A. Avron
B. Konikowska

Proof Systems for Reasoning about Computation Errors

Abstract. In the paper we examine the use of non-classical truth values for dealing with computation errors in program specification and validation. In that context, 3-valued McCarthy logic is suitable for handling lazy sequential computation, while 3-valued Kleene logic can be used for reasoning about parallel computation. If we want to be able to deal with both strategies without distinguishing between them, we combine Kleene and McCarthy logics into a logic based on a non-deterministic, 3-valued matrix, incorporating both options as a non-deterministic choice. If the two strategies are to be distinguished, Kleene and McCarthy logics are combined into a logic based on a 4-valued deterministic matrix featuring two kinds of computation errors which correspond to the two computation strategies described above. For the resulting logics, we provide sound and complete calculi of ordinary, two-valued sequents.

Keywords: three-valued logics, four-valued logics, parallel computation, lazy sequential computation, computation errors, non-deterministic matrices, sequent calculi.

1. Introduction

The use of computer software is ubiquitous in the present-day world. As a result, most of everyday activities, not only in the business or public spheres, but also in our private lives, rely — directly or indirectly — on the correct operation of some software. However, to ensure the correct, reliable operation of programs, we must first specify them in a correct and precise way, and then validate them, proving that they will operate fault-free and give the expected results.

Yet, as the computing practice clearly shows, instead of meeting the objectives set for them, programs do sometimes run into error states — so any logic used for program specification and validation must take this fact into consideration too. In the existing literature, this has been done in two ways. The first is based on using a partial logic, with formulas getting no value in case of a computation error — like in [BCJ84, Ho87, Owe85]. The second employs a three-valued logic with the third, "undefined" value representing a computation error — see e.g. [MC67, Bli91, KTB91, Ko93].

The second approach has become much more popular over years, as it provides more flexibility in reasoning by allowing us to define a three-valued semantics of the considered program logic in the way best tailored to a given application and the intended handling of errors. However, the drawback is that all possible computation errors in various computing scenarios are

Studia Logica **0**: 1–**??**, 2009.

usually bundled together under a single error value and are handled in the same way. Yet in fact there are two distinct types of computation errors, of inherently different characters:

- *critical errors* which make the whole computation stop, or "hang up", causing a total failure of the program
- *non-critical errors* which stop only part of the computation, and can be remedied by a success elsewhere in it

Clearly, the difference between them is quite fundamental from the practical viewpoint, especially for mission-critical software supporting the fundamental business processes of an enterprise. Hence to ensure the optimum specification and validation of programs we should distinguish between critical and non-critical errors, and treat them in different ways. The aim of our paper is to provide logics able to achieve the above.

A typical example of a critical error occurs in lazy, sequential computation, when we proceed from left to right, and the whole computation process stops after encountering the first error. For example, if we are computing the value $v(\alpha \vee \beta)$ of the disjunction of α and β, and encounter an error in computing $v(\alpha)$, then because of the sequential order of the computation we cannot proceed any further. As result, even if the computation of $v(\beta)$ would yield $\mathbf{t}$ (true), we will never learn this – and so we must necessarily assign an error value to $v(\alpha \vee \beta)$.

In turn, a non-critical error can be encountered in a parallel computation, where an error encountered in one branch of a computation stops this branch only, while the computation along other branches continues, and can still give the desired result if one of those branches is a valid alternative to the error-involving one. Hence in the preceding example this time we would get $v(\alpha \vee \beta) = \mathbf{t}$, for to compute $v(\alpha \vee \beta)$ we compute in parallel $v(\alpha)$ and $v(\beta)$, and as soon as either computation yields the value $\mathbf{t}$, we assign this value to the disjunction $\alpha \vee \beta$.

Another example of critical and non-critical errors are the so-called machine error and the error resulting from infinite computation in a sequential computation mode. Machine error, which consists in e.g. a syntax error in the program, or the use of an argument outside a function domain, is immediately signalled by the computer, which allows us to undertake some corrective actions and continue the computation. In turn, an error resulting from an infinite computation gives us no such chance, since the computer just keep churning on and we cannot tell if it will complete the computation in a moment or if it has got stuck in an infinite loop — for the halting problem is undecidable.

Obviously, a good tool for handling critical errors is a well-known logic describing lazy, sequential computation, namely, the three-valued *McCarthy logic* [MC67] with asymmetric conjunction and disjunction, represented by the following truth tables:

$$\begin{array}{c|ccc} \neg & \mathbf{t} & \mathbf{f} & \mathbf{e} \\ \hline & \mathbf{f} & \mathbf{t} & \mathbf{e} \end{array} \qquad \begin{array}{c|ccc} \vee & \mathbf{t} & \mathbf{f} & \mathbf{e} \\ \hline \mathbf{t} & \mathbf{t} & \mathbf{t} & \mathbf{t} \\ \mathbf{f} & \mathbf{t} & \mathbf{f} & \mathbf{e} \\ \mathbf{e} & \mathbf{e} & \mathbf{e} & \mathbf{e} \end{array} \qquad \begin{array}{c|ccc} \wedge & \mathbf{t} & \mathbf{f} & \mathbf{e} \\ \hline \mathbf{t} & \mathbf{t} & \mathbf{f} & \mathbf{e} \\ \mathbf{f} & \mathbf{f} & \mathbf{f} & \mathbf{f} \\ \mathbf{e} & \mathbf{e} & \mathbf{e} & \mathbf{e} \end{array} \tag{1}$$

where **e** denotes computation error.

McCarthy logic was originally developed for the purpose of describing the phenomenon of computation, including computation errors, and has found application in programming languages like Euclid, Ada and Algol-W. In [KTB91] and [Ko93], McCarthy connectives combined with Kleene quantifiers were used as a foundation for developing two versions of a three-valued logic for specification and validation of programs.

In turn, non-critical errors are adequately handled by three-valued *Kleene logic* [Kl52]. This is maybe the most famous logic used for reasoning about undefinedness in general, which is also known to describe parallel computation (with unlimited parallelism). Unlike McCarthy logic, Kleene logic has symmetric disjunction and conjunction, and is given by the following truth tables:

$$\begin{array}{c|ccc} \neg & \mathbf{t} & \mathbf{f} & \mathbf{u} \\ \hline & \mathbf{f} & \mathbf{t} & \mathbf{u} \end{array} \qquad \begin{array}{c|ccc} \vee & \mathbf{t} & \mathbf{f} & \mathbf{u} \\ \hline \mathbf{t} & \mathbf{t} & \mathbf{t} & \mathbf{t} \\ \mathbf{f} & \mathbf{t} & \mathbf{f} & \mathbf{u} \\ \mathbf{u} & \mathbf{t} & \mathbf{u} & \mathbf{u} \end{array} \qquad \begin{array}{c|ccc} \wedge & \mathbf{t} & \mathbf{f} & \mathbf{u} \\ \hline \mathbf{t} & \mathbf{t} & \mathbf{f} & \mathbf{u} \\ \mathbf{f} & \mathbf{f} & \mathbf{f} & \mathbf{f} \\ \mathbf{u} & \mathbf{u} & \mathbf{f} & \mathbf{u} \end{array} \tag{2}$$

where **u** denotes undefinedness.

Accordingly, from now on the computation strategy tailored to handling critical errors based on McCarthy logic will be referred to as the (MC) strategy, while the strategy tailored to handling non-critical errors based on Kleene logic will be termed the (K) strategy

In the paper we examine three computing scenarios involving both critical and non-critical errors. The first two of them correspond to the case when we do not know whether the strategy used by the computer in handling errors is the (MC) or the (K) strategy. To describe this scenario, we use a three-valued non-deterministic logical matrix [AL01, AL05] where each element is the union of the corresponding elements of the matrices for McCarthy and Kleene logics. The logic generated by the static semantics of that matrix describes the situation when the computer uses the same strategy in all cases (but we do not know which one). In turn, the logic generated by the dynamic semantics of that matrix describes the situation when the computer can use a different strategy in each particular case. Finally, the last scenario is one where the computer distinguishes between critical and non-critical errors, and hence chooses the right strategy — the (MK) or the (K) one — for either kind of error. This corresponds to a four-valued deterministic matrix featuring two kinds of error values, with the critical error handled like in McCarthy logic, and the non-critical one — like in Kleene logic.

For all the three considered logics, we provide sound and complete calculi of ordinary sequents. The systems for the static semantics of the three-valued Nmatrix and for the four-valued ordinary matrix are a great improvement over those developed earlier in [AK05, Ko08], respectively. Indeed, some of the rules in the latter systems relied on the use of constants, and such rules had to be used also in reasoning about constant-free formulas. In opposition to the above, the systems provided here do not employ any constants.

2. Non-deterministic matrices

We begin with recalling from [AL01, AL05, AK05] the fundamentals of the notions of a non-deterministic matrix and its semantics, needed to handle the first two computing scenarios mentioned in the introduction.

In what follows, $\mathcal{L}$ is a propositional language, $O_n\,(n \geq 0)$ is the set of its n-ary connectives, $\mathcal{W}$ is its set of wffs, p, q, r denote propositional variables, $\varphi, \psi, \phi, \tau$ denote arbitrary formulas (of $\mathcal{L}$), and Γ, Δ denote finite sets of formulas.

Definition 1. *A* non-deterministic matrix *(*Nmatrix*) for $\mathcal{L}$ is a triple $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (containing its designated values), and $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective $\diamond \in O_n$.*

Definition 2. *Let $\mathcal{M} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$ be an Nmatrix.*

1. *A* dynamic valuation *in $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{V}$ such that for each n-ary connective $\diamond \in O_n$, the following holds for all $\psi_1, \ldots, \psi_n \in \mathcal{W}$:*

$$\text{(S)} \qquad v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

2. *A* static valuation *in $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{V}$ which satisfies Condition (S) together with the following compositionality principle:*

for each $\diamond \in O_n$ *and for every* $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n \in \mathcal{W}$,

$$\text{(C)} \qquad v(\diamond(\psi_1, \ldots, \psi_n)) = v(\diamond(\varphi_1, \ldots, \varphi_n)) \text{ if } (\forall i)(v(\psi_i) = v(\varphi_i))$$

NOTE 1. The ordinary (deterministic) matrices correspond to the case when each $\widetilde{\diamond}$ is a function taking singleton values only. Then it can be treated as a function $\widetilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$; thus there is no difference between static and dynamic valuations, and we have full determinism.

NOTE 2. Both dynamic and static valuations depend solely on the set of truth values assigned to the given values of some formulas by the (nondeterministic) interpretation of the connective combining those formulas and not on the formulas themselves. Hence both types of valuation can be termed extensional.

By the above definitions, the dynamic semantics corresponds to selecting the value of $v(\diamond(\psi_1, \ldots, \psi_n))$ out of the whole set $\widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ separately and independently for each tuple $(v(\psi_1), \ldots, v(\psi_n))$, which means that $v(\psi_1), \ldots, v(\psi_n)$ do not uniquely determine $v(\diamond(\psi_1, \ldots, \psi_n))$. This semantics corresponds to the maximum level of non-determinism possible in the context of an Nmatrix.

In case of the static semantics, this choice is made globally, system-wide. Namely, by Condition (C), the value of $v(\diamond(\psi_1, \ldots, \psi_n))$ is now uniquely determined by $v(\psi_1), \ldots, v(\psi_n)$. Hence the interpretation of $\diamond$ is a function $f^v_\diamond : \mathcal{V}^n \to \mathcal{V}$ such that, for any $(t_1, \ldots, t_n) \in \mathcal{V}$, and any $\psi_1, \ldots, \psi_n \in \mathcal{W}$:

$$f^v_\diamond(t_1, \ldots, t_n) \in \widetilde{\diamond}(t_1, \ldots, t_n), \quad v(\diamond(\psi_1, \ldots, \psi_n)) = f^v_\diamond(v(\psi_1) \ldots v(\psi_n)) \quad (3)$$

Clearly $f^v_\diamond$ represents a "determinisation" of $\widetilde{\diamond}$ applied in computing the value of any formula under the given valuation, and the non-determinism is now limited to choosing $f^v_\diamond$ among all functions compatible with the nondeterministic interpretation $\widetilde{\diamond}$ of $\diamond$, with the selection being performed before any computation begins.

DEFINITION 3. *A valuation* v *in* $\mathcal{M}$ *satisfies a formula* ψ *(*$v \models \psi$*) if* $v(\psi) \in \mathcal{D}$*, and is a model of* Γ *(*$v \models \Gamma$*) if it satisfies every formula in* Γ*.*

DEFINITION 4. *We say that:*

- ψ *is dynamically (statically) valid in* $\mathcal{M}$*, in symbols* $\models^d_{\mathcal{M}} \psi$ *(*$\models^s_{\mathcal{M}} \psi$*), if* $v \models \psi$ *for each dynamic (static) valuation* v *in* $\mathcal{M}$*.*

- *We say that* Δ *dynamically (statically) follows from* Γ *in* $\mathcal{M}$*, in symbols* $\Gamma \vdash^d_{\mathcal{M}} \Delta$ ($\Gamma \vdash^s_{\mathcal{M}} \Delta$)*, if for every dynamic (static) model* v *of* Γ *in* $\mathcal{M}$ *we have* $v \models \phi$ *for some* $\phi \in \Delta$.

- *The relation* $\vdash^d_{\mathcal{M}}$ ($\vdash^s_{\mathcal{M}}$) *is called the dynamic (static) consequence relation induced by* $\mathcal{M}$.

Definition 5.

- *By a sequent over the language* $\mathcal{L}$ *we mean an expression* Σ *of the form* $\Gamma \Rightarrow \Delta$*, where* Γ, Δ *are finite sets of formulas of* $\mathcal{L}$.

- *A valuation* v *in an Nmatrix* $\mathcal{M}$ *satisfies the sequent* $\Sigma = \Gamma \Rightarrow \Delta$*, written* $v \models \Sigma$*, if either* $v \not\models \gamma$ *for some* $\gamma \in \Gamma$ *or* $v \models \delta$ *for some* $\delta \in \Delta$.

- *A sequent* Σ *is said to be dynamically (statically) valid in an Nmatrix* $\mathcal{M}$*, in symbols* $\models^d_{\mathcal{M}} \Sigma$ ($\models^s_{\mathcal{M}} \Sigma$) *if* $v \models \Sigma$ *for every dynamic (static) valuation* v *in* $\mathcal{M}$.

3. Combining Kleene and McCarthy logics

3.1. Combination based on a 3-valued non-deterministic matrix

Assume $\mathcal{V} = \{\mathbf{f}, \mathbf{e}, \mathbf{t}\}, \mathcal{D} = \{\mathbf{t}\}$, and $\mathcal{L}$ has a unary connective $\neg$ and a binary connective $\vee$. If we combine the matrices for Kleene and McCarthy logics restricted to $\neg, \vee$ [1] (by taking their union), we obtain the Nmatrix $\mathcal{M}^3_{MK} = (\mathcal{V}, \mathcal{D}, \mathcal{O})$, with $\mathcal{O} = \{\tilde{\neg}, \tilde{\vee}\}$, where[2]:

$\tilde{\neg}$	$\mathbf{f}$	$\mathbf{e}$	$\mathbf{t}$
	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{f}$

$\tilde{\vee}$	$\mathbf{f}$	$\mathbf{e}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{e}$	$\mathbf{t}$
$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\{\mathbf{e}, \mathbf{t}\}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$

(4)

One can easily see that $\mathcal{M}^3_{MK}$ with the static semantics describes the situation where the computer uses either strategy (K) or strategy (MC) all the time. Indeed: if in the static semantics we take $f^K_{\vee}(\mathbf{e}, \mathbf{t}) = \mathbf{t}$, we obtain Kleene logic, while by choosing $f^M_{\vee}(\mathbf{e}, \mathbf{t}) = \mathbf{e}$ we get McCarthy logic. In turn, the dynamic semantics corresponds to the situation where the computer can

[1]The standard language of these logics includes the connective $\wedge$ as well, but in both of them $\wedge$ is definable in terms of $\neg, \vee$ by De Morgan laws.

[2]For simplicity, in the truth tables below we omit the set brackets in singleton sets.

apply a different strategy in each particular case. Thus the two calculi related to the static and dynamic semantics of $\mathcal{M}_{MK}$ yield results applicable to both strategies.

The above Nmatrix and the logics corresponding to its dynamic and static semantics were studied in [AK05]. In that paper, complete sequent calculi for both static and dynamic semantics of $\mathcal{M}_{MK}$ were provided. They were obtained by translating systems of 3-sequent inference rules and axioms for both semantics obtained using the general method introduced in [AK05] into ordinary, 2-sided sequent calculi with help of the general translation method described in [ABK06]. However, the proof system for the static semantics had the drawback of using constants which were employed even in rules applied to constant-free formulas. In this paper we will remedy this deficiency by providing a proof system for the static semantics without constants, obtained by extending a cosmetically modified version of the proof system for the dynamic semantics introduced in [AK05]. More precisely, we provide here the following sequent calculi for the dynamic and static semantics of $\mathcal{M}^3_{MK}$:

Dynamic semantics: The system SC^d_3 for the dynamic semantics of $\mathcal{M}^3_{MK}$ is defined as follows:

Axioms:

(A1) $\Gamma, \alpha \Rightarrow \Delta, \alpha$ (A2) $\Gamma, \alpha, \neg\alpha \Rightarrow \Delta$

Inference rules: Cut plus the following rules:

$(r1)$ $\dfrac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \neg\neg\alpha \Rightarrow \Delta}$ $(r2)$ $\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg\neg\alpha}$

$(r3)$ $\dfrac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta}$ $(r4)$ $\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$

$(r5)$ $\dfrac{\Gamma \Rightarrow \Delta, \neg\alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$

$(r8)$ $\dfrac{\Gamma, \neg\alpha \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta}$ $(r9)$ $\dfrac{\Gamma \Rightarrow \Delta, \neg\alpha \quad \Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta)}$

$(r10)$ $\dfrac{\Gamma, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta}$

Note that the gap in the numbering of the rules follows from the fact that the numbering is aligned with that of the rules in a bigger system for the four-valued logic mentioned in the introduction, which will be provided later on.

The nonstandard character of some of the sequent rules given above follows from the nonstandard properties of the matrix $\mathcal{M}^3_{MK}$, especially with regard to disjunction. As those properties make some standard sequent rules unsound, they have forced us to replace them with sound, but nonstandard analogues. For example, rule (r5) has been introduced as a sound counterpart of the standard rule

$$\frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

which is unsound in the semantics based on $\mathcal{M}^3_{MK}$. Indeed: for $\Gamma = \Delta = \emptyset$ and any valuation v such that $v(\alpha) = \mathbf{e}$ and $v(\beta) = \mathbf{t}$ we have $v \models (\Rightarrow \beta)$ but $v \not\models (\Rightarrow \alpha \vee \beta)$, since $v(\alpha \vee \beta) = \mathbf{e} \vee \mathbf{t} = \mathbf{e} \notin \mathcal{D}$.

Static semantics: The system SC^s_3 for the static semantics of $\mathcal{M}^3_{MK}$ is obtained by supplementing the system SC^d_3 for the dynamic semantics with the rule:

$$(S)\ \frac{\Gamma \Rightarrow \Delta, \beta \qquad \Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta, \alpha \vee \beta}$$

Roughly speaking, the above rule ensures that if a valuation v selects $\mathbf{t}$ as the value of $\varphi \vee \psi$ for *some* φ, ψ such that $v(\varphi) = \mathbf{e}$, then it will choose the value $\mathbf{t}$ for $\alpha \vee \beta$ *whenever* $v(\beta) = \mathbf{t}$ — which amounts to choosing a deterministic function as the interpretation of $\vee$.

As we have noted above, the advantage of SC^s_3 over the system developed in [AK05] is that, unlike the former, it does not require the use of constants.

3.2. Combination based on a 4-valued deterministic matrix

The situation when the computer distinguishes between the two kinds of errors — critical and non-critical ones — and uses the appropriate strategy for each of them is described by an ordinary (deterministic) four-valued matrix $\mathcal{M}^4_{MK} = (\mathcal{T}, \mathcal{D}, \mathcal{I})$, where $\mathcal{T} = \{\mathbf{f}, \mathbf{t}, \mathbf{u}, \mathbf{e}\}$, $\mathcal{D} = \{\mathbf{t}\}$, and the interpretations of the connectives restricted to $\mathbf{t}$, $\mathbf{f}$ and $\mathbf{u}$ behave like Kleene connectives, and restricted to $\mathbf{t}$, $\mathbf{f}$ and $\mathbf{e}$ — like McCarthy ones. As to the interplay between the critical and non-critical errors, we have adopted here the assumption that **e** prevails over **u** whatever their order. More exactly, the interpretations $\widetilde{\neg}, \widetilde{\vee} \in \mathcal{I}$ of negation and disjunction[3] are given by:

$\widetilde{\neg}$	**t**	**f**	**u**	**e**
	f	**t**	**u**	**e**

$\widetilde{\vee}$	**t**	**f**	**u**	**e**
t	**t**	**t**	**t**	**t**
f	**t**	**f**	**u**	**e**
u	**t**	**u**	**u**	**e**
e	**e**	**e**	**e**	**e**

(5)

The logic generated by this matrix was considered in [Ko08]. However, the proof system provided there relied heavily on the use of the "finiteness operator" $\circ$, used to distinguish between **e** and **u** and defined with help of the constant added for that purpose to the original language. In this paper, we will improve on [Ko08] by providing a complete proof system for the considered logic without the use of any additional operator or constant.

The proof system for $\mathcal{M}^4_{MK}$ is the sequent calculus SC_4 consisting of the following elements:

Axioms:

(A1) $\Gamma, \alpha \Rightarrow \Delta, \alpha$ (A2) $\Gamma, \alpha, \neg\alpha \Rightarrow \Delta$

Inference rules:

Cut, plus the following rules:

$$(r1)\quad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \neg\neg\alpha \Rightarrow \Delta} \qquad (r2)\quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg\neg\alpha}$$

$$(r3)\quad \frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta} \qquad (r4)\quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

$$(r5)\quad \frac{\Gamma \Rightarrow \Delta, \neg\alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

$$(r6)\quad \frac{\Gamma \Rightarrow \Delta, \beta \quad \Gamma, \alpha \vee \beta \Rightarrow \Delta}{\Gamma, \neg\alpha \vee \beta \Rightarrow \Delta} \qquad (r7)\quad \frac{\Gamma \Rightarrow \Delta, \beta \quad \Gamma \Rightarrow \Delta, \alpha \vee \beta}{\Gamma \Rightarrow \Delta, \neg\alpha \vee \beta}$$

[3] As before, we disregard $\wedge$ in our considerations, since it can be obtained out of $\neg$ and $\vee$ using De Morgan laws.

$$(r8)\quad \frac{\Gamma, \neg\alpha \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta} \qquad (r9)\quad \frac{\Gamma \Rightarrow \Delta, \neg\alpha \quad \Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta)}$$

$$(r10)\quad \frac{\Gamma, \neg\beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta}$$

$$(r11)\quad \frac{\Gamma \Rightarrow \Delta, \gamma \quad \Gamma, \alpha \vee \gamma \Rightarrow \Delta}{\Gamma, (\alpha \vee \beta) \vee \gamma \Rightarrow \Delta} \qquad (r12)\quad \frac{\Gamma \Rightarrow \Delta, \alpha \vee \gamma \quad \Gamma \Rightarrow \Delta, \beta \vee \gamma}{\Gamma \Rightarrow \Delta, (\alpha \vee \beta) \vee \gamma}$$

$$(r13)\quad \frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \vee \gamma \Rightarrow \Delta}{\Gamma, (\alpha \vee \beta) \vee \gamma \Rightarrow \Delta}$$

As we see, the system SC_4 is again obtained out of the basic system SC_3^d for the dynamic semantics of $\mathcal{M}_{MK}^3$ — this time, by adding rules (r6–r7) and (r11–r13). Note also that since Axioms A1, A2 are given in the form which incorporates weakening, the weakening rules are admissible in SC_4, as well as in the other systems considered here, of which fact we will make use in the sequel.

4. Soundness and completeness of the proof systems

Let us denote provability under any sequent calculus SC considered in this paper by $\vdash_{SC}$.

4.1. Four-valued logic

We start with the largest system of all the three introduced here, i.e. SC_4:

THEOREM 1. *The system SC_4 is sound, i.e. each sequent provable in SC_4 is valid in $\mathcal{M}_{MK}^4$.*

PROOF. The validity of axioms is obvious. By way of example, we will show the soundness of three inference rules: (r6), (r11) and (r13). In what follows, v denotes an arbitrary valuation in $\mathcal{M}_{MK}^4$, $P_i, i = 1, 2$, premises of the considered rule, and C its conclusion.

Rule (r6) : $\dfrac{\Gamma \Rightarrow \Delta, \beta \quad \Gamma, \alpha \vee \beta \Rightarrow \Delta}{\Gamma, \neg\alpha \vee \beta \Rightarrow \Delta}$

Assume $v \models P_i$, $i = 1, 2$. If $v \not\models \gamma$ for some $\gamma \in \Gamma$ or $v \models \delta$ for some $\delta \in \Delta$, then obviously $v \models C$. Otherwise $v \models \beta$, whence $v(\beta) = \mathbf{t}$, and $v \not\models \alpha \vee \beta$, which in view of $v(\beta) = \mathbf{t}$ implies $v(\alpha) = \mathbf{e}$ by (5). Hence also $v(\neg\alpha) = \mathbf{e}$ and $v(\neg\alpha \vee \beta) = \mathbf{e}$, so $v \models C$.

(r11) $\dfrac{\Gamma \Rightarrow \Delta, \gamma \quad \Gamma, \alpha \vee \gamma \Rightarrow \Delta}{\Gamma, (\alpha \vee \beta) \vee \gamma \Rightarrow \Delta}$ Assume again that $v \models P_i$, $i = 1, 2$. As Γ, Δ, are handled like above, suppose $v \models \gamma$ and $v \not\models \alpha \vee \gamma$. Then, reasoning like for (r6) above, we get $v(\gamma) = \mathbf{t}$ and $v(\alpha) = \mathbf{e}$. Hence $v((\alpha \vee \beta) \vee \gamma) = (\mathbf{e} \vee v(\beta)) \vee \mathbf{t} = \mathbf{e} \vee \mathbf{t} = \mathbf{e}$ and $v \models C$.

(r13) $\dfrac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \vee \gamma \Rightarrow \Delta}{\Gamma, (\alpha \vee \beta) \vee \gamma \Rightarrow \Delta}$

Disregarding Γ, Δ, assume $v \not\models \alpha$ and $v \not\models \beta \vee \gamma$. Then $v(\alpha) \neq \mathbf{t}$ and $v(\beta \vee \gamma) \neq \mathbf{t}$, whence $v(\beta) \neq \mathbf{t}$ and either $v(\gamma) \neq \mathbf{t}$ or $v(\beta) = \mathbf{e}$. Accordingly, $v(\alpha \vee \beta) \neq \mathbf{t}$ and since — in view of $v(\alpha) \neq \mathbf{t}$ — $v(\beta) = \mathbf{e}$ implies $v(\alpha \vee \beta) = \mathbf{e}$, in both foregoing cases we get $v((\alpha \vee \beta) \vee \gamma) \neq \mathbf{t}$, whence $v \models C$. □

It remains to prove the converse result, i.e. the completeness of SC_4:

THEOREM 2. *The system SC_4 is complete, i.e. each sequent valid in $\mathcal{M}^4_{MK}$ is provable in SC_4.*

PROOF. Assume $\Sigma = \Gamma \Rightarrow \Delta$ is a sequent over $\mathcal{W}$ valid in $\mathcal{M}^4_{MK}$. To prove that $\vdash_{SC_4} \Sigma$, we argue by contradiction.

Suppose $\not\vdash_{SC_4} \Sigma$. We carry out an analogue of a maximal consistent set construction in the Lindenbaum style by extending Γ, Δ to a pair of sets $T, S \subseteq \mathcal{W}$ such that:

$$\begin{array}{ll} \text{(TS1)} \quad \Gamma \subseteq T, \Delta \subseteq S \qquad \text{(TS2)} \quad T \cap S = \emptyset \qquad \text{(TS3)} \quad T \cup S = \mathcal{W} & \\ \text{(TS4)} \quad \text{For any finite sets } \Gamma' \subseteq T, \Delta' \subseteq S, \ \not\vdash_{SC_4} (\Gamma' \Rightarrow \Delta') & \end{array} \tag{6}$$

The sets T, S with the above properties are constructed inductively, as the unions of two monotonically increasing sequences of sets $T_i, S_i \subseteq \mathcal{W}, i = 0, 1, \ldots$, satisfying conditions TS1, TS2, TS4 above. For the purposes of that construction, let $\varphi_1, \varphi_2, \ldots$ be an ordering of $\mathcal{W}$.

Let $T_0 = \Gamma, S_0 = \Delta$. Since $\Gamma \cap \Delta = \emptyset$ (for otherwise $\Sigma = \Gamma \Rightarrow \Delta$ would be an instance of Axiom A1), T_0, S_0 satisfy conditions TS1, TS2. Condition TS4 is also satisfied, for in the opposite case Σ would be derivable from $\Gamma' \Rightarrow \Delta'$ by weakening.

Assume now we have constructed the sequences $\{T_i\}, \{S_i\}$ with the desired properties up to $i = k$. Let $\varphi = \varphi_l$ be the first formula in the considered ordering of $\mathcal{W}$ such that $\varphi \notin (T_k \cup S_k)$. Then one of the following holds:

(1) $\not\vdash_{SC_4} \Gamma', \varphi \Rightarrow \Delta'$ for any finite sets $\Gamma' \subseteq T_k, \Delta' \subseteq S_k$; or

(2) $\nvdash_{SC_4} \Gamma'' \Rightarrow \Delta'', \varphi$ for any finite sets $\Gamma'' \subseteq T_k, \Delta'' \subseteq S_k$.

Indeed: otherwise we would have $\vdash_{SC_4} \Gamma', \varphi \Rightarrow \Delta'$ and $\vdash_{SC_4} \Gamma'' \Rightarrow \Delta'', \varphi$ for some finite $\Gamma', \Delta', \Gamma'', \Delta''$ as above, whence by weakening and cut we would get $\vdash_{SC_4} \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''$. However, as $\Gamma' \cup \Gamma'', \Delta' \cup \Delta''$ are finite and $\Gamma' \cup \Gamma'' \subseteq T_k, \Delta' \cup \Delta'' \subseteq S_k$, this would contradict condition TS3 for T_k, S_k, which is satisfied by the inductive hypothesis.

If (1) holds, we put $T_{k+1} = T_k \cup \{\varphi\}, S_{k+1} = S_k$; otherwise we put $S_{k+1} = S_k \cup \{\varphi\}, T_{k+1} = T_k$. Then $T_k \subseteq T_{k+1}, S_k \subseteq S_{k+1}$, and T_{k+1}, S_{k+1} obviously satisfy conditions TS1 and TS2. What is more, since T_k, S_k satisfy condition TS4, then from the definition of T_{k+1}, S_{k+1} it can be easily deduced that those sets also satisfy condition TS4.

For the monotonic sequences $\{T_i\}, \{S_i\}$ constructed in this way, we take $T = \bigcup_{i=1}^{\infty} T_i, S = \bigcup_{i=1}^{\infty} S_i$. Then it is easy to see that T, S satisfy conditions TS1, TS2, TS4. As $\Gamma \cup \Delta \subseteq (T \cup S)$ by TS1 and each formula $\varphi \in \mathcal{W}$ outside $\Gamma \cup \Delta$ is added either to T_i or S_i at some step i of the construction process described above, then T, S satisfy also condition TS4.

Based on the sets T, S, we now define a valuation v as follows:

1. For any atomic formula p in $\mathcal{W}$:

$$v(p) = \begin{cases} \mathbf{t} & \text{if (i) } p \in T \\ \mathbf{f} & \text{if (ii) } \neg p \in T \\ \mathbf{e} & \text{if (iii) } (\exists \beta \in T)(p \vee \beta \in S) \\ \mathbf{u} & \text{(iv) otherwise} \end{cases} \tag{7}$$

2. The valuation v is extended to complex formulas in $\mathcal{W}$ according to the truth tables (5) of the matrix $\mathcal{M}^4_{MK}$.

To see the reasons behind the rather peculiar $\mathbf{e}$ clause in (7), let us note that our intention is to have v satisfy the formulas in T and not satisfy the formulas in S. As in $\mathcal{M}^4_{MK}$ we have $\mathbf{e} \vee \mathbf{t} = \mathbf{e}$ and $a \vee \mathbf{t} = \mathbf{t}$ for $a \neq \mathbf{e}$, this means $\beta \in T$ and $p \vee \beta \in S$ must necessarily imply $v(p) = \mathbf{e}$.

As in (7) condition (iv) is the complement of the union of conditions (i)–(iii), to prove that the valuation v is well-defined we have to show that conditions (i)–(iii) are disjoint. In fact, we can prove this holds for arbitrary formulas, not only atomic ones:

Lemma 1. *For any formula $\alpha \in \mathcal{W}$, the following conditions are disjoint:*

$$\textit{(i)} \quad \alpha \in T \qquad \textit{(ii)} \quad \neg\alpha \in T \qquad \textit{(iii)} \quad (\exists \beta \in T)(\alpha \vee \beta \in S) \tag{8}$$

Proof. It is obvious that (i), (ii) are disjoint, for if both α and $\neg\alpha$ belonged to T, then by axiom (A2) for $\Gamma' = \{\alpha, \neg\alpha\}$, $\Delta' = \Delta$ we would have $\Gamma' \subseteq T, \Delta' \subseteq S$ and $\vdash_{SC_4} \Gamma' \Rightarrow \Delta'$, which would contradict condition TS4 holding for T, S.

In turn, if (i) and (iii) were not disjoint, then we would have both $\alpha \in T$ and $\alpha \vee \beta \in S$ for some $\beta \in T$. However, as from axiom (A1) and rule (r4) of SC_4 we can easily deduce that $\vdash_{SC_4} (\alpha \Rightarrow \alpha \vee \beta)$, this would again contradict condition TS4 holding for T, S.

Finally, if (ii) and (iii) were not disjoint, then for some $\beta \in T$ we would have $\neg\alpha \in T, \beta \in T$ and $\alpha \vee \beta \in S$. Considering that by axiom (A1) and rule (r5) of SC_4 we have $\vdash_{SC_4} (\neg\alpha, \beta \Rightarrow \alpha \vee \beta)$, this would once more contradict condition TS4 for T, S. Hence (i)–(iii) are all disjoint, which ends the proof of the lemma.

Next we deduce from Lemma 1 the following key lemma:

Lemma 2. [Truth Lemma]

1. *The implications "if" in points (i)–(iv) of Equation (7) can be replaced with equivalences "iff",*
2. *Equation (7) in the "iff" version holds for any formula $\alpha \in \mathcal{W}$.*

Proof. The first part follows directly from Lemma 1. To prove the second one, we argue by structural induction on the complexity of α. As by Lemma 1 it is enough to actually prove that (i)–(iv) in (7) hold in one direction, we will prove it in the "if" direction only. However, in view of the same Lemma, in applying the inductive assumption to formulas from the structural levels preceding the current one we will be able to assume that (i)–(iv) hold in the "only if" direction too.

First, by (7) and Lemma 1, the thesis holds for atomic formulas. Assume now it holds for formulas φ of rank $r(\varphi) \leq k$, with the rank of a formula defined in the usual way as the maximum nesting of operators, and let ψ be a formula of rank $k+1$. To prove that (7) holds also for ψ, we examine the two possible forms of ψ, and for each of them prove, based on the inductive hypothesis, that ψ satisfies each of the conditions (i)–(iv) in (7).

$\psi = \neg\alpha$, where $r(\alpha) \leq k$.

(i) If $\neg\alpha \in T$, then by (ii) of the inductive hypothesis for α we have $v(\alpha) = \mathbf{f}$, whence $v(\psi) = v(\neg\alpha) = \tilde{\neg}\mathbf{f} = \mathbf{t}$ by the truth tables (5), so (i) holds for ψ.

(ii) If $\neg(\neg\alpha) \in T$, then $\alpha \notin S$. Indeed: as $\neg\neg\alpha \in T$, and by axiom (A1) and rule (r1) we have $\vdash_{SC_4} \neg\neg\alpha \Rightarrow \alpha$, this would contradict property (TS4) of T, S. Hence by TS3 $\alpha \in T$, which by (i) of the inductive hypothesis for α implies $v(\alpha) = \mathbf{t}$. This yields $v(\psi) = \tilde{\neg}\mathbf{t} = \mathbf{f}$, so (ii) holds too.

(iii) Assume now there is a $\beta \in T$ such that $\neg\alpha \vee \beta \in S$. As by axiom A1 and rule (r7) we have $\vdash_{SC_4} (\beta, \alpha \vee \beta \Rightarrow \neg\alpha \vee \beta)$, this implies $\alpha \vee \beta \notin T$, whence $\alpha \vee \beta \in S$. By (iii) of the inductive hypothesis for α, this implies $v(\alpha) = \mathbf{e}$, whence $v(\psi) = \tilde{\neg}\mathbf{e} = \mathbf{e}$, and (iii) holds as well.

(iv) Assume finally that none of the foregoing holds, i.e. (A) $\neg\alpha \notin T$, (B) $\neg(\neg\alpha) \notin T$, and (C) $\neg(\exists\beta \in T)(\neg\alpha \vee \beta \in S)$. By the inductive assumption on α, (A) implies that $v(\alpha) \neq \mathbf{f}$, and so $v(\psi) \neq \mathbf{t}$. Further, as by rule (r2) $\vdash_{SC_4} (\alpha \Rightarrow \neg\neg\alpha)$, then (B) implies $\alpha \notin T$, whence $v(\alpha) \neq \mathbf{t}$ by the inductive assumption, and so $v(\psi) \neq \mathbf{f}$. Suppose now $v(\psi) = \mathbf{e}$. Then, as $\psi = \neg\alpha$, we have also $v(\alpha) = \mathbf{e}$. Hence (by the "only if" direction of point (iii) of the inductive assumption for α) there exists $\beta \in T$ such that $\alpha \vee \beta \in S$. Now TS3 and (C) imply that $\neg\alpha \vee \beta \in T$. However, by A1 and rule (r6) $\vdash_{SC_4} (\beta, \neg\alpha \vee \beta \Rightarrow \alpha \vee \beta)$, and so $\alpha \vee \beta \in T$ too. This contradicts property TS4 of T, S. Accordingly, $v(\psi) \neq \mathbf{e}$, which in view of $v(\psi) \neq \mathbf{f}, \mathbf{t}$ yields $v(\psi) = \mathbf{u}$. Thus (iv) holds too.

$\psi = \alpha \vee \beta$, where $r(\alpha), r(\beta) \leq k$

(i) Assume first $\alpha \vee \beta \in T$. As by axiom (A1) and rule (r3) we have $\vdash_{SC_4} (\alpha \vee \beta \Rightarrow \alpha, \beta)$, then either $\alpha \in T$ or $\beta \in T$. Indeed: in the opposite case, we would have $\alpha, \beta \in S$, and the provability of the above sequent would contradict property TS4 of T, S. Now, if $\alpha \in T$, then by (i) of the inductive hypothesis for α we have $v(\alpha) = \mathbf{t}$, whence $v(\psi) = \mathbf{t}\tilde{\vee}v(\beta) = \mathbf{t}$.
If $\beta \in T$, then $v(\beta) = \mathbf{t}$ by the inductive hypothesis for β. Hence $v(\psi) = v(\alpha)\tilde{\vee}\mathbf{t} \in \{\mathbf{t}, \mathbf{e}\}$. Suppose $v(\psi) = \mathbf{e}$; then $v(\alpha) = \mathbf{e}$, and so by the inductive hypothesis for α there exists $\beta' \in T$ such that $\alpha \vee \beta' \in S$. However, as $\vdash_{SC_4} (\alpha \vee \beta \Rightarrow (\alpha \vee \beta) \vee \beta')$ by rule (r4), and from rule (r11) we can derive that $\vdash_{SC_4} ((\alpha \vee \beta) \vee \beta', \beta' \Rightarrow \alpha \vee \beta')$, then by cut we get $\vdash_{SC_4} (\alpha \vee \beta, \beta' \Rightarrow \alpha \vee \beta')$, which contradicts property TS4 of T, S. Thus $v(\psi) = \mathbf{t}$, and condition (i) holds.

(ii) Assume now $\neg(\alpha \vee \beta) \in T$. As by (A1) and rules (r8), (r10) we have $\vdash_{SC_4} (\neg(\alpha \vee \beta) \Rightarrow \neg\alpha)$ and $\vdash_{SC_4} (\neg(\alpha \vee \beta) \Rightarrow \neg\beta)$, respectively, then by property TS4 of T, S we must have $\neg\alpha, \neg\beta \in T$. By (ii) of the inductive hypothesis for α, β, this implies $v(\alpha) = v(\beta) = \mathbf{f}$, whence by $v(\psi) = \widetilde{\neg}\mathbf{t}\widetilde{\vee}\widetilde{\neg}\mathbf{t} = \mathbf{f}\widetilde{\vee}\mathbf{f} = \mathbf{f}$, so (ii) holds too.

(iii) Assume next there is a $\gamma \in T$ such that $(\alpha \vee \beta) \vee \gamma \in S$. As by (A1) and rule (12) we have $\vdash_{SC_4} (\alpha \vee \gamma, \beta \vee \gamma \Rightarrow (\alpha \vee \beta) \vee \gamma)$, then by TS4 either $\alpha \vee \gamma \notin T$ or $\beta \vee \gamma \notin T$. As this amounts to either (1) $\alpha \vee \gamma \in S$ or (2) $\beta \vee \gamma \in S$ with $\gamma \in T$, then by (iii) of the inductive hypothesis for α, β, we have correspondingly either (1′) $v(\alpha) = \mathbf{e}$ or (2′) $v(\beta) = \mathbf{e}$. Clearly, (1′) implies $v(\psi) = \mathbf{e}\widetilde{\vee}v(\beta) = \mathbf{e}$. In turn, if (2′) holds, then $v(\psi) = v(\alpha)\widetilde{\vee}\mathbf{e} = \mathbf{t}$ if $v(\alpha) = \mathbf{t}$, and $\mathbf{e}$ otherwise. Suppose $v(\alpha) = \mathbf{t}$; then by the "only if" direction of point (i) of the inductive hypothesis for α we have $\alpha \in T$. By axiom (A1) and rule (r4) applied twice we obtain $\vdash_{SC_4} (\alpha \Rightarrow (\alpha \vee \beta) \vee \gamma)$ — which, considering that $\alpha \in T$ and $(\alpha \vee \beta) \vee \gamma \in S$ contradicts property TS4 of T, S. Hence $v(\alpha) \neq \mathbf{t}$, and accordingly $v(\psi) = \mathbf{e}$, so (iii) holds as well.

(iv) Finally, assume that none of (i)—(iii) holds, whence $\alpha \vee \beta \in S, \neg(\alpha \vee \beta) \in S$ and $(*)$ $(\forall\gamma)(\gamma \in S$ or $(\alpha \vee \beta) \vee \gamma \in T)$. We will show that in this case $v(\psi) = v(\alpha \vee \beta)$ cannot be $\mathbf{t}, \mathbf{f}$ or $\mathbf{e}$, whence $v(\psi) = \mathbf{u}$.

$v(\alpha \vee \beta) \neq$ **f:**

Suppose otherwise. Then we have $v(\alpha) = v(\beta) = \mathbf{f}$, which by (ii) of the inductive hypothesis for α, β yields $\neg\alpha, \neg\beta \in T$. However, by (A1) and (r9), $\vdash_{SC_4} (\neg\alpha, \neg\beta \Rightarrow \neg(\alpha \vee \beta))$, which in view of $\neg(\alpha \vee \beta) \in S$ contradicts property TS4 of T, S. Thus indeed $v(\psi) \neq \mathbf{f}$.

$v(\alpha \vee \beta) \neq$ **t:**

Suppose to the contrary. Then either
(1) $v(\alpha) = \mathbf{t}$, or
(2) $v(\beta) = \mathbf{t}$ and $v(\alpha) \neq \mathbf{e}$.
If (1), then $\alpha \in T$ by the "only if" implication of point (i) of the inductive hypothesis for α. However, as by A1 and rule (r4) we have $\vdash_{SC_4} (\alpha \Rightarrow \alpha \vee \beta)$, in view of $\alpha \vee \beta \in S$ we get a contradiction with property TS4 of T, S. In turn, (2) implies $\beta \in T$ by (i) of the inductive hypothesis for β. Together with $\alpha \vee \beta \in S$, this yields $v(\alpha) = \mathbf{e}$ by (iii) of the inductive assumption for α – which contradicts the second part of (2).

$v(\alpha \vee \beta) \neq$ **e:**

Assume otherwise. Then either
(1) $v(\alpha) = \mathbf{e}$, or
(2) $v(\alpha) \neq \mathbf{t}$ and $v(\beta) = \mathbf{e}$.
By the inductive hypothesis for α, (1) implies $(\exists\gamma \in T)(\alpha \vee \gamma \in S)$. However, in view of $(*)$ at the beginning of this point (iv), $\gamma \in T$ implies $(\alpha \vee \beta) \vee \gamma \in T$. In turn, by (A1) and rule (11), $\vdash_{SC_4} ((\alpha\vee\beta)\vee\gamma, \gamma \Rightarrow \alpha\vee\gamma)$ — which in view of $\alpha\vee\gamma \in S$ contradicts property TS4 of T, S. In turn, (2) implies (2') $\alpha \in S$ and (2") $(\exists\gamma \in T)(\beta \vee \gamma \in S)$, with $\gamma \in T$ implying $(\alpha \vee \beta) \vee \gamma \in T$ like in (1). However, by axiom (A1) together with rule (r13) we have $\vdash_{SC_4} ((\alpha \vee \beta) \vee \gamma \Rightarrow \alpha, \beta \vee \gamma)$. Since $\alpha \in S$ by $(2')$ and $\beta\vee\gamma \in S$ by $(2'')$, this once more contradicts property TS4 of T, S. Hence indeed $v(\alpha \vee \beta) \neq \mathbf{e}$, which in view of the fact that $v(\alpha \vee \beta) \neq \mathbf{f}, \mathbf{t}$ yields $v(\alpha \vee \beta) = \mathbf{u}$ as the only possibility. Thus (iv) also holds.

This ends the proof of Lemma 2.

To complete the proof of Theorem 2, note that as $\Gamma \subseteq T, \Delta \subseteq S$ by TS1, $T \cap S = \emptyset$ by TS2, and the only designated value in the matrix $\mathcal{M}^4_{MK}$ is $\mathbf{t}$, then Lemma 2 implies that $v \not\models (\Gamma \Rightarrow \Delta)$. This contradicts the validity of $\Gamma \Rightarrow \Delta$ in $\mathcal{M}^4_{MK}$. □

4.2. Three-valued nondeterministic logics

4.2.1. Dynamic semantics

We start with the system SC^d_3 for the dynamic semantics of the matrix $\mathcal{M}^3_{MK}$. Obviously, axioms (A1), (A2) are valid in that semantics. Reasoning exactly like in case of the four-valued logic above, we can easily show that the axioms and inference rules of SC^d_3 — which are a subset of those of SC^d_4 — are sound in that semantics too. Thus we obtain:

THEOREM 3. *The system SC^d_3 is sound for the dynamic semantics of $\mathcal{M}^3_{MK}$, i.e. each sequent provable in SC^d_3 is dynamically valid in $\mathcal{M}^3_{MK}$.*

The completeness theorem for SC^d_3 (presented in a slightly different version) was already proved in [AK05]. However, for the sake of uniformity, we will now give a new proof of that theorem, based on the new ideas used in the completeness proof for SC_4 given above.

THEOREM 4. *The system SC^d_3 is complete for the dynamic semantics of $\mathcal{M}^3_{MK}$, i.e. each sequent dynamically valid in $\mathcal{M}^3_{MK}$ is provable in SC^d_3.*

Proof We again argue by contradiction. Assuming that $\Sigma = \Gamma \Rightarrow \Delta$ is dynamically valid in $\mathcal{M}^3_{MK}$ and $\not\vdash_{SC^d_3} \Sigma$, we use the same method as in case of SC_4 to extend Γ, Δ to a pair of sets $T, S \subseteq \mathcal{W}$ such that:

$$\begin{array}{llllll} \text{(TS1)} & \Gamma \subseteq T, \Delta \subseteq S & \text{(TS2)} & T \cap S = \emptyset & \text{(TS3)} & T \cup S = \mathcal{W} \\ \text{(TS4)} & \multicolumn{5}{l}{\text{For any finite sets } \Gamma' \subseteq T, \Delta' \subseteq S, \ \not\vdash_{SC^d_3} (\Gamma' \Rightarrow \Delta')} \end{array} \tag{9}$$

Then we define a valuation v as follows: for any formula α in $\mathcal{W}$,

$$v(\alpha) = \begin{cases} \mathbf{t} & \text{iff (i) } \alpha \in T \\ \mathbf{f} & \text{iff (ii) } \neg\alpha \in T \\ \mathbf{e} & \text{iff (iii) } \alpha, \neg\alpha \in S \end{cases} \tag{10}$$

It is easy to see that v is well-defined since the cases (i)–(iii) above are mutually exclusive and exhaustive. Indeed: by TS3, each of $\alpha, \neg\alpha$ must belong to either T or S. However, by TS1 neither α nor $\neg\alpha$ can belong both to T and S, while by (A1) and TS4 α and $\neg\alpha$ cannot both belong to T.

What is more, in view of (TS1), (TS2), $v \models \gamma$ for each $\gamma \in \Gamma$ and $v \not\models \delta$ for each $\delta \in \Delta$, whence $v \not\models \Sigma$. Thus in order to contradict the assumed validity of Σ in the dynamic semantics of $\mathcal{M}^3_{MK}$, and hence prove the completeness of SC^d_3, it suffices to show that v is a legal dynamic valuation in $\mathcal{M}^3_{MK}$, i.e. that it is compliant with the truth tables (4) of that Nmatrix. To this end, we check what values are assigned by v to negation and disjunction of formulas with given logical values.

Negation

1. If $v(\alpha) = \mathbf{t}$, then by (10)(i) we have $\alpha \in T$. As $\vdash_{SC^d_3} (\alpha \Rightarrow \neg\neg\alpha)$ by (r2), from TS4 and TS3 we obtain $\neg(\neg\alpha) = \neg\neg\alpha \in T$, whence (10)(ii) for $\neg\alpha$ yields $v(\neg\alpha) = \mathbf{f}$.
2. If $v(\alpha) = \mathbf{f}$, then $\neg\alpha \in T$, whence by (10)(i) applied to $\neg\alpha$ we get $v(\neg\alpha) = \mathbf{t}$.
3. If $v(\alpha) = \mathbf{e}$, then by (10)(ii) we have $\alpha, \neg\alpha \in S$. As $\vdash_{SC^d_3} (\neg\neg\alpha \Rightarrow \alpha)$ by (r1), then in view of TS4 we get $\neg\neg\alpha \notin T$. Hence $\neg\alpha, \neg(\neg\alpha)$ are both in S and by (10)(iii) applied to $\neg\alpha$ we get $v(\neg\alpha) = \mathbf{e}$.

Disjunction

1. If $v(\alpha) = \mathbf{t}$, then $\alpha \in T$ by (10)(i). As $\vdash_{SC^d_3} (\alpha \Rightarrow \alpha \vee \beta)$ by rule (r4), in view of TS4 and TS3, $\alpha \in T$ implies $\alpha \vee \beta \in T$, whence $v(\alpha \vee \beta) = \mathbf{t}$ by (10)(i) applied to $\alpha \vee \beta$.

2. If $v(\alpha) = v(\beta) = \mathbf{f}$, then $\neg\alpha, \neg\beta \in T$ by (10)(i). Since by (r9) we have $\vdash_{SC_3^d} (\neg\alpha, \neg\beta \Rightarrow \neg(\alpha \vee \beta))$, then $\neg(\alpha \vee \beta) \in T$, whence $v(\alpha \vee \beta) = \mathbf{f}$ by (10)(ii).
3. If $v(\alpha) = \mathbf{f}$ and $v(\beta) = \mathbf{t}$, then by (10)(i), (ii) we have $\neg\alpha \in T, \beta \in T$. As by (r5) $\vdash_{SC_3^d} (\neg\alpha, \beta \Rightarrow \alpha \vee \beta)$, this yields $\alpha \vee \beta \in T$ and $v(\alpha \vee \beta) = \mathbf{t}$.
4. If $v(\alpha) = \mathbf{f}$ and $v(\beta) = \mathbf{e}$, then by (10)(ii), (iii) we get $\neg\alpha \in T$ and $\beta, \neg\beta \in S$. As by (r10) $\vdash_{SC_3^d} (\neg(\alpha \vee \beta) \Rightarrow \neg\beta)$, then $\neg\beta \in S$ implies $\neg(\alpha \vee \beta) \in S$. Further, as by axioms (A1), (A2) and rule (r3) $\vdash_{SC_3^d} (\neg\alpha, \alpha \vee \beta \Rightarrow \beta)$, then $\neg\alpha \in T$ and $\beta \in S$ imply $\alpha \vee \beta \in S$. Applying (10)(iii) for $\alpha \vee \beta$ to the last two results, we obtain $v(\alpha \vee \beta) = \mathbf{e}$.
5. Assume finally $v(\alpha) = \mathbf{e}$, which yields $(*)$ $\alpha, \neg\alpha \in S$ by (10)(iii). As by (r8) $\vdash_{SC_3^d} (\neg(\alpha \vee \beta) \Rightarrow \neg\alpha)$, then the former implies $(**)$ $\neg(\alpha \vee \beta) \in S$. Accordingly, $\neg(\alpha \vee \beta) \notin T$, and by (10) we obtain $v(\alpha \vee \beta) \neq \mathbf{f}$, whence $v(\alpha \vee \beta) \in \{\mathbf{t}, \mathbf{e}\}$. As by (4) $\widetilde{\vee}(\mathbf{e}, \mathbf{t}) = \{\mathbf{t}, \mathbf{e}\}$, then $v(\alpha \vee \beta)$ is compliant with the truth table for $\vee$ in (4) if $v(\beta) = \mathbf{t}$. If $v(\beta) = \mathbf{f}$, then $\neg\beta \in T$. As by (A1), (A2) and (r3) we have $\vdash_{SC_3^d} (\neg\beta, \alpha \vee \beta \Rightarrow \alpha)$, then $(*)$ implies $\alpha \vee \beta \in S$, which together with $(**)$ yields $v(\alpha \vee \beta) = \mathbf{e}$. Lastly, if $v(\beta) = \mathbf{e}$, then $\beta, \neg\beta \in S$, which again yields $\alpha \vee \beta \in S$ in view of $(*)$ and the fact that by (r3) $\vdash_{SC_3^d} (\alpha \vee \beta \Rightarrow \alpha, \beta)$. Thus once more $v(\alpha \vee \beta) = \mathbf{e}$.

Clearly, the above implies that $v(\neg\alpha) \in \widetilde{\neg} v(\alpha)$ and $v(\alpha \vee \beta) \in \widetilde{\vee}(v(\alpha), v(\beta))$, where $\widetilde{\neg}, \widetilde{\vee}$ are the interpretations of $\neg, \vee$ given by the truth tables (4) of $\mathcal{M}^3_{MK}$. Thus v is indeed a legal dynamic valuation in that matrix, which ends the proof. $\square$

Finally, let us remark that by adding to SC_3^d the rule

$$\frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

we obtain a proof system for Kleene logic, while by adding the rule

$$\frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \neg\alpha \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta}$$

we get a proof system for Kleene logic. The proofs of Theorems 3, 4 can be easily extended to soundness and completeness proofs for the systems obtained in the above way.

4.2.2. Static semantics

The last remaining soundness and completeness proofs are those for the system SC_3^s for the static semantics of the matrix $\mathcal{M}^3_{MK}$.

THEOREM 5. *The system SC_3^s is sound for the static semantics of $\mathcal{M}^3_{MK}$, i.e. each sequent provable in SC_3^s is statically valid in $\mathcal{M}^3_{MK}$.*

PROOF. The system SC_3^s differs from the system SC_3^d for the dynamic semantics of $\mathcal{M}^3_{MK}$ in just one inference rule

$$(S)\ \frac{\Gamma \Rightarrow \Delta, \beta \qquad \Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta, \alpha \vee \beta}$$

Since each static valuation in $\mathcal{M}^3_{MK}$ is also a dynamic valuation in that Nmatrix, from the soundness theorem for SC_3^d it follows that all the axioms and inference rules of SC_3^d are also statically valid in $\mathcal{M}^3_{MK}$. Hence to prove the soundness of SC_3^s it suffices to prove the soundness of rule (S).

Thus suppose v is an arbitrary static valuation in $\mathcal{M}^3_{MK}$ satisfying the premises of rule (S). First, if either $v \not\models \gamma$ for some $\gamma \in \Gamma$ or $v \models \delta$ for some $\delta \in \Delta$, then obviously v satisfies the conclusion of rule (S). Otherwise, $v \models \beta, v \not\models \varphi, v \not\models \neg\varphi$, whence $v(\varphi) \neq \mathbf{t}$ and $v(\varphi) \neq \mathbf{f}$, which yields $v(\varphi) = \mathbf{e}$. To show that v satisfies the conclusion of rule (S), assume $v \models \varphi \vee \psi$. Since v as a static valuation interprets $\vee$ as a function $f_\vee : \mathcal{T}^2 \to \mathcal{T}$ compliant with the non-deterministic interpretation $\widetilde{\vee}$ of $\vee$ in $\mathcal{M}^3_{MK}$ (see (3)), this means $\mathbf{t} = v(\varphi \vee \psi) = f_\vee(v(\varphi), v(\psi)) = f_\vee(\mathbf{e}, v(\psi))$. As $f_\vee$ is compliant with $\widetilde{\vee}$ given by (4), this implies $v(\psi) = \mathbf{t}$ and $f_\vee(\mathbf{e}, \mathbf{t}) = \mathbf{t}$. Hence, in view of $v(\beta) = \mathbf{t}$ and $f_\vee(\mathbf{f}, \mathbf{t}) = f_\vee(\mathbf{t}, \mathbf{t}) = \mathbf{t}$ by (4), we conclude that $v(\alpha \vee \beta) = f_\vee(v(\alpha), \mathbf{t}) = \mathbf{t}$. Thus v satisfies also the premises of rule S, and the rule is sound for the static semantics. □.

Now we can pass to proving the completeness of SC_3^s:

THEOREM 6. *The system SC_3^s is complete for the static semantics of $\mathcal{M}^3_{MK}$, i.e. each sequent statically valid in $\mathcal{M}^3_{MK}$ is provable in SC_3^s.*

PROOF The proof is exactly analogous to that for the system SC_3^d, plus one additional element: proving that the valuation v defined in the proof is a static valuation in $\mathcal{M}^3_{MK}$. In other words, we have to show that v interprets each connective $\diamond$ as a function $f_\diamond : \mathcal{T}^2 \to \mathcal{T}$ consistent with the non-deterministic interpretation $\widetilde{\diamond}$ of $\diamond$ in $\mathcal{M}^3_{MK}$ (see (3)). As the interpretation of negation in $\mathcal{M}^3_{MK}$ is deterministic, the above task reduces to proving this fact for disjunction.

Assume the sets T, S are constructed in exactly the same way as before, but using the provability in SC_3^s in condition TS4. Let v be defined by (10) in the completeness proof for SC_3^d. As the only nondeterminacy in the truth tables for disjunction in $\mathcal{M}_{MK}^3$ is $\widetilde{\vee}(\mathbf{e}, \mathbf{t}) = \{\mathbf{t}, \mathbf{e}\}$, in order to prove that v is indeed a static valuation, it suffices to show that, for any $\alpha, \beta, \varphi, \psi \in \mathcal{W}$

$$\begin{array}{c}\text{If } v(\varphi) = \mathbf{e}, v(\psi) = \mathbf{t} \text{ and } v(\varphi \vee \psi) = \mathbf{t}, \\ \text{then } v(\alpha) = \mathbf{e}, v(\beta) = \mathbf{t} \text{ implies } v(\alpha \vee \beta) = \mathbf{t}\end{array} \tag{11}$$

Thus assume that the antecedents of both implications in (11) hold. Then by the definition (10) of v we have $\beta, \varphi \vee \psi \in T$ and $\varphi, \neg\varphi \in S$. However, from rule (S) we can deduce that $\vdash_{SC_3^s} (\beta, \varphi \vee \psi \Rightarrow \alpha \vee \beta, \varphi, \neg\varphi)$. Accordingly, $\alpha \vee \beta \in S$ would contradict property $TS4$ of sets T, S, whence $\alpha \vee \beta \in T$. Thus $v(\alpha \vee \beta) = \mathbf{t}$, and (11) holds. $\square$.

5. Role of the cut rule in the presented proof systems

The cut rule is an official rule of the systems presented above. This may justly be taken as a drawback. Therefore we devote the last section of this paper to a short discussion of this issue.

First, it should be noted that though we have included the cut in the system SC_3^d, and used it in constructing the sets T, S on which the current completeness proof relies, the cut rule can be eliminated from that system, as shown in [AK05].

Moreover, from [AK05] and the results of the present paper it also follows that a sequent $\Gamma \Rightarrow \Delta$ is statically valid in $\mathcal{M}_{MK}^3$ iff both $\Gamma \Rightarrow \Delta, \overline{\mathbf{e}} \vee \overline{\mathbf{t}}$ and $\Gamma, \overline{\mathbf{e}} \vee \overline{\mathbf{t}} \Rightarrow \Delta$ have cut-free proofs in the system obtained from SC_3^s by augmenting its language with the constants $\overline{\mathbf{e}}, \overline{\mathbf{t}}$ representing $\mathbf{e}$ and $\mathbf{t}$ (respectively), and adding the axioms $(\Rightarrow \overline{\mathbf{t}}), (\overline{\mathbf{e}} \Rightarrow), (\neg\overline{\mathbf{e}} \Rightarrow)$ to the system itself. This means that cut in this variant of SC_3^d can be confined to the formula $\overline{\mathbf{e}} \vee \overline{\mathbf{t}}$. As for SC_3^d itself — it is still an open problem whether the cut rule can completely be eliminated from it or not (we believe it can).

Finally, we also do not know whether elimination of the cut rule from the system SC_4 is possible or not. This issue will be the subject of further research, and if the cut cannot be eliminated, the goal will be to find out how to control its use in an acceptable way.

Acknowledgment

This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No 809-06).

References

[AK05] Avron A., and Konikowska B., *Proof Systems for Logics Based on Non-deterministic Multiple-valued Structures*, Logic Journal of the IGPL, **13** (2005), pp. 365-387,

[ABK06] Avron A., Ben-Naim J., and Konikowska B., *Cut-free Ordinary Sequent Calculi for Logics Having Generalized Finite-Valued Semantics*, Logica Universalis **1** (2006), pp. 41–69,

[AL01] Avron A., and Lev I., *Canonical Propositional Gentzen-Type Systems*, in **Proceedings of the 1st International Joint Conference on Automated Reasoning (IJCAR 2001)** (R. Goré, A Leitsch, T. Nipkow, Eds), LNAI 2083, Springer Verlag, 2001, pp. 529–544,

[AL05] Avron A., and Lev I., *Non-deterministic Multiple-valued Structures*, Journal of Logic and Computation, **15** (2005), pp. 241–261,

[BCJ84] Barringer H., Chang J.H., Jones C.B., *A logic covering undefinedness in program proofs*, Acta Informatica **21** (1984), pp. 251–269,

[Bli91] Blikle A., *Three-valued predicates for software specification and validation*, Fundamenta Informaticae **14**(1991), pp. 387–410,

[Ho87] Hogevijs A., *Partial predicate logic in computer science*, Acta Informatica **24**, (1987), pp. 381–393,

[Kl52] Kleene S.C., *Introduction to Metamathematics*, North-Holland, 1952,

[KTB91] Konikowska, B., Tarlecki, A., Blikle, J., *A three-valued logic for software specification and validation*, Fundamenta Informaticae 14(4), 1991, pp. 411–453,

[Ko93] Konikowska, B. *Two over three: a two-valued logic for software specification and validation over a three-valued predicate calculus*, Journal for Applied Nonclassical Logic 3(1), 1993, pp. 39–71,

[Ko08] Konikowska B., *A Four-Valued Logic for Reasoning about Finite and Infinite Computation Errors in Programs*, accepted to: **The Many Sides of Logic**, W. A. Carnielli, M. E. Coniglio, I. M. Loffredo D' Ottaviano eds., Series *Studies in Logic*, College Publications, London,

[MC67] McCarthy J., *A basis for a mathematical theory of computation* Computer Programming and Formal Systems, North-Holland, 1967,

[Owe85] Owe O., *An approach to program reasoning based on first order logic for partial functions*, Res. Rep. Institute of Informatics, University of Oslo, no. **89**, 1985,