OPTIMAL REPAIR AND REPLACEMENT

IN MARKOVIAN SYSTEMS

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ABSTRACT

A deteriorating system is inspected at equally spaced points in time. After each inspection a repair to a better state or a full replacement are possible. We introduce a *generalized* control limit policy and show that, under reasonable conditions on the system's law of evolution and on the operating, repair and replacement costs, this policy is optimal for the expected total discounted cost, as well as for the (long-run) average cost criteria. Furthermore, we allow for *uncertain* repair actions (that may even end up in a worse state), and show that the generalized control limit rule is still optimal. The work extends, generalizes and unifies many models in the area of optimal control of repairable systems.

Key Words: maintenance policies, replacement, Markovian deterioration, uncertain repair

1. INTRODUCTION

A classical replacement problem is to find the optimal replacement policy for a system under Markovian deterioration. The system is inspected at times $t = 1, 2, \ldots$ and is classified (in a discrete case) into one of a finite number of states $\{0, 1, 2, \ldots N\}$, or (in a continuous case) into some state x within a given interval. The system suffers deterioration over time and the dichotomy after each inspection is whether to replace it or not. For the discrete case Derman [4] and Kolesar [11] established conditions on the transition probabilities and the cost functions under which the optimal policy is a control limit rule. Under such a rule, a replacement takes place if and only if the system is observed at state $i > i^*$, where i^* is the control limit. Ross [16] generalized the above models and extended the results to the continuous state-space case. These models have wide applicability and were extended to other areas, including reorganization of data bases (cf. Mendelson and Yechiali [13]).

In many real-world systems, however, replacement is not the only action possible. Often, one considers several degrees of repair, where a full replacement is only one of many options. In recent years various models for repairable systems with imperfect repair have been suggested (to mention a few, see Brown and Prochan [2], Block et al. [1], Yeh [19], [20], Kijima [10], Rangan and Grace [14]). In these models, *upon failure*, a maintenance action is performed and its outcome is either (with probability p) a *perfect* repair (equivalent to a full replacement), or (with probability $1 - p$) an *imperfect* repair which restores the failed system to its condition just prior to failure. Other models that allow for maintenance actions apart from full replacement were examined by Chikte and Deshmukh [3] and Zuckerman [21]. In these models the system is subjected to a shock-process causing deterioration over time. The system fails when the accumulated deterioration exceeds some threshold, whereupon it is replaced. Several maintenance actions are allowed which reduce the rate of damage accumulation. Kijima et al. [9] considered a periodical replacement problem where a system is replaced only at scheduled times kT $(k = 0, 1, \ldots)$ and is repaired whenever it fails in between. Upon failure, the repair may restore the system to its functioning condition just prior to failure (minimal repair), or reduce the system's age. The objective is to find an optimal replacement interval which minimizes the long-run expected cost. Recently, Stadje and Zuckerman [17] studied maintenance strategies with general degree of repair. Actions are taken *only* when the system fails. A failed system whose *age* is x can be restored to an operative system with an equivalent age $x-d$, where the degree of repair $0 \leq d \leq x$ is a decision variable determined by the controller. Using reasonable assumptions on the repair-upon-failure cost function and the system's law of deterioration, they established analytical and numerical methods for determining optimal maintenance strategies, and examined their structure.

In this paper we generalize and extend the above models for both the discrete and the continuous state spaces. We allow for a general-degree repair action from *any* state to *any* better state at *any* time of inspection. We consider state-dependent operating costs, as well as repair costs depending on the degree of repair. Two classical criteria are investigated: (i) minimizing the total expected discounted cost and (ii) minimizing the (long-run) average cost per unit of time.

We introduce a *generalized* control limit rule defined as follows: repair to a better state (or replace) if and only if the state of the system exceeds some control-limit state. We show that, under reasonable conditions on the system's transition laws and the cost functions, the optimal policy has the structure of a generalized control limit rule. We further extend the model (generalizing the concept of imperfect repair) to include situations where the result of a repair action is *uncertain* and a planned repair to a given state eventually ends up in some *other* state (which can even be a *worse* state). Surprisingly, the generalized control limit rule remains optimal in this case too.

The structure of the paper is as follows. In section 2 we study the discrete state-space case and derive the optimality of the generalized control limit policy for certain, as well as uncertain, repair actions. A brief presentation of the continuous state-space case is given in Section 3. This work generalizes, extends and unifies many studies in the area of optimal control of repairable systems.

2. DISCRETE STATE-SPACE

The Model

Consider a system (a unit, a piece of operating equipment, etc.) which is inspected at equally spaced points in time. After each inspection the system is classified into one of $N+1$ states: $0, 1, \ldots, N$. State 0 denotes a new (or functioning as good as new) system, whereas state N denotes a failed system. State i is better than state j if $i < j$. Let the times of inspection be $t = 1, 2, \ldots,$ and let X_t denote the observed state of the unit at time t. The infinite sequence $\{X_t \mid t = 1, 2, \ldots,\}$ is a finite-state Markov chain with stationary transition probabilities

$$p_{ij} = P(X_{t+1} = j \mid X_t = i) \quad \text{for all} \quad i, j \quad \text{and} \quad t\ .$$

A failed system must be replaced immediately by a new one, and the replacement is instantaneous. We suppose that, for each $i = 0, 1 \ldots N$, $p_{iN}^{(t)} > 0$ for some t. This condition assures that the system eventually reaches the failure state regardless of its initial state.

At each state $0 < i < N$ the system can be replaced by a new one, or be repaired such that its state after the repair is $k < i$. We assume that a repair (as well as an initiated replacement) takes no time. Clearly, the motivation for performing a repair or a replacement is to prevent the severe consequences of a failure, or of letting the unit operate under 'bad' conditions.

Denote by A_i the collection of maintenance actions which are possible when the system is at state i $(1 \le i \le N-1)$. We assume that, for each i, $A_i = \{0, 1, 2, \ldots, i\}$. An action m $(m \in A_i)$ means a repair of the system m stages "backwards", i.e. changing its state from i to $i - m$ ($m = 0$ means no repair at all, $m = i$ means a replacement of the system by a new one). A maintenance rule, denoted by R, consists of the maintenance activities to be performed at the various states. The maintenance rule R controls the behavior of the system and results in a modified Markov chain $\{X_t(R) \mid t = 1, 2, \ldots,\}$, governing the evolution of the system according to modified transition probabilities $p_{ij}(R)$:

$$p_{ij}(R) = p_{ij} \qquad \text{if no activity is performed at state } i\ .$$

$$p_{ij}(R) = p_{kj} \qquad \text{if a repair is performed from state } i \text{ to state } k < i\ .$$

Suppose that when the system is observed at state i, and action $0 \leq m \leq i$ is taken (changing the state to $k = i - m$), an expected operating cost $r_k \geq 0$ is incurred until the next inspection. The repair action itself costs $c_{ik} \geq 0$ (c_{i0} is the cost of replacing the system by a new one, and $c_{ii} = 0$). The goal of this work is to derive and characterize optimal maintenance (repair and replacement) rules for the following two criteria:

1. Total expected discounted cost for unbounded horizon.
2. Long-run average expected cost per unit of time.

Conditions

We impose the following conditions on the costs and the transition probabilities:

Condition 2.1. For each i, j, k such that $N - 1 \geq j > i \geq k \geq 0$, $c_{jk} \geq c_{ik}$. That is, the cost of a repair to a certain state, $k > 0$, as well as the cost of initiated replacement, is an increasing function of the state from which the repair is performed. Furthermore, $c_{N0} \geq c_{i0}$ for $N > i$, i.e., an initiated replacement costs less than the mandatory replacement in the case of a failure.

Condition 2.2. $r_0 \leq r_1 \leq \cdots \leq r_{N-1}$. That is, as the state of the system deteriorates, the operating cost increases.

Condition 2.3. For each i, $i < N$, and for each $k < i$, $c_{ik} + r_k \geq r_i$.

Condition 2.3 implies that for a one-period horizon, it does not pay to perform any repair. This condition is valid for systems in which the cost of a repair (composed of spare parts, repair-crew costs, etc.) is large relative to the operating cost.

Condition 2.4 (an IFR assumption). For each $k = 0, 1, 2, \ldots, N$ the function

$$D_k(i) = \sum_{j=k}^{N} p_{ij}$$

is nondecreasing in i ($i = 0, 1, 2, \ldots, N - 1$). From Derman [4], Condition 2.4 is *equivalent* to the following:

Condition 2.5. For every nondecreasing function $h(j)$, $j = 0, 1, \ldots, N$, the function

$$K(i) = \sum_{j=0}^{N} p_{ij} h(j)$$

is also nondecreasing $(i = 0, 1, 2, \ldots, N-1)$.

A Generalized Control Limit Rule

We focus our attention on the class of non-randomized stationary maintenance rules. It is known that for a finite state and a finite action space, there exists an optimal policy which depends only on the state of the system at decision epochs. We shall direct our attention to a subclass of the non-randomized stationary rules, which we call *generalized control limit rules*. A generalized control limit rule is a maintenance rule of the form:
"Repair (or replace) the system, at time t, if and only if $X_t \geq i^*$", where i^* is the control limit $(0 \leq i^* \leq N)$.

Remark. The rule is termed "generalized" because it is a generalization of the well known control limit rule (see Derman [4], Kolesar [11], Ross [16]), by which the unit is *replaced* whenever the state of the system exceeds some control limit. The "generalization" is the addition of repair actions which are not necessarily full replacement. The knowledge that the optimal maintenance policy is a generalized control limit rule is highly valuable, since it reduces considerably the search for an optimal policy.

Denote by $g_R(X_t)$ the one-step *expected cost* when the system is in state X_t at time t under the maintenance rule R. Let α be a discount factor, $0 < \alpha < 1$. Denote by $\phi_R(i, \alpha)$ the total expected discounted cost for unbounded horizon if the system starts in state i and maintenance rule R is used. Then,

$$\phi_R(i, \alpha) = \left\{ \sum_{t=1}^{\infty} \alpha^{t-1} g_R(X_t) \mid X_1 = i \right\} \ . \tag{2.1}$$

For a given discount factor α denote the optimal maintenance policy for criterion (2.1) by R^*_α, and denote the total discounted minimal cost by $\phi(i, \alpha)$, that is,

$$\phi(i, \alpha) = \min_R \phi_R(i, \alpha) = \phi_{R^*_\alpha}(i, \alpha) \ . \tag{2.2}$$

When the operation horizon is finite (T periods, say), denote by $\phi(i, \alpha, T)$ the minimal total expected discounted cost when the system starts in state i:

$$\phi(i, \alpha, T) = \min_R \left\{ \sum_{t=1}^{T} \alpha^{t-1} g_R(X_t) \mid X_1 = i \right\} . \tag{2.3}$$

In a problem with no discount, denote by ϕ_R the long-run average expected cost per time unit under the maintenance rule R,

$$\phi_R = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} g_R(X_t) . \tag{2.4}$$

Using standard arguments of Dynamic Programming, it can be shown that $\phi(i, \alpha)$ satisfies the functional set of equations:

$$\begin{aligned}
\phi(0, \alpha) &= r_0 + \alpha \sum_{j=0}^{N} p_{0j} \phi(j, \alpha) \\
\phi(i, \alpha) &= \min \left\{ r_i + \alpha \sum_{j=0}^{N} p_{ij} \phi(j, \alpha), \min_{0 \le k \le i-1} \left[c_{ik} + r_k + \alpha \sum_{j=0}^{N} p_{kj} \phi(j, \alpha) \right] \right\} , \\
&\qquad 0 < i < N \\
\phi(N, \alpha) &= c_{N0} + r_0 + \alpha \sum_{j=0}^{N} p_{0j} \phi(j, \alpha) .
\end{aligned} \tag{2.5}$$

In a similar manner the following set of successive approximations can be derived:

$$\begin{aligned}
\phi(0, \alpha, T) &= r_0 + \alpha \sum_{j=0}^{N} p_{0j} \phi(j, \alpha, T-1) \\
\phi(i, \alpha, T) &= \min \left\{ r_i + \alpha \sum_{j=0}^{N} p_{ij} \phi(j, \alpha, T-1), \right. \\
&\qquad \left. \min_{0 \le k \le i-1} \left[c_{ik} + r_k + \alpha \sum_{j=0}^{N} p_{kj} \phi(j, \alpha, T-1) \right] \right\} , \ 0 < i < N \\
\phi(N, \alpha, T) &= c_{N0} + r_0 + \alpha \sum_{j=0}^{N} p_{0j} \phi(j, \alpha, T-1) .
\end{aligned} \tag{2.6}$$

The initial conditions are derived by the use of condition 2.3 and the fact that at state N the system must be replaced:

$$\begin{aligned} &\phi(i,\alpha,1) = r_i \qquad 0 \le i \le N-1 \\ &\phi(N,\alpha,1) = c_{N0} + r_0 \ . \end{aligned} \tag{2.7}$$

The following lemma exhibits the monotonic property of $\phi(i,\alpha,T)$.

Lemma 2.1. *For fixed α and T $(0 < \alpha < 1,\ T \ge 1)$, $\phi(i,\alpha,T)$ is a nondecreasing function of i, $(i = 0,1,2,\ldots,N)$.*

Proof: The proof is by induction. For $T = 1$, the set (2.7) applies. The monotonicity in $0 \le i \le N-1$ follows trivially from condition 2.2. Condition 2.1 implies $c_{N0} \ge c_{N-1,0}$, and condition 2.3 implies $c_{N-1,0} + r_0 \ge r_{N-1}$. Therefore, $c_{N0} + r_0 \ge r_{N-1}$, which completes the proof for the case $T = 1$.

We suppose now that $\phi(i,\alpha,T-1)$ is a nondecreasing function of i and show that $\phi(i,\alpha,T) \le \phi(i+1,\alpha,T)$. Let $0 \le i \le N-1$. Then, for every maintenance policy R, only one of two possibilities exists at state $i+1$: either a repair (or a replacement) is performed, or no maintenance activity is performed at all. We consider each case separately.

1. If at state $i+1$ no maintenance action is taken then

$$\begin{aligned} \phi(i+1,\alpha,T) &= r_{i+1} + \alpha \sum_{j=0}^{N} p_{i+1,j}\phi(j,\alpha,T-1) \\ &\ge r_i + \alpha \sum_{j=0}^{N} p_{ij}\phi(j,\alpha,T-1) \ge \phi(i,\alpha,T) \ . \end{aligned}$$

The first inequality follows from condition 2.2 and condition 2.5 (using the monotonicity of $\phi(i,\alpha,T-1)$). The second inequality follows trivially from equations (2.6).

2. If at state $i+1$ a repair (or replacement) is performed, bringing the system to state k, $0 \le k < i+1$, it follows from condition 2.1 that

$$\begin{aligned} \phi(i+1,\alpha,T) &= c_{i+1,k} + r_k + \alpha \sum_{j=0}^{N} p_{kj}\phi(j,\alpha,T-1) \\ &\ge c_{ik} + r_k + \alpha \sum_{j=0}^{N} p_{kj}\phi(j,\alpha,T-1) \ . \end{aligned}$$

Again, (2.6) implies that, for $0 \le k \le i$,

$$c_{ik} + r_k + \alpha \sum_{j=0}^{N} p_{kj} \phi(j, \alpha, T-1) \ge \phi(i, \alpha, T)$$

(where $c_{ii} \equiv 0$). Therefore, $\phi(i+1, \alpha, T) \ge \phi(i, \alpha, T)$. Q.E.D.

A sufficient condition for the existence of an optimal rule, which is a generalized control limit policy, is given in the following theorem:

Theorem 2.1. *Suppose that for all i, k, v, such that $k < i < v < N$, $c_{vk} - c_{ik} \le r_v - r_i$. Then, for both optimality criteria, the optimal policy has the form of a generalized control limit rule.*

We first need the following proposition regarding $\phi(i, \alpha, T)$.

Proposition. *Under the condition of theorem 2.1, if at state $0 < i < N$, the minimum of (2.6) is achieved either by a repair to some state $1 \le k < i$ or by a replacement, then, for every state $j > i$, the minimum of (2.6) is also achieved by a repair or a replacement.*

(For convenience we shall regard 'replacement' in the sequel, as a 'repair' to the 0-state).

Proof: (by induction on T).
$T = 1$: By the initial conditions (2.7), N is the only state at which a repair (i.e., replacement) is beneficial. Suppose that the proposition holds for $T-1$. We shall show that it is also true for T. Suppose that for state i, $i \le N-1$, the minimum of equations (2.6) is achieved by repairing the system to a state k, $k < i$ (recall, $k = 0$ means a replacement). From equations (2.6) it follows that

$$\phi(i, \alpha, T) = c_{ik} + r_k + \alpha \sum_{j=0}^{N} p_{kj} \phi(j, \alpha, T-1) \le r_i + \alpha \sum_{j=0}^{N} p_{ij} \phi(j, \alpha, T-1) \ . \tag{2.8}$$

Now, if $i = N-1$, it is clear that the proposition is true since the only larger state than i is N, for which a repair is mandatory. If $i < N-1$, let v be such that $i < v < N$ and $c_{vk} - c_{ik} \le r_v - r_i$. Adding this last expression to both sides of the inequality in (2.8) results in

$$c_{vk}+r_k+\alpha\sum_{j=0}^{N}p_{kj}\phi(j,\alpha,T-1) \le r_v+\alpha\sum_{j=0}^{N}p_{ij}\phi(j,\alpha,T-1)\ , \qquad i<v<N\ . \tag{2.9}$$

As $\phi(i,\alpha,T)$ is nondecreasing in $i=0,1,2,\ldots,N$ (Lemma 2.1), it follows from condition 2.5 that

$$\sum_{j=0}^{N}p_{ij}\phi(j,\alpha,T-1) \le \sum_{j=0}^{N}p_{vj}\phi(j,\alpha,T-1)\ .$$

Substituting the above in the right hand side of equation (2.9) we get

$$c_{vk}+r_k+\alpha\sum_{j=0}^{N}p_{kj}\phi(j,\alpha,T-1) \le r_v+\alpha\sum_{j=0}^{N}p_{vj}\phi(j,\alpha,T-1)\ .$$

The latter expression says that at state v it is better to repair the system to state k, rather than doing nothing. Therefore, the action which minimizes the right hand side of (2.6) at state v, $i<v<N$, is to perform a repair (not necessarily to state k). The fact that at state N we always repair (to state 0) completes the proof of the proposition. Q.E.D.

Proof of Theorem 2.1 Let $0<i_T\le N$ be the smallest state at which a repair is beneficial when T steps are left for the operating horizon (such a state always exists because at state 0 no maintenance is needed, and at state N, replacement is mandatory).

Therefore, it readily follows from the Proposition above that $\phi(i,\alpha,T)$ has the form:

$$\begin{aligned}
\phi(i,\alpha,T) &= r_i+\alpha\sum_{j=0}^{N}p_{ij}\phi(j,\alpha,T-1) \quad 0\le i<i_T \\
\phi(i,\alpha,T) &= c_{ik_i}+r_{k_i}+\alpha\sum_{j=0}^{N}p_{k_ij}\phi(j,\alpha,T-1) \quad i_T\le i<N \\
\phi(N,\alpha,T) &= c_{N0}+r_0+\alpha\sum_{j=0}^{N}p_{0j}\phi(j,\alpha,T-1)
\end{aligned} \tag{2.10}$$

where k_i is the optimal state to which the system is repaired from state i, $i \geq i_T$. Since $\phi(i,\alpha) = \lim_{T\to\infty} \phi(i,\alpha,T)$ and, for each T, $\phi(i,\alpha,T)$ is nondecreasing in $i = 0,1,\ldots,N$ (lemma 2.1), then $\phi(i,\alpha)$ is also a nondecreasing function of i, $i = 0,1,\ldots,N$. Using condition 2.5 and equations (2.5) it is easy to show (in a similar way as before) that there exists a state i_α, $0 < i_\alpha \leq N$, such that $\phi(i,\alpha)$ has the form:

$$
\begin{aligned}
\phi(i,\alpha) &= r_i + \alpha \sum_{j=0}^{N} p_{ij}\phi(j,\alpha) \qquad 0 \leq i < i_\alpha \\
\phi(i,\alpha) &= c_{ik_i} + r_{k_i} + \alpha \sum_{j=0}^{N} p_{k_i j}\phi(j,\alpha) \qquad i_\alpha \leq i < N \qquad (2.11) \\
\phi(N,\alpha) &= c_{N0} + r_0 + \alpha \sum_{j=0}^{N} p_{0j}\phi(j,\alpha)
\end{aligned}
$$

where k_i is the optimal state to which the system is repaired from state i, $i \geq i_\alpha$.

To summarize, it has been shown so far that, when the optimization criterion is total expected discounted cost for unbounded horizon, there exists an optimal policy which has the form of generalized control limit rule.

We proceed now to the average cost criterion. For each α, $0 < \alpha < 1$, denote by i_α the control limit determined by R^*_α for criterion (2.1). Let $\{\alpha_v\}$ be an increasing sequence of distinct discount factors such that

$$
\lim_{v\to\infty} \alpha_v = 1, \qquad \text{and for each } v, \quad i_{\alpha_v} = \widehat{i}\ .
$$

That is, all the elements of the sequence α_v generate the same control limit $\widehat{i}$. (Since there is a finite number of states, such a sequence and $\widehat{i}$ exist). We claim that the optimal policy for criterion (2.4) has also the form of a generalized control limit rule.

To show this, let R be any policy which is *not* a generalized control limit rule. Let $\widehat{R}$ be a generalized control limit policy with $\widehat{i}$ as its control limit (i.e. $\widehat{R}$ is optimal for α_v, and $\widehat{R} = R^*_{\alpha_v}$ for all v). From the definition of $\phi(i,\alpha)$ it follows that

$$
\phi_R(i,\alpha_v) \geq \phi(i,\alpha_v) = \phi_{R^*_{\alpha_v}}(i,\alpha_v) \qquad v = 1,2,3,\ldots, \qquad (2.12)
$$

Using (see e.g. Derman [5] p. 25) $\lim_{\alpha \to 1}(1-\alpha)\phi_R(i,\alpha) = \phi_R$, we get, by letting $\alpha_v \to 1$

$$\phi_R = \lim_{v\to\infty}(1-\alpha_v)\phi_R(i,\alpha_v) \geq \lim_{v\to\infty}(1-\alpha_v)\phi(i,\alpha_v) = \phi_{\widehat{R}} \ .$$

Therefore, considering criterion (2.4), for every policy which is not a generalized control limit, there is a better one which is a generalized control limit rule with $\widehat{i}$ as its control limit. That is, for the average cost criterion as well, there exists an optimal policy $\widehat{R}$ which is a generalized control limit rule. Q.E.D.

A Special Case

Theorem 2.1 states that a sufficient condition for the existence of an optimal policy which has the form of a generalized control limit rule is: $c_{vk} - c_{ik} \leq r_v - r_i$, for all i, k, v such that $k < i < v < N$. We shall now present a special case in which this condition has a more intuitive meaning.

Suppose that the cost of a repair (or a replacement) is composed of a fixed cost c, and a variable cost δ_{ik} which is the additional cost of repairing from state i to state k. Then $c_{ik} = c+\delta_{ik}$. Further, suppose that the fixed cost c is higher than the operation cost r_i, $i = 0, 1, \ldots, N-1$. This assumption trivially satisfies condition 2.3, since $c \geq r_i$ implies $c + \delta_{ik} + r_k \geq r_i$, that is, $c_{ik} + r_k \geq r_i$, $0 \leq i \leq N-1$, $k < i$. In such a system the condition of Theorem 2.1, $c_{vk} - c_{ik} \leq r_v - r_i$, reduces to $\delta_{vk} - \delta_{ik} \leq r_v - r_i$. This sufficient condition has a direct intuitive meaning: if for all $i < v$ the difference between the marginal cost of repairing from state v to state k and the cost of repairing from state i to state k, is no more than the difference $r_v - r_i$ between the operating costs, then there exists an optimal policy which has the form of a generalized control limit rule.

To further demonstrate the applicability of the generalized control limit rule, we present a 5-state example.

Example

Let $\{0, 1, 2, 3, 4\}$ be the state space, where 0 is the “new” state and 4 is the “failure” state requiring a mandatory replacement. At each state $1 \leq i \leq 3$ the system can be repaired (or replaced) to some state j, $0 \leq j < i$.

The underlying transition probabilities p_{ij} for $0 \le i \le 3$, $0 \le j \le 4$ are

$i \backslash j$	0	1	2	3	4
0	0.1	0.7	0.1	0.05	0.05
1	0	0.8	0.1	0.05	0.05
2	0	0	0.5	0.25	0.25
3	0	0	0	0.5	0.5

The expected one period operation costs (satisfying condition 2.2) are $r_0 = 1$, $r_1 = 1$, $r_2 = 4$, $r_3 = 6$. The repair (or replacement) costs (satisfying conditions 2.1 and 2.3) are

$$c_{10} = 17\,, \quad c_{20} = 18\,, \quad c_{30} = 20\,, \quad c_{40} = 21\,; \quad c_{21} = 7\,, \quad c_{31} = 9\,; \quad c_{32} = 7\,.$$

It can be easily verified that

(i) the function $D_k(i) = \sum_{j \ge k} p_{ij}$ is nondecreasing in i ($i = 0, 1, 2, \ldots, N-1$) for each $k = 0, 1, 2, \ldots, N$, thus satisfying condition 2.4 (and the equivalent condition 2.5), and

(ii) for every i, v, k such that $0 \le k < i < v < N$, $c_{vk} - c_{ik} \le r_v - r_i$.

By Theorem 2.1 there exists an optimal policy which has the form of a generalized control limit rule. Indeed, by using Howard's policy improvement algorithm (cf. Tijms [18]) one can readily show that for the average cost criterion the optimal policy is the following: Do nothing when in state 0 or 1. Repair to state 1 when in state 2 or 3. Replace when in state 4. The above optimal policy has a simple intuitive explanation: Examining the transition probabilities, it is seen that when the system is new (state 0), it moves with a high probability to state 1. State 1 is more stable, and the system stays there with a high probability. When out of state 1 the probability of failure increases. Since the operating costs of states 0 and 1 are equal, whereas the cost of repairing the system to state 0 (i.e, a replacement) is higher than the cost of repairing to state 1, it is better, when in states 2 or 3, to repair to state 1 (which is stable and cheap) instead of either waiting for the eventual failure of the system (which requires a mandatory replacement), or performing an expensive replacement.

Uncertain Repair

In previous sections it was implicitly assumed that the result of a repair action is certain. That is, a repair from state i to state k is always successful. We wish to relax this assumption and consider the case where the result of a repair action is *un*certain. One can conceive a system at which any attempt to repair causes unstableness, and the end result might be a different state from the one planned for (including a worse state). Surprisingly, the generalized control limit rule remains optimal even in such a situation. Specifically, for every $i = 1, 2, \ldots, N$ denote by q_{jv} $(j = 0, 1, \ldots, N)$ the probability that a planned repair to state j ends up at state v. All other conditions remain the same as before. Equations (2.5) now take the form:

$$
\begin{aligned}
\phi(0,\alpha) &= r_0 + \alpha \sum_{j=0}^{N} p_{0j}\phi(j,\alpha) \\
\phi(i,\alpha) &= \min\left\{ r_i + \alpha \sum_{j=0}^{N} p_{ij}\phi(j,\alpha), \right. \\
&\qquad \left. \min_{0\le k\le i-1} \left[c_{ik} + \sum_{v=0}^{N} q_{kv} \left[r_v + \alpha \sum_{j=0}^{N} p_{vj}\phi(j,\alpha) \right] \right] \right\}, \quad 0 < i < N \\
\phi(N,\alpha) &= c_{N0} + \sum_{v=0}^{N} q_{0v} \left[r_v + \alpha \sum_{j=0}^{N} p_{v_j}\phi(j,\alpha) \right] . \qquad (2.13)
\end{aligned}
$$

In a similar manner we modify equations (2.6) for the finite horizon case. Theorem 2.2 below shows that, *in this case too*, and *without any conditions* on the values of the probabilities q_{jv}, the optimal policy has the structure of a generalized control limit rule.

Theorem 2.2. *Suppose that for every i, k, v such that $k < i < v < N$, $c_{vk} - c_{ik} \le r_v - r_i$. Then, for the uncertain repair case the optimal policy has the form of a generalized control limit rule, for both optimality criteria: total expected discounted cost and average cost per unit of time.*

Proof: Lemma 2.1 remains true: case 1 is unchanged, and case 2 remains true after substituting everywhere the expression

$$\sum_{v=0}^{N} q_{kv} \left[r_v + \alpha \sum_{j=0}^{N} p_{vj} \phi(j, \alpha, T-1) \right]$$

instead of $\quad r_k + \alpha \sum_{j=0}^{N} p_{kj} \phi(j, \alpha, T-1\ )\ .$

The proof continues in the same lines as the proof of Theorem 2.1 (and the proposition preceding it), where at each place where a certain repair is performed from state i to k, resulting in expected future cost of

$$c_{ik} + r_k + \alpha \sum_{j=0}^{N} p_{kj} \phi(j, \alpha, T-1)\ ,$$

the modified expression

$$c_{ik} + \sum_{v=0}^{N} q_{kv} \left[r_v + \alpha \sum_{j=0}^{N} p_{vj} \phi(j, \alpha, T-1) \right]$$

is substituted. Q.E.D.

3. CONTINUOUS STATE-SPACE

The Model

In this section we extend the results of the previous section to the case of a continuous state-space. Detailed proofs are omitted and may be found in Douer and Yechiali [6].

Suppose that the system is inspected at equally spaced points in time, and after each inspection it is classified into some state $x \in S$, $S = [0, \infty) U \{f\}$. The system is at state 0 if it is new (or functions as good as new), and it is at state f if it has failed.

Let X_t denote the observed state of the system at time t. Let $F_x(y)$ be the distribution function of a transition from state x to a state y which is not f,

$$F_x(y) = P\left(X_{t+1} \le y \mid X_t = x\right),\ y \ne f,\ y \in S\ .$$

Let μ_x be a continuous function $0 \leq \mu_x < 1$ which is defined on $S \setminus \{f\}$, such that, if $X_t = x$, then $X_{t+1} = f$ with probability μ_x. The definition of μ_x implies that for every $x \in [0, \infty)$:

$$\int_{y \geq 0} dF_x(y) = 1 - \mu_x \ . \tag{3.1}$$

Remark. It should be noted that the transition law $F_x(y)$ is general in the sense that the system can either deteriorate ($y > x$) or improve ($y < x$) without any external intervention. In a pure deterioration process, $F_x(y) = 0$ for $y < x$.

For every state $x \in [0, \infty)$, a maintenance action (repair or replacement) may be taken. Any maintenance action bringing the system to state $y < x$ is admissible, and it takes no time. A failed system (state f) must be replaced by a new one, and the replacement is instantaneous. At state 0 (new system) no maintenance action is done.

At each period in which the system operates at state x an expected cost $r(x)$ is incurred (in case of a repair, the cost of operating the system while in the new state is incurred). Each repair from state x to state y ($y < x$) costs $c_x(y)$ (a replacement costs $c_x(0)$), and a mandatory replacement from state f costs $c_f < \infty$.

Again, we are interested in finding optimal maintenance rules for each of the two optimality criteria:

1. Total expected discounted cost for unbounded horizon.
2. Long-run average expected cost per unit of time.

Conditions

We impose the following conditions on the costs and transition probabilities.

Condition 3.1. For every $x \in [0, \infty)$, $c_x(y)$ is a continuous non-increasing function of y, $y \in [0, x)$. That is, one has to pay more when a repair to a better state is performed.

Condition 3.2. $c_x(y) \geq c_z(y)$ for every $x, z, y \in [0, \infty)$ such that $x > z > y$. This condition is similar to Condition 2.1.

Condition 3.3. $c_x(0)$ is a bounded function of x, $x \in [0, \infty)$. That is, the cost of replacing the system, from any state, is bounded.

Condition 3.4. $r(x)$ is a continuous non-decreasing bounded function of x, $x \in [0, \infty)$. See Condition 2.2.

Condition 3.5. For every $x \in [0, \infty)$, $r(x) < \inf_{0 \leq z < x} [c_x(z) + r(z)]$. Therefore, there exists $\varepsilon_x > 0$ such that $r(x) + \varepsilon_x \leq c_x(z) + r(z)$ for every $z \in [0, x)$. As does Condition 2.3, Condition 3.5 implies that for a one-period horizon the best action is to do nothing.

Condition 3.6. μ_x is a continuous non-decreasing function of x, $x \in [0, \infty)$. That is, as the state of the system deteriorates, the probability of a failure increases.

Condition 3.7. For every $x \in [0, \infty)$, the density function $dF_x(y)$ is continuous in x, and is bounded.

Condition 3.8. For every $y \in [0, \infty)$, $T_y(x) \equiv 1 - F_x(y)$ is a non-decreasing function of x. This is similar to the IFR assumption of Condition 2.4, and from known results of stochastic domination (cf. Lehmann [12]) is equivalent to:

Condition 3.9. For every bounded function $h(y)$, non-decreasing in $y \in [0, \infty)$, the function $K(x) = \int_{y \geq 0} h(y) dF_x(y)$ is also non-decreasing. See Condition 2.5.

Optimality of the Generalized Control Limit Maintenance Rule

Let $\phi_R(x, \alpha)$, $\phi(x, \alpha)$, $\phi(x, \alpha, T)$ and ϕ_R be as defined in (2.1), (2.2), (2,3) and (2.4), respectively ($x \in S$, $0 < \alpha < 1$). Then, it is known that $\phi(x, \alpha)$ satisfies the following (compare with equations (2.5)):

$$\phi(0, \alpha) = r(0) + \alpha \left[\int_{y \geq 0} \phi(y, \alpha) dF_0(y) + \mu_0 \phi(f, \alpha) \right]$$

$$\phi(x, \alpha) = \min \left[r(x) + \alpha \left[\int_{y \geq 0} \phi(y, \alpha) dF_x(y) + \mu_x \phi(f, \alpha) \right] \right. ,$$

$$\inf_{0\le z<x}\left\{c_x(z)+r(z)+\alpha\left[\int\limits_{y\ge 0}\phi(y,\alpha)dF_z(y)+\mu_z\phi(f,\alpha)\right]\right\}\Bigg] \ , \qquad x>0$$

$$\phi(f,\alpha)=c_f+\phi(0,\alpha)\ . \tag{3.2}$$

Now, $\phi(x,\alpha,T)$ is derived from (3.2) in the same manner as (2.6) and (2.7) are derived from (2.5).

One can now show that for fixed α, T $(0<\alpha<1,\ T\ge 1)$, $\phi(x,\alpha,T)$ is a bounded non-decreasing function of x, $x\in[0,\infty)$. The proof follows the same arguments as in Lemma 2.1.

The following theorem presents a sufficient condition which ensures that, for the criterion of total expected discounted cost, the optimal policy has the form of a generalized control limit rule (see the discounted section in the proof of Theorem 2.1).

Theorem 3.1. *Consider the optimization criterion (2.3). Then, a sufficient condition for the optimal policy to be a generalized control limit rule is*

$$c_w(z)-c_x(z)\le r(w)-r(x) \quad \textit{for all} \quad 0\le z<x<w\ .$$

We now study the criterion of average expected cost per unit of time. We impose an additional condition for the adaptation of the previous results to this case.

Condition 3.10. For every $y\in[0,\infty)$, $c_x(y)$ is a continuous function of $x(x\ge y)$.

Now, observe that Condition 3.7 implies that

$$\lim_{x'\to x}\int\limits_{y\ge 0}\big|dF_{x'}(y)-dF_x(y)\big|dy=0\ . \tag{3.3}$$

Define

$$h_\alpha(x)=\phi(x,\alpha)-\phi(f,\alpha) \quad x\in[0,\infty)U\{f\} \tag{3.4}$$

and

$$g_\alpha=(1-\alpha)\phi(f,\alpha)\ . \tag{3.5}$$

Then it is easy to show (using equations (3.4), (3.5)) that the following equations are equivalent to equations (3.2).

$$g_\alpha + h_\alpha(x) = \min \Big[r(x) + \alpha \int\limits_{y\geq 0} h_\alpha(y) dF_x(y) \ ,$$

$$\inf_{0\leq z<x} \Big\{ c_x(z) + r(z) + \alpha \int\limits_{y\geq 0} h_\alpha(y) dF_z(y) \Big\} \Big] \quad x \in (0,\infty)$$

$$g_\alpha + h_\alpha(f) = c_f + r(0) + \alpha \int\limits_{y\geq 0} h_\alpha(y) dF_0(y) \tag{3.6}$$

$$g_\alpha + h_\alpha(0) = r(0) + \alpha \int\limits_{y\geq 0} h_\alpha(y) dF_0(y) \ .$$

As in Theorem 4 of Ross [15], Conditions 3.4 and 3.10 together with equations (3.3) and the fact that $-c_f \leq \phi(x,\alpha) - \phi(f,\alpha) \leq c_x(0) < \infty$, imply that $\{h_\alpha(x) \mid x \in [0,\infty) U \{f\}\}$ is uniformly bounded and equicontinuous for $x \neq f$, and therefore has a convergent subsequence. Thus, by Theorems 1 and 2 of Ross [15] we have:

Theorem 3.2. *There exists a bounded function $h(x)$ and a constant g such that*

(i) $h(f) = 0$

(ii) for every x, $x \in [0,\infty) U \{f\}$ the limit $\lim\limits_{\alpha\to 1^-} (1-\alpha)\phi(x,\alpha) = g$ exists

(iii) g and h satisfy

$$g + h(x) = \min \Big\{ r(x) + \int\limits_{y\geq 0} h(y) dF_x(y) \ ,$$

$$\inf_{0\leq z<x} \Big[c_x(z) + r(z) + \int\limits_{y\geq 0} h(y) dF_z(y) \Big] \Big\}, \quad x \in (0,\infty)$$

$$g + h(f) = c_f + r(0) + \int\limits_{y\geq 0} h(y) dF_0(y) \tag{3.7}$$

$$g + h(0) = r(0) + \int\limits_{y\geq 0} h(y) dF_0(y) \ .$$

(iv) $g = c_f + r(0) + \int_{y \geq 0} h(y) dF_0(y)$.

(v) *Any policy which, when in state $x \in (0, \infty)$, takes the action that minimizes the right hand side of (3.7) is optimal (for $x = 0$ and $x = f$ the decision is predetermined for all possible policies).*

(vi) $h(x) = \lim\limits_{\alpha \to 1^-} h_\alpha(x)$

(vii) *For every x, $g = \min\limits_R \{\phi_R \mid X_1 = x\}$.*
That is, g is the minimal expected cost per unit of time.

Using Theorem 3.2 we can finally conclude that the condition of Theorem 3.1 is also a sufficient condition for the optimization criterion (2.4).

Theorem 3.3. *Under the average expected cost per unit of time criterion (2.4), a sufficient condition for the optimality of a generalized control limit rule is:*

$$c_w(z) - c_x(z) \leq r(w) - r(x) \quad \textit{for every} \quad 0 \leq z < x < w \ .$$

Remarks.

1. The generalized control limit policy exists and is optimal for quite a general form of transition probabilities which enable the system, at any given state x, to either deteriorate or improve, with no external intervention.

2. Conditions 3.4 and 3.10 may be utilized so as to express the sufficient condition of theorem 3.3 in an equivalent form involving derivatives. Since $c_w(z) - c_x(z) \leq r(w) - r(x)$, $0 \leq z < x < w$, then $c_{x+\Delta x}(z) - c_x(z) \leq r(x + \Delta x) - r(x)$, $0 \leq z < x$, $\Delta x > 0$. Dividing by Δx and letting Δx approach 0, gives $\frac{\partial}{\partial x} c_x(z) \leq r'(x)$, $0 \leq z < x$, as a necessary condition which is easily seen to be sufficient as well.

3. Explicit computation of the optimal policy for the discounted case can be done by using the known method of successive approximation. This method can be used for the average cost criterion too by a simple transformation of the problem (cf. Ross [15]). Methods for accelerating the convergence of the Value Iteration Algorithm (by choosing a 'good' relaxation factor) are suggested by Herzberg and Yechiali [7], [8].

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