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POLLING SYSTEMS WITH PERMANENT AND TRANSIENT JOBS

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ABSTRACT

Almost all polling models studied in the literature deal with 'open' systems where all jobs are *transient*, i.e., they arrive to, are served by, and leave the system. Recently Altman & Yechiali [1994] introduced and analyzed models for 'closed' polling networks in which a *fixed* number of *permanent* jobs always reside in the system, such that each job, after being served in one station, is immediately (and randomly) routed to another station where it waits to be served again by the rotating server. Boxma and Cohen [1991] analyzed a single-node $M/G/1$ configuration with regular (transient) Poisson jobs and with a fixed number of permanent jobs who immediately return to the end of the queue each time they receive a service.

In this work we study *hybrid* multi-node polling systems with *both* transient and permanent jobs, operated under the Gated, Exhaustive, or Globally-Gated service regime. We define the laws of motion governing the evolution of such systems and derive the multi-dimensional generating functions of the number of jobs at the various queues at polling instants, and at arbitrary points in time. For each regime we investigate the interaction between the two populations of jobs and derive formulae for the means, as well as expressions for calculating the

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second moments, of the number of jobs at the various queues. Waiting times and throughput rates are calculated and the systems are compared with each other.

Keywords: polling. open and closed systems. transient and permanent jobs.

1. INTRODUCTION

Polling systems are queueing networks where N stations (queues, channels) are attended by a single server that moves, following some 'routing table' (most commonly, cyclic), between the stations and serves the jobs in each queue according to some service discipline. Due to their wide range of application, these systems have been studied extensively in the literature (cf. Takagi [1986], [1990], Browne and Yechiali [1989], Levy and Sidi [1990], Yechiali [1993] and many others). However, almost all investigated models dealt with 'open' networks where hexogenous streams of *transient* jobs flow into a system, get served and leave never to return. Recently Altman and Yechiali [1994] analyzed pure 'closed' polling systems in which a fixed number of *permanent* jobs always reside in a system in a way that each job, after being served in one station, is immediately routed (randomly) to another station where it waits to be served again by the rotating server, etc. Boxma and Cohen (1991] studied a single-node $M/G/1$ queue with the ordinary transient jobs and with K permanent jobs who immediately return to the end of the queue each time they receive a service. In particular they showed that, for the case where the service times of the Poisson jobs and the permanent ones are all negative exponential with *identical* means, the queue length and sojourn time distributions of the Poisson (transient) jobs are the $(K+1)$-fold convolutions of those for the case *without* permanent jobs.

In this work we study *hybrid* Polling systems with *both* transient and permanent jobs, operated under the Gated, Exhaustive, or Globally-Gated service regime. We define the laws of motion governing the evolution of such systems and derive the multi-dimensional generating functions of the number of jobs at the various queues at polling instants and at arbitrary points in time. For each regime we investigate the interaction between the two populations of jobs and derive formulae for the means, as well as expressions for calculating the second moments, of the number of jobs at the various queues. Waiting times and throughput rates are calculated and the systems are compared with each other.

We begin section 2 with a description of the assumptions leading to the hybrid model, and introduce the required notation. Section 3 deals with the Gated service regime. We derive a set of generating functions (GFs) from which we obtain explicit formulae for the mean number of transient, as well as permanent, jobs residing in the system at the various queues at polling instants. We also calculate the mean cycle time and the *throughput* of such a system. Section 4 is devoted to the analysis of the Exhaustive service regime. Again, we derive a set of multi-dimensional GFs, from which we obtain explicit formulae for the mean number of jobs (transient and permanent) at the various queues at polling instants and calculate the throughput. The Globally-Gated service regime (introduced by Boxma, Levy and Yechiali [1992]) is studied in section 5. The GF of the system-state (i.e. number of jobs) at the start of a cycle is derived and the Laplace Stieltjes Transform (LST) of the cycle duration is expressed in terms of this GF. Mean numbers of transient, as well as of permanent, jobs at the various stations are explicitly calculated, together with the mean cycle time and throughput. For a special case where the set of N stations is partitioned into two separate sets of 'transient only' and 'permanent only', we derive expressions for the mean waiting time of an arbitrary transient job in a given station. Section 6 is devoted to calculating second moments for all 3 regimes. We conclude with section 7, where, for each regime, the GFs and moments of the numbers of jobs at an *arbitrary* instant are obtained.

2. MODEL AND NOTATION

A single server attends N stations (channels, queues, nodes), denoted by $Q_1, Q_2, \ldots, Q_i, \ldots, Q_N$, in a cyclic manner and reside in each queue according to some service discipline (e.g. Gated, Exhaustive or Globally-Gated). External jobs flow into Q_i according to a homogeneous Poisson process with intensity λ_i, $i = 1, 2, \ldots, N$. These jobs, after being served, leave the system never to return, and are therefore called *transient*. In addition, there are M *permanent* jobs that always reside in the system, each one randomly routed from one queue to another every time it completes a service.

Service time of a transient job served in Q_i is a random variable B_i with mean b_i, second moment $b_i^{(2)}$ and LST $\widetilde{B}_i(s) \equiv E\left[e^{-sB_i}\right]$. The corresponding characteristics of a permanent job served in Q_i are $B_i^P, \beta_i, \beta_i^{(2)}$ and $\widetilde{B}_i^P(s)$, respectively. A permanent job, after being served in Q_i, is immediately routed to Q_j ($j = 1, 2, \ldots, N$) with probability P_{ij}, $\sum_{j=1}^{N} P_{ij} = 1$. The routed job then waits in Q_j to be served once more by the moving server (requiring service time duration B_j^P) and rerouted further to another queue, etc. Transient and permanent jobs in each queue are mixed and their service order follows the First Come, First Served (FCFS) discipline.

When moving from station i to station $i+1$, the server incurs a random switch-over time D_i, having mean d_i, second moment $d_i^{(2)}$ and LST $\widetilde{D}_i(s)$. Total switching time during a full cycle is $D = \sum_{i=1}^{N} D_i$, with mean $d = \sum_{i=1}^{N} d_i$, second moment $d^{(2)}$ and LST $\widetilde{D}(s)$.

We invoke the usual independence assumptions between stations, arrival processes, service times and switch-over times, and use the following

Notation.

$A_j(t)$ = number of external (transient) Poisson arrivals to station j during a time interval of length t.

Polling instant = the moment when the server 'enters' a station.

X_i^j = number of transient jobs at Q_j when Q_i is polled.

Y_i^j = number of permanent jobs at Q_j when Q_i is polled.

$(\underline{X}_i;\underline{Y}_i) = \left(X_i^1, X_i^2, \ldots, X_i^j, \ldots, X_i^N; Y_i^1, Y_i^2, \ldots, Y_i^j, \ldots, Y_i^N\right)$ = system state at a polling instant to station i (in steady state).

$f_i^j = E[X_i^j]$, $g_i^j = E[Y_i^j]$.

C = Cycle time.

$K_i^j(Y_i^i)$ = number of permanent jobs, out of Y_i^i, routed from Q_i to Q_j while the server is visiting station i.

3. THE GATED REGIME

Under the Gated service regime, at each visit to a station, the server attends only those jobs, transient and permanent, residing in the queue at the polling instant.

Laws of Motion.

The laws of motion describing the evolution of the system-state when the server moves from Q_i to Q_{i+1} are given by

$$X_{i+1}^j = \begin{cases} X_i^j + A_j\left(\sum\limits_{k=1}^{X_i^i} B_{ik} + \sum\limits_{m=1}^{Y_i^i} B_{im}^P + D_i\right), & i \neq j \\ A_i\left(\sum\limits_{k=1}^{X_i^i} B_{ik} + \sum\limits_{m=1}^{Y_i^i} B_{im}^P + D_i\right), & i = j \end{cases} \tag{3.1}$$

$$Y_{i+1}^j = \begin{cases} Y_i^j + K_i^j(Y_i^i), & i \neq j \\ K_i^i(Y_i^i), & i = j \end{cases} \tag{3.2}$$

Generating Functions.

The multi-dimensional GF of the system state, $(\underline{X}_i;\underline{Y}_i)$ at polling instant to Q_i ($i = 1, 2, \ldots, N$) is defined as

$$G_i(\underline{z};\underline{\omega}) = E\left[\prod_{j=1}^{N} z_j^{X_i^j} \cdot \prod_{j=1}^{N} \omega_j^{Y_i^j}\right] \tag{3.3}$$

Define also

$$BXB^PYD_i \equiv \sum_{k=1}^{X_i^i} B_{ik} + \sum_{m=1}^{Y_i^i} B_{im}^P + D_i; \quad BX_i \equiv \sum_{k=1}^{X_i^i} B_{ik}; \quad B^PY_i \equiv \sum_{m=1}^{Y_i^i} B_{im}^P$$

Then, by using (3.1) and (3.2), we have

$$\begin{aligned}
G_{i+1}(\underline{z};\underline{\omega}) &= E\left[\left(\prod_{\substack{j=1\\j\neq i}}^{N} z_j^{X_i^j}\right)\cdot\left(\prod_{j=1}^{N} z_j^{A_j(BXB^PYD_i)}\right)\cdot\left(\prod_{\substack{j=1\\j\neq i}}^{N} \omega_j^{Y_i^j}\right)\left(\prod_{j=1}^{N}\omega_j^{K_i^j(Y_i^i)}\right)\right]\\
&= E\left[\left(\prod_{\substack{j=1\\j\neq i}}^{N} z_j^{X_i^j}\right)\cdot\left(\prod_{j=1}^{N} z_j^{A_j(BX_i)}\right)\left(\prod_{j=1}^{N} z_j^{A_j(B^PY_i)}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{N} \omega_j^{Y_i^j}\right)\left(\prod_{j=1}^{N}\omega_j^{K_i^j(Y_i^i)}\right)\left(\prod_{j=1}^{N} z_j^{A_j(D_i)}\right)\right]\\
&= E\Bigg[\left(\prod_{\substack{j=1\\j\neq i}}^{N} z_j^{X_i^j}\right)\left(\widetilde{B}_i\Big(\sum_{j=1}^{N}\lambda_j(1-z_j)\Big)\right)^{X_i^i}\left(\widetilde{B}_i^P\Big(\sum_{j=1}^{N}\lambda_j(1-z_j)\Big)\right)^{Y_i^i}\Big(\prod_{\substack{j=1\\j\neq i}}^{N}\omega_j^{Y_i^j}\Big)\\
&\qquad\cdot\left(\prod_{j=1}^{N}\omega_j^{K_i^j(Y_i^i)}\right)\Bigg]\cdot\widetilde{D}_i\left(\sum_{j=1}^{N}\lambda_j(1-z_j)\right) \qquad (3.4)\\
&= E\left[\left(\prod_{\substack{j=1\\j\neq i}}^{N} z_j^{X_i^j}\right)\left(\widetilde{B}_i(\delta(\underline{z}))\right)^{X_i^i}\left(\widetilde{B}_i^P(\delta(\underline{z}))\right)^{Y_i^i}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\omega_j^{Y_i^j}\right)\left(\sum_{j=1}^{N}P_{ij}\omega_j\right)^{Y_i^i}\right]\cdot\widetilde{D}_i(\delta(\underline{z}))
\end{aligned}$$

where, $\quad \delta(\underline{z}) = \sum_{j=1}^{N}\lambda_j(1-z_j)$

Equation (3.4) implies:

$$G_{i+1}(\underline{z};\underline{\omega}) = G_i\Bigg(z_1,z_2,\ldots,z_{i-1},\widetilde{B}_i(\delta(\underline{z})),z_{i+1},\ldots,z_N;$$
$$\omega_1,\omega_2,\ldots,\omega_{i-1},\left[\widetilde{B}_i^P(\delta(\underline{z}))\cdot\left(\sum_{j=1}^{N}P_{ij}\omega_j\right)\right],\omega_{i+1},\ldots,\omega_N\Bigg)\cdot\widetilde{D}_i(\delta(\underline{z}))\,. \qquad (3.5)$$

First Moments.

Since

$$f_i^j = E[X_i^j] = \left.\frac{\partial G_i(\underline{z};\underline{\omega})}{\partial z_j}\right|_{\underline{z}=\underline{\omega}=\underline{1}} \quad\text{and}\quad g_i^j = E[Y_i^j] = \left.\frac{\partial G_i(\underline{z};\underline{\omega})}{\partial \omega_j}\right|_{\underline{z}=\underline{\omega}=\underline{1}} \qquad (3.6)$$

from (3.5) (also, from (3.1) and (3.2)) we obtain a set of $2N^2$ equations in the N^2 unknowns f_i^j and the N^2 unknowns g_i^j:

$$f_{i+1}^j = \begin{cases} f_i^j + \lambda_j\left[b_if_i^i + \beta_ig_i^i\right] + \lambda_jd_i & i\neq j\\ \lambda_i\left[b_if_i^i + \beta_ig_i^i\right] + \lambda_id_i & i=j\end{cases} \qquad (3.7)$$

$$g_{i+1}^j = \begin{cases} g_i^j + P_{ij}g_i^i & i\neq j\\ P_{ii}g_i^i & i=j\end{cases} \qquad (3.8)$$

It is important to observe that the solution of (3.8) for the permanent jobs is *independent* of the transient jobs. However, the solution of (3.7) does depend on (3.8).

From Altman and Yechiali [1994], we have

$$g_i^i = c \cdot \pi_i \tag{3.9}$$

where the π_i's are the limiting distribution of the routing Markov chain $[P_{ij}]$ (if such a limit does not exist, one may take the Cezaro limit). The π_i' are calculated from

$$\pi_j = \sum_{i=1}^{N} \pi_i P_{ij}, \quad j = 1, 2, \ldots, N; \quad \sum_{j=1}^{N} \pi_j = 1 \ .$$

The constant c is given by

$$c = \frac{M}{\sum_{k=1}^{N} \pi_k \sum_{j=1}^{k} P_{kj}} \tag{3.10}$$

and it represents physically the expected number of permanent jobs served within a cycle. This follows since $\sum_{i=1}^{N} g_i^i = c \sum_{i=1}^{N} \pi_i = c$.

For the transient jobs we get

$$f_i^i = \lambda_i E[C] \tag{3.11}$$

where the mean cycle time, $E[C]$, is given by

$$E[C] = \sum_{i=1}^{N} \left[b_i f_i^i + \beta_i g_i^i + d_i \right] \tag{3.12}$$

Using (3.11), the mean cycle time is expressed as

$$E[C] = \sum_{i=1}^{N} b_i \lambda_i E[C] + \sum_{k=1}^{N} \beta_i g_i^i + d \ .$$

With the aid of (3.9) it follows that

$$E[C] = \frac{\sum_{i=1}^{N} \beta_i g_i^i + d}{1 - \rho} = \frac{c \sum_{i=1}^{N} \beta_i \pi_i + d}{1 - \rho} \tag{3.13}$$

where $\rho_i = \lambda_i b_i$; $\rho = \sum_{i=1}^{N} \rho_i$.

Define ξ_i, the arrival intensity (and hence the throughput) of permanent jobs to (from) channel i, by $\xi_i = g_i^i \big/ E[C]$, and set $\rho_i^P = \xi_i \beta_i$, $\rho^P = \sum_{i=1}^{N} \rho_i^P$. Now,

$$E[C] = \rho E[C] + \sum_{i=1}^{N} g_i^i \beta_i + d$$

i.e.,

$$E[C] = \frac{d}{1-\rho-\rho^P} . \tag{3.14}$$

It should be noted, however, that the necessary and sufficient condition for steady state is $\rho < 1$ (*not* $\rho + \rho^P < 1$), since $d + \sum_{i=1}^{N} \beta_i g_i^i$ is the expected 'overhead' time per cycle (see (3.13)). Indeed,

$$\rho^P = \frac{c \sum_{i=1}^{N} \beta_i \pi_i}{d + c \sum_{i=1}^{N} \beta_i \pi_i} (1-\rho) . \tag{3.15}$$

Thus, $\rho^P < 1 - \rho$ whenever $\rho < 1$.

An explanation from the system's point of view is that, near saturation condition, the server works on transient jobs most of the time so that the existence of permanent jobs does not come into play in the stability condition.

As for the $g_i^j = E[Y_i^j]$, from Altman and Yechiali [1994]:

$$g_i^j = c \sum_{k=j}^{i-1} P_{kj} \pi_k . \tag{3.16}$$

Finally, for the transient jobs we get

$$f_i^j = \lambda_j \left[\sum_{k=j}^{i-1} \left[(\rho_k + \rho_k^P) E[C] + d_k \right] \right] . \tag{3.17}$$

Result (3.17) is readily verified by substituting in (3.7).

Throughput.

The mean number of jobs processed per unit time is

$$\Lambda^G = \frac{\sum_{i=1}^{N} f_i^i + \sum_{i=1}^{N} g_i^i}{E[C]} = \lambda + \frac{c}{E[C]} = \lambda + \frac{1-\rho}{\sum_{i=1}^{N} \pi_i \beta_i + d/c} . \tag{3.18}$$

where $\lambda = \sum_{i=1}^{N} \lambda_i$. Thus, by (3.10),

$$\Lambda^G = \lambda + \frac{1-\rho}{\sum_{i=1}^{N} \pi_i \beta_i + \frac{d}{M} \sum_{k=1}^{N} \pi_k \sum_{j=1}^{k} P_{kj}} \tag{3.19}$$

It is interesting to observe that the *overall* throughput in a hybrid system is *larger* than λ, the arrival rate and throuput in a *pure* Gated System with *no* permanent jobs.

However, queue sizes and waiting times of transient jobs *increase* due to the presence of the permanent jobs. Indeed, $f_i^i = \lambda_i E[C]$, as in the pure Gated case, but $E[C]$ in the hybrid process is larger, i.e. $E[C(\text{hybrid})] > d/(1-\rho) = E[C(\text{pure Gated})]$.

Symmetric System.

In a fully *symmetric* system, where all stations and jobs are (probabilistically) identical, i.e. $P_{ij} = \frac{1}{N}$, $\pi_i = \frac{1}{N}$, $b_i = \beta_i = b_1$, $\lambda_i = \lambda_1$, we obtain

$$\Lambda^G = \lambda + \frac{1-\rho}{\frac{d}{M}\sum_{k=1}^{N}\frac{1}{N}\cdot\frac{k}{N} + b_1} = N\lambda_1 + \frac{1-\rho}{\frac{d}{M}\cdot\frac{N+1}{2N} + b_1} \tag{3.20}$$

4. THE EXHAUSTIVE REGIME

Under the Exhaustive service regime, the server leaves a visited station only when there is no work left and the queue is empty.

Laws of Motion.

The law of motion for transient jobs under the Exhaustive regime is

$$X_{i+1}^j = \begin{cases} X_i^j + A_j\left(\sum_{k=1}^{X_i^i}\theta_{ik} + \sum_{m=1}^{\widehat{Y}_i^i}\Delta_{im} + D_i\right) & i \neq j \\ A_i(D_i) & i = j \end{cases} \tag{4.1}$$

where $\theta_{ik} \sim \theta_i$, and θ_i is a regular busy period in a regular $M/G/1$ queue with arrival rate λ_i and service times B_i. $\Delta_{ik} \sim \Delta_i$, where Δ_i is a *delay* busy period in an $M/G/1$ queue as above, but the first service time takes B_i^P units of time. It is well known that

$$E(\theta_i) = \frac{b_i}{1-\rho_i} \quad \text{and} \quad E(\Delta_i) = \frac{\beta_i}{1-\rho_i}\ .$$

$\widehat{Y}_i^i$ is a Negative-Binomial variable counting the number of Bernoulli trials required to obtain Y_i^i successes when the probability of success in each individual trial is $(1-P_{ii})$.

Note that (4.1) holds since each time a permanent job is served in queue i, it requires service time of length B_i^P.

The law of motion for *permanent* jobs is

$$Y_{i+1}^j = \begin{cases} Y_i^j + \widehat{K}_i^j(Y_i^i) & i \neq j \\ 0 & i = j \end{cases} \tag{4.2}$$

where $\widehat{K}_i^j(Y_i^i)$ is the number of jobs, out of Y_i^i, routed from i to j when the server visits station i under the Exhaustive regime.

Following the arguments in Altman & Yechiali [1994] we have,

$$P\left[\widehat{K}_i^j(Y_i^i) = m_j,\ j = 1,2,\ldots,N \middle| Y_i^i\right] = \frac{Y_i^i!}{\prod_{j=1}^{N} m_j!} \cdot \prod_{j=1}^{N} \widehat{P}_{ij}^{m_j}$$

where $\widehat{P}_{ij} = \begin{cases} P_{ij}/(1-P_{ii}), & i \neq j \quad (P_{ii} \neq 1) \\ 0, & i = j \end{cases}$

and $\sum_{j=1}^{N} m_j = Y_i^i$.

Generating Functions.

Define the multi-dimensional GF of the system-state vector $(\underline{X}_i; \underline{Y}_i)$ as in (3.3), i.e.,

$$G_i(\underline{z};\underline{\omega}) = E\left[\prod_{j=1}^{N} z_j^{X_i^j} \cdot \prod_{j=1}^{N} \omega_j^{Y_i^j}\right].$$

Then, by using the laws (4.1) and (4.2) and defining $\theta X\widehat{Y} \equiv \sum_{k=1}^{X_i^i} \theta_{ik} + \sum_{m=1}^{\widehat{Y}_i^i} \Delta_{im}$ we obtain,

$$G_{i+1}(\underline{z};\underline{\omega}) = E\left[\left(\prod_{\substack{j=1\\j\neq i}}^{N} z_j^{X_i^j}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{N} z_j^{A_j(\theta X\widehat{Y})}\right)\left(\prod_{j=1}^{N} z_j^{A_j(D_i)}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{N} \omega_j^{Y_i^j}\right)\left(\prod_{\substack{j=1\\j\neq i}}^{N} \omega_j^{\widehat{K}_i^j(Y_i^i)}\right)\right]$$

$$= E\left[\left(\prod_{\substack{j=1\\j\neq i}}^{N} z_j^{X_i^j}\right)\left(\widetilde{\theta}_i\left[\sum_{\substack{j=1\\j\neq i}}^{N}\lambda_j(1-z_j)\right]\right)^{X_i^i} \cdot \left(\widetilde{\Delta}_i\left[\sum_{\substack{j=1\\j\neq i}}^{N}\lambda_j(1-z_j)\right]\right)^{\widehat{Y}_i^i} \cdot \left(\prod_{\substack{j=1\\j\neq i}}^{N} \omega_j^{Y_i^j}\right)\right.$$

$$\left. \cdot \left(\sum_{\substack{j=1\\j\neq i}}^{N}\left(\frac{P_{ij}}{1-P_{ii}}\right)\omega_j\right)^{Y_i^i}\right] \cdot \widetilde{D}_i\left[\sum_{j=1}^{N}\lambda_j(1-z_j)\right]$$

Thus,

$$G_{i+1}(\underline{z};\underline{\omega}) = E\left[\left(\prod_{\substack{j=1\\j\neq i}}^{N} z_j^{X_i^j}\right) \cdot (\widetilde{\theta}_i(\gamma(\underline{z})))^{X_i^i} \cdot \right.$$

$$\left. \cdot \left(\frac{\widetilde{\Delta}_i(\gamma(\underline{z}))\cdot(1-P_{ii})}{1-\widetilde{\Delta}_i(\gamma(\underline{z}))P_{ii}}\right)^{Y_i^i}\left(\prod_{\substack{j=1\\j\neq i}}^{N}\omega_j^{Y_i^j}\right)\left(\sum_{\substack{j=1\\j\neq i}}^{N}\left(\frac{P_{ij}}{1-P_{ii}}\right)\omega_j\right)^{Y_i^i}\right] \cdot \widetilde{D}_i(\delta(\underline{z}))$$

where

$$\gamma(\underline{z}) = \sum_{\substack{j=1\\j\neq i}}^{N}\lambda_j(1-z_j) = \delta(\underline{z}) - \lambda_i(1-z_i)$$

Finally,

$$G_{i+1}(\underline{z};\underline{\omega}) = G_i\left(z_1, z_2, \ldots, z_{i-1}, \widetilde{\theta}_i(\gamma(\underline{z})), z_{i+1}, \ldots z_N;\right. \tag{4.3}$$

$$\left.\omega_1, \omega_2, \ldots, \omega_{i-1}, \left[\left(\frac{\widetilde{\Delta}_i(\gamma(\underline{z}))}{1-\widetilde{\Delta}_i(\gamma(\underline{z}))P_{ii}}\right)\left(\sum_{\substack{j=1\\j\neq i}}^{n} P_{ij}\omega_j\right)\right], \omega_{i+1}, \ldots, \omega_N\right) \cdot \widetilde{D}_i(\delta(\underline{z}))$$

First Moments.

From (4.1) and (4.2), or from (4.3), we obtain

$$f_{i+1}^j = \begin{cases} f_i^j + \lambda_j E(\theta_i) f_i^i + \lambda_j E[\Delta_i]\left(\frac{g_i^i}{1-P_{ii}}\right) + \lambda_j d_i & i \neq j \\ \lambda_i d_i & i = j \end{cases} \tag{(4.4)}$$

$$g_{i+1}^j = \begin{cases} g_i^j + \frac{P_{ij}}{1-P_{ii}} g_i^i \, , & i \neq j \\ 0 \, , & i = j \end{cases} \tag{4.5}$$

Following Altman and Yechiali [1994], the solution of (4.5) is

$$g_i^j = \begin{cases} \widehat{c} \cdot \sum_{k=j}^{i-1} \widehat{P}_{kj} \widehat{\pi}_k \, , & i \neq j \\ \widehat{c} \cdot \widehat{\pi}_i \, , & i = j \end{cases} \tag{4.6}$$

with $\{\widehat{\pi}_i\}_1^N$ determined from $\widehat{\pi}_j = \sum_{i=1}^N \widehat{\pi}_i \widehat{P}_{ij}$, $j = 1, 2, \ldots, N$ and $\sum_{j=1}^N \widehat{\pi}_j = 1$. (A direct substitution readily shows that $\widehat{\pi}_j = \pi_j$). The constant $\widehat{c}$ is given by

$$\widehat{c} = M \Big/ \Big(\sum_{k=1}^N \widehat{\pi}_k \sum_{j=1}^k \widehat{P}_{kj} \Big) \tag{4.7}$$

In order to solve (4.4) we proceed as follows. We rewrite (4.4) for $k \neq i$:

$$f_{k+1}^i - f_k^i = \lambda_i E(\theta_k) f_k^k + \lambda_i E[\Delta_k] \left(\frac{g_k^k}{1 - P_{kk}} \right) + \lambda_i d_k$$

Summing over k from j to $i-1$ ($j > i$) yields

$$f_i^i - f_j^i = \lambda_i \sum_{k=j}^{i-1} E(\theta_k) f_k^k + \lambda_i \sum_{k=j}^{i-1} E[\Delta_k] \left(\frac{g_k^k}{1 - P_{kk}} \right) + \lambda_i \sum_{k=j}^{i-1} d_k \tag{4.8}$$

Setting $j = i+1$ in (4.8) and using (4.4) for $i = j$, we get

$$f_i^i = \lambda_i \sum_{\substack{k=1 \\ k \neq i}}^N \left[E(\theta_k) f_k^k + E[\Delta_k] \left(\frac{g_k^k}{1 - P_{kk}} \right) \right] + \lambda_i \sum_{k=1}^N d_k \tag{4.9}$$

Clearly,

$$E[C] = \sum_{i=1}^N E[\theta_i] f_i^i + \sum_{i=1}^N E[\Delta_i] \left(\frac{g_i^i}{1 - P_{ii}} \right) + d \tag{4.10}$$

Proposition 1. *We claim*

$$f_i^i = \lambda_i (1 - \rho_i - \rho_i^P) E[C] \tag{4.11}$$

$$E[C] = \frac{d}{1 - \rho - \rho^P} \tag{4.12}$$

$$\text{where,} \quad \rho_i^P = \left[\left(\frac{g_i^i}{1 - P_{ii}} \right) / E[C] \right] \beta_i \, . \tag{4.13}$$

Proof: Substituting (4.11) and (4.12) in both sides of (4.9) leads to

$$\lambda_i(1-\rho_i-\rho_i^P)E[C] = \lambda_i \left\{ \sum_{\substack{k=1\\k\neq i}}^{N} \frac{b_k\lambda_k}{1-\rho_k}(1-\rho_k-\rho_k^P)E[C] + \sum_{\substack{k=1\\k\neq i}}^{N} \frac{\beta_k}{1-\rho_k}\frac{\rho_k^P}{\beta_k}E[C] \right\} + \lambda_i d$$

or,

$$\begin{aligned}(1-\rho_i-\rho_i^P) &= \sum_{\substack{k=1\\k\neq i}}^{N} \frac{\rho_k}{1-\rho_k}(1-\rho_k-\rho_k^P) + \sum_{\substack{k=1\\k\neq i}}^{N} \frac{\rho_k^P}{1-\rho_k} + \frac{d}{E[C]} \\ &= \sum_{\substack{k=1\\k\neq i}}^{N} \left[\rho_k + \frac{\rho_k^P}{1-\rho_k}(1-\rho_k)\right] + \frac{d}{E[C]} \\ &= \sum_{\substack{k=1\\k\neq i}}^{N} (\rho_k+\rho_k^P) + \frac{d}{E[C]}\,. \end{aligned}$$

That is,

$$1-\sum_{k=1}^{N}(\rho_k+\rho_k^P) = 1-\rho-\rho^P = \frac{d}{E[C]}\,,$$

which verifies the assertions (4.11) and (4.12).

Substituting (4.11) and (4.13) in (4.10) and using $E[\theta_i] = \frac{b_i}{1-\rho_i}$, $E[\Delta_i] = \frac{\beta_i}{1-\rho_i}$, yields

$$\begin{aligned} E[C] &= \sum_{i=1}^{N} \left(\frac{b_i}{1-\rho_i}\right)\lambda_i(1-\rho_i-\rho_i^P)E[C] + \sum_{i=1}^{N}\left(\frac{\beta_i g_i^i}{1-P_{ii}}\right)\left(\frac{1}{1-\rho_i}\right) + d \\ &= \sum_{i=1}^{N} \rho_i E[C] - \sum_{i=1}^{N}\left(\frac{\beta_i g_i^i}{1-P_{ii}}\right)\left(\frac{\rho_i}{1-\rho_i}\right) + \sum_{i=1}^{N}\left(\frac{\beta_i g_i^i}{1-P_{ii}}\right)\left(\frac{1}{1-\rho_i}\right) + d \\ &= \sum_{i=1}^{N} \rho_i E[C] + \sum_{i=1}^{N} \frac{\beta_i g_i^i}{1-P_{ii}} + d \end{aligned}$$

Thus,

$$E[C] = \frac{d + \sum_{i=1}^{N} \frac{\beta_i g_i^i}{1-P_{ii}}}{1-\rho} \tag{4.14}$$

It follows, as in the Gated case, that a *necessary* and *sufficient* condition for stability of the system is that the *traffic intensity* of the hexogenous (transient) customers is less than 1, i.e. $\rho < 1$.

Proposition 2.

$$f_i^j = \lambda_j \left[\sum_{k=j+1}^{i-1} (\rho_k+\rho_k^P)E[C] + \sum_{k=j}^{i-1} d_k \right] \quad i \neq j \tag{4.15}$$

Proof: Indeed, using (4.8), interchanging the indices i and j, substituting (4.9) for f_j^j and applying (4.10) and (4.11) gives:

$$
\begin{aligned}
f_i^j &= f_j^j - \lambda_j \sum_{k=i}^{j-1} E[\theta_k]\left(\lambda_k(1-\rho_k)E[C] - \lambda_k \frac{g_k^k}{1-P_{kk}}\beta_k + \frac{\beta_k}{b_k}\frac{g_k^k}{1-P_{kk}}\right) - \lambda_j \sum_{k=i}^{j-1} d_k \\
&= f_j^j - \lambda_j \sum_{k=i}^{j-1}\left[\rho_k E[C] + \rho_k^P E[C]\right] - \lambda_j \sum_{k=i}^{j-1} d_k \\
&= \lambda_j(1-\rho_j)E[C] - \lambda_j \sum_{k=i}^{j-1}\rho_k E[C] - \lambda_j \rho_j^P E[C] - \lambda_j \sum_{k=i}^{j-1}\rho_k^P E[C] - \lambda_j \sum_{k=i}^{j-1} d_k \\
&= \lambda_j\left(1 - \sum_{k=i}^{j}\rho_k\right)E[C] - \lambda_j \sum_{k=i}^{j}\rho_k^P E[C] - \lambda_j \sum_{k=i}^{j-1} d_k \\
&= \lambda_j\left[\left(\frac{d}{1-\rho-\rho^P}\right)\left[1 - \left(\rho - \sum_{k=j+1}^{i-1}\rho_k\right) - \left(\rho^P - \sum_{k=j+1}^{i-1}\rho_k^P\right)\right] - \left(d - \sum_{k=j}^{i-1} d_k\right)\right] \\
&= \lambda_j\left[d + \left(\frac{d}{1-\rho-\rho^P}\right)\left(\sum_{k=j+1}^{i-1}\rho_k + \sum_{k=j+1}^{i-1}\rho_k^P\right) - d + \sum_{k=j}^{i-1} d_k\right] \\
&= \lambda_j\left[\sum_{k=j+1}^{i-1}\left(\rho_k + \rho_k^P\right)E[C] + \sum_{k=j}^{i-1} d_k\right]
\end{aligned}
$$

Throughput.

The mean number of jobs processed per unit time is

$$
\Lambda^E = \frac{\sum_{i=1}^{N}\left[\left(f_i^i + \frac{g_i^i}{1-P_{ii}}\right)\Big/\left(1-\rho_i\right)\right]}{E[C]} = \lambda + \frac{\sum_{i=1}^{N}\frac{g_i^i}{1-P_{ii}}}{E[C]} . \tag{4.16}
$$

Note again that the overall throughput *increases* (with respect to the 'only transient jobs' Exhaustive case) due to the presence of the permanent jobs.

Symmetric System. In a fully symmetric system we have $\lambda_i = \lambda_1; b_i = \beta_i = b_1$; $P_{ij} = \frac{1}{N}$, $1 - P_{ii} = 1 - \frac{1}{N} = \frac{N-1}{N}$. Thus, for $i \neq j$,

$$
\widehat{P}_{ij} = \frac{P_{ij}}{1-P_{ii}} = \frac{1/N}{(N-1)/N} = \frac{1}{N-1}
$$

$$
\widehat{\pi}_j = \frac{1}{N} \quad \left(\text{since } \widehat{\pi}_j = \sum_{i=1}^{N}\widehat{\pi}_i\widehat{P}_{ij} = \sum_{\substack{i=1\\ i\neq j}}^{N}\widehat{\pi}_i\frac{1}{N-1} = \frac{1}{N-1}(1-\widehat{\pi}_j)\right.
$$

$$
\widehat{c} = \frac{M}{\sum_{k=1}^{N}\widehat{\pi}_k \sum_{j=1}^{k}\widehat{P}_{kj}} = \frac{M}{\sum_{k=1}^{N}\frac{1}{N}\sum_{j=1}^{k-1}\frac{1}{N-1}} = \frac{MN(N-1)}{\sum_{k=1}^{N}(k-1)} = \frac{M\cdot N(N-1)}{N(N-1)/2} = 2M
$$

$$
g_i^i = \widehat{c}\cdot\widehat{\pi}_i = \frac{2M}{N}
$$

$$
g_i^i/(1-P_{ii}) = \frac{2M}{N-1}
$$

$$E[C] = \frac{d + Nb_1 \frac{2M}{N-1}}{1-\rho}.$$

$$f_i^i = \lambda_i\left(1 - \rho_i - \rho_i^P\right)E[C] = \lambda_1\left(1 - \rho_1 - \rho_1^P\right)E[C]$$

$$\sum_{i=1}^{N} f_i^i = \lambda_1 E[C]\left[N - \rho - \frac{2Mb_1}{N-1}N/E[C]\right]$$

$$= \lambda_1\left[(N-\rho)E[C] - \frac{2M\rho_1 \cdot N}{N-1}\right].$$

$$\sum_{i=1}^{N} g_i^i/(1-P_{ii}) = \frac{2MN}{N-1}\ .$$

The throughput in such a symmetric system is given by

$$\Lambda^E = \frac{\sum_{i=1}^{N} f_i^i + \sum_{i=1}^{N} g_i^i\Big/(1-P_{ii})}{E[C]} = \frac{\lambda_1\left[(N-\rho)E[C] - \frac{2MNb_1}{N-1}\right] + \frac{2MN}{N-1}}{E[C]}$$

$$= \lambda_1(N-\rho) + \frac{(1-\rho_1)(1-\rho)}{\frac{d(N-1)}{2MN} + b_1} \tag{4.17}$$

5. THE GLOBALLY GATED SERVICE REGIME

Cycle Time and Generating Functions.

According to the Globally Gated (GG) service regime (see Boxma, Levy and Yechiali [1992] and Boxma, Weststrate and Yechiali [1993]), at the beginning of each cycle by the server (starting, say, from station 1), *all* gates are closed *simultaneously*, and only jobs present in the system at that closing moment will be served during this cycle.

Let $(\underline{X}_1(n); \underline{Y}_1(n))$ be the state of the system at the start of the n-th cycle. To ease notation we write

$$(\underline{X}_1(n); \underline{Y}_1(n)) = \left(X_1(n), X_2(n), \ldots, X_N(n); Y_1(n); Y_2(n), \ldots, Y_N(n)\right) = \left(\underline{X}(n); \underline{Y}(n)\right).$$

Now, the n^{th} cycle duration is given by

$$C(n) = \sum_{j=1}^{N}\left[\sum_{k=1}^{X_j(n)} B_{jk} + \sum_{m=1}^{Y_j(n)} B_{jm}^P + D_j\right]. \tag{5.1}$$

The conditional LST of $C(n)$ is

$$E\left[e^{-uC(n)}\Big|(\underline{X}(n); \underline{Y}(n))\right] = \left(\prod_{j=1}^{N}[\widetilde{B}_j(u)]^{X_j(n)}\right)\left(\prod_{j=1}^{N}[\widetilde{B}_j^P(u)]^{Y_j(n)}\right)\widetilde{D}(u) \tag{5.2}$$

where

$$\widetilde{D}(u) = E\left[e^{-uD}\right] = E\left[e^{-u\sum_{j=1}^{N} D_j}\right] = \prod_{j=1}^{N}\widetilde{D}_j(u)\ .$$

Let $G^n(\underline{z};\underline{\omega})$ be the GF of the state $\big(\underline{X}(n);\underline{Y}(n)\big)$. Then,

$$E\Big[e^{-uC(n)}\Big] = G^n\Big(\widetilde{B}_1(u),\widetilde{B}_2(u),\ldots,\widetilde{B}_N(u);\widetilde{B}_1^P(u),\ldots,\widetilde{B}_N^P(u)\Big)\cdot\widetilde{D}(u) \tag{5.3}$$

Now,

$$G^{n+1}\Big(\underline{z};\underline{\omega}\Big|\big(\underline{X}(n);\underline{Y}(n)\big)\Big) = E\left[\Big(\prod_{j=1}^{N} z_j^{X_j(n+1)}\Big)\Big(\prod_{j=1}^{N}\omega_j^{Y_j(n+1)}\Big)\Big|\big(\underline{X}(n);\underline{Y}(n)\big)\right]$$

Clearly, $X_j(n+1) = A_j(C(n))$, $Y_j(n+1) = \sum_{i=1}^{N} K_i^j(Y_i(n))$, where $K_i^j(Y_i(n))$ is the number of permanent jobs, out of $Y_i(n)$, routed from station i to station j. Hence,

$$G^{n+1}\Big(\underline{z};\underline{\omega}\big|\big(\underline{X}(n);\underline{Y}(n)\big)\Big) = E\left[\left(e^{-\sum_{j=1}^{N}\lambda_j(1-z_j)C(n)}\right)\Big(\prod_{i=1}^{N}\Big(\prod_{j=1}^{N}\omega_j^{K_i^j(Y_i(n))}\Big)\Big)\Big|\big(\underline{X}(n);\underline{Y}(n)\big)\right]$$

$$= E\left[\left(e^{-\delta(\underline{z})\left(D+\sum_{j=1}^{N}\sum_{k=1}^{X_j(n)}B_{jk}+\sum_{j=1}^{N}\sum_{m=1}^{Y_j(n)}B_{jm}^P\right)}\right)\Big(\prod_{i=1}^{N}\Big(\prod_{j=1}^{N}\omega_j^{K_i^j(Y_i(n))}\Big)\Big)\Big|\big(\underline{X}(n);\underline{Y}(n)\big)\right]$$

$$= \widetilde{D}\big(\delta(\underline{z})\big)\cdot\Big(\prod_{j=1}^{N}\big[\widetilde{B}_j(\delta(\underline{z}))\big]^{X_j(n)}\Big)\cdot\Big(\prod_{j=1}^{N}\big[\widetilde{B}_j^P(\delta(\underline{z}))\big]^{Y_j(n)}\Big)\cdot\Big(\prod_{i=1}^{N}E\Big[\prod_{j=1}^{N}\omega_j^{K_i^j(Y_i(n))}\Big|\underline{Y}(n)\Big]\Big)$$

$$= \widetilde{D}\big(\delta(\underline{z})\big)\cdot\Big(\prod_{j=1}^{N}\big[\widetilde{B}_j(\delta(\underline{z}))\big]^{X_j(n)}\Big)\Big(\prod_{j=1}^{N}\big[\widetilde{B}_j^P(\delta(\underline{z}))\big]^{Y_j(n)}\Big)\Big(\prod_{i=1}^{N}\Big[\sum_{k=1}^{n}P_{ik}\omega_k\Big]^{Y_i(n)}\Big)$$

Thus,

$$G^{n+1}(\underline{z};\underline{\omega}) = \widetilde{D}(\delta(\underline{z}))G^n\Big(\widetilde{B}_1(\delta(\underline{z})),\widetilde{B}_2(\delta(\underline{z})),\ldots,\widetilde{B}_N(\delta(\underline{z}));$$
$$\Big[\widetilde{B}_1^P(\delta(\underline{z}))\big(\sum_{k=1}^{N}P_{1k}\omega_k\big)\Big],\Big[\widetilde{B}_2^P(\delta(\underline{z}))\Big(\sum_{k=1}^{N}P_{2k}\omega_k\Big)\Big],\ldots,\Big[\widetilde{B}_N^P(\delta(\underline{z}))\Big(\sum_{k=1}^{N}P_{Nk}\omega_k\Big)\Big]\Big)\ . \tag{5.4}$$

Passing to limit we derive

$$G(\underline{z};\underline{\omega}) = G\big(\underline{h}(\underline{z});\underline{s}(\underline{z},\underline{\omega})\big)\cdot\widetilde{D}\big(\delta(\underline{z})\big) \tag{5.5}$$

where $\underline{h}(\underline{z}) = \big(\underline{h}_1(\underline{z}),h_2(\underline{z}),\ldots,h_N(\underline{z})\big)$, with $h_i(\underline{z}) = \widetilde{B}_i\big(\delta(\underline{z})\big)$, and $\underline{s}(\underline{z},\underline{\omega}) = \big(s_1(\underline{z},\underline{\omega}), s_2(\underline{z},\underline{\omega}),\ldots,s_N(\underline{z},\underline{\omega})\big)$, with $s_i(\underline{z},\underline{\omega}) = \Big[\widetilde{B}_i^P(\delta(\underline{z}))\cdot\Big(\sum_{k=1}^{N}P_{ik}\omega_k\Big)\Big]$. It now follows from (5.3) that the LST of C is given by

$$\widetilde{C}(u) = \widetilde{D}(u)\cdot G(\widetilde{B}_1(u),\widetilde{B}_2(u),\ldots,\widetilde{B}_N(u);\widetilde{B}_1^P(u)\widetilde{B}_2^P(u),\ldots,\widetilde{B}_N^P(u)) \tag{5.6}$$

where $G(\cdot,\cdot)$ is given implicitly by (5.5).

First Moments.

From (5.1) one obtains:

$$E[C] = \sum_{j=1}^{N} (b_j f_j + \beta_j g_j) + d \tag{5.7}$$

where $f_j = E(X_j) = \frac{\partial G(\underline{z};\underline{\omega})}{\partial z_j}\big|_{\underline{z}=\underline{\omega}=\underline{1}}$, $g_j = E(Y_j) = \frac{\partial G(\underline{z};\underline{\omega})}{\partial \omega_j}\big|_{\underline{z}=\underline{\omega}=\underline{1}}$.
By differentiating both sides of (5.5) we derive

$$g_j = \sum_{i=1}^{N} P_{ij} g_i \ .$$

Since $\sum_{j=1}^{N} g_j = M$ it readily follows that

$$g_j = M\pi_j \tag{5.8}$$

where $\{\pi_j\}_1^N$ are determined from $\pi_j = \sum_{i=1}^{M} \pi_i P_{ij}$, $\sum_{i=1}^{N} \pi_j = 1$. Also

$$f_j = \lambda_j \cdot \sum_{i=1}^{N} (b_i f_i + \beta_i g_i) + d_i] = \lambda_j E[C] \tag{5.9}$$

Thus, $E[C] = \sum_{j=1}^{N} (b_j f_j + \beta_j g_j) + d = (\sum_{j=1}^{N} \rho_j) E[C] + M \sum_{j=1}^{N} \beta_j \pi_j + d$. Hence

$$E[C] = \frac{d + M\sum_{j=1}^{N} \beta_j \pi_j}{1-\rho} \tag{5.10}$$

Again, the necessary and sufficient condition for stability is $\rho < 1$.

It should be noted that the mean cycle time $E[C^G]$ under the Gated regime is *larger* than the mean cycle time $E[C^{GG}]$ under the GG regime. This follows since $c \geq M$ (see (5.10)), as $\sum_{k=1}^{N} \pi_k \sum_{j=1}^{k} P_{kj} \leq \sum_{k=1}^{N} \pi_k = 1$.

Also, letting $\xi_i^P = \frac{g_i}{E[C]}$, $\rho_i^P = \xi_i^P \beta_i$, equation (5.7) is written as

$$E[C] = \sum_{j=1}^{N} \rho_j E[C] + \sum_{j=1}^{N} \rho_j^P E[C] + d = \frac{d}{1-\rho-\rho^P} \tag{5.11}$$

Throughput.

The mean number of jobs processed per unit time is

$$\Lambda^{GG} = \frac{\sum_{j=1}^{N} f_j + M}{E[C]} = \lambda + \frac{M(1-\rho)}{d + M\sum_{j=1}^{N} \beta_j \pi_j} \tag{5.12}$$

Since $c \geq M$, it follows (see (3.18)) that

$$\Lambda^{GG} \leq \Lambda^{G} \ .$$

Nevertheless, here too, the throughput is *larger* than its counterpart in a pure GG regime.

Calculation of $E[C^2]$.

To obtain $E[C^2]$ we differentiate twice equation (5.6) and derive

$$E[C^2] = d^{(2)} + 2d\sum_{j=1}^{N}(b_j f_j + \beta_j g_j) + \sum_{j=1}^{N}[b_j^{(2)} f_j + \beta_j^{(2)} g_j]$$
$$+\sum_{j=1}^{N}\sum_{k=1}^{N}[b_j b_k f(j,k) + 2b_j\beta_k r(j,k) + \beta_j\beta_k g(j,k)] \tag{5.13}$$

where

$$f(i,k) = \frac{\partial^2 G(\underline{z};\underline{\omega})}{\partial z_i \partial z_k}\Big|_{\underline{z}=\underline{\omega}=\underline{1}} = \begin{cases} E[X_i X_k]\ , & i \neq k \\ E[X_i(X_i-1)]\ , & i = k \end{cases}$$

$$g(i,k) = \frac{\partial^2 G(\underline{z};\underline{\omega})}{\partial \omega_i \partial \omega_k}\Big|_{\underline{z}=\underline{\omega}=\underline{1}} = \begin{cases} E[Y_i Y_k], & i \neq k \\ E[Y_i(Y_i-1)], & i = k \end{cases} \tag{5.14}$$

$$r(i,k) = \frac{\partial^2 G(\underline{z};\underline{\omega})}{\partial z_i \partial \omega_k}\Big|_{\underline{z}=\underline{\omega}=\underline{1}} = E[X_i Y_k]$$

The second and mixed moments $f(i,k)$, $g(i,k)$ and $r(i,k)$ are calculated by solving the set of $3N^2$ equations in $3N^2$ unknowns (see section 6), obtained by twice differentiating both sides of (5.5).

Since at polling instants the locations of the permanent customers are independent of the transient jobs, we can use the results in Altman and Yechiali [1994] to write

$$g(i,k) = M(M-1)\pi_i\pi_k \tag{5.15}$$

where $\{\pi_i\}$ is the limiting distribution of the routing matrix $[P_{ij}]$.

Waiting Times for a Special Case.

Consider the case where the set of N stations is partitioned into two sets: a set OP, with only 'open' stations, and a set CL, with only 'closed' stations. Let $|\text{OP}| = N_0$, $|\text{CL}| = N_c$, such that $N = N_0 + N_c$. The single server visits the stations in the regular cyclic manner. Consider an arbitrary transient job, K, that arrives at station $k \in \text{OP}$. Its waiting time is composed of the following parts:

(i) the remainder part, C_R, of the cycle

(ii) all switchover times $\sum_{j=1}^{k-1} D_j$ from station 1 to station k.

(iii) service times of all jobs in stations $1, 2, \ldots, k-1$. Those jobs either arrived or were routed to these stations during the cycle in which K has arrived. (The length of this cycle is $C_P + C_R$, where C_P is the past part of the cycle).

(iv) service of transient jobs in station k that arrived during the past part of the cycle, C_P, before the arrival instant of job K.

We therefore have, for $k \in$ OP,

$$E[W_k] = E[C_R] + \sum_{j=1}^{k-1} d_j + \sum_{\substack{j=1 \\ j\in OP}}^{k-1} \lambda_j b_j \big(E[C_p] + E[C_R]\big) + \left[\sum_{\substack{j=1 \\ j\in CL}}^{k-1} g_j \beta_j \right] + \rho_k E[C_P] \quad (5.16)$$

It is well known (see for example Boxma, Levy and Yechiali [1992]) that

$$E[C_R] = E[C_P] = \frac{E[C^2]}{2E[C]} \quad (5.17)$$

In our special case

$$E[C] = \sum_{i\in OP} b_i f_i + \sum_{i\in CL} \beta_i g_i + d \quad (5.18)$$

Still, we have $f_i = \lambda_i EC$, $i \in$ OP; $g_i = M\pi_i$, $i \in$ CL.

The mean cycle time is

$$E[C] = \frac{d + M \sum_{i\in CL} \beta_i \pi_i}{1-\rho} \quad (5.19)$$

The expression for $E[C^2]$ now takes the form

$$E[C^2] = 2d\Big(\sum_{i\in OP} f_i b_i + \sum_{i\in CL} g_i\beta_i\Big) + \sum_{i\in OP}\sum_{k\in OP} b_i b_k f(i,k) + \sum_{i\in CL}\sum_{k\in CL} \beta_i\beta_k g(i,k) + 2\sum_{i\in OP}\sum_{k\in CL} b_i \beta_k r(i,k) + \sum_{i\in OP} f_i b_i^{(2)} + \sum_{i\in CL} g_i \beta_i^{(2)} + d^{(2)} . \quad (5.20)$$

Thus, with (5.17), (5.19) and (5.20), $E[W_k]$ is given *explicitly*.

It also follows that if i and k are two successive 'open' stations ($k > i$), then $E[W_k] \geq E[W_i]$. This is true since

$$\begin{aligned} E[W_k] - E[W_i] &= \sum_{j=i}^{k-1} d_j + 2\rho_i E[C_R] + \sum_{\substack{j=i \\ j\in CL}}^{k-1} g_j\beta_j + (\rho_k - \rho_i) E[C_P] \\ &= \sum_{j=i}^{k-1} d_j + (\rho_i + \rho_k) E[C_R] + \sum_{\substack{j=i+1 \\ j\in CL}}^{k-1} g_j\beta_j > 0 . \end{aligned} \quad (5.21)$$

6. SECOND-ORDER MOMENTS

In this section we derive expressions for the calculations of second-order moments of the system state at polling instants for each of the service regimes: Gated, Exhaustive or Globally-Gated.

6.1 The Gated Regime. We use equation (3.5) and write:

$$f_i(j,k) = \frac{\partial^2 G_i(\underline{z};\underline{\omega})}{\partial z_j \partial z_k}\Big|_{\underline{z}=\underline{\omega}=\underline{1}} = \begin{cases} E\big[X_i^j(X_i^j-1)\big], & j=k \\ E\big[X_i^j X_i^k\big], & \text{otherwise} \end{cases}$$

$$g_i(j,k) = \frac{\partial^2 G_i(\underline{z};\underline{\omega})}{\partial \omega_j \partial \omega_k}\Big|_{\underline{z}=\underline{\omega}=\underline{1}} = \begin{cases} E\big[Y_i^j(Y_i^j-1)\big], & j=k \\ E\big[Y_i^j Y_i^k\big], & \text{otherwise} \end{cases} \tag{6.1}$$

$$r_i(j,k) = \frac{\partial^2 G_i(\underline{z};\underline{\omega})}{\partial z_j \partial \omega_k}\Big|_{\underline{z}=\underline{\omega}=\underline{1}} = E\big[X_i^j Y_i^k\big]$$

After lengthy calculations we obtain

$$f_{i+1}(j,k) = \begin{cases} d_i^{(2)}\lambda_j\lambda_k + f_i^j\lambda_k d_i + f_i^k\lambda_j d_i \\ +\lambda_j\lambda_k\big[f_i^i(2d_ib_i+b_i^{(2)}) + g_i^i(2d_i\beta_i+\beta_i^{(2)})\big] + f_i(j,k) \\ +b_i\big[\lambda_k f_i(i,j)+\lambda_j f_i(i,k)\big] + \beta_i\big[\lambda_k r_i(j,i)+\lambda_j r_i(k,i)\big] \\ +\lambda_j\lambda_k\big[f_i(i,i)b_i^2 + 2r_i(i,i)b_i\beta_i + g_i(i,i)\beta_i^2\big], & i\neq j, i\neq k \\ d_i^{(2)}\lambda_i\lambda_k + f_i^k\lambda_i d_i + \lambda_i\lambda_k\big[f_i^i(2d_ib_i+b_i^{(2)}) + g_i^i(2d_i\beta_i+\beta_i^{(2)})\big] \\ +\lambda_i\big[f_i(i,k)b_i + r_i(k,i)\beta_i\big] \\ +\lambda_i\lambda_k\big[f_i(i,i)b_i^2 + 2r_i(i,i)b_i\beta_i + g_i(i,i)\beta_i^2\big] & i=j, i\neq k \\ d_i^{(2)}\lambda_i^2 + \lambda_i^2\big[f_i^i(2d_ib_i+b_i^{(2)}) + g_i^i(2d_i\beta_i+\beta_i^{(2)}) \\ +f_i(i,i)b_i^2 + 2r_i(i,i)b_i\beta_i + g_i(i,i)\beta_i^2\big], & i=j=k \end{cases} \tag{6.2}$$

$$g_{i+1}(j,k) = \begin{cases} g_i(j,k) + g_i(j,i)P_{ik} + g_i(i,k)P_{ij} + g_i(i,i)P_{ij}P_{ik}\ , & i\neq j, i\neq k \\ g_i(i,k)P_{ii} + g_i(i,i)P_{ii}P_{ik}\ , & i=j, i\neq k \\ g_i(i,i)P_{ii}^2\ , & i=j=k \end{cases} \tag{6.3}$$

$$r_{i+1}(j,k) = \begin{cases} d_i\lambda_j(g_i^k + g_i^iP_{ik}) + r_i(j,k) + r_i(j,i)P_{ik} \\ +\lambda_j\big[r_i(i,k)b_i + g_i(i,k)\beta_i\big] \\ +\lambda_jP_{ik}\big[r_i(i,i)b_i + \big(g_i^i + g_i(i,i)\big)\beta_i\big], & i\neq j, i\neq k \\ d_i\lambda_i(g_i^k + g_i^iP_{ik}) + \lambda_i\big[r_i(i,k)b_i + g_i(i,k)\beta_i\big] \\ +\lambda_iP_{ik}\big[r_i(i,i)b_i + \big(g_i^i + g_i(i,i)\big)\beta_i\big], & i=j\ ,\ i\neq k \\ d_i\lambda_jg_i^iP_{ii} + r_i(j,i)P_{ii} \\ +\lambda_jP_{ii}\big[r_i(i,i)b_i + \big(g_i^i + g_i(i,i)\big)\beta_i\big], & i\neq j, i=k \\ d_i\lambda_ig_i^iP_{ii} + \lambda_iP_{ii}\big[r_i(i,i)b_i + \big(g_i^i + g_i(i,i)\big)\beta_i\big], & i=j=k \end{cases} \tag{6.4}$$

The collection (6.2), (6.3) and (6.4) gives a set of $3N^3$ equations in the $3N^3$ variables $f_i(j,k)$, $g_i(j,k)$ and $r_i(j,k)$. Notice that the solution of (6.3) for the permanent jobs is independent of the transient jobs. Moreover, the solution of (6.4) for the mixed moments depends only on the moments of the permanent jobs. However, the solution of (6.2) for the transient jobs depends *both* on the moments of the permanent jobs and the mixed moments. In fact, the solution for (6.3) is somewhat redundant, as we get a direct solution for the permanent jobs' second moments from Equation (15) and Section 3.4 of Altman and Yechiali [1994]:

$$G_i(\underline{1};\underline{\omega}) = \left(\sum_{j=1}^{N} \pi_j(i)\omega_j\right)^M . \tag{6.5}$$

$$g_i(j,k) = M(M-1)\pi_j(i)\pi_k(i) . \tag{6.6}$$

where $\pi_j(i)$ is the steady-state probability that an arbitrary job resides in station j when station i is polled. In particular,

$$g_i(i,i) = M(M-1)\pi_i(i)^2 .$$

Now, g_i^i is given by (3.9) and (3.10) so that the collection is reduced to $2N^3$ equations in the $2N^3$ variables $f_i(j,k)$ and $r_i(j,k)$.

The Symmetric Case. If all stations are (probabilistically) alike, and $P_{ij} = \frac{1}{N}$, then

$$b_i = \beta_i = b_1,\ d_i = d_1,\ \lambda_i = \lambda_1, f_i^i = f^{(1)},\ g_i^i = g^{(1)}$$

$$f_i(i,i) = f^{(2)},\ g_i(i,i) = g^{(2)},\ r_i(i,i) = r^{(2)}$$

Equation (6.2) becomes

$$f_{i+1}(j,k) = \begin{cases} \alpha + 2\beta\left[f_i^j + f_i^k\right] + \gamma\left[f^{(1)} + g^{(1)}\right] + f_i(j,k) \\ +\delta\left[f_i(i,j) + f_i(i,k) + r_i(j,i) + r_i(k,i)\right] \\ +\delta^2\left[f^{(2)} + g^{(2)} + 2r^{(2)}\right], & i \neq j\ ,\ i \neq k \\ \alpha + 2\beta f_i^j + \gamma\left[f^{(1)} + g^{(1)}\right] + f_i(j,j) \\ +2\delta\left[f_i(i,j) + r_i(j,i)\right] + \delta^2\left[f^{(2)} + g^{(2)} + 2r^{(2)}\right], & i \neq j\ ,\ i = k \\ \alpha + \beta f_i^k + \gamma\left[f^{(1)} + g^{(1)}\right] \\ +\delta\left[f_i(i,k) + r_i(k,i)\right] + \delta^2\left[f^{(2)} + g^{(2)} + 2r^{(2)}\right] & i = j,\ i \neq k \\ \alpha + \gamma\left[f^{(1)} + g^{(1)}\right] + \delta^2\left[f^{(2)} + g^{(2)} + 2r^{(2)}\right], & i = j = k \end{cases} \tag{6.7}$$

where

$$\alpha = \lambda_1^2 d_1^{(2)}\ ,\ \beta = \lambda_1 d_1$$

$$\gamma = \lambda_1^2(2b_1 d_1 + b_1^{(2)})\ ,\ \delta = \lambda_1 b_1$$

From Altman and Yechiali [1994], for the symmetric case, (see Eq. (14) there) $\pi_i(i) = \frac{2}{N+1}$

$$g^{(1)} = M\pi_i(i) = \frac{2M}{N+1}$$

$$g^{(2)} = M(M-1)[\pi_i(i)]^2 = M(M-1)\left(\frac{2}{N+1}\right)^2 = \frac{4M(M-1)}{(N+1)^2}$$

$$f^{(1)} = \frac{N\lambda[2Mb_1 + (N+1)d_1}{(N+1)(1-N\lambda_1 b_1)} .$$

Now, $f^{(1)} = \lambda_1 E[C]$, while

$$E[C] = Nb_1\left[f^{(1)} + g^{(1)}\right] + d = Nb_1\left[\lambda_1 E[C] + g^{(1)}\right] + d\ .$$

Thus,

$$E[C] = \frac{Nb_1 g^{(1)} + Nd_1}{1 - N\lambda_1 b_1} = \frac{Nb_1 \frac{2M}{N+1} + Nd_1}{1 - N\lambda_1 b_1}$$

$$= \frac{2NMb_1 + (N+1)Nd_1}{(N+1)(1 - N\lambda_1 b_1)} = \frac{N[2Mb_1 + (N+1)d_1]}{(N+1)(1 + N\lambda_1 b_1)}$$

The mixed moments are given by:

$$r_{i+1}(j,k) = \begin{cases} d_1\lambda_1\left(g_i^k + g^{(1)}\cdot\frac{1}{N}\right) \\ +r_i(j,k) + r_i(j,i)\frac{1}{N} + \left[r_i(i,k) + g_i(i,k)\right]b_1\lambda_1 \\ +\left[g^{(1)} + r^{(2)} + g^{(2)}\right]b_1\lambda_1\frac{1}{N} & i \neq j,\ i \neq k \\ d_1\lambda_1\left(g_i^k + g^{(1)}\cdot\frac{1}{N}\right) \\ +\left[r_i(i,k) + g_i(i,k)\right]b_1\lambda_1 + \left[g^{(1)} + r^{(2)} + g^{(2)}\right]b_1\lambda_1\frac{1}{N} & i = j,\ i \neq k \\ d_1\lambda_1 g^{(1)}\frac{1}{N} + r_i(j,i)\frac{1}{N} + \left[g^{(1)} + r^{(2)} + g^{(2)}\right]b_1\lambda_1\frac{1}{N} & i \neq j,\ i = k \\ d_1\lambda_1 g^{(1)}\frac{1}{N} + \left[g^{(1)} + r^{(2)} + g^{(2)}\right]b_1\lambda_1\frac{1}{N} & i = j = k \end{cases} \tag{6.8}$$

6.2 The Exhaustive Regime.

From the GF (4.3) we derive $f_{i+1}(j,k)$ and $r_{i+1}(j,k)$:

$$f_{i+1}(j,k) = \begin{cases} d_i^{(2)}\lambda_j\lambda_k + f_i^j\lambda_k d_i + f_i^k\lambda_j d_i \\ +\lambda_j\lambda_k\left[f_i^i(2d_i\theta_i^{(1)} + \theta_i^{(2)}) + g_i^i\left(\frac{2d_i\Delta_i^{(1)} + \Delta_i^{(2)}}{1-P_{ii}} + 2P_{ii}\left[\frac{\Delta_i^{(1)}}{1-P_{ii}}\right]^2\right)\right. \\ +f_i(j,k) \\ +\theta_i^{(1)}\left[\lambda_j f_i(i,k) + \lambda_k f_i(j,i)\right] + \left(\frac{\Delta_i^{(1)}}{1-P_{ii}}\right)\left[\lambda_j r_i(i,k) + \lambda_k r_i(j,i)\right] \\ +\lambda_j\lambda_k\left[f_i(i,i)(\theta_i^{(1)})^2 + 2r_i(i,i)\theta_i^{(1)}\left(\frac{\Delta_i^{(1)}}{1-P_{ii}}\right) + g_i(i,i)\left(\frac{\Delta_i^{(1)}}{1-P_{ii}}\right)^2\right], & i \neq j,\ i \neq k \\ d_i^{(2)}\lambda_i\lambda_k + \lambda_i d_i\left[f_i^k + \lambda_k\left(f_i^i\theta_i^{(1)} + g_i^i\left(\frac{\Delta_i^{(1)}}{1-P_{ii}}\right)\right)\right], & i = j,\ i \neq k \\ d_i^{(2)}\lambda_i\lambda_j + \lambda_k d_i\left[f_i^j + \lambda_j\left(f_i^i\theta_i^{(1)} + g_i^i\left(\frac{\Delta_i^{(1)}}{1-P_{ii}}\right)\right)\right], & i \neq j,\ i = k \\ \lambda_i^2 d_i^{(2)}, & i = j = k \end{cases} \tag{6.9}$$

where $\theta_i^{(1)}, \theta_i^{(2)}$ are the first and second moments, respectively, of a regular $M/G/1$ busy period, and $\Delta_i^{(1)}, \Delta_i^{(2)}$ are the first and second moments, respectively, of a delay busy period as defined in section 4. We repeat the well known that

$$\theta_i^{(1)} = E(\theta_i) = \frac{b_i}{1-\rho_i}, \quad \Delta_i^{(1)} = E[\Delta_i] = \frac{\beta_i}{1-\rho_i}$$
$$\theta_i^{(2)} = E(\theta_i^2) = \frac{b_i^{(2)}}{(1-\rho_i)^3}, \quad \Delta_i^{(2)} = E[\Delta_i^2] = \frac{\beta_i^{(2)}}{(1-\rho_i)^2} + \frac{\lambda_i \beta_i b_i^{(2)}}{(1-\rho_i)^3}$$

$$r_{i+1}(j,k) = \begin{cases} d_i\lambda_j\left(g_i^k + g_i^i\frac{P_{ik}}{1-P_{ii}}\right) + r_i(j,k) + r_i(j,k) + r_i(j,i)\frac{P_{ik}}{1-P_{ii}} \\ +\lambda_j\left[r_i(i,k)\theta_i^{(1)} + g_i(i,k)\frac{\Delta_{(1)}}{1-P_{ii}}\right] \\ +\frac{\lambda_j P_{ik}}{1-P_{ii}}\left[r_i(i,i)\theta_i^{(1)} + (g_i^i + g_i(i,i))\frac{\Delta_i^{(1)}}{1-P_{ii}}\right], & i \neq j,\ i \neq k \\ d_i\lambda_i\left(g_i^k + g_i^i\frac{P_{ik}}{1-P_{ii}}\right), & i = j, i \neq k \\ 0 \quad (\text{since } Y_{i+1}^i = 0) & i \neq j, i = k \\ 0 & i = j = k \end{cases} \tag{6.10}$$

g_i^i and $g_i(i,i)$ are taken from (4.6) and (6.6). Altogether, (6.9) and (6.10) determine a set of $2N^2$ equations in the $2N^2$ variables $f_i(i,k)$, and $r_i(j,k)$, where the solution of (6.10) for the mixed moments is independent of the transients' second moments in (6.9). Equation (6.9), however, does depend on (6.10). Substituting for $\theta_i^{(1)}, \theta_i^{(2)}, \Delta_i^{(1)}, \Delta_i^{(2)}$ we get

$$f_{i+1}(j,k) = \begin{cases} d_i^{(2)}\lambda_j\lambda_k + f_i^j\lambda_k d_i + f_i^k\lambda_j d_i + \lambda_j\lambda_k\left[f_i^i\left(\frac{2d_ib_i}{1-\rho_i} + \frac{b_i^{(2)}}{(1-\rho_i)^3}\right)\right. \\ \left. +g_i^i\left(\frac{2d_i\beta_i+\beta_i^{(2)}}{(1-\rho_i)(1-P_{ii})} + \frac{\lambda_i\beta_i b_i^{(2)}}{(1-\rho_i)^3(1-P_{ii})} + \frac{2P_{ii}\beta_i^2}{(1-\rho_i)^2(1-P_{ii})^2}\right)\right] \\ +f_i(j,k) + [\lambda_j f_i(i,k) + \lambda_k f_i(j,i)]\frac{b_i}{1-\rho_i} \\ +[\lambda_j r_i(i,k) + \lambda_k r_i(j,i)]\frac{b_i\beta_i}{(1-\rho_i)^2(1-P_{ii})^2} \\ +\lambda_j\lambda_k[f_i(i,i)\frac{b_i^2}{(1-\rho_i)^2} + \frac{2r_i(i,i)b_i\beta_i}{(1-\rho_i)^2(1-P_{ii})} + \frac{g_i(i,i)\beta_i^2}{(1-\rho_i)^2(1-P_{ii})^2}], & i \neq j,\ i \neq k \\ d_i^{(2)}\lambda_i\lambda_k + \lambda_i d_i\left[f_i^k + \frac{\lambda_k}{1-\rho_i}\left(f_i^i b_i + \frac{g_i\beta_i}{1-P_{ii}}\right)\right], & i=j,\ i \neq k \\ d_i^{(2)}\lambda_i\lambda_j + \lambda_i d_i\left[f_i^j + \frac{\lambda_j}{1-\rho_i}\left(f_i^i b_i + \frac{g_i^i\beta_i}{1-P_{ii}}\right)\right], & i \neq j,\ i=k \\ \lambda_i^2 d_i^{(2)}, & i=j=k \end{cases} \tag{6.11}$$

$$r_{i+1}(j,k) = \begin{cases} d_i\lambda_j\left(g_i^k + g_i^i\frac{P_{ik}}{1-P_{ii}}\right) + r_i(j,k) + r_i(j,i)\frac{P_{ik}}{1-P_{ii}} \\ +\frac{\lambda_j}{1-\rho_i}\left[r_i(i,k)b_i + g_i(i,k)\frac{\beta_i}{1-P_{ii}}\right] \\ +\frac{\lambda_j P_{ik}}{(1-\rho_i)(1-P_{ii})}\left[r_i(i,i)b_i + (g_i^i + g_i(i,i))\frac{\beta_i}{1-P_{ii}}\right], & i \neq j,\ i \neq k \\ d_i\lambda_i\left(g_i^k + g_i^i\frac{P_{ik}}{1-P_{ii}}\right), & i = j,\ i \neq k \\ 0 & i \neq j,\ i = k \\ 0 & i = j = k \end{cases} \tag{6.12}$$

6.3 The Globally Gated Regime.

The second moments (as defined in (5.14)) are derived from the GF (5.5):

$$f(j,k) = \lambda_j \lambda_k d^{(2)} + \lambda_j d \lambda_k \sum_{i=1}^{N} (b_i f_i + \beta_i g_i)$$
$$+ \lambda_k d \lambda_j \sum_{i=1}^{N} (b_i f_i + \beta_i g_i) + \sum_{i=1}^{N} \sum_{\ell=1}^{N} [f(i,\ell) \lambda_j \lambda_k b_i^2 + g(i,\ell) \lambda_j \lambda_k \beta_i^2$$
$$+ r(i,\ell) \lambda_j \lambda_k b_i \beta_\ell + r(\ell,i) \lambda_j \lambda_k b_\ell \beta_i] + \sum_{i=1}^{N} [f_i \lambda_j \lambda_k b_i^{(2)} + g_i \lambda_j \lambda_k \beta_i^{(2)}] \tag{6.13}$$
$$= \lambda_j \lambda_k \big[d^{(2)} + 2d \sum_{i=1}^{N} (b_i f_i + \beta_i g_i)$$
$$+ \sum_{i=1}^{N} \sum_{\ell=1}^{N} \big(b_i b_\ell f(i,\ell) + \beta_i \beta_\ell g(i,\ell) + 2 b_i \beta_\ell r(i,\ell)\big) + \sum_{i=1}^{N} \big(b_i^{(2)} f_i + \beta_i^{(2)} g_i\big)\big]$$
$$= \lambda_j \lambda_k E[C^2]$$

Since $\frac{\partial \widetilde{D}(\delta(z))}{\partial \omega_k} = 0$,

$$r(j,k) = \frac{\partial \widetilde{D}(\delta(\underline{z}))}{\partial z_j} \cdot \frac{\partial G(\underline{h}(\underline{z}); \underline{s}(\underline{z}, \underline{\omega}))}{\partial \omega_k} \Big|_{\underline{z}=\underline{\omega}=\underline{1}}$$
$$+ \widetilde{D}(\delta(\underline{z})) \cdot \frac{\partial^2 G(\widetilde{h}(\underline{z}); \underline{s}(\underline{z}, \underline{\omega}))}{\partial z_j \partial \omega_k} \Big|_{\underline{z}=\underline{\omega}=\underline{1}} \tag{6.14}$$
$$= d \lambda_j \Big(\sum_{i=1}^{N} P_{ik} g_i\Big) + \sum_{i=1}^{N} \sum_{\ell=1}^{N} [r(i,\ell) b_i \lambda_j P_{\ell k} + g(i,\ell) \beta_i \lambda_j P_{\ell k}]$$
$$= \lambda_j d \sum_{i=1}^{N} P_{ik} g_i + \sum_{i=1}^{N} \sum_{\ell=1}^{N} P_{\ell k} [b_i r(i,\ell) + \beta_i g(i,\ell)]$$

$$g(j,k) = \frac{\partial^2 G(\underline{h}(\underline{z}); \underline{s}(\underline{z}, \underline{\omega}))}{\partial \omega_j \partial \omega_k} \Big|_{\underline{z}=\underline{\omega}=\underline{1}} = \sum_{i=1}^{N} \sum_{\ell=1}^{N} g(i,\ell) P_{ij} P_{i\ell} \tag{6.15}$$
$$= M(M-1) \pi_i \pi_k \qquad (\text{see} \quad (5.15)) \ .$$

Again, we have a set of $3N^2$ equations in $3N^2$ unknowns. Since the solution for $g(j,k)$ is derived independently, the set is reduced to $2N^2$ equations in $2N^2$ unknowns where the solution of $f(i,k)$ depends on the solution for the $r(i,k)$, but the $r(i,k)$ are independent of the $f(i,k)$.

7. NUMBER OF CUSTOMERS AT AN ARBITRARY MOMENT

In this section we derive formulae for the GF and moments of the number of jobs in the system at an arbitrary instant. This is done for all three regimes studied previously, namely the Gated, Exhaustive and Globally-Gated. We use the following notation:
(X_i^j, Y_i^j) = number of transient and permanent jobs, respectively, in queue j (Q_j) at a *polling instant* to channel i. $f_i^j = E[X_i^j]$, $g_i^j = E[Y_i^j]$.

$(\overline{X}_i^j, \overline{Y}_i^j)$ = number of jobs in queue j at a server's visit-completion instant of channel i.
$\overline{f}_i^j = E[\overline{X}_i^j]$, $\overline{g}_i^j = E[\overline{Y}_i^j]$.
$\overline{G}_i(\underline{z};\underline{\omega})$ = GF of $(\overline{X}_i^j, \overline{Y}_i^j)_{j=1}^N$.
(VX_i^j, VY_i^j) = number of transient and permanent jobs, respectively, in queue j at a job's service-beginning instant in Q_i.
$V_i(\underline{z};\underline{\omega}) = GF$ of $(VX_i^j, VY_i^j)_{j=1}^N$.
$(\overline{VX}_i^j, \overline{VY}_i^j)$ = number of transient and permanent jobs, respectively, in queue j at a job's service-completion instant in queue i.
$\overline{V}_i(\underline{z};\underline{\omega})$ = GF of $(\overline{VX}_i^j, \overline{VY}_i^j)_{j=1}^N$.

Clearly,

$$E[VX_i^j] = \frac{\partial V_i(\underline{z};\underline{\omega})}{\partial z_j}\Big|_{\underline{z}=\underline{\omega}=\underline{1}}, \quad E[VY_i^j] = \frac{\partial V_i(\underline{z};\underline{\omega})}{\partial \omega_j}\Big|_{\underline{z}=\underline{\omega}=\underline{1}}$$

$$E[\overline{VX}_i^j] = \frac{\partial \overline{V}_i(\underline{z};\underline{\omega})}{\partial z_j}\Big|_{\underline{z}=\underline{\omega}=\underline{1}}, \quad E[\overline{VY}_i^j] = \frac{\partial \overline{V}_i(\underline{z};\underline{\omega})}{\partial \omega_j}\Big|_{\underline{z}=\underline{\omega}=\underline{1}}$$

(X^j, Y^j) = number of transient and permanent jobs, respectively, in queue j $(j = 1, 2, \dots, N)$ at an *arbitrary moment*.
$(\underline{X};\underline{Y}) = (X^1, X^2, \dots, X^N; Y^1, Y^2, \dots, Y^N)$ = system state at an arbitrary moment.
$F(\underline{z};\underline{\omega})$ = GF of $(\underline{X};\underline{Y})$, with $E[X^j] = \frac{\partial F(\underline{z};\underline{\omega})}{\partial z_j}\big|_{\underline{z}=\underline{\omega}=\underline{1}}$, $E[Y^j] = \frac{\partial F(\underline{z};\underline{\omega})}{\partial \omega_j}\big|_{\underline{z}=\underline{\omega}=\underline{1}}$.

Finally, let $1/\gamma_i$ be the expected number of jobs served in queue i during a visit, and $1/\gamma$ be the expected total number of jobs served during a full cycle.

Adapting Eisenberg's [1972] result (13) (see also Sidi et al [1992] and Altman and Yechiali [1994]), we have, for all three service regimes,

$$\gamma_i G_i(\underline{z};\underline{\omega}) + \overline{V}_i(\underline{z};\underline{\omega}) = V_i(\underline{z};\underline{\omega}) + \gamma_i \overline{G}_i(\underline{z},\underline{\omega}) \, . \tag{7.1}$$

7.1 The Gated Regime.

$$\frac{1}{\gamma_i} = f_i^i + g_i^i$$

$$\frac{1}{\gamma} = \sum_{i=1}^N f_i^i + \sum_{i=1}^N g_i^i = \lambda E[C] + c \, .$$

Let the probabilities P_1^i and P_2^i be

$$P_1^i = P \text{ (served job is transient} \mid S \text{ at } Q_i, \text{ service completion)}$$
$$= \frac{E[\text{number of transient jobs served during a visit of } Q_i]}{E[\text{total number of jobs served during a visit of } Q_i]} = f_i^i \gamma_i$$

$$P_2^i = P \text{ (served job is permanent} \mid S \text{ at } Q_i \text{ , service completion)}$$
$$= \frac{E[\text{number of permanent jobs served during a visit of } Q_i]}{E[\text{total number of jobs served during a visit of } Q_i]} = g_i^i \gamma_i$$

where S stands for Server. Using the above definitions, and $\delta(\underline{z}) = \sum_{j=1}^{N} \lambda_j(1 - z_j)$, $P_i(\underline{\omega}) = \sum_{j=1}^{N} P_{ij}\omega_j$, the following relations exist between the various GF:

$$\overline{V}_i(\underline{z};\underline{\omega}) = V_i(\underline{z};\underline{\omega}) \cdot [P_1^i(\frac{\widetilde{B}_i(\delta(\underline{z}))}{z_i}) + P_2^i(\frac{P_i(\underline{\omega})}{\omega_i}\widetilde{B}_i^P(\delta(\underline{z}))]$$
$$G_{i+1}(\underline{z};\underline{\omega}) = \overline{G}_i(\underline{z};\underline{\omega})\widetilde{D}_i(\delta(\underline{z})) \tag{7.2}$$

so that (see (3.5)),

$$\overline{G}_i(\underline{z};\underline{\omega}) = G_i(z_1, z_2, \ldots, z_{i-1}, h_i(\underline{z}), z_{i+1}, \ldots, z_N; \omega_1, \omega_2, \ldots, \omega_{i-1}, s_i(\underline{z},\underline{\omega}), \omega_{i+1}, \ldots, \omega_N) \tag{7.3}$$

where $h_i(\underline{z}) = \widetilde{B}_i(\delta(\underline{z}))$, $s_i(\underline{z},\underline{\omega}) = \widetilde{B}_i^P(\delta(\underline{z})) \cdot P_i(\underline{\omega})$. Now,

$$F\Big(\underline{z};\underline{\omega}\Big| \begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\Big) = \overline{G}_i(\underline{z};\underline{\omega})\frac{1 - \widetilde{D}_i(\delta(\underline{z}))}{d_i\delta(\underline{z})} \tag{7.4}$$

$$F\Big(\underline{z};\underline{\omega}\Big| \begin{matrix}\text{service}\\ \text{period } i\end{matrix}\Big) = V_i(\underline{z};\underline{\omega})\,[\widehat{P}_1^i(\frac{1 - \widetilde{B}_i(\delta(\underline{z}))}{b_i\delta(\underline{z})}) + \widehat{P}_2^i(\frac{1 - \widetilde{B}_i^P(\delta(\underline{z}))}{\beta_i\delta(\underline{z})})] \tag{7.5}$$

where,

$$\widehat{P}_1^i = P(\text{served job is transient}|S \text{ in } Q_i) = \frac{b_i f_i^i}{b_i f_i^i + \beta_i g_i^i} = \frac{b_i P_1^i}{b_i P_1^i + \beta_i P_2^i}$$
$$\widehat{P}_2^i = P\big(\text{served job is permanent}|S \text{ in } Q_i\big) = \frac{\beta_i g_i^i}{b_i f_i^i + \beta_i g_i^i} = \frac{\beta_2 P_2^i}{b_i P_1^i + \beta_i P_2^i}$$

Finally,

$$F(\underline{z},\underline{\omega}) = \frac{1}{E[C]}\Bigg\{\sum_{i=1}^{N}\Big[(f_i^i b_i + g_i^i \beta_i)F(\underline{z};\underline{\omega} \mid \text{service period } i) + d_i F(\underline{z};\underline{\omega} \mid \text{switch over period } i)\Big]\Bigg\} \tag{7.6}$$

where $E[C]$ is given by (3.12), and equation (7.3) is substituted in (7.4). Combining (7.1), (7.2), (7.3), (7.4) and (7.5) with (7.6) yields $F(\underline{z};\underline{\omega})$.

First Moments.

First moments are derived by differentiating the appropriate GFs. From (7.1)

$$\gamma_i E[X_i^j] + E[\overline{VX}_i^j] = E[VX_i^j] + \gamma_i E[\overline{X}_i^j] \tag{7.7}$$

$$\gamma_i E[Y_i^j] + E[\overline{VY}_i^j] = E[VY_i^j] + \gamma_i E[\overline{Y}_i^j] \tag{7.8}$$

From (7.2)

$$E[\overline{VX}_i^j] = \begin{cases} E[VX_i^j] + \lambda_j[P_1^i b_i + P_2^i \beta_i] & i \neq j \\ E[VX_i^i] + P_1^i(\rho_i - 1) + P_2^i \lambda_i \beta_i & i = j \end{cases} \tag{7.9}$$

$$E[\overline{VY}_i^j] = \begin{cases} E[VY_i^j] + P_2^i P_{ij} & i \neq j \\ E[VY_i^i] + P_2^i(P_{ii} - 1) & i = j \end{cases} \tag{7.10}$$

From (7.3)

$$f_i^j = E[X_i^j] = E[\overline{X_{i-1}^j}] + \lambda_j d_{i-1} \tag{7.11}$$

$$g_i^j = E[Y_i^j] = E[\overline{Y_{i-1}^j}] \tag{7.12}$$

From (7.4)

$$E\left[X^j \middle| \begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right] = E[\overline{X}_i^j] + \lambda_j \frac{di^{(2)}}{2d_i} \tag{7.13}$$

$$E\left[Y^j \middle| \begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right] = E[\overline{Y}_i^j] \tag{7.14}$$

From (7.5) we get

$$E\left[X^j \middle| \begin{matrix}\text{service}\\ \text{period } i\end{matrix}\right] = E[VX_i^j] + \lambda_j \left[\widehat{P}_1^i \frac{b_i^{(2)}}{2b_i} + \widehat{P}_2^i \frac{\beta_i^{(2)}}{2\beta_i}\right]$$

$$E\left[Y^j \middle| \begin{matrix}\text{service}\\ \text{period } i\end{matrix}\right] = E[VY_i^j] \tag{7.15}$$

Now, $V_i(\underline{z};\underline{\omega})$ can be expressed as follows. From (7.1)

$$\overline{V}_i(\underline{z};\underline{\omega}) - V_i(\underline{z};\underline{\omega}) = \gamma_i[\overline{G}_i(\underline{z};\underline{\omega}) - G_i(\underline{z};\underline{\omega})]$$

Using (7.2) and substituting for $\overline{V}_i(\underline{z};\underline{\omega})$ we get

$$V_i(\underline{z};\underline{\omega})\left[P_1^i\left(\frac{\widetilde{B}_i(\delta(\underline{z}))}{z_i}\right) + P_2^i\left(\frac{P_i(\underline{\omega})}{\omega_i}\right)\widetilde{B}_i^P(\delta(z)) - 1\right] = \gamma_i[\overline{G}_i(\underline{z};\underline{\omega}) - G_i(\underline{z};\underline{\omega})]$$

$$V_i(\underline{z};\underline{\omega})\left[\omega_i P_1^i \widetilde{B}_i^P(\delta(\underline{z})) + z_i P_2^i P_i(\underline{\omega})\widetilde{B}_i^P(\delta(\underline{z})) - z_i\omega_i\right] = \gamma_i z_i \omega_i[\overline{G}_i(\underline{z};\underline{\omega}) - G_i(\underline{z};\underline{\omega})]$$

$$V_i(\underline{z};\underline{\omega}) = \frac{\gamma_i z_i \omega_i\left[\overline{G}_i(\underline{z};\underline{\omega}) - G_i(\underline{z};\underline{\omega})\right]}{\omega_i P_1^i \widetilde{B}_i(\delta(\underline{z})) + z_i P_2^i P_i(\underline{\omega})\widetilde{B}_i^P(\delta(\underline{z})) - z_i\omega_i} \tag{7.16}$$

From (7.13), (7.11) and (3.7), and from (7.14), (7.12) and (3.8), we get

$$E\left[X^j \middle| \begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right] = \begin{cases} f_i^j + \lambda_j(f_i^i b_i + g_i^i \beta_i) + \frac{\lambda_j d_i^{(2)}}{2di} & i \neq j \\ \lambda_i(f_i^i b_i + g_i^i \beta_i) + \frac{\lambda_i d_i^{(2)}}{2di} & i = j \end{cases} \tag{7.17}$$

$$E\left[Y^j \middle| \begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right] = \begin{cases} g_i^j + g_i^i P_{ij} & i \neq j \\ g_i^i P_{ii} & i = j \end{cases} \tag{7.18}$$

From (3.3), (7.3), (7.15) and (7.16), we derive

$$E\left[X^j \Big| {\text{service} \atop \text{period } i}\right] = \begin{cases} \begin{aligned} &\gamma_i \Bigg[\frac{b_i f_i(i,j) + \beta_i r_i(j,i)}{P_1^i b_i + P_2^i \beta_i} \\ &+ \frac{\lambda_j \left[f_i(i,i) b_i^2 + 2 r_i(i,i) b_i \beta_i + g_i(i,i) \beta_i^2 + f_i^i b_i^{(2)} + g_i^i \beta_i^{(2)} \right]}{2(P_1^i b_i + P_2^i \beta_i)} \\ &- \frac{(f_i^i b_i + g_i^i \beta_i)(P_1^i b_i^{(2)} + P_2^i \beta_i^{(2)})}{2(P_1^i b_i + P_2^i \beta_i)} \Bigg] \\ &+ \frac{\lambda_j \left[P_1^i b_i^{(2)} + P_2^i \beta_i^{(2)} \right]}{2(P_1^i b_i + P_2^i \beta_i)} \end{aligned} & i \neq j \\ \begin{aligned} &\gamma_i \Bigg\{ \frac{[f_i^i(\rho_i - 1) + g_i^i \lambda_i \beta_i]}{P_1^i(\rho_i - 1) + P_2^i \lambda_i \beta_i} \\ &+ \frac{\lambda_j^2 [f_i^i b_i^{(2)} + g_i^i \beta_i^{(2)} + 2 r_i(i,i) b_i \beta_i + g_i(i,i) \beta_i^2]}{2[P_1^i(\rho_i - 1) + P_2^i \lambda_i \beta_i]} \\ &+ \frac{f_i(i,i)[\rho_i^2 - 1]}{2[P_1^i(\rho_i - 1) + P_2^i \lambda_i \beta_i]} \\ &- \frac{[f_i^i(\rho_i - 1) + g_i^i \lambda_i \beta_i][P_1^i \lambda_i^2 b_i^{(2)} + P_2^i(2\lambda_i \beta_i + \lambda_i^2 \beta_i^{(2)})]}{2[P_1^i(\rho_i - 1) + P_2^i \lambda_i \beta_i]^2} \Bigg\} \\ &+ \lambda_i \left[\frac{P_1^i b_i^{(2)} + P_2^i \beta_i^{(2)}}{2(P_1^i b_i + P_2^i \beta_i)} \right] \end{aligned} & i = j \end{cases} \tag{7.19}$$

$$E\left[Y^j \Big| {\text{service} \atop \text{period } i}\right] = \begin{cases} \frac{\gamma_i M(M-1) \widehat{\pi}_i(i)}{P_2^i} \left[\widehat{\pi}_j(i) + \frac{\widehat{\pi}_i(i) P_{ij}}{2} \right] & i \neq j \\ \frac{\gamma_i M(M-1)(\widehat{\pi}_i(i))^2 (1 + P_{ii})}{2 P_2^i} & i = j \end{cases} \tag{7.20}$$

Finally, substituting the above appropriately in the following (7.21) and (7.22), we get the first moments of X^j and Y^j:

$$E\left[X^j\right] = \frac{\sum_{i=1}^{N} \left\{ E\left[X^j \Big| {\text{service} \atop \text{period } i}\right] E[T_i] + E\left[X^j \Big| {\text{switch-over} \atop \text{period } i}\right] E[D_i] \right\}}{E[C]} \tag{7.21}$$

and

$$E\left[Y^j\right] = \frac{\sum_{i=1}^{N} \left\{ E\left[Y^j \Big| {\text{service} \atop \text{period } i}\right] E[T_i] + E\left[Y^j \Big| {\text{switch-over} \atop \text{period } i}\right] E[D_i] \right\}}{E[C]} \tag{7.22}$$

where

$$E[T_i] = f_i^i b_i + g_i^i \beta_i = (\lambda_i b_i + \xi_i \beta_i) E[C] = (\rho_i + \rho_i^P) E[C] \ ,$$

$E[D_i] = d_i$, and $E[C] = \frac{d}{1-\rho-\rho^P}$, as given by (3.14). The mean residence time of an arbitrary transient job is given, using Little's law, by

$$E[W_j] = \frac{E[X_j] + E[Y_j]}{\lambda_j + \xi_j} \ . \tag{7.23}$$

The waiting time of such a job is $E[W_j] - b_j$.

7.2 The Exhaustive Regime.

Expected number of permanent jobs served during a visit of $Q_i = \dfrac{g_i^i}{1 - P_{ii}}$.

Expected number of transient jobs served during a visit of $Q_i = (f_i^i + \lambda_i\beta_i \frac{g_i^i}{1-P_{ii}}/(1-\rho_i))$.

Hence, the mean number of all jobs served during a visit of Q_i is

$$\frac{1}{\gamma_i} = \Big[f_i^i + (\lambda_i\beta_i + 1 - \rho_i)\frac{g_i^i}{1-P_{ii}}\Big]/(1-\rho_i), \quad \text{and} \quad \frac{1}{\gamma} = \sum_{i=1}^{N} \frac{1}{\gamma_i} .$$

Similarly to the Gated regime, define the probabilities

$$P_1^i = \frac{f_i^i + \lambda_i\beta_i(g_i^i/(1-P_{ii}))}{f_i^i + (\lambda_i\beta_i + 1 - \rho_i)(g_i^i/(1-P_{ii}))}$$
$$P_2^i = \frac{(g_i^i/(1-P_{ii}))(1-\rho_i)}{f_i^i + (\lambda_i\beta_i + 1 - \rho_i)(g_i^i/(1-P_{ii}))}$$

We use the notation: $\gamma(\underline{z}) = \sum_{\substack{j=1\\ j\neq i}}^{N} \lambda_j(1-z_j)$, $g_i(\underline{\omega}) = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{P_{ij}\omega_j}{1-P_{ii}}$, and $\delta(\underline{z}), P_i(\underline{\omega})$ as for the Gated regime. The following relations exist between the different GFs:

$$\overline{V}_i(\underline{z},\underline{\omega}) = V_i(\underline{z},\underline{\omega})\Big[P_1^i\big(\frac{\widetilde{B}_i(\delta(\underline{z}))}{z_i}\big) + P_2^i\big(\frac{P_i(\underline{\omega})}{\omega_i}\big)\widetilde{B}_i^P(\delta(\underline{z}))\Big] \tag{7.24}$$

$$G_{i+1}(\underline{z};\underline{\omega}) = \overline{G}_i(\underline{z};\underline{\omega})\widetilde{D}_i(\delta(\underline{z})) \tag{7.25}$$

so that

$$\overline{G}_i(\underline{z};\underline{\omega}) = G_i(z_1, z_2, \ldots, z_{i-1}, h_i(\underline{z}), z_{i+1}, \ldots, z_N;$$
$$\omega_1, \omega_2, \ldots, \omega_{i-1}, s_i(\underline{z},\underline{\omega}), \omega_{i+1}, \ldots, \omega_N)$$

where, see (4.3),

$$h_i(\underline{z}) = \widetilde{\theta}_i(\gamma(\underline{z}))$$
$$s_i(\underline{z},\underline{\omega}) = \frac{\widetilde{\Delta}_i(\gamma(\underline{z}))(1-P_{ii})}{1-\widetilde{\Delta}_i(\gamma(\underline{z}))P_{ii}} \cdot g_i(\underline{\omega})$$

$$F\Big(\underline{z};\underline{\omega}\Big|{\text{switch-over} \atop \text{period } i}\Big) = \overline{G}_i(\underline{z};\underline{\omega})\frac{1-\widetilde{D}_i(\delta(\underline{z}))}{d_i\delta(\underline{z}))} \tag{7.26}$$

$$F\Big(\underline{z};\underline{\omega}\Big|{\text{service} \atop \text{period } i}\Big) = V_i(\underline{z};\underline{\omega})\Big[\widehat{P}_1^i\Big(\frac{1-\widetilde{B}_i(\delta(\underline{z}))}{b_i\delta(\underline{z}))}\Big) + \widehat{P}_2^i\Big(\frac{1-\widetilde{B}_i^P(\delta(\underline{z}))}{\beta_i\delta(\underline{z})}\Big)\Big] \tag{7.27}$$

where $\widehat{P}_1^i, \widehat{P}_2^i$ are defined as for the Gated regime, but f_i^i, g_i^i are those corresponding to the Exhaustive regime.

$$F(\underline{z};\underline{\omega}) = \frac{1}{E[C]}\Bigg\{\sum_{i=1}^{N}\Big[f_i^i E[\theta_i] + g_i^i E[\Delta_i]\Big] F\Big(\underline{z};\underline{\omega}\Big|{\text{service} \atop \text{period } i}\Big)$$

$$+ d_i F\left(\underline{z};\underline{\omega}\middle|\begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right)\Bigg\} \tag{7.28}$$

where $E[\theta_i] = \frac{b_i}{1-\rho_i}$, $E[\Delta_i] = \frac{\beta_i}{1-\rho_i}$, $\rho_i = \lambda_i b_i$ as in previous sections.

First Moments. Differentiating the GFs, we get from (7.1) equations (7.7) and (7.8), as for the Gated regime, but with γ_i and all other parameters modified for the Exhaustive case. From (7.24)

$$E\left[\overline{VX_i^j}\right] = \begin{cases} E\left[VX_i^j\right] + \lambda_j\left[P_1^i b_i + P_2^i \beta_i\right] & i \neq j \\ E\left[VX_i^i\right] + P_1^i(\rho_i - 1) + P_2^i \beta_i & i = j \end{cases} \tag{7.29}$$

$$E\left[\overline{VY_i^j}\right] = \begin{cases} E\left[VY_i^j\right] + P_2^i P_{ij} & i \neq j \\ E\left[VY_i^i\right] + P_2^i(P_{ii} - 1) & i = j \end{cases} \tag{7.30}$$

From (7.25)

$$E\left[X_{i+1}^j\right] = E\left[\overline{X_i^j}\right] + \lambda_j d_i$$
$$E\left[Y_{i+1}^j\right] = E\left[\overline{Y_i^j}\right] \tag{7.31}$$

Specifically, for the Exhaustive regime,

$$\overline{X_i^i} = 0\ , \quad X_{i+1}^i = \lambda_i d_i$$
$$\overline{Y_i^i} = 0\ , \quad Y_{i+1}^i = 0$$

From (7.26)

$$E\left[X^j\middle|\begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right] = \begin{cases} E\left[\overline{X_i^j}\right] + \frac{\lambda_j d_i^{(2)}}{2d_i} & i \neq j \\ \frac{\lambda_i d_i^{(2)}}{2d_i} & i = j \end{cases} \tag{7.32}$$

$$E\left[Y^j\middle|\begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right] = \begin{cases} E\left[\overline{Y_i^j}\right] & i \neq j \\ 0 & i = j \end{cases} \tag{7.33}$$

From (7.27)

$$E\left[X^j\middle|\begin{matrix}\text{service}\\ \text{period } i\end{matrix}\right] = E\left[VX_i^j\right] + \lambda_j\left[\widehat{P}_1^i \frac{b_i^2}{2b_i} + \widehat{P}_2^i \frac{\beta_i^{(2)}}{2\beta_i}\right] \tag{7.34}$$

$$E\left[Y^j\middle|\begin{matrix}\text{service}\\ \text{period } i\end{matrix}\right] = E\left[VY_i^j\right] \tag{7.35}$$

$V_i(\underline{z};\underline{\omega})$ has the same form as (7.16), with the relevant modifications for the Exhaustive regime. Using (7.32) and (7.33) we get

$$E\left[X^j\middle|\begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right] = \begin{cases} f_i^j + \frac{\lambda_j}{1-\rho_i}\left(f_i^i b_i + g_i^i \beta_i\right) + \frac{\lambda_j d_i^{(2)}}{2d_i} & i \neq j \\ \frac{\lambda_i d_i^{(2)}}{2d_i} & i = j \end{cases} \tag{7.36}$$

$$E\left(Y^j\middle|\begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\right) = \begin{cases} g_i^j + \frac{P_{ij}}{1-P_{ii}} g_i^i & i \neq j \\ 0 & i = j \end{cases} \tag{7.37}$$

From (7.34) and (7.35)

$$E\left(X^j\,\Big|\,\begin{matrix}\text{service}\\ \text{period } i\end{matrix}\right) = \begin{cases} \gamma_i\left\{ \dfrac{\frac{\lambda_j}{1-\rho_i}\left(b_i[f_i(i,j)+f_i(j,i)]+\frac{\beta_i}{1-P_{ii}}[r_i(i,j)+r_i(j,i)]\right)}{2[P_1^i(\rho_i-1)+P_2^i\lambda_i\beta_i]} \right. \\ \quad + \dfrac{\frac{\lambda_j^2}{1-\rho_i}\left(f_i^i\frac{b_i^{(2)}}{(1-\rho_i)^2}+\frac{g_i^i}{1-P_{ii}}\left[\beta_i^{(2)}+\frac{\lambda_i\beta_i b_i^{(2)}}{(1-\rho_i)^2}+\frac{2P_{ii}\beta_i^2}{(1-\rho_i)(1-P_{ii})}\right]\right)}{2[P_1^i(\rho_i-1)+P_2^i\lambda_i\beta_i]} \\ \quad + \dfrac{\frac{\lambda_j^2}{(1-\rho_i)^2}\left[f_i(i,i)b_i^2+\frac{2r_i(i,i)b_i\beta_i}{1-P_{ii}}+\frac{g_i(i,i)\beta_i^2}{(1-P_{ii})^2}\right]}{2[P_1^i(\rho_i-1)+P_2^i\lambda_i\beta_i]} \\ \quad \left. - \dfrac{\left(f_i^i b_i+\frac{g_i^i\beta_i}{1-P_{ii}}\right)\left(P_1^i b_i^{(2)}+P_2^i\beta_i^{(2)}\right)}{2(1-\rho_i)(P_1^i b_i+P_2^i\beta_i)} \right\} + \dfrac{\lambda_j\left(P_1^i b_i^{(2)}+P_2^i\beta_i^{(2)}\right)}{2(P_1^i b_i+P_2^i\beta_i)} & i \neq j \\[2ex] \gamma_i\left\{ \dfrac{f_i^i\left[P_1^i\lambda_i^2 b_i^{(2)}+P_2^i\left(2\lambda_i\beta_i+\lambda_i\beta_i^{(2)}\right)\right]}{2\left[P_1^i(\rho_i-1)+P_2^i\lambda_i\beta_i\right]^2} \right. \\ \quad \left. - \dfrac{2f_i^i+f_i(i,i)}{2\left[P_1^i(\rho_i-1)+P_2^i\lambda_i\beta_i\right]} \right\} + \dfrac{\lambda_i\left(P_1^i b_i^{(2)}+P_2^i\beta_i^{(2)}\right)}{2\left(P_1^i b_i+P_2^i\beta_i\right)} & i = j \end{cases} \tag{7.38}$$

$$E\left(Y^j\,\Big|\,\begin{matrix}\text{service}\\ \text{period } i\end{matrix}\right) = \begin{cases} (M-1)\left[\widehat{\pi}_j(i) + \frac{\widehat{\pi}_i(i)}{2}\frac{P_{ij}}{1-P_{ii}}\right] & i \neq j \\ 1 + \frac{(M-1)\widehat{\pi}_i(i)}{2} & i = j \end{cases} \tag{7.39}$$

The first moments $E\left[X^j\right]$, $E\left[Y^j\right]$ are obtained as in (7.21), (7.22), but with the parameters of the Exhaustive regime, where

$$\begin{aligned} E\left[T_i\right] &= f_i^i E\left[\theta_i\right] + g_i^i E\left[\Delta_i\right] = f_i^i\frac{b_i}{1-\rho_i} + g_i^i\frac{\beta_i}{1-\rho_i} \\ &= \left[\rho_i(1-\rho_i-\rho_i^P) + (1-P_{ii})\rho_i^P\right]\frac{E[C]}{1-\rho_i}\ . \end{aligned}$$

$E[W_j]$ is calculated as for the Gated regime, but with $\lambda_j, \xi_j, E[X^j], E[Y^j]$ of the Exhaustive case.

7.3 The Globally Gated Regime.

In order to calculate the mean number of customers at an arbitrary moment under the GG regime we first have to derive the GF $G_i(\underline{z};\underline{\omega})$ for the system-state at polling instants of *each* station i, in addition to the derivation of $G_1(\underline{z};\underline{\omega})$ as given by (5.5). We define

$$G_i(\underline{z};\underline{\omega}) = E\left[\Pi_{j=1}^N z_j^{X_i^j}\,\Pi_{j=1}^N \omega_j^{Y_i^j}\right] \tag{7.40}$$

The laws of motion describing the evolution of the system-state are given by

$$X_i^j = \begin{cases} X_1^j + \sum\limits_{n=1}^{i-1}\left[A_j\left(\sum\limits_{k=1}^{X_1^n} B_{nk} + \sum\limits_{m=1}^{Y_1^n} B_{nm}^P + D_n\right)\right] & 1 \le i \le j \le N \\ \sum\limits_{n=1}^{i-1}\left[A_j\left(\sum\limits_{k=1}^{X_1^n} B_{nk} + \sum\limits_{m=1}^{Y_1^n} B_{nm}^D + D_n\right)\right] & i > j \end{cases} \tag{7.41}$$

$$Y_i^j = \begin{cases} Y_1^j + \sum\limits_{n=1}^{i-1} K_n^j(Y_1^n) & \underset{1\le i,j\le N}{i\le j} \\ \sum\limits_{n=1}^{i-1} K_n^j(Y_1^n) & i > j \end{cases} \tag{7.42}$$

Define, similarly to the definitions in Section 3,

$$BXB^PYD_n \equiv \sum_{k=1}^{X_1^n} B_{nk} + \sum_{m=1}^{Y_1^n} B_{nm}^P + D_n$$

$$BX_n \equiv \sum_{k=1}^{X_1^n} B_{nk}; \quad B^PY_n \equiv \sum_{m=1}^{Y_1^n} B_{nm}^P$$

Then, by using (7.41) and (7.42), we have

$$\begin{aligned}
G_i(\underline{z};\underline{\omega}) &= E\Big[\Big(\prod_{j=1}^{i-1} z_j^{\sum_{n=1}^{i-1} A_j(BXB^PYD_n)}\Big)\cdot\Big(\prod_{j=i}^{N} z_j^{X_1^j+\sum_{n=1}^{i-1} A_j(BXB^PYD_n)}\Big) \\
&\quad \cdot\Big(\prod_{j=1}^{i-1} \omega_j^{\sum_{n=1}^{i-1} K_n^j(Y_1^n)}\Big)\Big(\prod_{j=i}^{N} \omega_j^{Y_1^j+\sum_{n=1}^{i-1} K_n^j(Y_1^n)}\Big)\Big] \\
&= E\Big[\Big(\prod_{j=1}^{N} z_j^{\sum_{n=1}^{i-1} A_j(BXB^PYD_n)}\Big)\Big(\prod_{j=i}^{N} z_j^{X_1^j}\Big)\cdot\Big(\prod_{j=1}^{N} \omega_j^{\sum_{n=1}^{i-1} K_n^j(Y_1^n)}\Big)\Big(\prod_{j=i}^{N} \omega_j^{Y_1^j}\Big)\Big] \\
&= E\Big[\Big(\prod_{j=1}^{N} z_j^{\sum_{n=1}^{i-1} A_j(BX_n)}\Big)\Big(\prod_{j=1}^{n} z_j^{\sum_{n=1}^{i-1} A_j(B^PY_n)}\Big)\Big(\prod_{j=i}^{N} z_j^{X_1^j}\Big) \\
&\quad \cdot\Big(\prod_{j=1}^{N} \omega_j^{\sum_{n=1}^{i-1} K_n^j(Y_1^n)}\Big)\Big(\prod_{j=i}^{N} \omega_j^{Y_1^j}\Big)\Big]\cdot E\Big[\prod_{j=1}^{N} z_j^{\sum_{n=1}^{i-1} A_j(D_n)}\Big] \\
&= E\Big[\Big(\prod_{j=1}^{N}\Big(\prod_{n=1}^{i-1} z_j^{A_j(BX_n)}\Big)\Big)\Big(\prod_{j=1}^{N}\Big(\prod_{n=1}^{i-1} z_j^{A_j(B^PY_n)}\Big)\Big)\Big(\prod_{j=i}^{N} z_j^{X_1^j}\Big) \\
&\quad \cdot\Big(\prod_{j=1}^{N}\Big(\prod_{n=1}^{i-1} \omega_j^{K_n^j(Y_1^n)}\Big)\Big)\Big(\prod_{j=i}^{N} \omega_j^{Y_1^j}\Big)\Big]E\Big[\prod_{j=1}^{N}\Big(\prod_{n=1}^{i-1} z_j^{A_j(D_n)}\Big)\Big] \\
&= E\Big[\Big(\prod_{n=1}^{i-1}\Big(\widetilde{B}_n(\delta(\underline{z}))\Big)^{X_1^n}\Big)\Big(\prod_{j=i}^{N} z_j^{X_1^j}\Big)\Big(\prod_{n=1}^{i-1}\Big(\widetilde{B}_n^P(\delta(\underline{z}))\Big)^{Y_1^n}\Big) \\
&\quad \cdot\Big(\prod_{n=1}^{i-1}\Big(P_n(\underline{\omega})\Big)^{Y_1^n}\Big)\Big(\prod_{j=i}^{N} \omega_j^{Y_1^j}\Big)\Big]\cdot\Big(\prod_{n=1}^{i-1} \widetilde{D}_n(\delta(\underline{z}))\Big)
\end{aligned} \tag{7.43}$$

where

$$\delta(\underline{z}) = \sum_{j=1}^{N} \lambda_j(1-z_j)\ , \quad P_n(\underline{\omega}) = \sum_{j=1}^{N} P_{nj}\omega_j$$

as used in previous sections. Equation (7.43) implies

$$\begin{aligned}
G_i(\underline{z};\underline{\omega}) = G_1\Big(&\widetilde{B}_1(\delta(\underline{z})), \widetilde{B}_2(\delta(\underline{z})), \ldots, \widetilde{B}_{i-1}(\delta(\underline{z})), z_i, z_{i+1}, \ldots, z_N; \\
&\widetilde{B}_1^P(\delta(\underline{z}))\underline{P}_1(\underline{\omega}), \widetilde{B}_2^P(\delta(\underline{z}))P_2(\underline{\omega}), \ldots, \widetilde{B}_{i-1}^P(\delta(\underline{z}))P_{i-1}(\underline{\omega}), \omega_i, \omega_{i+1}, \ldots, \omega_n\Big) \\
&\cdot \Pi_{n=1}^{i-1} \widetilde{D}_n(\delta(\underline{z}))
\end{aligned} \tag{7.44}$$

Setting $i = N+1$ in (7.44) we get $G_1(\underline{z};\underline{\omega}) = G(\underline{z};\underline{\omega})$ as in (5.5), the GF of the system state at the beginning of a cycle. f_i^j, g_i^j are defined as before. From (7.44), (also from (7.41) and (7.42)) we obtain

$$f_i^j = \begin{cases} f_1^j + \sum_{n=1}^{i-1} \lambda_j \Big[b_n f_1^n + \beta_n g_1^n + d_n \Big] & i \le j \\ \sum_{n=1}^{i-1} \lambda_j \Big[b_n f_1^n + \beta_n g_1^n + d_n \Big] & i > j \end{cases} \tag{7.45}$$

$$g_i^j = \begin{cases} g_1^j + \sum_{n=1}^{i-1} g_1^n P_{nj} & i \le j \\ \sum_{n=1}^{i-1} g_1^n P_{nj} & i > j \end{cases} \tag{7.46}$$

For the Globally Gated regime we have

$$\frac{1}{\gamma_i} = f_1^i + g_1^i, \quad \frac{1}{\gamma} = \sum_{i=1}^{N} f_1^i + \sum_{i=1}^{N} g_1^i = \lambda E[C] + M$$

The probabilities P_1^i, P_2^i are defined as before, so that

$$P_1^i = \frac{f_1^i}{f_1^i + g_1^i} = f_1^i \cdot \gamma_i, \quad P_2^i = \frac{g_1^i}{f_1^i + g_1^i} = g_1^i \cdot \gamma_i$$

The connections between the GF's are

$$\overline{V}_i(\underline{z};\underline{\omega}) = V_i(\underline{z};\underline{\omega}) \Big[P_1^i \frac{\widetilde{B}_i(\delta(\underline{z}))}{z_i} + P_2^i \Big(\frac{P_i(\underline{\omega})}{\omega_i} \Big) \widetilde{B}_i^P(\delta(\underline{z})) \Big] \tag{7.47}$$

$$G_i(\underline{z};\underline{\omega}) = \overline{G}_{i-1}(\underline{z};\underline{\omega}) \widetilde{D}_{i-1}(\delta(\underline{z})) \tag{7.48}$$

where

$$\begin{aligned} \overline{G}_i(\underline{z};\underline{\omega}) = G_1\Big(& \widetilde{B}_1(\delta(\underline{z})), \widetilde{B}_2(\delta(\underline{z})), \ldots, \widetilde{B}_i(\delta(\underline{z})), z_{i+1}, \ldots, z_N; \\ & \widetilde{B}_1^P(\delta(\underline{z})) P_1(\underline{\omega}), \ldots, \widetilde{B}_i^P(\delta(\underline{z})) P_i(\underline{\omega}), \omega_{i+1}, \ldots, \omega_N \Big) \\ & \Pi_{n=1}^{i-1} \widetilde{D}_n(\delta(\underline{z})) \end{aligned}$$

$$\begin{aligned} F(\underline{z};\underline{\omega}) = \sum_{i=1}^{N} \Big[& F\Big(\underline{z};\underline{\omega} \Big| \begin{matrix}\text{service}\\ \text{period } i\end{matrix}\Big) \frac{f_1^i b_i + g_1^i \beta_i}{E[C]} \\ & + F\Big(\underline{z};\underline{\omega} \Big| \begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\Big) \frac{d_i}{E[C]} \Big] \end{aligned} \tag{7.49}$$

$$F\Big(\underline{z};\underline{\omega} \Big| \begin{matrix}\text{switch-over}\\ \text{period } i\end{matrix}\Big) = \overline{G}_i(\underline{z};\underline{\omega}) \frac{1 - D_i(\delta(\underline{z}))}{d_i \delta(\underline{z})} \tag{7.50}$$

$$F\Big(\underline{z};\underline{\omega} \Big| \begin{matrix}\text{service}\\ \text{period } i\end{matrix}\Big) = V_i(\underline{z};\underline{\omega}) \Big[\widehat{P}_1^i \Big(\frac{1 - \widetilde{B}_i(\delta(\underline{z}))}{b_i \delta(\underline{z})} \Big) + \widehat{P}_2^i \Big(\frac{1 - \widetilde{B}_i^P(\delta(\underline{z}))}{\beta_i \delta(\underline{z})} \Big) \Big] \tag{7.51}$$

where

$$\widehat{P}_1^i = \frac{b_i f_1^i}{b_i f_1^i + \beta_i g_1^i}, \quad \widehat{P}_2^i = \frac{\beta_i g_1^i}{b_i f_1^i + \beta_i g_1^i}.$$

$V_i(\underline{z};\underline{\omega})$ is expressed as in (7.16) with γ_i, GF's, P_1^i and P_2^i corresponding to the Globally Gated regime. First moments are obtained as in (7.7) through (7.15). By differentiating (7.47) through (7.51) we get the conditioned first moments of X^j and Y^j (similarly to (7.17)-(7.20)) and the unconditioned first moments (as in (7.21) and (7.22)). $E[W_j]$ is calculated as in (7.23) with the appropriate parameters.

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References

1. Altman, E. and Yechiali, U. (1994). "Polling in a Closed Network". *Probability in the Engineering and Informational Sciences*, **8**, 327-343.
2. Boxma, O.J. and Cohen, J.W. (1991). "The $M|G|1$ Queue with Permanent Customers". *IEEE Journal on Selected Areas in Communications*, **9**, 179-184.
3. Boxma, O.J., Levy, H. and Yechiali, U. (1992). "Cyclic Reservation Schemes for Efficient Operation of Multiple-Queue Single-Server Systems". *Annals of Operations Research*, **35**, 187-208.
4. Boxma, O.J., Weststrate, J.A., and Yechiali, U. (1993). "A Globally Gated Polling System with Server Interruptions, and Applications to the Repairman Problem". *Probability in the Engineering and Informational Sciences* **7**, 187-208.
5. Browne, S. and Yechiali, U. (1989). "Dynamic Priority rules for Cyclic-Type Queues". *Advances in Applied Probability*, **21**, 432-450.
6. Eisenberg, M. (1972). "Queues with Periodic Service and Changeover Time". *Operations Research*, **20**, 440-451.
7. Levy, H. and Sidi, M. (1990). "Polling Systems: Applications, Modeling and Optimization". *IEEE Transactions on Communications*, **8**, 1750-1760.
8. Sidi, M., Levy, H. and Fuhrmann, S.W. (1992). "A Queueing Network with a Single Cyclically Roving Server". *Queueing Systems*, **11**, 121-144.
9. Takagi, H. (1986). *Analysis of Polling Systems*. MIT Press.
10. Takagi, H. (1990). "Queueing Analysis of Polling Models: an Update". In: *Stochastic Analysis of Computer and Communications Systems*, H. Takagi, (ed.), North Holland, pp. 267-318.

11. Yechiali, U. (1993). "Analysis and Control of Polling Systems". In: *Performance Evaluation of Computer and Communication Systems*, L. Donatiello and R. Nelson (Eds.), Springer Verlag, pp. 630-650.

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