-arrival time of negative jobs to station i. $T_i \sim \text{Exp}(\gamma_i)$.

B_i Service requirement of a positive job in channel i (with mean b_i, second moment $b_i^{(2)}$ and density function $f_{B_i}(.)$).

$M_i = \min\{B_i, T_i\}$ Attained service time of a customer in queue i (with mean m_i and second moment $m_i^{(2)}$).

$A_i(t)$ Number of (Poisson with rate λ_i) positive arrivals to queue i during a time interval of length t.

$G_i(\underline{z}) = E\left[\prod_{j=1}^{N} z_j^{X_i^j}\right]$ The joint probability generating function (PGF) of the system state vector $\underline{X_i}$, where $\underline{z} = (z_1, \ldots, z_N)$ and $\underline{X_i} = (X_i^1, X_i^2, \ldots, X_i^N)$.

D_i Switch-over duration when moving from channel i to the next channel (with mean d_i and second moment $d_i^{(2)}$).

$D = \sum_{i=1}^N D_i$ Total switch-over duration during a cycle (with mean d and second moment $d^{(2)}$.

$\rho_i = \lambda_i m_i$ Traffic load of queue i.

$\rho = \sum_{i=1}^N \rho_i$ Total traffic load of the system.

$\theta^i_{X_i^i} = \theta_i$ Sojourn time (busy period) of the server in station i, starting with X_i^i jobs.

Φ_i Length of a busy period in a *regular* M/G/1 queue with arrival rate λ_i and service times M_i.

$K_i = K_i(X_i^i)$ Total number of positive customers *successfully* served in channel i during a visit of the server to that channel, when starting with X_i^i customers. That is, K_i is the number of customers whose service was not interrupted by an arrival of a negative customer.

$\overline{K_i} = \overline{K_i}(X_i^i)$ Total number service *attempts* (including unsuccessful attempts interrupted by negative customers) in channel i during a visit of the server to that channel, when starting with X_i^i positive customers.

L_i Number of positive customers left behind by an arbitrary departing customer from channel i.
L_i also represents the steady-state number of positive customers at channel i at an arbitrary epoch of time.

N_i Queue size in channel i at the *end* of a busy period.

W_i Sojourn time of a positive customer in channel i.

Wq_i Waiting (queueing) time of an arbitrary positive customer (before start of service) at channel i.

$\underline{S_i}(z)$ *PGF* of queue size in channel i at service *beginning* instant.

$\overline{S_i}(z)$ *PGF* of queue size in channel i at service *termination* instant.

$S_i(z)$ *PGF* of queue size, S_i, in channel i at an arbitrary moment *during a service time*.

$V_i(z)$ *PGF* of queue size, V_i, in channel i at an arbitrary moment when the queue is *not* being served.

C Length of a cycle. That is, the time interval between two consecutive polling instants of a queue.

$\sigma = \sigma(\underline{z}) = \sum_{j=1}^{N} \lambda_j(1 - z_j)$

$\sigma_i = \sigma_i(\underline{z}) = \sum_{\substack{j \\ (j \neq i)}} \lambda_j(0 - z_j)$

$\delta_i = \gamma_i + \lambda_i(1 - z)$

$H_i = \sum_{\substack{j \\ (j \neq i)}} \theta_j + \sum_{j=1}^{N} D_j$

$\Gamma_i = \sum_{j=1}^{i-1} (\theta_j + D_j)$

$\Psi_i = \sum_{j=i+1}^{N} \theta_j + \sum_{j=i}^{N} D_j$

$\Theta_r^i(w, z) = E[e^{-w\theta_r^i} \cdot z^{N_i(r)}]$ Joint transform of θ_r^i (busy period at queue i starting with r jobs) and $N_i(r)$ (number of customers in queue i at the end of that busy period).

$\widetilde{\Lambda_i}(w, s) = E[e^{-wH_i} \cdot e_i^{-s\theta}]$

$\widetilde{\Delta_i}(s_1, s_2, s_3) = E[e^{-s_1\Gamma_i} \cdot e^{-s_2\theta_i} \cdot e^{-s_3\Psi_i}]$

$\widetilde{\Omega_i}(s_1, s_2) = E[e^{-s_1\Gamma_i} \cdot e^{-s_2\theta_i}] = \widetilde{\Delta_i}(s_1, s_2, 0)$

3. Gated Service Discipline

In this section we analyze the Gated regime. According to this regime, when a queue is polled, only customers present there at that moment are candidates to be served during the current server's visit.

As indicated in the Introduction, upon arrival of a negative customer to a queue *being served*, the service of the positive customer is interrupted and this customer leaves the system immediately. The server then starts serving the next customer in line (among the 'candidates'). This implies that the actual attained service of each gated customer in queue i is $M_i = \min(B_i, T_i)$.

However, when a negative customer arrives at an unattended queue, he removes a positive waiting customer from that queue (if any). If a negative customer arrives at an empty queue, it has no effect whatsoever.

3.1 Analysis

A key observation is the following: Since T_i, the inter-arrival time of negative customers to queue i, is exponentally distributed with parameter γ_i, the behavior of the system during the time when queue i is *not* attended is that of a time-dependent $M(\lambda_i)/M(\gamma_i)/1$ queue.

For an $M(\lambda)/M(\gamma)/1$ queue, let $R(k,t)$ be the number of customers in the queue at time $t > 0$, given that k customers were present there at time $t = 0$.

The distribution of $R(k,t)$ is given by (see Takacs [1962]) as:

$$P(R(k,t)=j) = P_{kj}(t) = e^{-(\lambda+\gamma)t} \cdot \begin{bmatrix} \left(\frac{\lambda}{\gamma}\right)^{(j-k)/2} \cdot I_{j-k}\ (2\sqrt{\lambda\gamma}t) \\ +\left(\frac{\lambda}{\gamma}\right)^{(j-k+1)/2} \cdot I_{j+k+1}\ (2\sqrt{\lambda\gamma}t) \\ +(1-\frac{\lambda}{\gamma}) \cdot \left(\frac{\lambda}{\gamma}\right)^{j} \cdot \sum\limits_{r=k+j+2}^{\infty} \left(\frac{\lambda}{\gamma}\right)^{-(r/2)} \cdot I_r\ (2\sqrt{\lambda\gamma}t) \end{bmatrix} \tag{3-1}$$

where $I_r(x)$ $(r = 0, \pm 1, \pm 2, \dots)$ is the Bessel function of order r defined by

$$I_r(x) = \sum_{j=0}^{\infty} \frac{(x/2)^{r+2j}}{j!(j+r)!}, \quad r \geq 0, \text{ and } I_{-r}(x) = I_r(x).$$

As a result of rejection of positive customers by negative arrivals, we cannot make use of the classical approach of writing 'law of motion' relating variables X_{i+1}^j: number of positive customers in queue j when queue $i+1$ is polled, with the variables X_i^j, and then express the PGF of the vector $\underline{X_{i+1}} = \left(X_{i+1}^1, X_{i+1}^2, \dots, X_{i+1}^N\right)$ as a function of the PGF of the vector $\underline{X_i} = \left(X_i^1, X_i^2, \dots, X_i^N\right)$. Instead, we derive the joint PGF of the N variables $X_i^i = X_i$ $(i = 1, 2, \dots, N)$ and then use interpolation theory to closely approximate the above $PGF's$.

We make use of the function $R(k,t)$ in writing the following law of motion:

$$X_i \stackrel{d}{=} R_i\left(A_i(\theta_i), H_i\right) \qquad i = 1, \dots, N \tag{3-2}$$

where $H_i = \sum\limits_{\substack{j \\ (j \neq i)}} \theta_j + \sum\limits_{j=1}^{N} D_j$ is the time interval from the end of a busy period in queue i until the next polling instant there.

Here $R_i(A_i(\theta_i), H_i)$ is the number of positive customers at time $t = H_i$ in a $M(\lambda_i)/M(\gamma_i)/1$ queue, starting with $A_i(\theta_i)$ customers at time $t = 0$. θ_i is the length of a busy period in queue i when starting with X_i positive customers.

The explanation to equation (3-2) is the following: at the end of a busy period in queue i (when the server leaves that queue), the queue size is equal to the number of arrivals during that busy period, which is $A_i(\theta_i)$. The server then returns to that queue after visiting all other queues, that is, after H_i units of time, and finds there X_i jobs. During the length of time H_i, queue i is not served, and therefore, due to arrivals of negative customers, it behaves like a time-dependent $M(\lambda_i)/M(\gamma_i)/1$ queue.

Note that X_i appears in both sides of equation (3-2) (in the right hand side implicitly, through $\theta_i = \theta^i_{X^i_i}$). We now derive its distribution.

For this goal, we define the following functions and obtain some relations between them.

The joint PGF for the system's queue sizes at their polling instants is denoted by

$$G(z_1, \ldots, z_N) = E\left[\prod_{i=1}^{N} z_i^{X_i}\right]$$

The joint LST of the busy periods in the N queues is denoted by

$$\widetilde{\theta}(s_1, \ldots, s_N) = E\left[\prod_{j=1}^{N} e^{-s_j\theta_j}\right]$$

We have:

$$\widetilde{\theta}(s_1, \ldots, s_N) = E\left[e^{-\sum\limits_{j=1}^{N} s_j\theta_j}\right] = E_{X_1,\ldots,X_N}\left[E\left[\exp\left\{-\sum_{j=1}^{N} s_j\theta_j\right\} |X_1, \ldots, X_N\right]\right]$$

$$= E_{X_1,\ldots,X_N}\left[E\left[\exp\left\{-\sum_{j=1}^{N} s_j \sum_{k=1}^{X_j} M_{jk}\right\} |X_1, \ldots, X_N\right]\right] = E_{X_1,\ldots,X_N}\left[\prod_{j=1}^{N}\left[\widetilde{M_j}(s_j)\right]^{X_j}\right]$$

This follows since, for each queue j, $\theta_j = \sum\limits_{k=1}^{X_j} M_{jk}$, where $M_{jk} \sim M_j$.

Hence,

$$\widetilde{\theta}(s_1, \ldots, s_N) = G\left(\widetilde{M_1}(s_1), \ldots, \widetilde{M_N}(s_N)\right) \tag{3-3}$$

Note that in the classical Gated model, with no negative arrivals, the corresponding joint LST is similar to (3-3) but with $B_j'\ s$ instead of $M_j'\ s$.

Next we define, for every queue i, the joint LST of the variables H_i and θ_i: $\widetilde{\Lambda_i}(w, s) = E\left[e^{-wH_i} \cdot e^{-s\theta_i}\right]$

We have:

$$\widetilde{\Lambda_i}(w, s) = E\left[\exp\left\{-w\left(\sum_{\substack{j \\ (j\neq i)}} \theta_j + \sum_{j=1}^{N} D_j\right) - s\theta_i\right\}\right] = E\left[e^{-w\sum\limits_{\substack{j \\ (j\neq i)}}\theta_j - s\theta_i}\right] \cdot$$

$$E\left[e^{-w\sum\limits_{j=1}^{N} D_j}\right] = \widetilde{\theta}(w, \ldots, w, s, w, \ldots, w) \cdot \prod_{j=1}^{N} \widetilde{D_j}(w) \tag{3-4}$$

where the $'s'$ in the expression $\widetilde{\theta}(w, ...w, s, w, \ldots, w)$ is located at the ith place.

We utilize equation (3-2) to express the PGF of X_i as a function of the (yet unknown) joint density function, $f_{H_i,\theta_i}(\cdot\,,\cdot)$, of the variables H_i and θ_i:

$$\widehat{X_i}(z) = E[z^{X_i}] = \sum_{x=0}^{\infty} z^x \cdot P(X_i = x) = \sum_{x=0}^{\infty} z^x \cdot P\left(R_i\left(A_i(\theta_i), H_i\right) = x\right)$$

$$= \sum_{x=0}^{\infty} z^x \cdot \int_{t=0}^{\infty} \int_{h=0}^{\infty} P\left(R_i\left(A_i(t), h\right) = x\right) \cdot f_{H_i,\theta_i}(h,t)dhdt$$

$$= \sum_{x=0}^{\infty} z^x \cdot \int_{t=0}^{\infty} \int_{h=0}^{\infty} \sum_{k=0}^{\infty} P\left(R_i\left(k,h\right) = x\right) \cdot \frac{e^{-\lambda_i t}(\lambda_i t)^k}{k!} \cdot f_{H_i,\theta_i}(h,t)dhdt \qquad (3\text{-}5)$$

In what follows we apply the modified polynomial interpolation theory to equation (3-5) in order to obtain an approximation to the PGF $G(z_1, \ldots, z_N)$.

3.2 Approximating $G(z_1, z_2, \ldots, z_N)$ using polynomial interpolation

Interpolation theory (see e.g. Atkinson [1989]) states that a function $h(x)$ with $n+1$ continuous derivatives can be approximated by a polynomial sum

$$P(x) = a_0 + a_1 x + \ldots + a_m x^m \quad (m \le n).$$

The coefficients $a_0, a_1, \ldots, a_m$ are uniquely evaluated when knowing the values of the function $h(x)$ at some distinct $m+1$ points $x_0, \ldots, x_m$. This is done by extracting the coefficients values from $m+1$ equations, each equating the polynomial expression to the function value $h(x_i)$, $i = 0, 1, \ldots, m$.

The approximation error at a point t is expressed by: $\dfrac{(t-x_0)\cdot \ldots \cdot (t-x_m)}{(m+1)!} \cdot h^{(m+1)}(\xi)$, where ξ is a point lying in the minimal interval connecting the points $t, x_0, x_1, \ldots, x_m$, and $h^{(m+1)}$ is the $(m+1)$st derivative of $h(\cdot)$. The error gets smaller as t approaches one of the points $x_0, \ldots, x_m$ and when m gets larger.

The method is extended to multi-variate functions and polynomials.

Since our goal is to approximate the multi-variate function $G\left(\underline{z}\right)$, we now introduce a modification to the interpolation method in order to solve cases in which, instead of knowing values of the function $G\left(\underline{z}\right)$ at distinct points, we have a formula (equation (3-5)), equating the value of the function $G\left(\underline{z}\right)$ at a certain point to an expression involving its values at the entire domain, as well as the parameters of the model.

Accordingly, a polynomial sum is substituted instead of $G\left(\cdot\right)$ in the above formula. Then, the coefficients $a_0, a_1, \ldots$ are calculated. When approximating a generating function, one of the equations must be the normalizing equation: $G(\underline{1}) = 1$. Furthermore, since the error of the approximation is smaller at the vicinity of any one of the sampled points, we will carry the sampling of those points as close as possible to points where the approximation is required.

The method consists of the following steps:

1. Construct a polynomial approximation of order n to the PGF $G(z_1, \dots, z_N)$ with unknown coefficients.

2. Replace the left-hand side of equation (3-5), namely $\widehat{X_i}(z)$, by the polynomial approximation to $G(1, \dots, 1, z, 1, \dots, 1)$.

3. Given the distribution of the $M_i\,'s$ and using equation (3-3), find $\widetilde{\theta}(s_1, \dots, s_N)$ in terms of the (unknown) coefficients of the polynomial from step 1.

4. Using $\widetilde{\theta}(s_1, \dots, s_N)$ from step 3, and equation (3-4), derive an expression for $\widetilde{\Lambda_i}(w, s)$ in terms of the polynomial coefficients.

5. Derive the joint density function $f_{H_i,\theta_i}(h,t)$, which is the inverse LST transform of $\widetilde{\Lambda_i}(w, s)$, namely: $f_{H_i,\theta_i}(h,t) = L^{-1}\{\widetilde{\Lambda_i}(w, s)\}$. (In some cases step 5 may be done by using the method of partial fraction expansion).

6. Insert the expression of $f_{H_i,\theta_i}(h,t)$ into the right-hand side of equation (3-5).

7. Substitute in equation (3-5) the coordinates $z_1, \dots, z_N$ of a chosen point.

8. Repeat step 7 several times and get a system of independent equations with unknown variables (the polynomial coefficients).
   Solve for these variables and get an approximation to $G(z_1, \dots, z_N)$.

We demonstrate these steps in the following numerical example:

Numerical example 1

For simplicity, we use a symmetric system with identical values for each station i. We assume exponential services and exponential switch-over times.

The following values for the system variables are used: Number of queues: $N = 3$

Poisson arrival rates of positive and negative customers: $\lambda_i = 1$, $\gamma_i = 1$

Service times: $B_i \sim \exp(4)$, with $E[B_i] = b_i = \frac{1}{4}$

Switch-over times: $D_i \sim \exp(1)$, with $E[D_i] = d_i = 1$

Since $M_i = \min\{B_i, T_i\}$ and $T_i \sim \exp(1)$, we have: $M_i \sim \exp(4+1)$ and $\widetilde{M_i}(s) = \frac{5}{5+s}$

Also, $\prod_{j=1}^{3} \widetilde{D_j}(s) = \left(\dfrac{1}{1+s}\right)^3$

We use in this example a polynomial of order 2 as an approximation to the PGF $G(z_1, \dots, z_N)$:

$$G(z_1, z_2, z_3) = \mu_0 + \mu_1 z_1 + \mu_2 z_2 + \mu_3 z_3 + \mu_{11} {z_1}^2 + \mu_{22} z_2^2 + \mu_{33} z_3 + \\ + \mu_{12} z_1 z_2 + \mu_{13} z_1 z_3 + \mu_{23} z_2 z_3 \quad (3\text{-}6)$$

Due to the symmetric assumption of all queues, we can further assume $\mu_1 = \mu_2 = \mu_3$, $\mu_{11} = \mu_{22} = \mu_{33}$ and $\mu_{12} = \mu_{13} = \mu_{23}$.

Hence we get,

$$G(z_1,z_2,z_3)=\mu_0+\mu_1\left(z_1+z_2+z_3\right)+\mu_{11}\left({z_1}^2+{z_2}^2+{z_3}^2\right)+\\ \mu_{12}\left(z_1z_2+z_1z_3+z_2z_3\right) \quad \text{(3-7)}$$

Now, equation (3-3) turns into

$$\widetilde{\theta}(s_1,s_2,s_3)=G\left(\widetilde{M_1}(s_1),\widetilde{M_2}(s_2),\widetilde{M_3}(s_3)\right)=G\left(\frac{5}{5+s_1}\ ,\ \frac{5}{5+s_2}\ ,\ \frac{5}{5+s_3}\right)$$

and from equation (3-4) we get

$$\widetilde{\Lambda_1}(w,s)=\widetilde{\theta}(s,w,w)\cdot\prod_{j=1}^{3}\widetilde{D_j}(w)=G\left(\frac{5}{5+s}\ ,\ \frac{5}{5+w}\ ,\ \frac{5}{5+w}\right)\cdot\left(\frac{1}{1+w}\right)^3 \quad \text{(3-8)}$$

By inserting the polynomial (3-7) into (3-8) we get

$$\widetilde{\Lambda_1}(w,s)=\left[\begin{array}{c}\mu_0+\mu_1\left(\frac{5}{5+s}+\frac{10}{5+w}\right)+\mu_{11}\left(\frac{25}{(5+s)^2}+\frac{50}{(5+w)^2}\right)\\ +\mu_{12}\left(\frac{50}{(5+s)\cdot(5+w)}+\frac{25}{(5+w)^2}\right)\end{array}\right]\cdot\left(\frac{1}{1+w}\right)^3$$

$$=\frac{\mu_0}{(1+w)^3}+\left(\frac{5\mu_1}{5+s}+\frac{25\mu_{11}}{(5+s)^2}\right)\cdot\frac{1}{(1+w)^3}+\frac{10\mu_1}{(5+w)\cdot(1+w)^3}$$

$$+\frac{50\mu_{11}+25\mu_{12}}{(5+w)^2\cdot(1+w)^3}+\frac{50\mu_{12}}{(5+s)\cdot(5+w)\cdot(1+w)^3} \quad \text{(3-9)}$$

Now, when calculating the density function $f_{H_1,\theta_1}(h,t)$ from its LST $\widetilde{\Lambda_1}(w,s)$, we use partial fraction expansion (see Melsa and Sage [1973]).

The method is applied to an expression of the form $F(w)=\dfrac{\prod_{k=1}^{m}(w+c_k)}{\prod_{k=1}^{n}(w+d_k)^{n_k}}$ having repeated poles of multiplicity n_k at $w=-d_k$.

If the numerator polynomial is of lower order than the denominator polynomial, the partial-fraction expansion is

$$F(w)=\sum_{i=1}^{n}\sum_{j=1}^{n_i}\frac{a_{ij}}{(w+d_i)^j} \text{ where } a_{k,n_k-i}=\left(\frac{1}{i!}\cdot\frac{d^i}{dw^i}\left[\ (w+d_k)^{n_k}F(w)\right]\right)|_{w=-d_k}$$

We adopt this formula to expand terms of equation (3-9). We get

$$\frac{1}{(5+w)\cdot(1+w)^3}=\frac{1/64}{1+w}+\frac{-1/16}{(1+w)^2}+\frac{1/4}{(1+w)^3}+\frac{-1/64}{5+w} \quad \text{(3-10)}$$

and

$$\frac{1}{(5+w)^2\cdot(1+w)^3}=\frac{3/256}{1+w}+\frac{-2/64}{(1+w)^2}+\frac{1/16}{(1+w)^3}+\frac{-3/256}{5+w}+\frac{-1/64}{(5+w)^2} \quad \text{(3-11)}$$

Substituting the expansions (3-10) and (3-11) in (3-9) and combining similar fractions yields,

$$
\begin{aligned}
\widetilde{\Lambda_1}(w,s) &= \left(\mu_0 + \frac{10\mu_1}{4} + \frac{50\mu_{11}+25\mu_{12}}{16}\right)\cdot\frac{1}{(1+w)^3} \\
&+ \left(5\mu_1 + \frac{50\mu_{12}}{4}\right)\cdot\frac{1}{(5+s)(1+w)^3} + \frac{25\mu_{11}}{(5+s)^2(1+w)^3} \\
&+ \left(\frac{10\mu_1}{64} + \frac{3\cdot(50\mu_{11}+25\mu_{12})}{256}\right)\cdot\frac{1}{1+w} \\
&+ \left(\frac{-10\mu_1}{16} + \frac{-2\cdot(50\mu_{11}+25\mu_{12})}{64}\right)\cdot\frac{1}{(1+w)^2} \\
&+ \left(\frac{-10\mu_1}{64} + \frac{-3\cdot(50\mu_{11}+25\mu_{12})}{256}\right)\cdot\frac{1}{5+w} - \frac{50\mu_{11}+25\mu_{12}}{64}\cdot\frac{1}{(5+w)^2} \\
&+ \frac{50\mu_{12}}{64}\cdot\frac{1}{(5+s)(1+w)} - \frac{50\mu_{12}}{16}\cdot\frac{1}{(5+s)(1+w)^2} \\
&- \frac{50\mu_{12}}{64}\cdot\frac{1}{(5+s)(5+w)} \qquad (3\text{-}12)
\end{aligned}
$$

In order to get the inverse Laplace transform of $\widetilde{\Lambda_1}(w,s)$ in (3-12), we recall the following properties of Laplace transform:

Let $L^{-1}\{F(s)\}$ denote the inverse Laplace transform of $F(s)$. If $F(s) = \int\limits_{t=-\infty}^{\infty} e^{-st}\cdot f(t)dt$ then, $f(t) = L^{-1}\{F(s)\}$ uniquely.

If the LST is a linear combination of LSTs, then the inverse LST is the same linear combination of the corresponding inverse LSTs. That is, if $F_1(s) = F_2(s) + c\cdot F_3(s)$, then $f(t) = L^{-1}\{F_1(s)\} = L^{-1}\{F_2(s)\} + c\cdot L^{-1}\{F_3(s)\} = f_2(t) + c\cdot f_3(t)$

By using the latter property, we apply the known inverse Laplace transforms to each of the terms in the sum (3-12) to obtain the desired density function. We use:

$$
L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}\ ;\ L^{-1}\left\{\frac{1}{(s+a)^2}\right\} = te^{-at}\ ; L^{-1}\left\{\frac{1}{(s+a)^3}\right\} = \frac{1}{2}t^2e^{-at} \qquad (3\text{-}13)
$$

The joint LST, $\widetilde{\Lambda}(w,s)$, and the density $f(t,h)$ are functions of two variables such that

$$
\widetilde{\Lambda}(w,s) = E[e^{-wH}\cdot e^{-s\theta}] = \int_{h=0}^{\infty}\int_{t=0}^{\infty} e^{-wh}\cdot e^{-st}\cdot f_{H,\theta}(h,t)dtdh
$$

In our case, the composite density function is a product of two functions: $f_{H,\theta}(h,t) = f_1(h) \cdot f_2(t)$, and therefore the corresponding LST is also a product of the $LSTs$ of the corresponding functions:

$$\widetilde{\Lambda}(w,s) = \int\limits_{h=0}^{\infty} \int\limits_{t=0}^{\infty} e^{-wh} \cdot e^{-st} \cdot f_1(h) \cdot f_2(t) dt dh = F_1(w) \cdot F_2(s).$$

We note that some of the terms in (3-12) lack the s variable. Therefore, in order to calculate their inverse transform, we use the following $\delta(\cdot)$ function.

<u>The unit impulse function $\delta(\cdot)$</u> (see Desoer and Kuh [1969])

The pulse function $P_\Delta(\cdot)$ is defined by

$$P_\Delta(t) = \left\{ \begin{array}{cc} 0 & t < 0 \\ \dfrac{1}{\Delta} & 0 < \ t < \Delta \\ 0 & \Delta < t \end{array} \right\}$$

Then, the unit impulse function $\delta(\cdot)$ is defined as $\delta(t) = \lim\limits_{\Delta \to 0} P_\Delta(t)$.

This function (also called the dirac delta function) is characterized by:

$$\delta(t) = \left\{ \begin{array}{ll} 0 & \text{for } \ t \neq 0 \\ \text{singular} & \text{at } \ t = 0 \end{array} \right\}$$

The singularity at the origin is such that for any $\varepsilon > 0$, $\int\limits_{-\varepsilon}^{\varepsilon} \delta(t) dt = 1$.

A useful property of the unit impulse, which stems from the first definition, is called the "sifting" property:

Let f be a continuous function. Then, for any positive ε,

$$\int\limits_0^\varepsilon f(t) \cdot \delta(t) dt = f(0) \tag{3-14}$$

From the last property we obtain that

$$L\left\{\delta(t)\right\} = \int\limits_{t=0}^{\infty} e^{-st} \cdot \delta(t) dt = e^{-st}|_{t=0} = 1$$

To conclude, $\delta(t)$ may be regarded as a density function whose LST equals 1.

Consider a function $f(\cdot)$ with LST $F(w) = \dfrac{1}{1+w}$. Then, $F(w)$ can be written as a joint LST :

$$F(w) \cdot 1 = \int\limits_{h=0}^{\infty} e^{-wh} \cdot f(h) dh \cdot \int\limits_{t=0}^{\infty} e^{-st} \cdot \delta(t) dt = \int\limits_{h=0}^{\infty} \int\limits_{t=0}^{\infty} e^{-wh} \cdot e^{-st} \cdot f(h) \cdot \delta(t) dt dh$$

Hence,

$$L^{-1}\left\{F(w)\right\} = L^{-1}\left\{\frac{1}{1+w} \cdot 1\right\} = f(h) \cdot \delta(t) = e^{-h} \cdot \delta(t)$$

Let $f_1(h,t)$ denote the inverse LST of the sum of all terms in $\widetilde{\Lambda_1}(w,s)$ that involve both s and w, and let $f_2(h) \cdot \delta(t)$ be the inverse LST of all terms there not involving s. Then, by using (3-13) and making use of $\delta(t)$ for all relevant terms in (3-12), we get,

$$f_{H_1,\theta_1}(h,t) = L^{-1}\ \left\{\widetilde{\Lambda_1}(w,s)\right\} = f_1(h,t) + \delta(t)\cdot f_2(h) \tag{3-15}$$

where,

$$f_1(h,t) = e^{-h-5t}\cdot\left[\tfrac{1}{2}h^2\left(5\mu_1 + \frac{50\mu_{12}}{4}\right) + h^2 t\frac{25}{2}\mu_{11} + \frac{50\mu_{12}}{64} - h\frac{50\mu_{12}}{16}\right]$$

$$-e^{-5h-5t}\cdot\frac{50\mu_{12}}{64} \tag{3-16}$$

and

$$f_2(h) = e^{-h}\cdot\left[\begin{array}{c}\tfrac{1}{2}h^2\left(\mu_0+\frac{10\mu_1}{4}+\frac{50\mu_{11}+25\mu_{12}}{16}\right)-h\left(\frac{10\mu_1}{16}+\frac{2\cdot(50\mu_{11}+25\mu_{12})}{64}\right)\\ +\left(\frac{10\mu_1}{64}+\frac{3\cdot(50\mu_{11}+25\mu_{12})}{256}\right)\end{array}\right]$$

$$-e^{-5h}\cdot\left[\left(\frac{10\mu_1}{64}+\frac{3\cdot(50\mu_{11}+25\mu_{12})}{256}\right)+h\ \left(\frac{50\mu_{11}+25\mu_{12}}{64}\right)\right] \tag{3-17}$$

Now, by substituting (3-15) into equation (3-5) we obtain,

$$\widehat{X_1}(z) = \sum_{x=0}^{\infty} z^x\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}\sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right)\cdot\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\cdot f_{H_1\ ,\theta_1}(h,t)dhdt$$

$$= \sum_{x=0}^{\infty} z^x\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}\sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right)\cdot\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\cdot f_1(h,t)dhdt$$

$$+\sum_{x=0}^{\infty} z^x\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}\sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right)\cdot\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\cdot\delta(t)\cdot f_2(h)dhdt \tag{3-18}$$

Using property (3-14) of $\delta(t)$ we have
$\int_{t=0}^{\infty} e^{-\lambda_1 t}\cdot\delta(t)dt = 1$, and $\int_{t=0}^{\infty}\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\cdot\delta(t)dt = 0$ for $k>0$
We get,

$$\int_{t=0}^{\infty}\sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right)\cdot\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\cdot\delta(t)dt =$$

$$= \int_{t=0}^{\infty} P\left(R_1\left(0\ ,\ h\right)=x\right)\cdot e^{-\lambda_1 t}\cdot\delta(t)dt + \sum_{k=1}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right)\cdot$$

$$\int_{t=0}^{\infty}\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\cdot\delta(t)dt = P\left(R_1(0,h)=x\right) \tag{3-19}$$

By substituting (3-19) in (3-18), we get,

$$\widehat{X_1}(z) = \sum_{x=0}^{\infty} z^x \cdot \int_{t=0}^{\infty} \int_{h=0}^{\infty} \sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right) \cdot \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \cdot f_1(h,t)dhdt$$

$$+\sum_{x=0}^{\infty} z^x \cdot \int_{h=0}^{\infty} f_2(h) \cdot \left(\int_{t=0}^{\infty} \sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right) \cdot \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \cdot \delta(t)dt \right) dh$$

$$= \sum_{x=0}^{\infty} z^x \cdot \int_{t=0}^{\infty} \int_{h=0}^{\infty} \sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right) \cdot \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \cdot f_1(h,t)dhdt$$

$$+\sum_{x=0}^{\infty} z^x \cdot \int_{h=0}^{\infty} P\left(R_1\left(0\ ,\ h\right)=x\right) \cdot f_2(h)dh \qquad \text{(3-20)}$$

Now, from equation (3-7),

$$\widehat{X_1}(z) = G(z,1,1) = \mu_0 + \mu_1\left(z+2\right) + \mu_{11}\left(z^2+2\right) + \mu_{12}\left(2z+1\right) \qquad \text{(3-21)}$$

By substituting $z = 1$ in (3-21), we get the normalized equation in the unknown parameters μ_0, μ_1, μ_{11} and μ_{12}:

$$1 = \mu_0 + 3(\mu_1 + \mu_{11} + \mu_{12})$$

Equating (3-20) to (3-21), we get the main equation:

$$\mu_0 + \mu_1\left(z+2\right) + \mu_{11}\left(z^2+2\right) + \mu_{12}\left(2z+1\right)$$

$$= \sum_{x=0}^{\infty} z^x \cdot \int_{t=0}^{\infty} \int_{h=0}^{\infty} \sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right) \cdot \frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!} \cdot f_1(h,t)dhdt$$

$$+\sum_{x=0}^{\infty} z^x \cdot \int_{h=0}^{\infty} P\left(R_1\left(0\ ,\ h\right)=x\right) \cdot f_2(h)dh \qquad \text{(3-22)}$$

To complete the approximation, three distinct values of z have to be substituted in (3-22) in order to solve for the unknown parameters μ_0, μ_1, μ_{11} and μ_{12}. Finally, we substitute equation (3-1) into equation (3-22), where $\lambda = 1$ and $\gamma = 1$. (Note that, since we chose $\lambda = \gamma$, the infinite sum in the expression of $P(R(k,t) = j)$ is zero). We get

$$P\left(R_1\left(k\ ,\ h\right)=x\right) = e^{-2h} \cdot \left[I_{|x-k|}\left(2h\right) + I_{x+k+1}(2h)\right] \qquad \text{(3-23)}$$

We turn to solve a system of 4 non linear equations by the Mathcad software:

<u>Equation 1</u>

$$1 = \mu_0 + 3(\mu_1 + \mu_{11} + \mu_{12})$$

<u>Equation 2</u> $(z = 0.7)$

$$\mu_0 + 2.7\mu_1 + 2.49\mu_{11} + 2.4\mu_{12} =$$

$$= \sum_{x=0}^{\infty} 0.7^x \cdot \int_{t=0}^{\infty} \int_{h=0}^{\infty} \sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right)=x\right) \cdot \frac{e^{-t}t^k}{k!} \cdot f_1(h,t)dhdt+$$

$$+\sum_{x=0}^{\infty} 0.7^x \cdot \int_{h=0}^{\infty} P\left(R_1\left(0\ ,\ h\right)=x\right) \cdot f_2(h)dh$$

Equation 3 $(z = 0.8)$

$\mu_0 + 2.8\mu_1 + 2.64\mu_{11} + 2.6\mu_{12} =$

$$= \sum_{x=0}^{\infty} 0.8^x \cdot \int_{t=0}^{\infty} \int_{h=0}^{\infty} \sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right) = x\right) \cdot \frac{e^{-t}t^k}{k!} \cdot f_1(h,t)dhdt +$$

$$+ \sum_{x=0}^{\infty} 0.8^x \cdot \int_{h=0}^{\infty} P\left(R_1\left(0\ ,\ h\right) = x\right) \cdot f_2(h)dh$$

Equation 4 $(z = 0.9)$

$\mu_0 + 2.9\mu_1 + 2.81\mu_{11} + 2.8\mu_{12} =$

$$= \sum_{x=0}^{\infty} 0.9^x \cdot \int_{t=0}^{\infty} \int_{h=0}^{\infty} \sum_{k=0}^{\infty} P\left(R_1\left(k\ ,\ h\right) = x\right) \cdot \frac{e^{-t}t^k}{k!} \cdot f_1(h,t)dhdt +$$

$$+ \sum_{x=0}^{\infty} 0.9^x \cdot \int_{h=0}^{\infty} P\left(R_1\left(0\ ,\ h\right) = x\right) \cdot f_2(h)dh$$

The solution is:

$\mu_0 = -284.468$; $\mu_1 = 187.459$; $\mu_{11} = 1.793$; $\mu_{12} = -94.096$.

After substituting these values in (3-21), we obtain an explicit approximated function for the PGF $\widehat{X_1}(z)$:

$$\widehat{X_1}(z) = -0.06 - 0.733 \cdot z + 1.793 \cdot z^2 \tag{3-24}$$

The (approximated) first and second moments of X_1 are given by

$$E[X_1] = \frac{d}{dz}\widehat{X_1}(z)|_{z=1} = \mu_1 + 2\mu_{11} + 2\mu_{12} = 2.853; E[X_1^2] = 3.586 \tag{3-25}$$

In this example, due to symmetry, the solution for $\widehat{X_i}(z)$, $i = 2, \ldots, N$ is the same as that of $\widehat{X_1}(z)$.

3.3 Waiting times

First we derive the PGF of the queue size at service termination instants of positive customers who got some (full or partial) service. Note that customers may depart from queues, due to arrival of negative customers, without even starting service.

We use the following equation (Eisenberg [1972]):

$$\underline{B}(z) + \overline{S}(z) \cdot E[\overline{K}] = \overline{B}(z) + \underline{S}(z) \cdot E[\overline{K}] \tag{3-26}$$

where:

$\underline{B}(z)$ and $\overline{B}(z)$ are the $PGFs$ of the queue size at the beginning and at the end of a busy period, respectively (for simplicity, we remove the symbol $\widehat{}$ indicating a PGF).

$\underline{S}(z)$ and $\overline{S}(z)$ are the $PGFs$ of the queue size at service beginning and at service termination instants, respectively.

$\overline{K}$ is the number of service attempts during a busy period.

Suppressing the index i we write:

$\underline{B}(z) = \widehat{X}(z)$

$\overline{B}(z) = E\left[z^{A(\theta)}\right] = E\left[z^{A\left(\sum\limits_{k=1}^{X} M_k\right)}\right] = \widehat{X}\left(\widetilde{M}(\lambda(1-z))\right)$

$E[\overline{K}] = E[X]$

$\overline{S}(z) = \underline{S}(z) \cdot \widetilde{M}(\lambda(1-z)) \cdot \dfrac{1}{z}$

The last equation holds since $\widetilde{M}(\lambda(1-z))$ stands for the PGF of the number of (positive) arrivals to the queue during a service duration, and the number of customers present at service termination is equal to those present at service beginning minus one, plus the number of arrivals during time M.

Inserting the above into equation (3-26) yields,

$$\widehat{X}(z) + \overline{S}(z) \cdot E[X] = \widehat{X}\left(\widetilde{M}(\lambda(1-z))\right) + \frac{\overline{S}(z) \cdot z}{\widetilde{M}(\lambda(1-z))} \cdot E[X] \qquad (3\text{-}27)$$

from which we get the PGF of the queue size in station i at service termination instants:

$$\overline{S_i}(z) = \frac{\widetilde{M_i}(\lambda_i(1-z))}{E[X_i] \cdot \left[z - \widetilde{M_i}(\lambda_i(1-z))\right]} \cdot \left[\widehat{X_i}(z) - \widehat{X_i}\left(\widetilde{M_i}(\lambda_i(1-z))\right)\right] \qquad (3\text{-}28)$$

The approximated PGF of X_i and its moments were already obtained in (3-24) and (3-25), and can be substituted in (3-28).

Next, we turn to obtain $\widehat{L_i}(z)$, the PGF of the queue size at departure instants. Note that, a positive customer departs either at the end of service completion or as a result of an arrival of a negative customer, regardless of the position of the server. Since the distribution of L_i, the number of customers in the system at departure epochs, is identical in distribution to the number of customers at epochs of arrivals, and as a result of PASTA, L_i also stands for the number of customers at channel i (under steady-state condition) at an arbitrary point of time.

To derive $\widehat{L_i}(z)$, we use the same approach as in Shomrony and Yechiali [2001]: Define $S_i(z)$ =: PGF of the queue size S_i *at an arbitrary moment* during service duration

$V_i(z)$ =: PGF of the queue size V_i *at an arbitrary moment* during time H_i, when the queue is not being served

$p_i = P$(the server is serving queue i)

Then,

$$\widehat{L_i}(z) = S_i(z) \cdot p_i + V_i(z) \cdot (1 - p_i) \qquad (3\text{-}29)$$

Furthermore, with C denoting the cycle time,

$$p_i = \frac{E[\theta_i]}{E[C]} = \frac{E\left[\sum\limits_{k=1}^{X_i} M_{ik}\right]}{E\left[\sum\limits_{i=1}^{N}\left(\sum\limits_{k=1}^{X_i} M_{ik} + D_i\right)\right]} = \frac{E[X_i]\cdot m_i}{\sum\limits_{i=1}^{N} E[X_i]\cdot m_i + d} \tag{3-30}$$

and

$$S_i(z)\cdot g_i(z)\cdot \frac{1}{z} = \overline{S_i}(z) \tag{3-31}$$

where $g_i(z)$ is the PGF of the number of (positive) customers arriving during the so called 'remaining' part of a service time, which is the time from the moment of an arbitrary observation within a service time until the instant of service termination.

It is known (see Cohen [1982]: P. 113) that the LST of the remaining time ξ in a renewal process with inter-renewal times Y is given by: $\widetilde{\xi}(s) = \dfrac{1-\widetilde{Y}(s)}{s\cdot E[Y]}$

Hence, in our case, the PGF of the number of arrivals during the remaining part of a service time is given by $g_i(z) = \dfrac{1-\widetilde{M_i}(\lambda_i(1-z))}{\lambda_i(1-z)\cdot m_i}$.

Substituting $g_i(z)$ and (3-28) in (3-31) yields

$$S_i(z) = \frac{\widetilde{M_i}(\lambda_i(1-z))}{E[X_i]\cdot\left[z-\widetilde{M_i}(\lambda_i(1-z))\right]}\cdot\frac{\left[\widehat{X_i}(z)-\widehat{X_i}\left(\widetilde{M_i}(\lambda_i(1-z))\right)\right]}{1-\widetilde{M_i}(\lambda_i(1-z))}\cdot z\lambda_i(1-z)m_i \tag{3-32}$$

Now, to compute $V_i(z)$ we define $H_{i,\,p}$ to be the 'past time' for the period H_i.

The following relation holds:

$$V_i = R_i(A_i(\theta_i), H_{i,p}) \tag{3-33}$$

from which we compute the PGF of V_i, using again the joint density function $f_{H_i,\theta_i}(\cdot,\cdot)$:

$$V_i(z) = E[z^{V_i}] = \sum_{v=0}^{\infty} z^v\cdot P(V_i = v) = \sum_{v=0}^{\infty} z^v\cdot P\left(R_i\left(A_i(\theta_i)\ ,\ H_{i,\,p}\right) = v\right)$$

$$= \sum_{v=0}^{\infty} z^v\cdot \int_{t=0}^{\infty}\int_{h=0}^{\infty} P\left(R_i\left(A_i(\theta_i)\ ,\ H_{i,\,p}\right) = v \mid H_i = h\ ,\ \theta_i = t\right)\cdot f_{H_i\ ,\theta_i}(h,t)\,dh\,dt \tag{3-34}$$

Since $\widetilde{H_{i,\,p}}(s) = \dfrac{1-\widetilde{H_i}(s)}{s\cdot E[H_i]} = \dfrac{1-E\left[e^{-sH_i}\right]}{s\cdot E[H_i]}$, when $H_i = h$ (constant), we get:

$$\widetilde{H_{i,\,p}}(s) = \frac{1-e^{-sh}}{s\cdot h}$$

It is also known that the LST of a continuous Uniform variable $U(a,b)$ is $\dfrac{e^{-sa}-e^{-sb}}{s\cdot(b-a)}$.

From the uniqueness of the LST we get that $(H_{i,p}|H_i=h)\sim U(0,h)$.

Using this distribution, while defining in (3-34) a continuous Uniform variable $h_p\sim U(0,h)$ with density function $\frac{1}{h}$, yields,

$$V_i(z)=\sum_{v=0}^{\infty}z^v\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}P\left(R_i\left(A_i(t)\ ,\ h_p\right)=v\right)\cdot f_{H_i\ ,\theta_i}(h,t)dhdt$$

$$=\sum_{v=0}^{\infty}z^v\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}\int_{r=0}^{h}P\left(R_i\left(A_i(t)\ ,\ r\right)=v\right)\cdot\frac{1}{h}\cdot f_{H_i\ ,\theta_i}(h,t)drdhdt$$

$$=\sum_{v=0}^{\infty}z^v\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}\int_{r=0}^{h}\sum_{k=0}^{\infty}\left(P\left(R_i\left(k\ ,\ r\right)=v\right)\cdot\frac{e^{-\lambda_i t}(\lambda_i t)^k}{k!}\right)\cdot\frac{1}{h}\cdot f_{H_i\ ,\theta_i}(h,t)drdhdt \tag{3-35}$$

Where $f_{H_i,\theta_i}(h,t)$ was derived in (3-15) as $f_{H_i,\theta_i}(h,t)=f_1(h,t)+\delta(t)\cdot f_2(h)$, with $f_1(h,t)$ and $f_2(h)$ given by (3-16) and (3-17), respectively.

We get from (3-35),

$$V_i(z)=\sum_{v=0}^{\infty}z^v\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}\int_{r=0}^{h}\sum_{k=0}^{\infty}\left(\begin{array}{c}\left(P\left(R_i\left(k\ ,\ r\right)=v\right)\cdot\dfrac{e^{-\lambda_i t}(\lambda_i t)^k}{k!}\right)\cdot\dfrac{1}{h}\\ \cdot\left(\ f_1(h,t)+\delta(t)\cdot f_2(h)\ \right)\end{array}\right)drdhdt$$

$$=\sum_{v=0}^{\infty}z^v\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}\int_{r=0}^{h}\sum_{k=0}^{\infty}\left(P\left(R_i\left(k\ ,\ r\right)=v\right)\cdot\frac{e^{-\lambda_i t}(\lambda_i t)^k}{k!}\right)\cdot\frac{1}{h}\cdot f_1(h,t)drdhdt+$$

$$+\sum_{v=0}^{\infty}z^v\cdot\int_{h=0}^{\infty}\int_{r=0}^{h}\frac{1}{h}\cdot f_2(h)\cdot\left(\int_{t=0}^{\infty}\sum_{k=0}^{\infty}\left(P\left(R_i\left(k\ ,\ r\right)=v\right)\cdot\frac{e^{-\lambda_i t}(\lambda_i t)^k}{k!}\right)\delta(t)dt\right)drdh \tag{3-36}$$

Using property (3-14) as in the derivation of (3-19) we get:

$$V_i(z)=\sum_{v=0}^{\infty}z^v\cdot\int_{t=0}^{\infty}\int_{h=0}^{\infty}\int_{r=0}^{h}\sum_{k=0}^{\infty}\left(P\left(R_i\left(k\ ,\ r\right)=v\right)\cdot\frac{e^{-\lambda_i t}(\lambda_i t)^k}{k!}\right)\cdot\frac{1}{h}\cdot f_1(h,t)drdhdt$$

$$+\sum_{v=0}^{\infty}z^v\cdot\int_{h=0}^{\infty}\int_{r=0}^{h}P\left(R_i\left(0\ ,\ r\right)=v\right)\cdot\frac{1}{h}\cdot f_2(h)drdh \tag{3-37}$$

We can now substitute the results (3-30), (3-32) and (3-37) in (3-29) to get an expression for $\widehat{L}_i(z)$, the PGF of the size of queue i at an arbitrary moment.

Computation of W_i – the sojourn time of a customer in queue i

We now derive the LST of W_i by using its relation to the queue size at departure instants. The classical argument that the number of customers left behind by a departing customer in queue i is exactly the number of customers that arrived there during his sojourn time, holds in our case as well. A departing customer can depart either due to its service completion (during θ_i), or due to an arrival of a negative customer (during θ_i^C). Thus, customers arriving after our customer's arrival will never leave the system due to service completion before his departure (FIFO), and on the other hand, these customers cannot be removed from the queue by negative arrivals before our customer's departure (because a negative customer removes a positive customer from the head of the queue).

For the same reasons, customers arriving before our customer will exit the queue before his departure.

Thus, since $W_i = Wq_i + M_i$, we have,
$\widehat{L_i}(z) = \widetilde{W_i}(\lambda_i(1-z)) = \widetilde{Wq_i}(\lambda_i(1-z)) \cdot \widetilde{M_i}(\lambda_i(1-z))$

That is,

$$\widetilde{W_i}(s) = \widehat{L}_i(1 - \frac{s}{\lambda_i}) \tag{3-38}$$

and

$$\widetilde{Wq_i}(s) = \frac{\widehat{L_i}(1 - \frac{s}{\lambda_i})}{\widetilde{M_i}(s)} \tag{3-39}$$

Finally, expressions for $E[L_i]$, $E[Wq_i]$ and $E[W_i]$ can be derived by differentiation, where $V_i(z)$ may be calculated by using, for example, the Mathcad software.

4. Exhaustive Service Discipline

In this section we consider the Exhaustive regime in which, at each visit, the server attends every queue until it becomes fully empty before switching on to the next queue.

4.1 Analysis

Negative customers flow constantly to *all* queues, at all times, independently of the location of the server. Upon arrival of a negative customer to a queue being served, the service is terminated and the interrupted customer leaves the system immediately. On the other hand, when a negative customer arrives at a queue not being attended by the server, he removes the first positive waiting customer from that queue (if any).

Therefore, when queue i is not being served, it behaves like a $M(\lambda_i)/M(\gamma_i)/1$ queue, since the negative flow may be looked upon as a sequence of 'service' times.

For an $M(\lambda_i)/M(\gamma_i)/1$ queue, let $R_i(k,t)$ be the number of jobs in queue i at time $t > 0$, given that at time $t = 0$, k jobs were present (see section 3.1).

When the server leaves queue i, there are no jobs left, and when he returns after $H_i = \sum\limits_{\substack{j \\ (j \neq i)}} \theta_j + \sum\limits_{j=1}^{N} D_j$ units of time, the queue size at polling instants is X_i.

Thus,

$$X_i \stackrel{d}{=} R_i(0, H_i) \tag{4-1}$$

Now, the length of the busy period in queue j is $\theta_j = \sum\limits_{k=1}^{X_j} \Phi_{jk}$, where Φ_{jk}, all distributed like Φ_j, denote i.i.d busy periods in a regular $M/G/1$ queue with service times M_j and arrival rate λ_j.

We make use of the property

$$\widetilde{\Phi_j}(s) = \widetilde{M_j}\left(s + \lambda_j(1 - \widetilde{\Phi_j}(s))\right) \tag{4-2}$$

If $\lambda_i E[M_i] \le 1$, (4-2) has a unique solution (Saaty [1961] P. 195). We denote the density of Φ_i by $f_{\Phi_i}(t)$.

The joint LST of the N separate busy periods in the N queues is given by

$$\widetilde{\theta}(s_1, \ldots, s_N) = E\left[\prod_{j=1}^{N} e^{-s_j\theta_j}\right] = E_{X_1,\ldots,X_N}\left[\prod_{j=1}^{N}\left[\widetilde{\Phi_j}(s_j)\right]^{X_j}\right] = G\left(\widetilde{\Phi_1}(s_1), \ldots, \widetilde{\Phi_N}(s_N)\right) \tag{4-3}$$

where $G(z_1, \ldots, z_N) = E\left[\prod\limits_{i=1}^{N} z_i^{X_i}\right]$

The PGF of X_i is derived by using equation (4-1) and the density function $f_{H_i}(\cdot)$ of H_i.

$\widehat{X_i}(z) = E[z^{X_i}] = \sum\limits_{x=0}^{\infty} z^x \cdot P(X_i = x) = \sum\limits_{x=0}^{\infty} z^x \cdot P\left(R_i\left(0\ ,\ H_i\right) = x\right)$

$$= \sum_{x=0}^{\infty} z^x \cdot \int_{h=0}^{\infty} P\left(R_i\left(0\ ,\ h\right) = x\right) \cdot f_{H_i}(h)dh \tag{4-4}$$

Also, using (4-3) yields,

$$\widetilde{H_i}(s) = E\left[e^{-sH_i}\right] = E\left[\exp\left\{-s\left(\sum_{\substack{j\\(j\neq i)}} \theta_j + \sum_{j=1}^{N} D_j\right)\right\}\right] = E\left[\prod_{\substack{j\\(j\neq i)}} \exp\left\{-s\theta_j\right\}\right] \cdot$$

$\prod\limits_{j=1}^{N} \widetilde{D_j}(s) =$

$$= \widetilde{\theta}(s, ...s, 0, s, \ldots, s) \cdot \prod_{j=1}^{N} \widetilde{D_j}(s) = G\left(\widetilde{\Phi}_1(s), \ldots, \widetilde{\Phi}_{i-1}(s), 1, \widetilde{\Phi}_{i+1}(s), \ldots, \widetilde{\Phi}_N(s)\right) \cdot \prod_{j=1}^{N} \widetilde{D}_j(s) \tag{4-5}$$

Approximation procedure for $G(z_1, \ldots, z_N)$

We now describe the steps of the procedure and present a numerical example demonstrating its applicability.

1. Construct a polynomial approximation of $G(z_1, \ldots, z_N) = E\left[\prod\limits_{i=1}^{N} z_i^{X_i}\right]$ with unknown coefficients.
2. Given the distribution of M_i, express $\widetilde{\Phi}_i(s)$ from (4-2), either explicitly (if possible) or as an approximation.

3. Insert $\widetilde{\Phi}_i(s)$ and the approximation of $G(z_1,\ldots,z_N)$ into (4-5) to get $\widetilde{H_i}(s)$.

4. Use the inverse Laplace transform formula to obtain $f_{H_i}(h)$:

$$f_{H_i}(h) = L^{-1}\left(\widetilde{H_i}(s)\right) = \frac{1}{2\pi j}\int\limits_{\sigma-j\cdot\infty}^{\sigma+j\cdot\infty}\widetilde{H_i}(s)\cdot e^{sh}ds \tag{4-6}$$

where $\sigma > \sigma_0$ and $\int\limits_0^\infty f_{H_i}(h)\cdot e^{-\sigma_1 h}\,dh$ exists for every $\sigma_1 > \sigma_0$ (see Appendix A of Melsa and Sage [1973]).

5. Substituting the above into (4-4) yields

$$G(1,\ldots,1,z,1,\ldots,1) = \widehat{X_i}(z) = \sum_{x=0}^{\infty} z^x \cdot \int\limits_{h=0}^{\infty} P\left(R_i\left(0\ ,\ h\right)=x\right)\cdot f_{H_i}\ (h)dh \tag{4-7}$$

6. Substitute several values of z into equation (4-7) and get a set of independent equations. Solve these equations for the unknown coefficients of $G(z_1,\ldots,z_N)$.

Numerical example 2

Consider again a symmetric system with equal parameter values for all queues, and assume exponential services and exponential switch-over times in each queue. The following values for the system variables are assumed:

Number of queues: $N = 3$

Poisson arrival rates of positive and negative customers: $\lambda_i = 1$, $\gamma_i = 1$

Service times: $B_i \sim \exp\left(\frac{1}{b_i}\right)$, with $E[B_i] = b_i = \frac{1}{2}$

Switch-over times: $D_i \sim \exp(10)$, with $E[D_i] = d_i = 0.1$

$M_i \sim \exp\left(\frac{1}{b_i}+\gamma_i\right)$, implying that $\widetilde{M_i}(s) = \dfrac{\frac{1}{b_i}+\gamma_i}{\frac{1}{b_i}+\gamma_i+s}$.

Also, $\prod\limits_{j=1}^{3}\widetilde{D_j}(s) = \left(\dfrac{10}{10+s}\right)^3$

We use a polynomial of order 2 as an approximation to the PGF $G(z_1,\ldots,z_N)$.

As a result of symmetry we get

$$G(z_1,z_2,z_3) = \mu_0+\mu_1\left(z_1+z_2+z_3\right)+\mu_{11}\left({z_1}^2+{z_2}^2+{z_3}^2\right)+\mu_{12}\left(z_1z_2+z_1z_3+z_2z_3\right) \tag{4-8}$$

Using (4-2): $\widetilde{\Phi_i}(s) = \dfrac{\frac{1}{b_i}+\gamma_i}{\frac{1}{b_i}+\gamma_i+s+\lambda_i(1-\widetilde{\Phi_i}(s))}$.

The solution for $\widetilde{\Phi_i}(s)$ is given by

$$\widetilde{\Phi_i}(s) = \frac{s+\lambda_i+\frac{1}{b_i}+\gamma_i-\sqrt{\left(s+\lambda_i+\frac{1}{b_i}+\gamma_i\right)^2-4\lambda_i\cdot\left(\frac{1}{b_i}+\gamma_i\right)}}{2\lambda_i} \quad (4\text{-}9)$$

Combining (4-8) and (4-9), equation (4-5) turns into (after omitting the indices),

$$\widetilde{H}_1(s) = G\left(1, \widetilde{\Phi_2}(s), \widetilde{\Phi_3}(s)\right)\cdot\prod_{j=1}^{3}\widetilde{D_j}(s) = G\left(1, \widetilde{\Phi}(s), \widetilde{\Phi}(s)\right)\cdot\prod_{j=1}^{3}\widetilde{D_j}(s)$$

$$= \left[\mu_0+\mu_1\left(1+2\widetilde{\Phi}(s)\right)+\mu_{11}\left(1+2\left(\widetilde{\Phi}(s)\right)^2\right)+\mu_{12}\left(2\widetilde{\Phi}(s)+\left(\widetilde{\Phi}(s)\right)^2\right)\right]\cdot \left(\frac{10}{10+s}\right)^3$$

$$= \left[\begin{array}{c} \mu_0+\mu_1\left(5+s-\sqrt{(s+4)^2-12}\right)+ \\ +\mu_{11}\left(1+(s+4)\cdot\left(s+4-\sqrt{(s+4)^2-12}\right)-6\right)+ \\ +\mu_{12}\left(\left(s+4-\sqrt{(s+4)^2-12}\right)\cdot\left(\frac{s+6}{2}\right)-3\right) \end{array}\right]\cdot\left(\frac{10}{10+s}\right)^3 \quad (4\text{-}10)$$

Next we derive the density function $f_{H_i}(h)$:

In our example, $\lambda=\gamma=1$. This implies that the infinite sum in (3-1) for the expression of $P(R(k,t)=j)$ vanishes $(1-\lambda/\gamma=0)$.

Thus,

$$P\left(R_i\left(0\ ,\ h\right)=x\right)=e^{-2h}\cdot\left[I_x\left(2h\right)+I_{x+1}(2h)\right] \quad (4\text{-}11)$$

Now, by writing $s=\sigma+j\omega$ (a complex number), and changing variables in (4-6) we get,

$$f_{H_i}(h)=\frac{1}{2\pi j}\cdot\int_{\sigma-j\cdot\infty}^{\sigma+j\cdot\infty}\widetilde{H_i}(s)\cdot e^{sh}\ ds=\frac{1}{2\pi j}\cdot\int_{-\infty}^{+\infty}\widetilde{H_i}(\sigma+j\omega)\cdot e^{(\sigma+j\omega)h}j\ dw$$

$$=\frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty}\widetilde{H_i}(1+j\omega)\cdot e^{(1+j\omega)h}\ d\omega \quad (4\text{-}12)$$

($\sigma=1$ was chosen after assuring stability of the results for several values of σ which are greater than 1).

Substituting (4-10) in (4-12) yields,

$$f_{H_i}(h) = \frac{1}{2\pi} \cdot$$

$$\int_{-\infty}^{+\infty} \left[\begin{array}{c} \left[\begin{array}{c} \mu_0 + \mu_1 \left(5 + (1+j\omega) - \sqrt{((1+j\omega)+4)^2 - 12} \right) + \\ + \mu_{11} \left(1 + ((1+j\omega)+4) \cdot \left((1+j\omega) + 4 - \sqrt{((1+j\omega)+4)^2 - 12} \right) - 6 \right) + \\ + \mu_{12} \left(\left((1+j\omega) + 4 - \sqrt{((1+j\omega)+4)^2 - 12} \right) \cdot \left(\frac{(1+j\omega)+6}{2} \right) - 3 \right) \end{array} \right] \\ \cdot \left(\frac{10}{10 + (1+j\omega)} \right)^3 \end{array} \right] \cdot e^{(1+j\omega)h} d\omega \tag{4-13}$$

We also have, from (4-8),

$$\widehat{X_i}(z) = G(z,1,1) = \mu_0 + \mu_1 (2+z) + \mu_{11} \left(2 + z^2 \right) + \mu_{12} (2z+1) \tag{4-14}$$

We now substitute $P(R_i(0,h) = x)$, $f_{H_i}(h)$ and $\widehat{X_i}(z)$ from (4-11), (4-13) and (4-14), respectively, in (4-7) and substitute successively 3 different values of z in order to solve for the unknown parameters μ_0, μ_1, μ_{11} and μ_{12}.

The following 4 non-linear equations, obtained from (4-7), are solved by the Mathcad software:

<u>Equation 1</u> $(z = 0.9)$

$$\mu_0 + 2.9\mu_1 + 2.8\mu_{11} + 2.8\mu_{12} = \sum_{x=0}^{\infty} 0.9^x \cdot \int_{h=0}^{\infty} P(R_i(0\ ,\ h) = x) \cdot f_{H_i}\ (h) dh$$

<u>Equation 2</u> $(z = 0.8)$

$$\mu_0 + 2.8\mu_1 + 2.64\mu_{11} + 2.6\mu_{12} = \sum_{x=0}^{\infty} 0.8^x \cdot \int_{h=0}^{\infty} P(R_i(0\ ,\ h) = x) \cdot f_{H_i}\ (h) dh$$

<u>Equation 3</u> $(z = 0.7)$

$$\mu_0 + 2.7\mu_1 + 2.49\mu_{11} + 2.4\mu_{12} = \sum_{x=0}^{\infty} 0.7^x \cdot \int_{h=0}^{\infty} P(R_i(0\ ,\ h) = x) \cdot f_{H_i}\ (h) dh$$

<u>Equation 4</u> (from (4-8)):

$1 = G(1,1,1) = \mu_0 + 3(\mu_1 + \mu_{11} + \mu_{12})$

The solution is:

$\mu_0 = -3.191$; $\mu_1 = 1.7$; $\mu_{11} = 1.569$; $\mu_{12} = -1.872$

By substituting these values in (4-14), an explicit approximated function for the PGF $\widehat{X_i}(z)$ is obtained:

$\widehat{X_i}(z) = 1.475 - 2.044 \cdot z + 1.569 \cdot z^2$

Therefore,

$$E[X_i] = \frac{d}{dz} \widehat{X_i}(z)|_{z=1} = \mu_1 + 2\mu_{11} + 2\mu_{12} = 1.094$$

4.2 Waiting times

We use again Eisenbergs equation [1972] (see definitions in section 3.3),

$$\underline{B_i}(z) + \overline{S_i}(z) \cdot E[\overline{K_i}] = \overline{B_i}(z) + \underline{S_i}(z) \cdot E[\overline{K_i}] \tag{4-15}$$

in order to derive $\overline{S_i}(z)$, the PGF of the queue size at service termination instants. In the exhaustive case we have, for queue i:

$$\overline{K_i}(X_i) = X_i + A_i(\theta_i) = X_i + A_i\left(\sum_{k=1}^{X_i} \Phi_{ik}\right).$$

Hence,

$$E\left[\overline{K_i}(X_i)\right] = E[X_i] + \lambda_i E[X_i] \cdot \frac{m_i}{1-\rho_i} = \frac{E[X_i]}{1-\rho_i} \tag{4-16}$$

Also,

$$\underline{B_i}(z) = \widehat{X_i}(z)$$

$$\overline{B_i}(z) = E\left[z^0\right] = 1$$

$$\overline{S_i}(z) = \underline{S_i}(z) \cdot \widetilde{M_i}(\lambda_i(1-z)) \cdot \frac{1}{z}$$

Inserting the above and (4-16) in equation (4-15) yields,

$$\widehat{X_i}(z) + \overline{S_i}(z) \cdot \frac{E[X_i]}{1-\rho_i} = 1 + \frac{\overline{S_i}(z) \cdot z}{\widetilde{M_i}(\lambda_i(1-z))} \cdot \frac{E[X_i]}{1-\rho_i} \tag{4-17}$$

from which,

$$\overline{S_i}(z) = \frac{\widetilde{M_i}(\lambda_i(1-z)) \cdot \left(\widehat{X_i}(z) - 1\right) \cdot (1-\rho_i)}{E[X_i] \cdot \left[z - \widetilde{M_i}(\lambda_i(1-z))\right]} \tag{4-18}$$

Note that $\overline{S_i}(z)$ is the PGF of the i-th queue size at service completion instants, and not at departure instants, since there are departing customers who don't receive service at all.

We now derive $\widehat{L_i}(z)$, the PGF of the queue size at an arbitrary point of time. We use again equation (3-29) (see definitions in section 3.3):

$$\widehat{L_i}(z) = S_i(z) \cdot p_i + V_i(z) \cdot (1-p_i) \tag{4-19}$$

Using $E[\Phi_j] = \frac{m_j}{1-\rho_j}$ we have,

$$p_i = \frac{E[\theta_i]}{E[C]} = \frac{E\left[\sum_{k=1}^{X_i} \Phi_{ik}\right]}{E\left[\sum_{j=1}^{N}\left(\sum_{k=1}^{X_j} \Phi_{jk} + D_j\right)\right]} = \frac{E[X_i] \cdot \dfrac{m_i}{1-\rho_i}}{\sum_{j=1}^{N} E[X_j] \cdot \dfrac{m_j}{1-\rho_j} + d} \tag{4-20}$$

Now, using the relation (3-31) and the expression for $g_i(z)$ in section 3.3, we finally get,

$$S_i(z) = \frac{\widetilde{M_i}(\lambda_i(1-z))}{E[X_i] \cdot \left[z - \widetilde{M_i}(\lambda_i(1-z))\right]} \cdot \frac{\left[\widehat{X_i}(z) - 1\right] \cdot (1-\rho_i)}{1 - \widetilde{M_i}(\lambda_i(1-z))} \cdot z\lambda_i(1-z)m_i \tag{4-21}$$

Derivation of $V_i(z)$

Recall that V_i stands for the number of customers in queue i at an arbitrary moment when queue i is *not* being served.

The following relation holds:

$$V_i \stackrel{d}{=} R_i(0, H_{i,p}) \tag{4-22}$$

where $H_{i,p}$ is the remaining time of the random variable H_i.

Using the density function $f_{H_i}(\cdot)$ we write

$$V_i(z) = E[z^{V_i}] = \sum_{v=0}^{\infty} z^v \cdot P(V_i = v) = \sum_{v=0}^{\infty} z^v \cdot P\left(R_i\left(0\ ,\ H_{i,\ p}\right) = v\right)$$

$$= \sum_{v=0}^{\infty} z^v \cdot \int_{h=0}^{\infty} P\left(R_i\left(0, H_{i,\ p}\right) = v \mid H_i = h\ \right) \cdot f_{H_i}\ (h) dh \tag{4-23}$$

Again, when $H_i = h$, $\widetilde{H}_{i,\ p}(s) = \dfrac{1 - e^{-sh}}{s \cdot h}$, which means that $H_{i,p} | H_i = h \sim U(0, h)$.

Using this distribution, while defining in (4-23) a continuous Uniform variable: $h_p \sim U(0, h)$ with density function $\frac{1}{h}$, yields

$$V_i(z) = \sum_{v=0}^{\infty} z^v \cdot \int_{h=0}^{\infty} P\left(R_i\left(0\ ,\ h_p\right) = v\right) \cdot f_{H_i}\ (h) dh$$

$$= \sum_{v=0}^{\infty} z^v \cdot \int_{h=0}^{\infty} \int_{r=0}^{h} P\left(R_i\left(0\ ,\ r\right) = v\right) \cdot \frac{1}{h} \cdot f_{H_i}\ (h) dr dh \tag{4-24}$$

where $f_{H_i}(h)$ is given by (4-13).

Substituting (4-20), (4-21) and (4-24) in (4-19) yields $\widehat{L}_i(z)$, the *PGF* of the size of queue i at an arbitrary moment.

Note that $\widehat{L}_i(z)$ depends on the distribution of X_i and on $G(\underline{z})$ which were derived by using interpolation theory (equations (4-8) and (4-14), respectively).

As in the Gated service discipline, in a specific queue, the number of customers left behind by a departing customer is exactly those that arrived there during his sojourn time. Therefore,

$$\widehat{L}_i(z) = \widetilde{W}_i(\lambda_i(1 - z)), \quad or \quad \widetilde{W}_i(s) = \widehat{L}_i(1 - \frac{s}{\lambda_i}) \tag{4-25}$$

5. Globally Gated Service Discipline

In the classical Globally Gated (GG) regime (Boxma, Levy and Yechiali [1992]) with only positive arrivals, the server moves cyclically among the queues and uses the instant of cycle beginning as a reference point of time. When it reaches a queue it serves only those customers who were present there at the start of the cycle. We extend the analysis to the case where both positive and negative customers flow into the system.

5.1 Analysis

Negative customers flow continuously to all queues, at all times, independently of the location of the server. During a busy period at station i, the server attempts service on all X_1^i customers who were present there at the cycle start, and only on them. Upon arrival of a negative customer to a queue being served, the service is terminated and the interrupted customer leaves the system immediately. The next customer (if any) enters service. On the other hand, when a negative customer arrives at a queue not attended by the server, he can remove only positive waiting customer (if any) that arrived after the start of the cycle. That is, the X_1^i customers are 'safe' as long as the server is not attending queue i. The rest of the queue behaves like a $M(\lambda_i)/M(\gamma_i)/1$ queue, starting with 0 jobs. Thus, at a moment of polling of queue i, the number of customers present is

$$X_i^i = X_1^i + R_i(0, \Gamma_i) \tag{5-1}$$

where $\Gamma_i = \sum\limits_{j=1}^{i-1} (\theta_j + D_j)$ is the time elapsing from the start of the cycle until that polling instant. θ_j is the relevant busy period. Also, since all X_1^i customers are served during the busy period θ_i at queue i, the number of jobs left when the server leaves is $R_i(0, \Gamma_i) + A_i(\theta_i) \equiv N_i$.

Now, from the server's departure until the start of the next cycle, the queue again behaves like a $M(\lambda_i)/M(\gamma_i)/1$ queue. Hence,

$$X_1^i = R_i\left(R_i(0, \Gamma_i) + A_i(\theta_i), \Psi_i\right) \tag{5-2}$$

where $\Psi_i = \sum\limits_{j=i+1}^{N} \theta_j + \sum\limits_{j=i}^{N} D_j$.

In order to solve equation (5-2) for X_1^i, we use again the following functions:

The joint Generating Function for the N queue sizes at the start of the cycle:

$$G_1(z_1, \ldots, z_N) = E\left[\prod_{j=1}^{N} z_j^{X_1^j}\right].$$

The joint LST of the N busy periods in the N queues: $\widetilde{\theta}(s_1, \ldots, s_N) = E\left[\prod\limits_{j=1}^{N} e^{-s_j\theta_j}\right]$. We have

$$\widetilde{\theta}(s_1, \ldots, s_N) = E\left[\exp\left\{-\sum_{j=1}^{N} s_j\theta_j\right\}\right] = E_{\underline{X_1}}\left[E\left[\exp\left\{-\sum_{j=1}^{N} s_j\theta_j\right\} | \underline{X_1}\right]\right]$$

$$= E_{\underline{X_1}}\left[E\left[\exp\left\{-\sum_{j=1}^{N} s_j \sum_{k=1}^{X_1^j} M_{jk}\right\} | \underline{X_1}\right]\right] = E_{\underline{X_1}}\left[\prod_{j=1}^{N}\left[\widetilde{M_j}(s_j)\right]^{X_1^j}\right]$$

Hence,

$$\widetilde{\theta}(s_1,\ldots,s_N) = G_1\left(\widetilde{M_1}(s_1),\ldots,\widetilde{M_N}(s_N)\right) \tag{5-3}$$

We need to define the joint LST of Γ_i, θ_i and Ψ_i :

$\widetilde{\Delta_i}(s_1,s_2,s_3) = E\left[e^{-s_1\Gamma_i}\cdot e^{-s_2\theta_i}\cdot e^{-s_3\Psi_i}\right]$

We have,

$\widetilde{\Delta_i}(s_1,s_2,s_3) = E\left[\exp\left\{-s_1\left(\sum\limits_{j=1}^{i-1}(\theta_j+D_j)\right) - s_2\theta_i - s_3\left(\sum\limits_{j=i+1}^{N}\theta_j + \sum\limits_{j=i}^{N}D_j\right)\right\}\right]$

$= E\left[\exp\left\{-s_1\left(\sum\limits_{j=1}^{i-1}\theta_j\right) - s_2\theta_i - s_3\left(\sum\limits_{j=i+1}^{N}\theta_j\right)\right\}\right]\cdot E\left[\exp\left\{-s_1\left(\sum\limits_{j=1}^{i-1}D_j\right) - s_3\left(\sum\limits_{j=i}^{N}D_j\right)\right\}\right]$

Hence,

$$\widetilde{\Delta_i}(s_1,s_2,s_3) = \widetilde{\theta}(s_1,\ldots,s_1,s_2,s_3,\ldots,s_3)\cdot\prod_{j=1}^{i-1}\widetilde{D_j}(s_1)\cdot\prod_{j=i}^{N}\widetilde{D_j}(s_3) \tag{5-4}$$

where s_2 in $\widetilde{\theta}(s_1,\ldots,s_1,s_2,s_3,\ldots,s_3)$ is positioned at the i-th coordinate.

The PGF of X_1^i is derived from equation (5-2):

$\widehat{X_1^i}(z) = E\left[z^{X_1^i}\right] = \sum\limits_{x=0}^{\infty} z^x\cdot P(X_1^i = x) = \sum\limits_{x=0}^{\infty} z^x\cdot P\left(R_i\left(R_i(0,\Gamma_i)+A_i(\theta_i),\Psi_i\right) = x\right)$

$= \sum\limits_{x=0}^{\infty} z^x\cdot\int\limits_{t_1=0}^{\infty}\int\limits_{t_2=0}^{\infty}\int\limits_{t_3=0}^{\infty}\left[\begin{array}{c} P\left(\,R_i\left(R_i(0,t_1)+A_i(t_2),t_3\,\right)=x\,\right)\cdot \\ \cdot f_{\Gamma_i\,,\theta_i\,,\Psi_i}(t_1,t_2,t_3)dt_3dt_2dt_1 \end{array}\right]$

$= \sum\limits_{x=0}^{\infty} z^x\cdot\int\limits_{t_1=0}^{\infty}\int\limits_{t_2=0}^{\infty}\int\limits_{t_3=0}^{\infty}\sum\limits_{k=0}^{\infty}\left[\begin{array}{c} P\left(\,R_i\left(k,t_3\,\right)=x\,\right)\cdot P\left(R_i(0,t_1)+A_i(t_2)=k\,\right)\cdot \\ \cdot f_{\Gamma_i\,,\theta_i\,,\Psi_i}(t_1,t_2,t_3)dt_3dt_2dt_1 \end{array}\right]$

$= \sum\limits_{x=0}^{\infty} z^x\cdot\int\limits_{t_1=0}^{\infty}\int\limits_{t_2=0}^{\infty}\int\limits_{t_3=0}^{\infty}\sum\limits_{k=0}^{\infty}\left[\begin{array}{c} P\left(\,R_i\left(k,t_3\,\right)=x\,\right)\cdot \sum\limits_{j=0}^{k} P\left(R_i(0,t_1)=j\,\right)\cdot \\ \cdot P\left(A_i(t_2)=k-j\,\right)\cdot f_{\Gamma_i\,,\theta_i\,,\Psi_i}(t_1,t_2,t_3)dt_3dt_2dt_1 \end{array}\right]$

$$\widehat{X_1^i}(z) = \sum_{x=0}^{\infty} z^x\cdot\int\limits_{t_1=0}^{\infty}\int\limits_{t_2=0}^{\infty}\int\limits_{t_3=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\begin{array}{c} P\left(\,R_i\left(k,t_3\,\right)=x\,\right)\cdot P\left(R_i(0,t_1)=j\,\right)\cdot \\ \cdot\dfrac{e^{-\lambda_i t_2}\cdot(\lambda_i t_2)^{k-j}}{(k-j)!}\cdot f_{\Gamma_i\,,\theta_i\,,\Psi_i}(t_1,t_2,t_3)dt_3dt_2dt_1 \end{array}\right] \tag{5-5}$$

We will apply again the modified polynomial interpolation theory (see section 3.2) to equation (5-5) in order to obtain an approximation for the joint PGF, $G_1(z_1,\ldots,z_N) = E\left[\prod\limits_{j=1}^{N} z_j^{X_1^j}\right]$.

The method consists of the following steps:

1. Construct a polynomial approximation of order n to the $PGF\ G_1(z_1, \ldots, z_N)$ with unknown coefficients.
2. Replace the left-hand side of equation (5-5), namely $\widehat{X_1^i}(z)$, by the polynomial approximation to $G_1(1, \ldots, 1, z, 1, \ldots, 1)$.
3. Given the distribution of the $M_i{}'s$ and using equation (5-3), find $\widetilde{\theta}(s_1, \ldots, s_N)$ in terms of the (unknown) coefficients of the polynomial from step 1.
4. Using the expression of $\widetilde{\theta}(s_1, \ldots, s_N)$ from step 3 and equation (5-4), derive an expression for $\widetilde{\Delta_i}(s_1, s_2, s_3)$ in terms of the polynomial coefficients.
5. Derive the joint density function $f_{\Gamma_i\ ,\theta_i\ ,\Psi_i}(t_1, t_2, t_3)$, which is the inverse LST transform of $\widetilde{\Delta_i}(s_1, s_2, s_3)$, namely: $f_{\Gamma_i\ ,\theta_i\ ,\Psi_i}(t_1, t_2, t_3) = L^{-1}\left\{\widetilde{\Delta_i}(s_1, s_2, s_3)\right\}$.
6. Insert the expression of $f_{\Gamma_i\ ,\theta_i\ ,\Psi_i}(t_1, t_2, t_3)$ from step 5, into the right-hand side of equation (5-5).
7. Substitute in equation (5-5) a chosen point z.
8. Repeat step 7 several times, with different values of z, and get a system of independent equations with unknown variables (the polynomial coefficients).
9. Solve for these variables and get the approximation to $G_1(z_1, \ldots, z_N)$.

We demonstrate these steps in the following numerical example:

Numerical example 3

Consider again a symmetric system with equal parameter values for all queues, and assume exponential services and exponential switch-over times in every queue. The following values for the system variables are assumed:

Number of queues: $N = 3$

Poisson arrival rates of positive and negative customers: $\lambda_i = 1$, $\gamma_i = 1$

Service times: $B_i \sim \exp(4)$, with $E[B_i] = b_i = \frac{1}{4}$

Switch-over times: $D_i \sim \exp(1)$, with $E[D_i] = d_i = 1$

$M_i \sim \exp(4+1)$, thus $\widetilde{M_i}(s) = \frac{5}{5+s}$. Also, $\widetilde{D_j}(s) = \frac{1}{1+s}$

We use in this example a polynomial of order 2 as an approximation to the $PGF\ G_1(z_1, \ldots, z_N)$:

$$G_1(z_1, z_2, z_3) = \mu_0 + \mu_1 (z_1 + z_2 + z_3) + \mu_{11}\left({z_1}^2 + {z_2}^2 + z_3^2\right) + \mu_{12}(z_1 z_2 + z_1 z_3 + z_2 z_3) \tag{5-6}$$

Now, equation (5-3) turns into

$$\widetilde{\theta}(s_1, s_2, s_3) = G_1\left(\frac{5}{5+s_1}\ ,\ \frac{5}{5+s_2}\ ,\ \frac{5}{5+s_3}\right)$$

Consequently, we get from equation (5-4)

$$\widetilde{\Delta_2}(s_1,s_2,s_3)=\widetilde{\theta}(s_1,s_2,s_3)\cdot\widetilde{D_1}(s_1)\cdot\prod_{j=2}^{3}\widetilde{D_j}(s_3)=G_1\left(\frac{5}{5+s_1},\ \frac{5}{5+s_2},\ \frac{5}{5+s_3}\right)\cdot\frac{1}{1+s_1}\cdot\left(\frac{1}{1+s_3}\right)^2 \tag{5-7}$$

Inserting the polynomial (5-6) into (5-7) results in

$$\widetilde{\Delta_2}(s_1,s_2,s_3)=\frac{1}{1+s_1}\cdot\left(\frac{1}{1+s_3}\right)^2\cdot$$

$$\cdot\left[\begin{array}{c}\mu_0+\mu_1\cdot\sum\limits_{i=1}^{3}\frac{5}{5+s_i}+\mu_{11}\cdot\sum\limits_{i=1}^{3}\left(\frac{5}{5+s_i}\right)^2+\\ +25\mu_{12}\cdot\left(\frac{1}{(5+s_1)\cdot(5+s_2)}+\frac{1}{(5+s_1)\cdot(5+s_3)}+\frac{1}{(5+s_2)\cdot(5+s_3)}\right)\end{array}\right] \tag{5-8}$$

In order to get the density function $f_{\Gamma_2\ ,\theta_2\ ,\Psi_2}(t_1,t_2,t_3)$ from its LST $\widetilde{\Delta_2}(s_1,s_2,s_3)$, we use again the method of partial fraction expansion (see section 3.2) and get the following representation,

(1) $\displaystyle\frac{1}{(5+s_1)\cdot(1+s_1)}=\frac{a_1}{5+s_1}+\frac{a_2}{1+s_1}$

(2) $\displaystyle\frac{1}{(5+s_3)\cdot(1+s_3)^2}=\frac{a_3}{5+s_3}+\frac{a_4}{1+s_3}+\frac{a_5}{(1+s_3)^2}$

(3) $\displaystyle\frac{1}{(5+s_1)^2\cdot(1+s_1)}=\frac{a_6}{5+s_1}+\frac{a_7}{(5+s_1)^2}+\frac{a_8}{1+s_1}$

(4) $\displaystyle\frac{1}{(5+s_3)^2\cdot(1+s_3)^2}=\frac{a_9}{5+s_3}+\frac{a_{10}}{(5+s_3)^2}+\frac{a_{11}}{1+s_3}+\frac{a_{12}}{(1+s_3)^2}$

The solution is given by

$a_1=-\frac{1}{4}$; $a_2=\frac{1}{4}$; $a_3=\frac{1}{16}$; $a_4=-\frac{1}{16}$;

$a_5=\frac{1}{4}$; $a_6=-\frac{1}{16}$; $a_7=-\frac{1}{4}$; $a_8=\frac{1}{16}$;

$a_9=\frac{1}{32}$; $a_{10}=\frac{1}{16}$; $a_{11}=-\frac{1}{32}$; $a_{12}=\frac{1}{16}$

By substituting (1) to (4) in (5-8), and combining similar fractions we get,

$$\widetilde{\Delta_2}(s_1,s_2,s_3)=\frac{5\mu_1+25\mu_{12}\cdot(a_2+a_5)}{(5+s_2)\cdot(1+s_1)\cdot(1+s_3)^2}+\frac{25\mu_{11}}{(5+s_2)^2\cdot(1+s_1)\cdot(1+s_3)^2}+$$
$$+\frac{25\mu_{12}a_1}{(5+s_2)\cdot(1+s_3)^2\cdot(5+s_1)}+\frac{25\mu_{12}a_3}{(5+s_2)\cdot(1+s_1)\cdot(5+s_3)}+$$
$$+\frac{25\mu_{12}a4}{(5+s_2)\cdot(1+s_1)\cdot(1+s_3)}+\frac{\mu_0+5\mu_1a_2+5\mu_1a_5+25\mu_{11}a_8+25\mu_{11}a_{12}+25\mu_{12}a_2a_5}{(1+s_1)\cdot(1+s_3)^2}+$$
$$+\frac{5\mu_1a_1+25\mu_{11}a_6+25\mu_{12}a_1a_5}{(1+s_3)^2\cdot(5+s_1)}+\frac{25\mu_{11}a_7}{(1+s_3)^2\cdot(5+s_1)^2}+$$

$$+\frac{5\mu_1 a_3+25\mu_{11}a_9+25\mu_{12}a_2a_3}{(1+s_1)\cdot(5+s_3)}+\frac{5\mu_1 a_4+25\mu_{11}a_{11}+25\mu_{12}a_2a_4}{(1+s_1)\cdot(1+s_3)}+$$

$$+\frac{25\mu_{11}a_{10}}{(1+s_1)\cdot(5+s_3)^2}+\frac{25\mu_{12}a_1a_3}{(5+s_1)\cdot(5+s_3)}+\frac{25\mu_{12}a_1a_4}{(5+s_1)\cdot(1+s_3)} \tag{5-9}$$

Now we can apply, as in the Gated model, the known inverse Laplace transforms to each of the members in the sum (see equation (3-3)), in order to obtain the density function. Some of the terms in (5-9) lack the s_2 variable. Hence, as was done previously (see section 3.2), we use the $\delta(\cdot)$ function and the values a_1 to a_{12} to get

$$f_{\Gamma_2\ ,\theta_2\ ,\Psi_2}(t_1,t_2,t_3)=L^{-1}\left\{\widetilde{\Delta_2}(s_1,s_2,s_3)\right\}=f_1(t_1,t_2,t_3)+\delta(t_2)\cdot f_2(t_1,t_3) \tag{5-10}$$

where

$$f_1(t_1,t_2,t_3)=e^{-t_1-5t_2-t_3}\cdot\left[(5\mu_1+12.5\mu_{12})\,t_3+25\mu_{11}t_2t_3-\frac{25}{16}\mu_{12}\right]+$$

$$+e^{-5t_1-5t_2-t_3}\cdot\left[-\frac{25}{4}t_3\mu_{12}\right]+e^{-t_1-5t_2-5t_3}\cdot\left[\frac{25}{16}\mu_{12}\right] \tag{5-11}$$

and $f_2(t_1,t_3)=e^{-t_1-t_3}\cdot\left[\left(-\frac{5}{16}\mu_1-\frac{25}{32}\mu_{11}-\frac{25}{64}\mu_{12}\right)+t_3\left(\mu_0+\frac{5}{2}\mu_1+\frac{25}{8}\mu_{11}+\frac{25}{16}\mu_{12}\right)\right]+$

$$+e^{-5t_1-t_3}\cdot\left[t_3\left(-\frac{5}{4}\mu_1-\frac{25}{16}\mu_{11}-\frac{25}{16}\mu_{12}\right)-\frac{25}{4}\mu_{11}t_3t_1+\frac{25}{64}\mu_{12}\right]+$$

$$+e^{-t_1-5t_3}\cdot\left[\left(\frac{5}{16}\mu_1+\frac{25}{32}\mu_{11}+\frac{25}{64}\mu_{12}\right)+\frac{25}{16}\mu_{11}t_3\right]+e^{-5t_1-5t_3}\cdot\left[-\frac{25}{64}\mu_{12}\right] \tag{5-12}$$

Now, substituting (5-10) into equation (5-5) leads to

$$\widehat{X_1^i}(z)=\sum_{x=0}^{\infty}z^x\cdot\int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty}\int_{t_3=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\begin{array}{c}P\left(\ R_i\left(k,t_3\ \right)=x\ \right)\cdot P\left(R_i(0,t_1)=j\ \right)\cdot\\ \cdot\dfrac{e^{-\lambda_i t_2}\cdot(\lambda_i t_2)^{k-j}}{(k-j)!}\cdot f_1(t_1,t_2,t_3)dt_3dt_2dt_1\end{array}\right]+$$

$$+\sum_{x=0}^{\infty}z^x\cdot\int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty}\int_{t_3=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\begin{array}{c}P\left(\ R_i\left(k,t_3\ \right)=x\ \right)\cdot P\left(R_i(0,t_1)=j\ \right)\cdot\\ \cdot\dfrac{e^{-\lambda_i t_2}\cdot(\lambda_i t_2)^{k-j}}{(k-j)!}\cdot\delta(t_2)\cdot f_2(t_1,t_3)dt_3dt_2dt_1\end{array}\right] \tag{5-13}$$

Using property (3-14) of $\delta(t)$ we have

$$\int_{t=0}^{\infty}e^{-\lambda_1 t}\cdot\delta(t)dt=1\quad\text{and}\quad\int_{t=0}^{\infty}\frac{e^{-\lambda_1 t}(\lambda_1 t)^k}{k!}\cdot\delta(t)dt=0\quad\text{for } k>0$$

Thus,

$$\int_{t_2=0}^{\infty}\sum_{j=0}^{k}\left[P\left(\ R_i\left(k,t_3\ \right)=x\ \right)\cdot P\left(R_i(0,t_1)=j\ \right)\cdot\frac{e^{-\lambda_i t_2}\cdot(\lambda_i t_2)^{k-j}}{(k-j)!}\cdot\delta(t_2)dt_2\right]=$$

$$=\sum_{j=0}^{k-1}\left[P\left(\ R_i\left(k,t_3\ \right)=x\ \right)\cdot P\left(R_i(0,t_1)=j\ \right)\cdot\int_{t_2=0}^{\infty}\frac{e^{-\lambda_i t_2}\cdot(\lambda_i t_2)^{k-j}}{(k-j)!}\cdot\delta(t_2)dt_2\right]$$

$$+P\left(\, R_i\left(k,t_3\,\right)=x\,\right)\cdot P\left(R_i(0,t_1)=k\,\right)\cdot\int\limits_{t_2=0}^{\infty}e^{-\lambda_i t_2}\cdot\delta(t_2)dt_2$$

$$=P\left(\, R_i\left(k,t_3\,\right)=x\,\right)\cdot P\left(R_i(0,t_1)=k\,\right) \tag{5-14}$$

By substituting (5-14) in (5-13), we get,

$$\widehat{X_1^i}(z)=\sum_{x=0}^{\infty}z^x\cdot\int\limits_{t_1=0}^{\infty}\int\limits_{t_2=0}^{\infty}\int\limits_{t_3=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\begin{array}{c} P\left(\, R_i\left(k,t_3\,\right)=x\,\right)\cdot P\left(R_i(0,t_1)=j\,\right)\cdot \\ \cdot\dfrac{e^{-\lambda_i t_2}\cdot(\lambda_i t_2)^{k-j}}{(k-j)!}\cdot f_1(t_1,t_2,t_3)dt_3dt_2dt_1\end{array}\right]+$$

$$+\sum_{x=0}^{\infty}z^x\cdot\int\limits_{t_1=0}^{\infty}\int\limits_{t_3=0}^{\infty}\sum_{k=0}^{\infty}P\left(\, R_i\left(k,t_3\,\right)=x\,\right)\cdot P\left(R_i(0,t_1)=k\,\right)\cdot f_2(t_1,t_3)dt_3dt_1 \tag{5-15}$$

Now, from equation (5-6),

$$\widehat{X_1^i}(z)=G_1(z,1,1)=\mu_0+\mu_1\left(z+2\right)+\mu_{11}\left(z^2+2\right)+\mu_{12}\left(2z+1\right) \tag{5-16}$$

By substituting $\underline{z}=1$ in (5-16), we get the normalized equation in the unknown parameters μ_0, μ_1, μ_{11} and μ_{12}:

$$1=\mu_0+3(\mu_1+\mu_{11}+\mu_{12})$$

Equating (5-15) to (5-16), we get the main equation:

$$\mu_0+\mu_1\left(z+2\right)+\mu_{11}\left(z^2+2\right)+\mu_{12}\left(2z+1\right)=$$

$$=\sum_{x=0}^{\infty}z^x\cdot\int\limits_{t_1=0}^{\infty}\int\limits_{t_2=0}^{\infty}\int\limits_{t_3=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left[\begin{array}{c} P\left(\, R_i\left(k,t_3\,\right)=x\,\right)\cdot P\left(R_i(0,t_1)=j\,\right)\cdot \\ \cdot\dfrac{e^{-\lambda_i t_2}\cdot(\lambda_i t_2)^{k-j}}{(k-j)!}\cdot f_1(t_1,t_2,t_3)dt_3dt_2dt_1\end{array}\right]$$

$$+\sum_{x=0}^{\infty}z^x\cdot\int\limits_{t_1=0}^{\infty}\int\limits_{t_3=0}^{\infty}\sum_{k=0}^{\infty}P\left(\, R_i\left(k,t_3\,\right)=x\,\right)\cdot P\left(R_i(0,t_1)=k\,\right)\cdot f_2(t_1,t_3)dt_3dt_1 \tag{5-17}$$

where, in our example, $f_1(t_1,t_2,t_3)$ and $f_2(t_1,t_3)$ are given in (5-11) and (5-12), respectively. Also, since we chose $\lambda=\gamma$, in equation (3-1) the infinite sum in the expression of $P(R(k,t)=j)$ is zero. Hence,

$$P\left(R_i\left(k,t_3\,\right)=x\right)=e^{-2t_3}\cdot\left[I_{|x-k|}\left(2t_3\right)+I_{x+k+1}(2t_3)\right]$$

and

$$P\left(R_i(0,t_1)=j\right)=e^{-2t_1}\cdot\left[I_j\left(2t_1\right)+I_{j+1}(2t_1)\right]$$

Three distinct values of z have to be substituted in (5-17) in order to solve for the unknown parameters μ_0, μ_1, μ_{11} and μ_{12}. Finally, we get from (5-16):

$$E\left[X_1^i\right]=\frac{d}{dz}\widehat{X_1^i}(z)|_{z=1}=\mu_1+2\mu_{11}+2\mu_{12} \tag{5-18}$$

5.2 Waiting times

We imitate the method used in sections 3.3 and 4.2. Employing Eisenberg's formula we have in the current case, $\underline{B_i}(z) = \widehat{X_i^i}(z)$; $\overline{B_i}(z) = E[z^{N_i}] \equiv \widehat{N_i}(z)$; $E[\overline{K_i}] = E[X_1^i]$ and $\overline{S_i}(z) = \underline{S_i}(z) \cdot \widetilde{M_i}(\lambda_i(1-z)) \cdot \frac{1}{z}$.
Substituting the above leads to the following expression for the PGF of the queue size at service termination instants:

$$\overline{S_i}(z) = \frac{\widetilde{M_i}(\lambda_i(1-z))}{z - \widetilde{M_i}(\lambda_i(1-z))} \cdot \frac{\widehat{X_i^i}(z) - \widehat{N_i}(z)}{E[X_1^i]} \tag{5-19}$$

Now,

$$\begin{aligned}
\widehat{X_i^i}(z) &= \sum_{x=0}^{\infty} z^x \cdot P(X_i^i = x) = \sum_{x=0}^{\infty} z^x \cdot P\big(X_1^i + R_i(0,\Gamma_i) = x\big) \\
&= \int_{t=0}^{\infty} \sum_{x=0}^{\infty} z^x \cdot P\big(X_1^i + R_i(0,t) = x\big) \cdot f_{\Gamma_i}(t)dt \\
&= \int_{t=0}^{\infty} \sum_{x=0}^{\infty} z^x \cdot \sum_{j=0}^{x} P(X_1^i = j) \cdot P\big(R_i(0,t) = x-j\big) \cdot f_{\Gamma_i}(t)dt
\end{aligned} \tag{5-20}$$

where $f_{\Gamma_i}(t) = L^{-1}\{\widetilde{\Gamma}_i(s)\}$. Using (5-3),

$$\widetilde{\Gamma}_i(s) = E\left[e^{-s\Gamma_i}\right] = E\left[\exp\left\{-s\sum_{j=1}^{i-1}(\theta_j + D_j)\right\}\right] = E\left[\exp\left\{-s\sum_{j=1}^{i-1}(\theta_j)\right\}\right] \cdot E\left[\exp\left\{-s\sum_{j=1}^{i-1}(D_j)\right\}\right]$$

$$= \widetilde{\theta}(s,\ldots,s,0,\ldots,0) \cdot \prod_{j=1}^{i-1} \widetilde{D_j}(s) = G_1(\widetilde{M_1}(s),\ldots,\widetilde{M}_{i-1}(s),1,\ldots,1) \cdot \prod_{j=1}^{i-1} \widetilde{D_j}(s) \tag{5-21}$$

We present the calculation of $\widetilde{\Gamma}_i(s)$ for $i = 2$, using the parameter values of numerical example 3.

$$\widetilde{\Gamma_2}(s) = G_1(\widetilde{M_1}(s),1,1) \cdot \widetilde{D_1}(s) =$$

$$= \frac{\mu_0 + \frac{13}{4}\mu_1 + \frac{57}{16}\mu_{11} + \frac{14}{4}\mu_{12}}{(1+s)} - \frac{\frac{5}{4}\mu_1 + \frac{10}{4}\mu_{12} + \frac{25}{16}\mu_{11}}{(5+s)} - \frac{\frac{25}{4}\mu_{11}}{(5+s)^2} \tag{5-22}$$

Hence,

$$\begin{aligned}
f_{\Gamma_2}(t) = L^{-1}\left\{\widetilde{\Gamma_2}(s)\right\} &= e^{-t} \cdot \left(\mu_0 + \frac{13}{4}\mu_1 + \frac{57}{16}\mu_{11} + \frac{14}{4}\mu_{12}\right) \\
&\quad - e^{-5t} \cdot \left(\frac{5}{4}\mu_1 + \frac{10}{4}\mu_{12} + \frac{25}{16}\mu_{11}\right) - te^{-5t} \cdot \left(\frac{25}{4}\mu_{11}\right)
\end{aligned} \tag{5-23}$$

Also,

$$P(X_1^2 = j) = \frac{1}{j!} \cdot \frac{d^{(j)}\widehat{X_1^2}(z)}{d^{(j)}z}|_{z=0} \tag{5-24}$$

where $\widehat{X_1^2}(z)$ is given by(5-16). We substitute equations (5-23) and (5-24) in (5-20) to get an expression for $\widehat{X_2^2}(z)$.

Now,

$$\widehat{N_2}(z) = \sum_{x=0}^{\infty} z^x \cdot P(N_2 = x) = \sum_{x=0}^{\infty} z^x \cdot P(R_2(0,\Gamma_2) + A_2(\theta_2) = x)$$

$$= \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \sum_{x=0}^{\infty} z^x \cdot P(R_2(0,t_1) + A_2(t_2) = x) \cdot f_{\Gamma_2\ ,\theta_2}\ (t_1,t_2)dt_2dt_1$$

$$= \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \sum_{x=0}^{\infty} z^x \cdot \sum_{j=0}^{x} P(R_2(0,t_1) = j) \cdot P\left(A_2(t_2) = x - j\right) \cdot f_{\Gamma_2\ ,\theta_2}\ (t_1,t_2)dt_2dt_1$$

$$= \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \sum_{x=0}^{\infty} z^x \cdot \sum_{j=0}^{x} P(R_2(0,t_1) = j) \cdot \frac{e^{-\lambda_2 t_2} \cdot (\lambda_2 t_2)^{x-j}}{(x-j)!} \cdot f_{\Gamma_2\ ,\theta_2}\ (t_1,t_2)dt_2dt_1 \tag{5-25}$$

where $f_{\Gamma_2\ ,\theta_2}\ (t_1,t_2) = L^{-1}\left\{\widetilde{\Omega_2}(s_1,s_2)\right\}$ and

$$\widetilde{\Omega_2}(s_1,s_2) = E\left[e^{-s_1\Gamma_2} \cdot e^{-s_2\theta_2}\right] = E\left[e^{-s_1(\theta_1+D_1)-s_2\theta_2}\right]$$

$$= E\left[e^{-s_1\theta_1-s_2\theta_2}\right] \cdot E\left[e^{-s_1 D_1}\right] = \widetilde{\theta}(s_1,s_2,0) \cdot \widetilde{D_1}(s_1) = G_1\left(\widetilde{M_1}(s_1),\widetilde{M_2}(s_2),1\right) \cdot$$

$$\widetilde{D_1}(s_1)$$

$$= \frac{\mu_0 + \frac{9}{4}\mu_1 + \frac{41}{16}\mu_{11} + \frac{5}{4}\mu_{12}}{(1+s_1)} - \frac{\frac{5}{4}\mu_1 + \frac{5}{4}\mu_{12} - \frac{25}{16}\mu_{11}}{(5+s_1)} - \frac{\frac{25}{4}\mu_{11}}{(5+s_1)^2} +$$

$$+ \frac{5\mu_1 + 5\mu_{12} + \frac{25}{4}\mu_{12}}{(5+s_2)\cdot(1+s_1)} + \frac{25\mu_{11}}{(5+s_2)^2\cdot(1+s_1)} - \frac{\frac{25}{4}\mu_{12}}{(5+s_2)\cdot(5+s_1)} \tag{5-26}$$

Since the first three terms in (5-26) lack the variable s_2 we write the density function $f_{\Gamma_2\ ,\theta_2}\ (t_1,t_2) = L^{-1}\left\{\widetilde{\Omega_2}(s_1,s_2)\right\}$ as a sum of two functions using the function $\delta(t)$.

$$f_{\Gamma_2\ ,\theta_2}\ (t_1,t_2) = f_1(t_1,t_2) + \delta(t_2)\cdot f_2(t_1) \tag{5-27}$$

where

$$f_1(t_1,t_2) = e^{-t_1-5t_2} \cdot \left[5\mu_1 + \frac{45}{4}\mu_{12} + 25\mu_{11}t_2\right] - e^{-5t_1-5t_2} \cdot \left[\frac{25}{4}\mu_{12}\right] \tag{5-28}$$

and

$$f_2(t_1) = e^{-t_1} \cdot \left[\mu_0 + \frac{9}{4}\mu_1 + \mu_{11} + \frac{5}{4}\mu_{12}\right] - e^{-5t_1} \cdot \left[\frac{5}{4}\mu_1 + \frac{5}{4}\mu_{12} - \frac{25}{4}\mu_{11}t_1\right] \tag{5-29}$$

Now, by substituting (5-27) and the probability $P(R_2(0,t_1)=j)$ (from (3-1)) in (5-25) we get $\widehat{N_2}(z)$. By substituting $\widehat{N_2}(z)$ and $\widehat{X_2^2}(z)$ in (5-19) we get $\overline{S_2}(z)$, the PGF of the queue size at service termination instants.

Our goal is to derive $\widehat{L_i}(z)$, the PGF of the queue size in station i at an arbitrary point of time. Following the definitions in section 3.3 we have,

$$\widehat{L_i}(z) = S_i(z)\cdot p_i + V_i(z)\cdot(1-p_i) \tag{5-30}$$

For the value of p_i we need:

$$E[C] = E\left[\sum_{j=1}^{N}(\theta_j + D_j)\right] = E\left[\sum_{j=1}^{N}\left(\sum_{k=1}^{X_1^j} M_{jk} + D_j\right)\right] = \sum_{j=1}^{N} E[X_1^j]\cdot m_j + d$$

Then,

$$p_i = \frac{E[\theta_i]}{E[C]} = \frac{E[X_1^i]\cdot m_i}{\sum\limits_{j=1}^{N} E[X_1^j]\cdot m_j + d} \tag{5-31}$$

Equation (3-31) holds here too: $S_i(z)\cdot g_i(z)\cdot \frac{1}{z} = \overline{S_i}(z)$, where $g_i(z)$ is given in section 3.3. That is,

$$S_i(z) = \frac{\overline{S_i}(z)\cdot z\cdot \lambda_i(1-z)m_i}{1-\widetilde{M_i}(\lambda_i(1-z))}$$

Thus, by substituting $\overline{S_i}(z)$ from (5-19) we obtain:

$$S_i(z) = \frac{\widetilde{M_i}(\lambda_i(1-z))}{z-\widetilde{M_i}(\lambda_i(1-z))}\cdot\frac{z\cdot\lambda_i(1-z)m_i}{1-\widetilde{M_i}(\lambda_i(1-z))}\cdot\frac{\widehat{X_i^i}(z)-\widehat{N_i}(z)}{E[X_1^i]} \tag{5-32}$$

<u>Derivation of $V_i(z)$</u>

Recall that V_i stands for the number of customers in queue i at an arbitrary moment when queue i is not being served. The following relation holds:

$$V_i \stackrel{d}{=} R_i(N_i, H_{i,p}) = R_i(R_i(0,\Gamma_i) + A_i(\theta_i), H_{i,p}) \tag{5-33}$$

where $H_{i,p}$ is the past time (see section 3.3) of the random variable H_i (recall that $C = H_i + \theta_i$). We have,

$$V_i(z) = E[z^{V_i}] = \sum_{v=0}^{\infty} z^v\cdot P(V_i = v) = \sum_{v=0}^{\infty} z^v\cdot P\left(R_i\left(R_i(0,\Gamma_i)+A_i(\theta_i)\ ,\ H_{i,\,p}\right)=v\right)$$

$$= \int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty}\int_{t_3=0}^{\infty}\sum_{v=0}^{\infty} z^v\cdot P\left(R_i\left(R_i(0,t_1)+A_i(t_2)\ ,\ H_{i,\,p}\right)=v \mid \Gamma_i = t_1\ ,\ \theta_i = t_2\ ,\ \Psi_i = t_3\right)\cdot$$

$$\cdot f_{\Gamma_i\ ,\theta_i\ ,\Psi_i}(t_1,t_2,t_3)dt_3dt_2dt_1 \tag{5-34}$$

Note that when $\Gamma_i = t_1$, $\theta_i = t_2$, $\Psi_i = t_3$, we get that $H_i = \Gamma_i + \Psi_i = t_1 + t_3$ and $H_{i,p}|H_i = t_1 + t_3 \sim U(0, t_1+t_3)$ (see section 3.3).

Using this distribution, while defining a continuous Uniform variable $h_p \sim U(0, t_1+t_3)$, with density function $\frac{1}{t_1+t_3}$, in (5-34) yields,

$$V_i(z) = \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \int_{t_3=0}^{\infty} \sum_{v=0}^{\infty} z^v \int_{u=o}^{t_1+t_3} P\left(R_i\left(R_i(0,t_1)+A_i(t_2)\ ,\ u\right)=v\right) \cdot \frac{1}{t_1+t_3} \cdot$$

$$\cdot f_{\Gamma_i\ ,\theta_i\ ,\Psi_i}(t_1,t_2,t_3)dudt_3dt_2dt_1 =$$

$$= \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \int_{t_3=0}^{\infty} \sum_{v=0}^{\infty} z^v \int_{u=o}^{t_1+t_3} \sum_{j=0}^{\infty} P\left(\ R_i\left(j,u\right)=v\right) \cdot P\left(R_i(0,t_1)+A_i(t_2)=j\right) \cdot$$

$$\cdot \frac{1}{t_1+t_3} \cdot f_{\Gamma_i\ ,\theta_i\ ,\Psi_i}(t_1,t_2,t_3)dudt_3dt_2dt_1 =$$

$$= \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \int_{t_3=0}^{\infty} \sum_{v=0}^{\infty} z^v \int_{u=0}^{t_1+t_3} \sum_{j=0}^{\infty} P\left(\ R_i\left(j,u\right)=v\right) \cdot \sum_{k=0}^{j} P\left(R_i(0,t_1)=k\ \right) \cdot \frac{e^{-\lambda_i t_2} \cdot (\lambda_i t_2)^{j-k}}{(j-k)!} \cdot$$

$$\cdot \frac{1}{t_1+t_3} \cdot f_{\Gamma_i,\theta_i\ ,\Psi_i}(t_1,t_2,t_3)dudt_3dt_2dt_1 \tag{5-35}$$

Now, $V_i(z)$ is determined by using $f_{\Gamma_i,\theta_i,\Psi_i}(t_1,t_2,t_3)$ from equation (5-10).

Substituting the results (5-31), (5-32) and (5-35) in (5-30) we get an expression for $\widehat{L}_i(z)$, the PGF of queue $i\ 's$ size at an arbitrary moment.

Note that $\widehat{L}_i(z)$ depends on the distribution of X_1^i which was derived under the interpolation theory. As in the Gated case, in a specific queue the number of customers left behind by a departing customer is exactly those that arrived there during his sojourn time. Thus, we get for the waiting time of a customer in queue i:

$$\widehat{L}_i(z) = \widetilde{W_i}(\lambda_i(1-z)) \quad \text{or} \quad \widetilde{W_i}(s) = \widehat{L}_i(1-\frac{s}{\lambda_i}).$$

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