Natural 3-valued Logics—Characterization and Proof Theory

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1 Introduction

Many-valued logics in general and 3-valued logic in particular is an old subject which had its beginning in the work of Lukasiewicz [Luk]. Recently there is a revived interest in this topic, both for its own sake (see, e.g. [Ho]), and also because of its potential applications in several areas of computer science, like: proving correctness of programs ([Jo]), knowledge bases ([CP]) and Artificial Intelligence ([Tu]). There are, however, a huge number of 3-valued systems which logicians have studied throughout the years. The motivation behind them and their properties are not always clear and their proof theory is frequently not well developed. This state of affairs makes both the use of 3-valued logics and doing fruitful research on them rather difficult.

Our first goal in this work is, accordingly, to identify and characterize a class of 3-valued logics which might be called *natural*. For this we use the general framework for characterizing and investigating logics which we have developed in [Av1]. Not many 3-valued logics appear as natural within this framework, but it turns out that those that do include some of the best known ones. These include the 3-valued logics of Lukasiewicz, Kleene and Sobociński, the logic LPF used in the VDM project, the Logic RM_3 from the relevance family and the paraconsistent 3-valued logic of [dCA]. Our presentation provides justifications for the introduction of certain connectives in these logics which are often regarded as ad-hoc. It also shows that they are all closely related to each other. It is shown, for example, that Lukasiewicz 3-valued logic and RM_3 (the strongest logic in the family of relevance logics) are in a strong sense dual to each other, and that both are derivable by the same general construction from, respectively, Kleene 3-valued logic and the 3-valued paraconsistent logic.

Our second goal is to provide a proof-theoretical analysis of all the 3-valued systems we discuss. This includes:

- Hilbert type representations with M.P. as the sole rule of inference of almost every system (or fragment thereof) which includes an appropriate implication connective in its language (including the purely implicational ones)[1].

- Cut-free Gentzen-type formulations of *all* the systems we discuss. In the cases of Lukasiewicz and RM_3 this will be possible only by employing a calculus of *hypersequents*, which are finite sequences of ordinary sequents. [2]

All the 3-valued systems we consider below are based on the following *basic structure*:

- Three truth-values :T, F and $\perp$. T and F correspond to the classical two truth values.

- An operation $\neg$, which is defined on these truth-values. It behaves like classical negation on $\{T, F\}$, while $\neg \perp = \perp$.

The language of all the systems we consider includes a negation connective, also denoted by $\neg$, which corresponds to the operation above. Most of them include also the connectives $\wedge$ and $\vee$. The corresponding truth tables are defined as follows: $a \wedge b = min(a, b)$, $a \vee b = max(a, b)$, where $F < \perp < T$. We shall see that the introduction of these connectives as well as the way they are defined are dictated by the interpretation of the operation $\neg$ as *negation*.

Traditionally, the differences between the various systems are with respect to:

[1] RM_3 and its fragments with either $\wedge$ or $\vee$ are the only exceptions.

[2] By a Gentzen-type system we mean here a system that treats only two-sided sequents and whose language is *not* tailored to the needs of a specific n-valued logic. In [Car], e.g., tableaux systems are introduced which use signed formulae so that to every truth-value of the logic corresponds a different "sign". Hence the *syntax* of any of the systems there already determines the fact that it corresponds to an n-valued logic and even determines the exact n. Our proof-theoretical investigations are done, in contrast, within an absolutely general framework (the method of hypersequents was used, e.g., in [Av5] to provide a cut-free Gentzen-type formulation of RM which is an *infinite-valued* logic).

- What other connectives are taken as basic. Especially: what is the official "implication" connective of the language.

- What truth-values are taken to be designated.

Examples:

Kleene 3-valued logic: This logic has, essentially, the basic connectives we describe above with the same truth tables. In addition its standard presentation includes also a connective $\rightarrow_K$ defined by

$$a \rightarrow_K b = \neg a \vee b$$

T is here the only designated value.

LPF: This is an extension of Kleene's logic which was developed within the VDM project (see [BCJ], [Jo]). On the propositional level it is obtained from Kleene by adding a connective Δ such that:

$$\Delta(a) = \begin{cases} F & a = \perp \\ T & a = T, F \end{cases}$$

Lukasiewicz: This was the first 3-valued logic ever to be invented. Besides the basic 3 connectives above it has also an implication connective $\rightarrow$ so that:

$$a \rightarrow b = \begin{cases} \neg a \vee b & a > b \\ T & \text{otherwise} \end{cases}$$

Again T is taken as the only designated value.

RM$_3$: This is the strongest logic in the family of relevance logics ([AB], [Du]). It has *both* T and $\perp$ as designated. Besides the 3 basic connectives above it has an implication connective $\rightarrow$ (first introduced in [Sob]) so that:

$$a \rightarrow b = \begin{cases} \perp & a = b = \perp \\ F & a > b \\ T & \text{otherwise} \end{cases}$$

3-valued paraconsistent logic: This logic also has both T and $\perp$ as designated. In addition to the 3 basic connectives above it has one extra implication connective $\supset$. It is defined as follows:

$$a \supset b = \begin{cases} T & a = F \\ b & \text{otherwise} \end{cases}$$

The truth table for this connective was first introduced in [OdC] and used also in [dC]. The corresponding logic was investigated and axiomatized in [Av3], where it is shown to be a maximal paraconsistent logic (i.e. a logic in which contradictions do not imply everything).

For obvious reasons, all these systems take T as designated and none takes F. This leads into two main directions, corresponding to whether or not we take $\perp$ as designated. The decision depends, of course, on the intended intuitive interpretation of $\perp$. If it corresponds to some notion of *incomplete* information, like "undefined" or "unknown" then usually it is not taken as designated. If, on the other hand, it corresponds to *inconsistent* information (so its meaning is something like "known to be both true and false") then it does. Accordingly, the logics below will be divided into two classes, corresponding to these two interpretations. We shall see that each class has one basic logic from which all the rest are derivable by general methods.

The above two criteria do not really suffice for characterizing the various logics we discuss. We shall see below, for example, that LPF and Lukasiewicz 3-valued logic have exactly the same expressive power: every primitive or definable connective of one is also a primitive or definable connective of the other. Also both have T as the only designated value. The only difference is therefore with respect to what connectives are taken as primitive. Usually this is not taken as an essential issue [3], unless this choice reflects something deeper. This can only be (especially when we are dealing with "implication" connectives) a difference with respect to the *consequence relation* associated with the logic. We shall begin therefore our discussion with this crucial notion as our starting point.

[3]In the literature one can find a lot of different formulations of classical logic with different choices of the primitive connectives— and they all are generally taken to be equivalent!

2 General Considerations

The notion of a consequence relation between finite sets of formulae and formulae was first introduced by Tarski. It was generalized to a relation between finite sets of formulae in [Sc1] and [Sc2]. Scott's notion is extensively used in [Ur] for characterizing many-valued logics. In what follows we shall need, however, a further generalization (from [Av1]) of Scott's definition (see there for explanations and motivations):

Definition: A *consequence relation* (C.R.) on a set Σ of formulae is a binary relation $\vdash$ between finite multisets of formulae s.t.:

(I) Reflexivity: $A \vdash A$ for every formula A.

(II) Transitivity, or "Cut": if $\Gamma_1 \vdash \Delta_1, A$ and $A, \Gamma_2 \vdash \Delta_2$, then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$.

It is more customary to take a C.R. to be a relation between *sets*, rather than multisets (which are "sets" in which an element may occur more than once). We define, accordingly, a C.R. to be *regular* if it can be viewed in this way (equivalently, if it is closed under contraction and its converse). There are, however, logics the full understanding of which requires us to make finer distinctions that only the use of multisets enable us to make. Examples are provided in [Av1] and below. Another standard condition that we find necessary to omit is closure under *weakening*. In what follows we shall call *ordinary* any regular C.R. which satisfies this condition[4] .

Other concepts from [Av1] that will be of great importance below are those of *internal* and *combining* connectives. The internal connectives are connectives that make it possible to transform a given sequent to an equivalent one that has a special required form. The combining connectives, on the other hand, make it possible to combine certain pairs of sequents into a single one, which is valid iff the original two are valid. In [Av1] we have characterized several logics (including classical, intuitionistic, relevant and linear logic) in terms of the internal and combining connectives available in them and the structural rules under which they are closed. We repeat here

[4]The concept of an ordinary C.R. coincides with the original concept of a C.R. due to Scott.

the definitions of the internal negation and implication and of the combining conjunction and disjunction:

Definition: Let $\vdash$ be a C.R.

Internal Negation: We call a unary connective $\neg$ a right internal negation if for all Γ, Δ, A:

$$\Gamma, A \vdash \Delta \quad \text{iff} \quad \Gamma \vdash \Delta, \neg A \ .$$

We call a unary connective $\neg$ a left internal negation if for all Γ, Δ, A:

$$\Gamma \vdash \Delta, A \quad \text{iff} \quad \Gamma, \neg A \vdash \Delta \ .$$

It can easily be shown that $\neg$ is a right internal negation iff it is a left one. We shall use therefore just the term *internal negation* to denote both. We shall call a C.R. which has an internal negation **symmetrical**.

Internal Implication: [5] We call a binary connective $\rightarrow$ an internal implication if for all Γ, Δ, A, B:

$$\Gamma, A \vdash \Delta, B \quad \text{iff} \quad \Gamma \vdash \Delta, A \rightarrow B \ .$$

Combining Conjunction: We call a connective $\wedge$ a combining conjunction iff for all Γ, Δ, A, B:

$$\Gamma \vdash \Delta, A \wedge B \quad \text{iff} \quad \Gamma \vdash \Delta, A \ \text{ and } \ \Gamma \vdash \Delta, B \ .$$

Combining Disjunction: We call a connective $\vee$ a combining disjunction iff for all Γ, Δ, A, B:

$$A \vee B, \Gamma \vdash \Delta \quad \text{iff} \quad A, \Gamma \vdash \Delta \ \text{ and } \ B, \Gamma \vdash \Delta \ .$$

The following facts were shown in [Av1]:

1. $\neg$ is an internal negation iff $\vdash$ is closed under the rules:

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \quad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \ .$$

[5] This was called strong intensional implication in [Av1]. We believe that the present terminology is better.

2. $\wedge$ is a combining conjunction iff $\vdash$ is closed under the rules:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}.$$

3. $\vee$ is a combining disjunction iff $\vdash$ is closed under the rules:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}.$$

4. $\rightarrow$ is an internal implication iff $\vdash$ is closed under the rules:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \quad \frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \vdash \Delta_1, \Delta_2}.$$

The most important of these connectives (for our present purposes) is the internal negation. Indeed, all the C.R.s we discuss in this work are either ordinary or symmetrical (i.e., have an internal negation), but not both. The only exception is, of course, classical logic (which can, in fact, be *characterized* by these two properties).

Suppose that $\vdash$ is a C.R., and that $\neg$ is a unary connective in its language. How can we reasonably change $\vdash$ to make $\neg$ an internal negationΓ There are two possible directions in which a solution to this problem may be sought. One involves weakening $\vdash$, the other involves strengthening it. Specifically, call a sequent $\Gamma' \vdash \Delta'$ a *version* of $\Gamma \vdash \Delta$ if it can be obtained from the later by finite number of steps, in each of which a formula is transferred from one side of a sequent to the other while removing a $\neg$ symbol from its beginning or adding one there. If we define a sequent to be w-valid iff some of its versions is valid in $\vdash$ then the minimal C.R. for which all w-valid sequents obtain is also the minimal C.R. which extends $\vdash$ and relative to which $\neg$ is an internal negation. Classical Logic is obtained from Intuitionistic Logic in this way. Alternatively we might try to restrict $\vdash$ by demanding a sequent to be strongly valid iff *every* version of it is valid. Unfortunately, this is *too* strong: Unless $\neg$ is already an internal negation even the reflexivity condition fails for this new relation. Nevertheless, if we demand the new relation to be a strengthening only of the *single-conclusioned* fragment of the old one (i.e., those sequents which have exactly one formula on the r.h.s of the $\vdash$) then under certain natural conditions we can do better:

Definition: Let $\vdash$ be a C.R. so that both $A \vdash \neg\neg A$ and $\neg\neg A \vdash A$ (these conditions will be called below *the symmetry conditions for negation*). Define $\vdash^S$, the *derived symmetrical version of* $\vdash$, as follows: $\Gamma \vdash^S \Delta$ iff every single-conclusioned version of $\Gamma \vdash \Delta$ obtains.

Proposition:

1. $\vdash^S$ is a C.R..
2. If $\Gamma \vdash^S A$ then $\Gamma \vdash A$.
3. $\neg$ is an internal negation with respect to $\vdash^S$.
4. $\vdash^S$ is the maximal C.R. having the above properties.
5. $\vdash$ and $\vdash^S$ have the same *logical theorems*, i.e. for any A, $\vdash A$ iff $\vdash^S A$.
6. $$A_1, \ldots, A_m \vdash^S B_1, \ldots, B_n$$
   iff for every $1 \leq i \leq m$ and $1 \leq j \leq n$ we have:
   $$A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_m, \neg B_1, \ldots, \neg B_n \vdash \neg A_i$$
   $$A_1, \ldots, A_m, \neg B_1, \ldots, \neg B_{j-1}, \neg B_{j+1}, \ldots, \neg B_n \vdash B_j$$

We leave the easy proof of this proposition to the reader. We note that the last claim in it provides an *effective* alternative definition of the derived symmetrical C.R.. It is also easy to see that the symmetry conditions for negation are in fact necessary for getting a C.R. from this construction. They are obviously satisfied by any C.R. based on the above 3-valued semantics (with respect, of course, to the connective $\neg$ defined there).

Our next goal is to find conditions on $\vdash$ which guarantee that $\vdash^S$ has the other connectives we have defined.

Proposition: Let $\wedge$ be a combining conjunction for $\vdash$. Suppose also that $\vdash$ is closed under the rules:

$$\frac{\Gamma, \neg A \vdash \Delta \quad \Gamma, \neg B \vdash \Delta}{\Gamma, \neg(A \wedge B) \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \neg A}{\Gamma \vdash \Delta, \neg(A \wedge B)} \quad \frac{\Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \wedge B)}$$

(we shall call these conditions *the symmetry conditions for conjunction*). Then $\wedge$ is a combining conjunction for $\vdash^S$.

Proof: Suppose $\Gamma \vdash^S \Delta, A$ and $\Gamma \vdash^S \Delta, B$. We want to show that $\Gamma \vdash^S \Delta, A \wedge B$. Let, accordingly, $\Gamma' \vdash C$ be a single-conclusioned version of $\Gamma \vdash \Delta, A \wedge B$. We want to prove that this sequent is true. There are two possible cases to consider:

1. C is $A \wedge B$.
   By our assumptions, $\Gamma' \vdash A$ and $\Gamma' \vdash B$ are both true. Hence also $\Gamma' \vdash A \wedge B$ is true, since $\wedge$ is a combining conjunction for $\vdash$.

2. $\neg(A \wedge B)$ is in Γ'.
   In this case our assumptions and the first symmetry condition for $\wedge$ easily entail that $\Gamma' \vdash C$.

For the converse, we should show that if $\Gamma \vdash^S \Delta, A \wedge B$ then $\Gamma \vdash^S \Delta, A$ and $\Gamma \vdash^S \Delta, B$. The proofs can again be split into two cases. The second symmetry condition for $\wedge$ is used for one of them, the other part of the definition of a combining conjunction— for the other. Details are left to the reader.

Analogous symmetry conditions for the existence of a combining disjunction can easily be formulated, but in the presence of an internal negation and a combining conjunction such a connective is available anyway.

We next turn our attention to the problem of having an internal implication for $\vdash^S$. If $\rightarrow$ is such a connective then $\vdash^S A \rightarrow B$ iff $A \vdash^S B$ iff $A \vdash B$ and $\neg B \vdash \neg A$. Suppose now that $\vdash$ has an internal implication $\supset$ and a combining conjunction $\wedge$. Then the last two conditions are together equivalent to $\vdash (A \supset B) \wedge (\neg B \supset \neg A)$. This, in turn, is equivalent to $\vdash^S (A \supset B) \wedge (\neg B \supset \neg A)$ (by 5. of the last proposition). Hence the last formula provides an obvious candidate for defining $\rightarrow$. Our next proposition contains natural conditions for this candidate to succeed.

Proposition: Suppose $\wedge$ is a combining conjunction for $\vdash$ which satisfies (in $\vdash$) the corresponding symmetry conditions. Suppose also that $\supset$ is an internal implication for $\vdash$ and that $\vdash$ is closed under the following rules:

$$\frac{\Gamma, A, \neg B \vdash \Delta}{\Gamma, \neg(A \supset B) \vdash \Delta} \quad \frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, \neg B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \neg(A \supset B)}$$

(These two rules will be called below *the symmetry conditions for implication*). Define:

$$A \to B =_{Df} (A \supset B) \wedge (\neg B \supset \neg A)$$

Then $\to$ is an internal implication for $\vdash^S$.

The proof of this proposition is left to the reader. We only note that the naturalness of the above symmetry conditions for implication can most clearly be seen by working out the details of this proof!

By collecting the various conditions at which we arrive in this section we get a Gentzen-type system for the minimal *ordinary* C.R. for which all these conditions obtain. This system, with or without its implicational rules, will be called below *The Basic System*, and it will provide the basis for all the formal representations of the *ordinary* C.R.s which we present below. It has a 4-valued semantics which will be discussed elsewhere. Cut-elimination can be shown for it rather easily.

3 Consequence Relations based on 3-valued Semantics

3.1 The "Undefined" Interpretation

In this section we investigate several C.R.s in which $\perp$ is taken as corresponding to a truth gap, and so T is the only designated value. We start with the basic relation which naturally corresponds to this interpretation. As we shall see, all the others are essentially based on it.

Definition: $\vdash_{Kl}$ is the C.R. defined by:

> $\Gamma \vdash_{Kl} \Delta$ iff any valuation v (in the basic 3-valued structure) which assigns (the designated value) T to all the sentences in Γ assigns it also to at least one of the sentences in Δ.

Two obvious facts about this C.R. are:

- $\vdash_{Kl}$ is an ordinary C.R..
- $\neg$ satisfies the symmetry conditions for negation, but it is not an internal negation.

We next check how can we define operations on the basic structure so that we get combining conjunction and internal implication, both satisfying the corresponding symmetry conditions. The main conclusion is that these requirements completely determine the truth-tables for such connectives.

Proposition:

1. The connective $\wedge$ which was described in the introduction ($a \wedge b = min(a, b)$ where $F < \perp < T$) is a combining conjunction for $\vdash_{Kl}$ which satisfies the symmetry conditions. Moreover, it is the only possible connective on this structure which has these properties. Similar results hold for $\vee$ from the introduction with respect to disjunction.

2. Define a connective $\supset$ on the basic 3-valued structure as follows:

$$a \supset b = \begin{cases} b & a = T \\ T & \text{otherwise} \end{cases}$$

   Then $\supset$ defines an internal implication for $\vdash_{Kl}$ which satisfies the symmetry conditions. Moreover, $\supset$ is the only possible connective on this structure which has these properties. [6]

The proof of the above proposition is straightforward, so its details are left to the reader (compare with section 3.2.2). We turn now to investigate some known logics that are obtained using the connectives that were introduced in it and the general constructions of the previous section.

[6] This table for implication has already been defined by Monteiro in [Mo] and Wojcicki in [Wo] (we thank the referee for this information).

3.1.1 Kleene's 3-valued logic

This logic can now be characterized as $\vdash_{Kl}$ in a language which has, besides $\neg$, also the above unique combining conjunction (or disjunction) that satisfies the symmetry conditions.

An important property of this logic is that *it has no logical theorems*: $\nvdash_{Kl} A$ for no A in its language. This means, first of all, that no corresponding internal implication exists in its language (since at least $A \to A$ should be a theorem for any possible candidate $\to$) [7]. Since an internal disjunction *is* available it follows also that no possible internal negation is definable (and so not only the official $\neg$ fails to be one).

The official "implication connective" usually associated with this logic is not an implication in any sense, and it is just one out of many connectives that are definable from $\neg$ and $\wedge$.

3.1.2 LPF

This logic is $\vdash_{Kl}$ in a language which has, in addition to Kleene's connectives, also the internal implication defined above. It is, of course, an ordinary conservative extension of the original logic of Kleene, and the basic connectives of Kleene retain in it their properties.

At the introduction we follow [BCJ] and define LPF in terms of another connective, Δ. We have, however, the following relations between this connective and our $\supset$:

$$\Delta A = \neg(A \equiv \neg A) \text{ where } A \equiv B =_{Df} (A \supset B) \wedge (B \supset A)$$

$$A \supset B = \Delta A \wedge A \to_K B = \neg\Delta A \vee \neg A \vee B.$$

These relations mean that the expressive powers of the two languages are the same. Since the C.R. associated with both is $\vdash_{Kl}$, [8] the two versions are equivalent. The present version seems to us more natural, though, and it opens the door to interesting observations, like the one given in our next proposition.

[7]The same consideration will apply to any possible C.R. which is based on Kleene's connectives.

[8]In the case of the original LPF this is obvious from the natural deduction system presented in [BCJ].

Proposition: The positive fragment of LPF (i.e. the $\{\vee, \wedge, \supset\}$-fragment) is identical to the corresponding classical one. In particular every classical positive tautology is valid in it.

The proof of this fact is by showing that every axiom and rule of the standard Gentzen-type representation of positive classical logic is valid in the 3-valued semantics (the converse is obvious). All these rules are included in the basic system of the previous section, the rules of which are all valid here.

3.1.3 The 3-valued C.R. of Lukasiewicz

As observed above, $\vdash_{Kl}$ is not symmetrical. Nevertheless, the various symmetry conditions concerning $\neg, \vee, \wedge$ obtain for it, and those concerning implication hold for $\supset$ in the extended version. We can apply therefore our general construction to get the symmetrical versions of both. We shall denote the symmetrical version of Kleene's basic logic by $\vdash_{WLuk}$ and that of its extension with $\supset$ by $\vdash_{Eluk}$. When we mean either we shall just use $\vdash_{Luk}$. We give first a semantical characterization of this C.R.:

Proposition: $\Gamma \vdash_{Luk} \Delta$ iff for every assignment, either one of the sentences in Δ gets T, or one of the sentences in Γ gets F, or at least two (occurrences of) sentences in Γ, Δ get $\perp$ [9].

Proof: Suppose first that the condition holds. Let $\Gamma' \vdash A$ be a single conclusioned version of $\Gamma \vdash \Delta$ and v an assignment for which all the sentences in Γ' get T. This means that the third possibility mentioned in the proposition does not obtain, since at most the ancestor of A can gets $\perp$. On the other hand, each of the other two possibilities obviously guarantees that A gets T in case all the sentences in Γ' get T.

For the converse, suppose that v is an assignment for which the condition above fails for the sequent $\Gamma \vdash \Delta$. If there is no sentence in Γ or Δ which gets $\perp$ then *no* single-conclusioned version of $\Gamma \vdash \Delta$ belongs to $\vdash_{Kl}$. Otherwise let $\Gamma' \vdash A$ be the single conclusioned version of $\Gamma \vdash \Delta$ in which A is the unique sentence in $\Gamma \vdash \Delta$ which gets $\perp$ (if it occurs in Δ) or its negation (if

[9]Note that they can both be in Γ or both be in Δ!

it occurs in Γ). The failure of the condition entails that all the sentences in Γ' get T, and so the resulting single-conclusioned version does not belong to $\vdash_{Kl}$, and $\Gamma \vdash \Delta$ does not belong to $\vdash_{Luk}$.

Our next proposition just summarizes the properties which $\vdash_{Luk}$ has according to the general discussion of the previous section:

Proposition:

1. If $\Gamma \vdash_{Luk} A$ then $\Gamma \vdash_{Kl} A$.
2. $\neg$ is an internal negation for $\vdash_{Luk}$.
3. $\wedge$ and $\vee$ are, respectively, combining conjunction and disjunction for $\vdash_{Luk}$.
4. $\vdash_{Eluk}$ is a conservative extension of $\vdash_{WLuk}$.
5. Define:
$$A \rightarrow B =_{DF} (A \supset B) \wedge (\neg B \supset \neg A)$$
   Then $\rightarrow$ is an internal implication for $\vdash_{Eluk}$.

The relation between the derived symmetrical version of $\vdash_{Kl}$ and Lukasiewicz 3-valued logic (which justifies the name $\vdash_{Luk}$) is given in the next proposition and its corollary:

Proposition: $\rightarrow$ of the previous proposition is exactly Lukasiewicz' implication.

Corollary: $A_1, \ldots, A_n \vdash_{Luk} B$ iff $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B)\ldots)$ is valid in Lukasiewicz 3-valued logic.

We show now that the difference between Lukasiewicz 3-valued logic and LPF is only with respect to the associated C.R.:

Proposition: Lukasiewicz 3-valued logic and LPF have the same expressive power.

Proof: We have seen already that Lukasiewicz implication is definable using $\neg, \wedge$ and $\supset$. for the converse something even stronger holds: $\supset$ is definable from $\rightarrow$ alone. In fact, we have:

$$a \supset b = a \rightarrow (a \rightarrow b)$$

It is worth to recall at this point that $\vee$ is also definable from $\rightarrow$ alone, since $a \vee b = (a \rightarrow b) \rightarrow b$. Hence the languages of $\{\neg, \rightarrow\}$ and that of LPF are equivalent.

We note, finally a quite remarkable property of $\vdash_{Luk}$:

Proposition: $\vdash_{Luk}$ is not closed under contraction. Hence it is not regular (note, however, that it is still closed under weakening).

Proof: We have, e.g., that $\neg A \wedge A, \neg A \wedge A \vdash_{WLuk} B$ is valid while $\neg A \wedge A \vdash_{WLuk} B$ is not.

The last proposition is reflected in the fact that $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ is not a theorem of Lukasiewicz logic. Note, however, that the example we gave is not connected with $\rightarrow$ at all, and applies also to $\vdash_{WLuk}$!

3.2 The "Inconsistent" Interpretation

In this section we investigate several C.R.s in which the meaning of $\perp$ is "both true and false", and so $\perp$ will be designated. The discussion will parallel that of the "unknown" case, and there will be a lot of similarities. We start it, as before, by introducing the basic associated C.R.:

Definition: $\vdash_{Pac}$ is the C.R. defined by:

> $\Gamma \vdash_{Pac} \Delta$ iff every valuation v (in the basic three-valued structure) which assigned either T or $\perp$ to all the sentences of Γ does the same to at least one sentence of Δ.

Again it is obvious that $\vdash_{Pac}$ is an ordinary C.R. in which $\neg$ satisfies the symmetry conditions (but is not an internal negation). Another aspect in which

$\vdash_{Pac}$ resembles $\vdash_{Kl}$ is that for $\vdash_{Pac}$ too there is exactly one possible way to define an internal implication and a combining conjunction (or disjunction) which satisfy the symmetry conditions. For the combining connectives exactly the same truth-tables do the job as before, with a very similar proof. We shall see, however, that for the implication a new truth-table will be needed.

We shall examine now the associated and derived logics.

3.2.1 The basic 3-valued paraconsistent logic

This logic is $\vdash_{Pac}$ in the language of the usual $\neg$ and $\wedge$. $\neg, \wedge$ and $\vee$ have in it exactly the same properties they have in Kleene's logic. On the other hand, unlike $\vdash_{Kl}$ (which has no logical theorems at all) $\vdash_{Pac}$ has a very distinguished set of logical theorems:

Proposition: $\vdash_{Pac} A$ iff A is a classical tautology.

Proof: One direction is trivial. For the converse, suppose that v is a 3-valued valuation. Let w be the two-valued valuation which assigns T to an atomic variable p iff $v(p)$ is designated. It is easy to prove by induction on the complexity of A that if $w(A) = T$ then $v(A) \in \{T, \bot\}$, and if $w(A) = F$ then $v(A) \in \{F, \bot\}$. It follows that if $w(A) = T$ for every two-valued valuation w then $v(A)$ is designated for every 3-valued v.

An alternative proof is to note that the classical equivalences which are used for reducing a sentence to its conjunctive normal form are valid in $\vdash_{Pac}$ in the strong sense that both sides of each equivalence always have the same truth-value. It is also easy to see that a sentence in such normal form is classically valid if it is valid in the present 3-valued semantics.

It is important to note that despite the last proposition classical logic and the basic $\vdash_{Pac}$ are *not* identical. In classical logic, e.g., contradictions entail everything. This is not the case for $\vdash_{Pac}$: in general $\neg A, A \not\vdash_{Pac} B$. This means that $\vdash_{Pac}$ is *paraconsistent* in the sense of [dC]. [10] Moreover, the basic $\vdash_{Pac}$ has no logical contradictions : $A \vdash_{Pac}$ for no A . This entails immediately (since we have an internal conjunction in the language) that no

[10] The relations between paraconsistent logics and many-valued logics in general are studied, e.g., in [dCA] and [Se].

definable internal negation is available. It is also possible to show that no internal implication is definable.

3.2.2 3-valued Paraconsistent Logic with Internal Implication

Like in the $\vdash_{Kl}$ case, our next goal is to enrich the language of $\vdash_{Pac}$ with an internal implication. Again, demanding also the symmetry conditions for this connective determines it completely:

- The condition $A, A \supset B \vdash_{Pac} B$ implies that $a \supset F = F$ if $a \in \{T, \bot\}$ (i.e., if a is designated).
- The conditions $B \vdash_{Pac} A \supset B$ and $\vdash_{Pac} A \supset A$ imply that $a \supset b$ is designated in all other cases.
- The conditions $\neg(A \supset B) \vdash_{Pac} A$ and $\neg(A \supset B) \vdash_{Pac} \neg B$ imply, respectively, that $F \supset a = T$ and $a \supset T = T$
- The condition $A, \neg B \vdash_{Pac} \neg(A \supset B)$ implies that if a is designated and $b = \bot$ then $a \supset b$ cannot be T. Since by the second fact it cannot be F either, it should be $\bot$.

The above facts leads us to a single candidate: the $\supset$ of the 3-valued paraconsistent logic which was described in the introduction. It is not difficult to show that this $\supset$ does really meet the requirements. The situation is therefore completely analogous to the one in the case of $\vdash_{Kl}$. This is clearly reflected also in the next proposition, which summarizes the main properties of $\vdash_{Pac}$ in the full language of the 3-valued paraconsistent logic:

Proposition: In the extended language for $\vdash_{Pac}$ we have:

1. $\neg$ satisfies the symmetry conditions (but again $A, \neg A \not\vdash_{Pac} B$).
2. $\wedge$ and $\vee$ are combining conjunction and disjunction, respectively. Both satisfy the symmetry conditions.
3. $\supset$ is an internal implication which satisfies the symmetry conditions.
4. The positive fragment of $\vdash_{Pac}$ is identical to the corresponding fragment of the classical, two-valued C.R..

It follows from the last proposition that $\vdash_{Pac}$ and $\vdash_{Kl}$ have quite similar properties concerning $\wedge, \vee, \supset$, and the differences are all connected with their negation connective!

3.2.3 RM_3 and Sobociński C.R..

Exactly like $\vdash_{Kl}$, $\vdash_{Pac}$ is not symmetrical, but all the needed symmetry conditions hold for it. Hence we can apply our general construction again to get the symmetrical versions of it in both the basic language and its extension with $\supset$. We shall denote these versions, respectively, by $\vdash_{WSob}$ and $\vdash_{ESob}$, and use $\vdash_{Sob}$ to denote either. The semantical characterization this time (the proof of which we leave to the reader) is the following:

Proposition: $\Gamma \vdash_{Sob} \Delta$ iff for every assignment, either one of the sentences in Γ gets F, or one of the sentences in Δ gets T, or the sequent is not empty and *all* its sentences get $\perp$.

$\vdash_{Sob}$ has the same basic properties of $\vdash_{Luk}$ which were described in the second proposition of **2.1.3**, and its internal implication was again known and used before:

Proposition: The internal implication of $\vdash_{ESob}$, defined as usual by

$$A \rightarrow B =_{Df} (A \supset B) \wedge (\neg B \supset \neg A)$$

is identical to the $\rightarrow$ of RM_3 (i.e., it is Sobociński 3-valued implication).

Corollary: $A_1, \ldots, A_n \vdash_{Sob} B$ iff $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$ is a theorem of RM_3.

Proposition: The languages of RM_3 and $\vdash_{Pac}$ have the same expressive power.

Proof: It is enough to note that $\supset$ is definable in RM_3 by:

$$a \supset b = b \vee (a \rightarrow b)$$

The most remarkable property of $\vdash_{Sob}$, and the main aspect in which it differs from $\vdash_{Luk}$ is given in the following

Proposition: $\vdash_{Sob}$ is a regular C.R. but it is not ordinary: Weakening fails for it.

The last proposition entails that $A \to (B \to A)$ is not a theorem of RM_3. This is a characteristic feature of a *Relevance* logic. RM_3 is indeed the strongest logic in the family of logics which were created by the relevantists' school (see [AB] and [Du]).

3.3 Merging The Two Interpretations

In this section we investigate C.R.s which are based on a *four*-valued structure, in which both the "undefined" and "inconsistent" interpretations of $\perp$ have a counterpart.

Definition: The lattice $KB4$ consists of the four elements $T, F, \perp_N, \perp_B$, together with the order relation $\leq$ defined by the following diagram:

$$T$$

$$\perp_N \qquad \perp_B$$

$$F$$

(i.e: $F \leq \perp_N, \perp_B \leq T$).

We define the operations $\neg, \vee, \wedge$ on $KB4$ as follows: $\vee$ and $\wedge$ are the usual lattice operations. $\neg T = F$, $\neg F = T$, $\neg \perp_N = \perp_N$, $\neg \perp_B = \perp_B$.

Historically a structure which closely resembles $KB4$ [11] was first introduced in order to characterize the valid relevant first-degree entailments (f.d.e.). These are the theorems of the usual relevant logics (R and E - see [AB] , [Du]) which have the form $A \to B$ where $\to$ (the "relevant implication") occurs in neither A nor B (i.e the only connectives occurring in A or in B are $\neg, \vee, \wedge$). The characterization is given in the following:

Fact: A f.d.e. $A \to B$ is provable in the relevance systems R and E iff

[11] But in which only T is taken as designated and $\to$ is differently defined.

$v(A) \leq v(B)$ for every valuation v in $KB4$.

In [Be1] and [Be2] Belnap suggests the use of this 4-valued structure for answering queries in knowledge bases. In his interpretation $\bot_N$ corresponds to Kleene's $\bot$ while $\bot_B$ corresponds to that of $\vdash_{Pac}$. Following him, we take T and $\bot_B$ as designated, and define the corresponding C.R. in the obvious way:

Definition: $\Gamma \vdash_{Be} \Delta$ iff every valuation which makes all the sentences in Γ true (i.e. assigns to them either T or $\bot_B$) makes at least one of the sentences in Δ true.

$\vdash_{Be}$ has the familiar properties of $\vdash_{Kl}$ and $\vdash_{Pac}$: it is ordinary. $\neg$ is not internal negation for it but it satisfies the symmetry conditions. $\wedge$ and $\vee$ are combining conjunction and disjunction for it which satisfies the symmetry conditions, and they are the only possible connectives with these properties (proof — as usual). Like $\vdash_{Kl}$, $\vdash_{Be}$ has no logical theorems, and like $\vdash_{Pac}$ it is paraconsistent. As for the existence of a well-behaved internal implication the situation is exactly like in the 3-valued fragments, with a similar proof:

Proposition: There is exactly one possible way to define an operation $\supset$ on $KB4$ so that the symmetry conditions for it obtain. It is characterized by the following two principles:

- If a is not designated (i.e. $a =\bot_N, F$) then $a \supset b = T$.
- If a is designated (i.e. $a =\bot_B, T$) then $a \supset b = b$.

Our next step is to introduce $\vdash^S_{Be}$— the symmetrical version of $\vdash_{Be}$. For this C.R. both weakening and contraction fail. This, and the fact that it has all the standard internal and combining connectives, makes it a very close relative of the *Linear* C.R. [12]. Accordingly, the internal implication of $\vdash^S_{Be}$, defined as usual, has a lot in common with the relevant implication of the Relevance Logic R, and even more— with the linear implication of Girard.[13].

[12]Linear Logic was introduced in [Gi]. Its C.R. is characterized in [Av1]. Its connections with Relevance Logic are explained in [Av2].

[13]One difference is that for $\vdash^S_{Be}$ the converse of contraction is valid, while for Linear Logic and the standard Relevance logics it is not. RM is the most famous exception in this respect. RMI of [Av4] is another.

As in the previous case, the two implications, $\supset$ and $\rightarrow$, are equivalent as far as expressive power goes. $A \rightarrow B$ is equivalent, as usual, to $(A \supset B) \wedge (\neg B \supset \neg A)$. $A \supset B$, on the other hand is equivalent this time to $B \vee (A \rightarrow (A \rightarrow B))$.

4 Proof Theory of The Ordinary C.R.s

4.1 Gentzen-type Systems

In this section we provide Gentzen-type systems for the *ordinary* C.R.s we introduce above. They are based on the basic system from the end of section 2.

Theorem:

1. By adding $A, \neg A \vdash$ to the basic system we get a Gentzen-type formulation for $\vdash_{Kl}$.

2. By adding $\vdash A, \neg A$ to the basic system we get a Gentzen-type formulation for $\vdash_{Pac}$.

3. By adding both $\vdash A, \neg A$ and $A, \neg A \vdash$ to the basic system we get a Gentzen-type formulation for classical logic.

Proof: The proof in all three cases is basicly the same. We first replace, in the usual way, each of the pairs of rules for $(\wedge \vdash)$, $(\neg\vee \vdash)$, $(\vdash \vee)$ and$(\vdash \neg\wedge)$ by a single rule (the possibility of doing so is due to the soundness of weakening and contraction). The rules of the resulting system are all easily seen to be invertible from both the semantical and the proof-theoretical point of view. By this we mean that the conclusion of each rule is valid iff all its premises are valid, and it is provable iff they are provable (cuts are needed for showing the last part!). It follows that for any given sequent we can construct a finite set of sequents, consisting only of atomic formulae or their negations, so that the given sequent is valid iff all the sequents in the set we construct are, and provable iff all of them are. It remains therefore to check that a sequent of this form is valid in one of the above C.R.s iff it is provable in the corresponding system. This is easy.

4.2 Hilbert-type formulations

The system HBe

Defined connective: $A \equiv B =_{Df} (A \supset B) \wedge (B \supset A)$

Axioms:

I1 $A \supset (B \supset A)$

I2 $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

I3 $((A \supset B) \supset A) \supset A$

C1 $A \wedge B \supset A$

C2 $A \wedge B \supset B$

C3 $A \supset (B \supset A \wedge B)$

D1 $A \supset A \vee B$

D2 $B \supset A \vee B$

D3 $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

N1 $\neg(A \vee B) \equiv \neg A \wedge \neg B$

N2 $\neg(A \wedge B) \equiv \neg A \vee \neg B$

N3 $\neg\neg A \equiv A$

N4 $\neg(A \supset B) \equiv A \wedge \neg B$

Rule of Inference:

$$\frac{A \qquad A \supset B}{B}$$

Note: The first nine axioms provide a standard axiomatization of classical positive logic.

Theorem: $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is provable in the basic system iff $A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m$ is a theorem of HBe.

The details of the proof of this theorem are standard and we leave them to the reader. We note only that the fact that any positive tautology is provable in HBe (as follows from the above note) makes the proof here particularly easy.

Theorems on extensions:

1. If we add either $\neg A \vee A$ or $(A \supset B) \supset (\neg A \supset B) \supset B$ to HBe we get a sound and complete Hilbert-type axiomatization of $\vdash_{Pac}$.
2. If we add either $\neg A \supset (A \supset B)$ or $(B \supset A) \supset (B \supset \neg A) \supset \neg B$ to HBe we get a sound and complete Hilbert-type axiomatization of $\vdash_{Kl}$.
3. By adding both $\neg A \vee A$ and $\neg A \supset (A \supset B)$ (say) to HBe we get classical logic.

Proof: Using the previous theorem, it is straightforward to show the equivalence of these Hilbert-type systems and the Gentzen-type system of the previous subsection.

5 Proof-theory of Lukasiewicz 3-valued Logic

5.1 A Hilbert-type formulation

A Hilbert-type formulation of Lukasiewicz 3-valued logic was first given in [Wa]. An axiomatization of the implicational fragment of this logic was provided in [MM]. Other Axiomatizations can be found in [Sch] and [Ep]. For the sake of completeness, and since we shall need the Hilbert-type formulations later, we include here a formulation and a completeness proof for it which is simpler than any other we were able to find in the literature. A special care was taken to provide a *well-axiomatization*. This means that any fragment of the logic which contains $\rightarrow$ is completely axiomatized by those axioms below which mention just the connectives of that fragment. This includes the implicational fragment itself[14].

[14]We believe that the present axiomatization is simpler and more transparent than the one given in [MM]. It is certainly shorter, since axiom I3 below is easily seen to be derivable from I1,I2 and I4 (we still prefer to include it as an axiom, since together with I1-I2 it provides a very natural subsystem). It was noted by the referee that our axiomatization, though independently found, is a special case of those given in [Sch].

5.1.1 The system HLuk

Axioms:

I1 $A \to (B \to A)$

I2 $(A \to B) \to ((B \to C) \to (A \to C))$

I3 $(A \to (B \to C)) \to (B \to (A \to C))$

I4 $((A \to B) \to B) \to ((B \to A) \to A)$

I5 $((((A \to B) \to A) \to A) \to (B \to C)) \to (B \to C)$

C1 $A \wedge B \to A$

C2 $A \wedge B \to B$

C3 $(A \to B) \to ((A \to C) \to (A \to B \wedge C))$

D1 $A \to A \vee B$

D2 $B \to A \vee B$

D3 $(A \to C) \to ((B \to C) \to (A \vee B \to C))$

N1 $(\neg B \to \neg A) \to (A \to B)$

Rule of Inference:

$$\frac{A \qquad A \to B}{B}$$

Theorem: HLuk is sound and complete for Lukasiewicz 3-valued logic. Moreover, T $\vdash_{HLuk} \phi$ iff $v(\phi) = T$ for any valuation v which assigns T to all the sentences in T.

Notes:

1. $\vdash_{HLuk}$ corresponds to (the single-conclusioned fragment of) $\vdash_{Kl}$, not to that of $\vdash_{Luk}$. Thus $A \to A \to B, A \vdash_{HLuk} B$ though $A \to A \to B, A \nvdash_{Luk} B$. Recall, however, that the two C.R.s have the same logical theorems!

2. It is a standard task to show that a sentence is derivable from I1–I3 alone (using M.P.) iff it has a proof *without contraction* in the intuitionistic Gentzen-type implicational calculus. Since the last criterion is very easy to apply, we shall feel free below to claim derivability using I1–I3 without giving the formal derivation.

3. Since $(A \to B) \to B$ is equivalent to $A \vee B$, Axioms I4 and I5 are just purely implicational formulations of, respectively, the more perspicuous propositions $A \vee B \to B \vee A$ and $A \vee (A \to B) \vee (B \to C)$.

Proof of the theorem: The soundness part is easy. The completeness is a special case of the second claim. Suppose therefore by contraposition that $\mathrm{T} \nvdash_{HLuk} \phi$. Let T_0 be a maximal extension of T such that $\mathrm{T}_0 \nvdash_{HLuk} \phi$. The main property of T_0 is:

$$\mathrm{T}_0 \nvdash_{HLuk} A \quad \text{iff} \quad \mathrm{T}_0, A \vdash_{HLuk} \phi$$

Define now:

$$v(A) = \begin{cases} T & \mathrm{T}_0 \vdash_{HLuk} A \\ F & \mathrm{T}_0 \vdash_{HLuk} A \to B \text{ for every } B \\ \bot & \text{otherwise} \end{cases}$$

Obviously $v(A) = T$ for every A in T while $v(\phi) \neq T$. It remains to show that v is really a valuation, i.e., it respects the operations. For this we need first two lemmas.
Lemma 1: If $\mathrm{T}_0 \vdash_{HLuk} (A \to B) \to B$ then either $\mathrm{T}_0 \vdash_{HLuk} A$ or $\mathrm{T}_0 \vdash_{HLuk} B$.
Proof: First use I1-I3 and an induction on the length of proofs to show that if $\mathrm{T}_0, A \vdash_{HLuk} B$ then $\mathrm{T}_0, (A \to C) \to C \vdash_{HLuk} (B \to C) \to C$. This and I4 (and the fact that $C \to C$ is derivable from I1–I3) easily imply that if $\mathrm{T}_0, A \vdash_{HLuk} C$ and $\mathrm{T}_0, B \vdash_{HLuk} C$ then $\mathrm{T}_0, (A \to B) \to B \vdash_{HLuk} C$. The Lemma follows from this fact and the above main property of T_0[15].
Lemma 2: For every A and B, either $v(A) = T$ or $v(B) = F$ or $v(A \to B) = T$.
Proof: Applying Lemma 1 to axiom I5 we get that either $\mathrm{T}_0 \vdash_{HLuk} B \to C$ for every C or $\mathrm{T}_0 \vdash_{HLuk} ((A \to B) \to A) \to A$. In the first case $v(B) = F$. In the second case $v(A) = T$ or $v(A \to B) = T$ by another application of Lemma 1.

We are ready now to prove that v respects the various operations. As an example we shall show that $v(A \to B) = v(A) \to v(B)$ (The other cases are easier and are left to the reader). Well, This equation is immediate if $v(A) = F$, or if $v(A) = T$, or if $v(B) = T$, while in case $v(A) = v(B) = \bot$ it

[15]The proof of this Lemma just follows standard proofs from [AB] that maximal theories with certain properties are *prime*, i.e., $A \vee B$ is provable in them iff either A or B is provable.

follows from Lemma 2. Suppose finally that $v(A) = \perp$ and $v(B) = F$. Then there exists D such that $T_0 \not\vdash_{HLuk} A \to D$, while $T_0 \vdash_{HLuk} B \to D$. Hence, by I2, $T_0 \not\vdash_{HLuk} A \to B$ and so $v(A \to B) \neq T$. Since neither A nor B are theorems of T_0, it follows by Lemma 1 that $T_0 \not\vdash_{HLuk} (A \to B) \to B$ and so $v(A \to B) \neq F$. Hence $v(A \to B) = \perp = v(A) \to v(B)$ in this case as well.

5.2 A Gentzen-type formulation

As was emphasized above the structural rule of contraction is not valid for $\vdash_{Luk}$. A natural first attempt to construct a Gentzen-type formalism for it would be, therefore, to delete this rule from (an appropriate version of) the corresponding classical system. The resulting formalism is equivalent to the Hilbert-type system which is obtained from HLuk above by dropping I4 and I5. To capture the whole system we need to employ a calculus of *Hypersequents* [16]. We start by recalling the definition of a Hypersequent in [Av5]:

Definition: Let L be a language. A hypersequent is a creature of the form:

$$\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2 | \ldots | \Gamma_n \Rightarrow \Delta_n$$

where Γ_i, Δ_i are finite sequences of formulae of L. The $\Gamma_i \Rightarrow \Delta_i$ -s will be called the *components* of the hypersequent. We shall use G, H as metavariables for (possibly empty, i.e., without components) hypersequents.

The intended semantics of hypersequents is given in the following natural generalization of the semantics of $\vdash_{Luk}$:

Definition: A hypersequent G is $\vdash_{Luk}$-valid if for every valuation v, there is a component of G which contains either a formula on its r.h.s. which gets T (under v), or a formula on its l.h.s. which gets F, or two different occurrences of formulae which get $\perp$.

We next provide a corresponding (generalized) Gentzen-type formalism.

Axioms:

$$A \Rightarrow A$$

[16] Such a calculus was first introduce in [Pot] for the modal S5, and independently in [Av5] for the semi-relevant RM.

External structural rules:

EW (**External Weakening**):

$$\frac{G}{G|H}$$

EC (**External Contraction**):

$$\frac{G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta}$$

EP (**External Permutation**):

$$\frac{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2 \Rightarrow \Delta_2|H}{G|\Gamma_2 \Rightarrow \Delta_2|\Gamma_1 \Rightarrow \Delta_1|H}$$

Internal structural rules:

IW (**Internal Weakening**):

$$\frac{G|\Gamma \Rightarrow \Delta}{G|A, \Gamma \Rightarrow \Delta} \qquad \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta, A}$$

IP (**Internal Permutation**):

$$\frac{G|\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{G|\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta} \qquad \frac{G|\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{G|\Gamma \Rightarrow \Delta_1, B, A, \Delta_2}$$

M (**Merging**):

$$\frac{G|\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3 \qquad G|\Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1', \Delta_2', \Delta_3'}{G|\Gamma_1, \Gamma_1' \Rightarrow \Delta_1, \Delta_1'|\Gamma_2, \Gamma_2' \Rightarrow \Delta_2, \Delta_2'|\Gamma_3, \Gamma_3' \Rightarrow \Delta_3, \Delta_3'}$$

Logical Rules: These are exactly like in Classical Logic, but with "side" sequents allowed. For example, the rules for conjunction are:

$$\frac{G|\Gamma, A \Rightarrow \Delta}{G|\Gamma, A \wedge B \Rightarrow \Delta} \quad \frac{G|\Gamma, B \Rightarrow \Delta}{G|\Gamma, A \wedge B \Rightarrow \Delta} \quad \frac{G|\Gamma \Rightarrow \Delta, A \qquad G|\Gamma \Rightarrow \Delta, B}{G|\Gamma \Rightarrow \Delta, A \wedge B}$$

The Cut-elimination Theorem: If $G|\Gamma_1 \Rightarrow \Delta_1, A$ and $G|A, \Gamma_2 \Rightarrow \Delta_2$ are both derivable in Gluk then so is also $G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.

The proof of this theorem uses the "history" technique of [Av5]. Like in the case of the hypersequential formulation of *RM* which was investigated there, external contraction is the source of the main difficulties. The proof, however, closely follows that in [Av5] and since it is rather tedious we shall not repeat it here (the lack of the *internal* contraction rule somewhat simplifies the proof in the present case, though).

Soundness Theorem: Every Hypersequent which is derivable in Gluk is valid.

Proof: By checking that every rule leads from valid hypersequents to a valid hypersequent. The only non-standard case is rule M. Given a valuation v, there is again only one intersting case to note: when both $\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3$ and $\Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1', \Delta_2', \Delta_3'$ from the premises of the rule have two occurrences of formulae which get $\perp$ under v. In this case, however, the pigeon-hole principle entails that one of the components of the conclusion will have this property as well.

Definition: Let G be a hypersequent, which is not the empty sequent. We define its translation, ϕ_G as follows:

- if G is of the form $A_1, \ldots, A_n \Rightarrow B$ then ϕ_G is $A_1 \to (A_2 \to \ldots \to (A_n \to B) \ldots)$.
- If G has a single nonempty component then ϕ_G is any translation of one of its single-conclusioned versions (recall that $\vdash_{Luk}$ is symmetric!).
- If G has the form $S_1|S_2|\ldots|S_n$, where the S_i's are ordinary sequents then ϕ_G is $\phi_{S_1} \vee \phi_{S_2} \vee \ldots \vee \phi_{S_n}$.

Lemma: G is provable in GLuk iff $\Rightarrow \phi_G$ is.

Proof: It is easy to see that if G is derivable so is ϕ_G. The converse is also not difficult, using the cut elimination theorem. The most significant step is to show that if $\Rightarrow A_1 \vee \ldots \vee A_n$ is provable then so is $\Rightarrow A_1| \ldots | \Rightarrow A_n$.

For this it is enough to show that in general, if $G|\Gamma \Rightarrow \Delta, A \vee B$ is provable then so is $G|\Gamma \Rightarrow \Delta, A|\Gamma \Rightarrow \Delta, B$. This can be done by using two cuts (followed by external contractions), if we start from the given provable hypersequent and $A \vee B \Rightarrow A|A \vee B \Rightarrow B$. The last hypersequent can be derived as follows: By applying rule M to $A \Rightarrow A$ and $B \Rightarrow B$ we can infer $A \Rightarrow B|B \Rightarrow A$. Two applications of $(\vee \Rightarrow)$ to this sequent and to its two premises give then the desired result.

Proof of the completeness of GLuk: By the last Lemma and the completeness of the Hilbert-type system HLuk it is enough to show that every theorem of the later is derivable in GLuk. The only problematic cases for this are I4 and I5. Now by applying rule M to $A \Rightarrow A$ and to $B \Rightarrow B$ we can obtain $B \Rightarrow \ | \ \Rightarrow A|A \Rightarrow B$. From each of the 3 components of this hypersequent one can easily derive both $\Rightarrow$ I4 and $\Rightarrow$ I5 in the classical system, *without using contractions or cuts* (for example: starting from $B \Rightarrow$ and the easily derived $\Rightarrow A, A \rightarrow B$ one can infer $(A \rightarrow B) \rightarrow B \Rightarrow A$ and then I4 by weakening and two applications of $\Rightarrow\rightarrow$). Since we can independently work with each component, we can use these three classical proofs and external contractions to obtain I4 and I5.

6 Proof-theory of RM_3

Hilbert-type representations of RM_3 and its various fragments were extensively investigated in the past. We refer the reader to [AB] and [Du] for details and references[17]. Gentzen-type formulations, on the other hand, were known so far only for the fragments without the combining connectives[18]. We remedy this now by introducing a Gentzen-type formulation for the full system. Again we find it necessary to employ hypersequents in order to achieve this purpose. The discussion closely resembles that of the previous section, and so we shall make it as brief as possible.

The system GRM3

Axioms, external structural rules and logical rules: Like in GLuk.

[17][Av6] includes an axiomatization of the pure implicational fragment which is more perspicuous than those mentioned in these two resources.

[18]Such a formulation appears, e.g., in [Av5], but was known long before.

Internal structural rules:

IC (Internal Contraction):

$$\frac{G|\Gamma, A, A \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta} \qquad \frac{G|\Gamma \Rightarrow \Delta, A, A}{G|\Gamma \Rightarrow \Delta, A}$$

IP (Internal Permutation): Like in GLuk.

Mi (Mingle):

$$\frac{G|\Gamma_1 \Rightarrow \Delta_1 \qquad G|\Gamma_2 \Rightarrow \Delta_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

WW (Weak Weakening):

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'}$$

provided $\Gamma_1 \Rightarrow \Delta_1$ is not empty.

The Soundness of GRM3 can easily be proved, while cut-elimination can again be proved with the help of the history technique. Having done this we can use the same method of translation as before in order to prove completeness (using the completeness of the Hilbert-type formulations). The use of hypersequents is necessary for proving the distribution axiom $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$ [19] and the characteristic axiom of RM_3: $A \vee A \rightarrow B$. The proof of the last formula uses, of course, the WW rule. Other details are left to the reader.

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