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Cut-Elimination and Quantification in Canonical Systems

Abstract. Canonical propositional Gentzen-type systems are systems which in addition to the standard axioms and structural rules have only pure logical rules with the subformula property, in which exactly one occurrence of a connective is introduced in the conclusion, and no other occurrence of any connective is mentioned anywhere else. In this paper we considerably generalize the notion of a "canonical system" to first-order languages and beyond. We extend the propositional coherence criterion for the non-triviality of such systems to rules with unary quantifiers and show that it remains constructive. Then we provide semantics for such canonical systems using 2-valued non-deterministic matrices extended to languages with quantifiers, and prove that the following properties are equivalent for a canonical system G: (1) G admits Cut-Elimination, (2) G is coherent, and (3) G has a characteristic 2-valued non-deterministic matrix.

Keywords: proof theory, cut elimination, canonical systems, non-deterministic matrices.

1. Introduction

There is a long tradition in the philosophy of logic, according to which the meaning of a connective is determined by the introduction and the elimination rules which are associated with it[1]. This tradition goes back to Gentzen, who made the following remark in his classical paper *Investigations Into Logical Deduction* ([9]):

> *The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.*

Now the supporters of this thesis of Gentzen usually have in mind Natural Deduction systems of an ideal type. In this type of "canonical systems" each connective $\Diamond$ has its own introduction and elimination rules, in each of which $\Diamond$ is mentioned exactly once, and no other connective is involved. The rules should also be pure in the sense of [1]. Unfortunately, already the handling of negation requires rules which are not canonical in this sense. This problem was solved by Gentzen himself by moving to what is now known as Gentzen-type calculi, which instead of introduction and elimination rules use left and

[1]See e.g. [11] for details.

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right introduction rules. The intuitive notions of a "canonical rule" can be adapted to such systems in a straightforward way, and it is well-known that the usual classical connectives can indeed be fully characterized by canonical Gentzen-type rules. Moreover: the cut-elimination theorem obtains in all the known Gentzen-type calculi for propositional classical logic (or some fragment of it) which employ only canonical rules.

In [3, 4] these facts were generalized by defining "canonical propositional Gentzen-type rules and systems" in precise terms. A constructive *coherence* criterion for the non-triviality of such systems was then provided, and it was shown that a system of this kind admits cut-elimination iff it is coherent. It was further proved that the semantics of such systems is provided by two-valued non-deterministic matrices (2Nmatrices), which form a natural generalization of the classical matrix. In fact, a characteristic 2Nmatrix was constructed for every coherent canonical propositional system.

The main difficulty in generalizing these notions and results to quantificational rules is the fact that such rules involve two different types of substitutions: internal substitution of terms for variables of the language, and external substitution of formulas for schematic variables. Another difficulty is caused by the fact that such rules are usually no longer pure in the strict sense of [1]. Still, this paper generalizes the propositional theory of canonical systems to predicate calculi with unary quantifiers. We propose a precise characterization of a canonical quantificational rule in such calculi (not surprisingly, the standard Gentzen-type rules for $\forall$ and $\exists$ are canonical according to it), and give a constructive extension of the coherence criterion of [3, 4] for canonical systems of this type. Then we prove that again a canonical Gentzen-type system of this type admits cut-elimination iff it is coherent, and that it is coherent iff it has a characteristic 2Nmatrix.

In addition to providing a better insight into the phenomenon of cut-elimination, the results of this paper also provide further evidence for the thesis that the meaning of a logical constant is given by the introduction (and "elimination") rules associated with it. In fact, we show that any reasonable set of canonical rules for some unary quantifier completely determines the semantics of that quantifier in the framework of 2Nmatrices.

2. Preliminaries

We start by reproducing the relevant definitions and results of [5] (where the notion of a non-deterministic martix was generalized to languages with quantifiers). In what follows, L is a language with unary quantifiers, Frm_L is its set of formulas, Frm_L^{cl} its set of closed formulas, Tr_L its set of terms and

Tr_L^{cl} its set of closed terms. *Var* is its set of variables. x, y are meta-variables ranging over the variables from Var, A, B denote L-formulas, $\mathbf{t}, \mathbf{t}'$ denote L-terms, Γ, Δ denote sets of formulas. $\equiv_\alpha$ is the α-equivalence relation between formulas, i.e identity up to the renaming of bound variables. We use [ ] for application of functions in the meta-language, leaving the use of ( ) to the object language. $A\{\mathbf{t}/x\}$ denotes the formula obtained from A by substituting $\mathbf{t}$ for x. Given a set of L-formulas Γ and an L-term $\mathbf{t}$, we denote the set $\{A\{\mathbf{t}/x\} \mid A \in \Gamma\}$ by $\Gamma\{\mathbf{t}/x\}$. Given an L-formula A, $Fv[A]$ is the set of variables occurring free in A. Given a set S, $P^+(S)$ is the set of all nonempty subsets of S.

DEFINITION 2.1. A formula A, a set of formulas Γ or a sequent $\Gamma \Rightarrow \Delta$ *satisfies the pure-variable condition* if its set of free variables is disjoint from its set of bound variables.

Note that the cut-elimination theorem for classical first-order logic holds only for sequents satisfying the pure-variable condition. We shall see that a similar limitation holds for all canonical systems.

DEFINITION 2.2. (**Non-deterministic matrix**) Given a language L based on connectives and unary quantifiers, a non-deterministic matrix (henceforth Nmatrix) for L is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{G}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ is a non-empty set of truth values.
- $\mathcal{G}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$.
- For every n-ary connective $\Diamond$ and for every quantifier Q of L, $\mathcal{O}$ includes the corresponding interpretation functions[2] :
  - $\tilde{\Diamond}_\mathcal{M} : \mathcal{V}^n \to P^+(\mathcal{V})$
  - $\tilde{\mathsf{Q}}_\mathcal{M} : P^+(\mathcal{V}) \to P^+(\mathcal{V})$

The set $\mathcal{V} - \mathcal{G}$ is denoted by $\mathcal{N}$.

DEFINITION 2.3. **(L-structure)** Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{G}, \mathcal{O} \rangle$ be an Nmatrix. An L-structure for $\mathcal{M}$ is a pair $S = \langle D, I \rangle$ where D is a (non-empty) domain, and I is a function interpreting the constants, predicate symbols and function symbols of L, so that $I[c] \in D$ if c is a constant, $I[p] : D^n \to \mathcal{V}$ if p is an n-ary predicate, and $I[f] : D^n \to D$ if f is an n-ary function. I is extended to interpret closed terms of L as follows:

$$I[f(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[f][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$$

[2]This is a generalization of distribution quantifiers introduced in [8].

DEFINITION 2.4. **(L(D))** Let S=$\langle D, I\rangle$ be an L-structure for an Nmatrix $\mathcal{M}$. $L(D)$ is the language obtained from L by adding to it the set of *individual constants* $\{\overline{a} \mid a \in D\}$. $S' = \langle D, I'\rangle$ is the $L(D)$-structure, such that I' is the extension of I satisfying: $I'[\overline{a}] = a$.

Given an L-structure $S = \langle D, I\rangle$, we shall refer to the extended $L(D)$-structure $\langle D, I'\rangle$ as S and to I' as I when the meaning is clear from the context.

DEFINITION 2.5. **(S-substitution)** Given an L-structure $S = \langle D, I\rangle$ for an Nmatrix $\mathcal{M}$ for L, an *S-substitution* is a function $\sigma : Var \rightarrow Tr^{\text{cl}}_{L(D)}$. It is extended to $\sigma : Tr_L \cup Frm_L \rightarrow Tr^{\text{cl}}_{L(D)} \cup Frm^{\text{cl}}_{L(D)}$ as follows: for a term $\mathbf{t}$ of $L(D)$, $\sigma[\mathbf{t}]$ is the closed term obtained from $\mathbf{t}$ by replacing every $x \in Fv[\mathbf{t}]$ by $\sigma[x]$. For a formula B, $\sigma[B]$ is the sentence obtained from B by replacing every $x \in Fv[B]$ by $\sigma[x]$. Given a set Γ of formulas, we denote the set $\{\sigma[A] \mid A \in \Gamma\}$ by $\sigma[\Gamma]$.

DEFINITION 2.6. **(Congruence of terms and formulas)** Let S be an L-structure for an Nmatrix $\mathcal{M}$. The relation $\sim^S$ is defined inductively as follows:

- $x \sim^S x$
- For closed terms $\mathbf{t}, \mathbf{t}'$ of $L(D)$: $\mathbf{t} \sim^S \mathbf{t}'$ when $I[\mathbf{t}] = I[\mathbf{t}']$.
- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, \ \dots \ , \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $f(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S f(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.
- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, \mathbf{t}_2 \sim^S \mathbf{t}'_2, \ \dots \ , \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $p(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S p(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.
- If $A_1 \sim^S B_1, \ \dots \ , A_n \sim^S B_n$, then $\Diamond(A_1, ..., A_n) \sim^S \Diamond(B_1, ..., B_n)$ for any n-ary connective $\Diamond$ of L.
- If $A\{z/x\} \sim^S B\{z/y\}$, where z is a new variable, then $\mathsf{Q}x\, A \sim^S \mathsf{Q}y\, B$ for any unary quantifier Q of L.

PROPOSITION 2.7. *For any L-structure S, $\sim^S$ is a congruence relation.*

LEMMA 2.8. *Let S be an L-structure for an Nmatrix $\mathcal{M}$. Let A, A' be formulas of $L(D)$. Let $\mathbf{t}, \mathbf{t}'$ be closed terms of $L(D)$, such that $\mathbf{t} \sim^S \mathbf{t}'$.*

1. *If $A \equiv_\alpha A'$, then $A \sim^S A'$.*
2. *$A\{\mathbf{t}/x\} \sim^S A\{\mathbf{t}'/x\}$.*

Definition 2.9. **(Legal valuation)** Let $S = \langle D, I \rangle$ be an L-structure for an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{G}, \mathcal{O} \rangle$. An S-valuation $v : Frm^{\text{cl}}_{L(D)} \to \mathcal{V}$ is *legal in* $\mathcal{M}$ if it satisfies the following conditions:

- $v[A] = v[A']$ for every two sentences A, A' of $L(D)$, such that $A \sim^S A'$.
- $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.
- $v[\Diamond(A_1, ..., A_n)] \in \tilde{\Diamond}_{\mathcal{M}}[v[A_1], ..., v[A_n]]$.
- $v[\mathsf{Q}x\ A] \in \tilde{\mathsf{Q}}_{\mathcal{M}}[\{v[A\{\overline{a}/x\}] \mid a \in D\}]$.

Definition 2.10. **(Model, $\mathcal{M}$-validity, $\vdash_{\mathcal{M}}$, characteristic Nmatrix)** Let $S = \langle D, I \rangle$ be an L-structure for an Nmatrix $\mathcal{M}$, A a sentence of $L(D)$ and Γ, Δ sets of formulas of $L(D)$.

1. An $\mathcal{M}$-legal S-valuation v is *a model* of A (Γ) in $\mathcal{M}$, denoted by $S, v \models_{\mathcal{M}} A$ ($S, v \models_{\mathcal{M}} \Gamma$), if $v[A] \in \mathcal{G}$ ($S, v \models_{\mathcal{M}} A$ for every $A \in \Gamma$).
2. A sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$*-valid in* S if for every S-substitution σ and every S-valuation v legal in $\mathcal{M}$, if $S, v \models_{\mathcal{M}} \sigma[\Gamma]$ then there exists some $B \in \Delta$ s.t. $S, v \models_{\mathcal{M}} \sigma[B]$. A sequent is $\mathcal{M}$*-valid* if it is $\mathcal{M}$-valid in every L-structure for $\mathcal{M}$.
3. $\vdash_{\mathcal{M}}$, the consequence relation induced by $\mathcal{M}$ is defined as follows: $\Gamma \vdash_{\mathcal{M}} \Delta$ if $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid.
4. An L-Nmatrix $\mathcal{M}$ is a *characteristic* Nmatrix of a calculus P over L if $\vdash_P = \vdash_{\mathcal{M}}$.

Lemma 2.11. *Let $\mathcal{M}$ be an Nmatrix for L and A, A' L-formulas. If $A \equiv_\alpha A'$, then the sequent $A \Rightarrow A'$ is $\mathcal{M}$-valid.*

3. Canonical systems with quantifiers

Henceforth we assume that L has an infinite countable set of constants *Con* and an infinite countable set of variables *Var*. In addition, we add to L an infinite set of propositional constants $\{q_1, q_2, ...,\}$ and p - a unary predicate symbol. Let x, y be meta-variables ranging over variables from *Var* and c, c' - over constants from *Con*.
By a clause we mean a sequent which consists of atomic formulas only. The following is a generalization of definition 3 from [3]:

DEFINITION 3.1. 1. A *canonical propositional rule* of arity n is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C$, where $m \geq 0$, and for every $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a non-empty clause such that $\Pi_i, \Sigma_i \subseteq \{q_1, ..., q_n\}$, and C is either $\Diamond(q_1, ..., q_n) \Rightarrow$ or $\Rightarrow \Diamond(q_1, ..., q_n)$ for some n-ary connective $\Diamond$ of L.

2. A *canonical quantificational rule* is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C$, where $m \geq 0$, C is either $\Rightarrow \mathsf{Q}x\, p(x)$ or $\mathsf{Q}x\, p(x) \Rightarrow$ for some quantifier Q of L and for every $1 \leq i \leq m$: $\Pi_i \Rightarrow \Sigma_i$ is a clause such that $|\Pi_i \cup \Sigma_i| = 1$, $\Pi_i, \Sigma_i \subseteq \{p(c) | \ c \in \mathit{Con}\} \cup \{p(y) \mid y \in \mathit{Var}\}$ and every $c \in \mathit{Con}$ and $y \in \mathit{Var}$ occurs in at most one of the premises.

3. An *application* of a canonical propositional rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/\Diamond(q_1, ..., q_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \Diamond(A_1, ..., A_n) \Rightarrow \Delta}$$

where for every $1 \leq i \leq m$, Π_i^* and Σ_i^* are obtained from Π_i and Σ_i respectively by substituting A_j for q_j for all $1 \leq j \leq n$, and Γ, Δ are any sets of formulas. An application of a canonical rule with a conclusion of the form $\Rightarrow \Diamond(q_1, ..., q_n)$ is defined similarly.

4. An *application* of a canonical quantificational rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/\mathsf{Q}x\ p(x) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \mathsf{Q}z\ A \Rightarrow \Delta}$$

where $z \in \mathit{Var}$, A is any L-formula, Γ, Δ are any sets of L-formulas and for every $1 \leq i \leq m$, Π_i^* and Σ_i^* are obtained from Π_i and Σ_i respectively by (i) for every $c \in \mathit{Con}$, substitute $A\{\mathbf{t}_c/z\}$ for $p(c)$, where $\mathbf{t}_c$ is any L-term free for z in A, and (ii) for every $y \in \mathit{Var}$, substitute $A\{w_y/z\}$ for $p(y)$, where w_y is any variable free for z in A, which does not occur free in $\Gamma \cup \Delta \cup \{\mathsf{Q}z\ A\}$.
An application of a canonical quantificational rule with a conclusion of the form $\Rightarrow \mathsf{Q}x\ p(x)$ is defined similarly.

Note that the constants in rules are used as term variables, while the variables in rules play the role of eigenvariables.

Example 3.2.

1. *([4]) The two standard introduction rules for classical conjunction can be formulated as follows:*

$$\{p_1, p_2 \Rightarrow\}/p_1 \wedge p_2 \Rightarrow \qquad \{\Rightarrow p_1, \Rightarrow p_2\}/\Rightarrow p_1 \wedge p_2$$

2. *The two standard introduction rules for* $\forall$ *can be formulated as follows:*

$$\{p(c) \Rightarrow\}/\forall x\, p(x) \Rightarrow \qquad \{\Rightarrow p(y)\}/\Rightarrow \forall x\, p(x)$$

Applications of these rules have the forms:

$$\frac{\Gamma, A\{\boldsymbol{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta}\ (\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}\ (\Rightarrow \forall)$$

where $z, w \in Var$*,* z *is free for* w *in* A*,* z *is not free in* $\Gamma \cup \Delta \cup \{\forall w A\}$*, and* $\boldsymbol{t}$ *is any term free for* w *in* A*.*

3. *An application of the rule* $\{\Rightarrow p(c)\ ,\ p(y) \Rightarrow\}/\Rightarrow \mathsf{Q}_1 x\ p(x)$ *is of the form:*

$$\frac{\Gamma \Rightarrow A\{\boldsymbol{t}/w\}, \Delta \quad \Gamma, A\{z/w\} \Rightarrow \Delta}{\Gamma \Rightarrow \mathsf{Q}_1 w\ A}$$

where $\boldsymbol{t}$ *is free for* w *in* A *and* z *is free for* w *in* A *and does not occur free in* $\Gamma \cup \Delta \cup \{\mathsf{Q}_1 w\ A\}$*.*

By lemma 2.11, the sequent $A \Rightarrow A'$, where $A \equiv_\alpha A'$, should be provable in any proof system which has a characteristic Nmatrix. What natural syntactic conditions guarantee its derivability from the standard axiom of Gentzen-type systems is still a question for further research. For now, we simply strengthen the standard axiom of Gentzen-type systems to $A \Rightarrow A'$, where $A \equiv_\alpha A'$. Henceforth we refer to this scheme as *the* α*-axiom.*

Definition 3.3. A Gentzen-type calculus G is *canonical* if in addition to the α-axiom and the standard structural rules, G has only canonical propositional and quantificational rules.

Note that the standard classical first-order calculus is a canonical calculus. Therefore it is not surprising that the following properties of the classical calculus (see e.g. [12]) hold for any canonical calculus:

Proposition 3.4. **(α-conversion)** *Let* $\Gamma' \Rightarrow \Delta'$ *be obtained from* $\Gamma \Rightarrow \Delta$ *by renaming of bound variables. Then any derivation of* $\Gamma \Rightarrow \Delta$ *in a canonical calculus* G *can be converted into a derivation in* G *of* $\Gamma' \Rightarrow \Delta'$ *by renaming of bound variables.*

PROPOSITION 3.5. **(Substitution)** *Let G be a canonical calculus. Let $\Gamma \Rightarrow \Delta$ be a sequent derivable in G. For any $\boldsymbol{t}$ free for x in any $A \in \Gamma \cup \Delta$, $\Gamma\{\boldsymbol{t}/x\} \Rightarrow \Delta\{\boldsymbol{t}/x\}$ is derivable in G with the same derivation height.*

The proofs of the propositions are rather standard and we omit them.

Now a natural question arises: what properties should a canonical Gentzen-type calculus G satisfy, so that the relation $\vdash_G$ defined by G is non-trivial (i.e., there exist some non-empty Γ, Δ, such that $\Gamma \not\vdash_G \Delta$)? The following definition, which is a generalization of definition 6 from [3], provides a constructive equivalent for the non-triviality condition:

DEFINITION 3.6. **(Coherence)** A canonical calculus G is *coherent* if for every two canonical rules of G of the form $S_1/ \Rightarrow A$ and $S_2/A \Rightarrow$, the set of clauses $S_1 \cup S_2$ is classically inconsistent (and so the empty sequent can be derived from it using first-order resolution).

By the soundness and completeness of the first-order resolution method, a set of clauses $S_1 \cup S_2$ is classically inconsistent iff there exists no L- structure, which classically satisfies $S_1 \cup S_2$. Note that if the clauses consist only of closed formulas, then the cut rule suffices for showing inconsistency (otherwise, unification is also needed).

EXAMPLE 3.7. *The two classical rules for $\forall$, $\{p(c) \Rightarrow\}/\forall x\, p(x) \Rightarrow$ and $\{\Rightarrow p(y)\}/ \Rightarrow \forall x\, p(x)$ form a coherent set of rules. Here, $S_1 = \{p(c_1) \Rightarrow\}$ and $S_2 = \{\Rightarrow p(y_1)\}$ and so $S_1 \cup S_2$ is the classically-inconsistent set $\{p(c_1) \Rightarrow, \Rightarrow p(y_1)\}$, from which the empty sequent can be derived using one step of resolution.*

PROPOSITION 3.8. **(Decidability of coherence)** *The coherence of a canonical calculus G is decidable.*

PROOF. For every pair of rules $S_1/ \Rightarrow A$ and $S_2/A \Rightarrow$ of G, it is sufficient to check the satisfiability of $S_1 \cup S_2$, a simple fragment of monadic logic, the satisfiability for which is decidable. ∎

Properties of canonical systems with quantifiers

Now we establish a deep connection between the possibility to eliminate cuts in a canonical Gentzen-type system G with quantifiers, the coherence of G, and the existence of a characteristic 2Nmatrix for it. We begin by showing that any canonical calculus can be transformed into an equivalent calculus in *normal form*, satisfying certain properties defined below.

Canonical systems in normal form

DEFINITION 3.9. **(Equal rules)** Two canonical quantificational rules, different only by the names of constants and/or variables, are equal.

DEFINITION 3.10. **(Normal form)** A canonical calculus G is *in normal form* if it satisfies the following conditions:

1. Every canonical propositional rule of G has the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq n}/C$, where $\Pi_i \cup \Sigma_i = \{q_i\}$ for every $1 \leq i \leq n$ and C is either $\Rightarrow \diamondsuit(q_1, ..., q_n)$ or $\diamondsuit(q_1, ..., q_n) \Rightarrow$ for some n-ary connective $\diamondsuit$.

2. Every canonical quantificational rule of G has one of the following forms (for some $y \in \mathit{Var}$ and $c, c' \in \mathit{Con}$ $(c \neq c')$) :

   (a) Type AT: $\{\Rightarrow p(y)\}/C$

   (b) Type AF: $\{p(y) \Rightarrow\}/C$

   (c) Type ETF: $\{p(c) \Rightarrow, \Rightarrow p(c')\}/C$

   where C is either $\Rightarrow \mathsf{Q}x\ p(x)$ or $\mathsf{Q}x\ p(x) \Rightarrow$ for some quantifier Q.

3. There is no pair of equal rules in G.

Note that not all of the standard quantificational rules from example 3.2 are in normal form. We shall show their transformation into normal form shortly.

Notation:[3] Let $-1 = 0, -0 = 1$ and $ite(1, A, B) = A$, $ite(0, A, B) = B$. Let Φ, A^s (where Φ may be empty) denote $ite(s, \Phi \cup \{A\}, \Phi)$. For instance, in this notation the rule $\{p(c) \Rightarrow\}/\forall x\ p(x) \Rightarrow$ has the form $\{p(c)^{-s} \Rightarrow p(c)^s\}/\forall x\ p(x)^{-r} \Rightarrow \forall x\ p(x)^r$ for $s = r = 0$.

In this notation each rule of a canonical calculus in normal form has one of the following forms (where $r, s, s_1, ..., s_n \in \{0, 1\}$):

1. Propositional rules: $\{q_i^{-s_i} \Rightarrow q_i^{s_i}\}_{1\leq i\leq m}/\diamondsuit(q_1, ..., q_n)^{-r} \Rightarrow \diamondsuit(q_1, ..., q_n)^r$. We denote a rule of this form by $[\diamondsuit(s_1, ..., s_n) : r]$.

2. Quantifier rules, types AF and AT: $\{p(y)^{-s} \Rightarrow p(y)^s\}/\mathsf{Q}x\ p(x)^{-r} \Rightarrow \mathsf{Q}x\ p(x)^r$. We denote a rule of this form by $[\mathsf{Q}(s) : r]$

[3] Following [3], notations 3-5.

3. Quantifier rules, type ETF:
$\{p(c) \Rightarrow, \Rightarrow p(c')\}/\mathsf{Q}x\ p(x)^{-r} \Rightarrow \mathsf{Q}x\ p(x)^r$. We denote a rule of this form by $[\mathsf{Q}(\mathcal{V}) : r]$.

THEOREM 3.11. **(Transformation into normal form)** *Let G be a canonical calculus. Then a canonical calculus G^n in normal form can be constructed, satisfying the following conditions: (i) $\vdash_G = \vdash_{G^n}$, (ii) if G is coherent, then G^n is coherent, (iii) if a sequent has a cut-free proof in G^n, then it has a cut-free proof in G, and (iv) the sets of constants in the different rules of G^n are disjoint.*

PROOF. We show the transformation of G to G^n, proceeding in the following stages.

1. Transform G to G_1, such that (i) $\vdash_G = \vdash_{G_1}$, (ii) if G is coherent, then G_1 is coherent, (iii) if a sequent has a cut-free proof in G_1, then it has a cut-free proof in G, and (iv) every propositional rule of G_1 of arity n has the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq n}/C$, where C is either $\Rightarrow \Diamond(q_1, ..., q_n)$ or $\Diamond(q_1, ..., q_n) \Rightarrow$ for some n-ary connective $\Diamond$ of L, $\Pi_i \cup \Sigma_i = \{p_i\}$ for every $1 \leq i \leq n$. The transformation is like in [4][4]

2. Obtain G_2 from G_1 by discarding all quantificational rules that include one of the following combinations in their premises: (i) $p(x)^{-s} \Rightarrow p(x)^s$ and $p(c)^s \Rightarrow p(c)^{-s}$ for some $x \in \mathit{Var}$, $c \in \mathit{Con}$ and $s \in \{0, 1\}$, or (ii) $\Rightarrow p(x)$ and $p(y) \Rightarrow$ for some $x, y \in \mathit{Var}$.
Let R be a rule including both $p(x)^{-s} \Rightarrow p(x)^s$ and $p(c)^s \Rightarrow p(c)^{-s}$ in its premises. Then an application of R is of the form (the premises are listed vertically):

$$\frac{\begin{array}{c}\Gamma, \Pi_1^* \Rightarrow \Delta, \Sigma_1^* \\ ... \\ \Gamma, A\{z/x\}^s \Rightarrow \Delta', A\{z/x\}^{-s} \\ \Gamma, A\{\mathbf{t}/x\}^{-s} \Rightarrow \Delta, A\{\mathbf{t}/x\}^s \\ ... \\ \Gamma, \Pi_m^* \Rightarrow \Delta, \Sigma_m^*\end{array}}{\Gamma, \mathsf{Q}x\ A^r \Rightarrow \Delta, \mathsf{Q}x\ A^{-r}}$$

where $\mathbf{t}$ is free for x in A and z is free for x in A and does not occur free in $\Gamma \cup \Delta \cup \{\mathsf{Q}x\ A\}$. By lemma 3.5[5], if $\Gamma, A\{z/x\}^s \Rightarrow \Delta, A\{z/x\}^{-s}$

[4] see proof of theorem 4.7 of [4].

[5] $\mathbf{t}$ is free for z in $A\{z/x\}$. Since z does not occur free in $\Gamma \cup \Delta$, $\mathbf{t}$ is also free for z in any formula of $\Gamma \cup \Delta$.

is derivable in G_2, then so is $\Gamma, A\{\mathtt{t}/z\}\{z/x\}^s \Rightarrow \Delta', A\{\mathtt{t}/z\}\{z/x\}^{-s}$, and since z is free for x in A and does not occur free in $\mathsf{Q}x\ A$, $A\{\mathtt{t}/z\}\{z/x\} = A\{\mathtt{t}/x\}$. Thus, an application of R can be simulated using cuts and weakening. The case of a rule including both $p(x)^{-s} \Rightarrow p(x)^s$ and $p(c)^s \Rightarrow p(c)^{-s}$ in its premises is similar. Hence, G_2 is equivalent to G_1. Obviously, any cut-free proof in G_2 is also a cut-free proof in G_1, and if G_1 is coherent, so is G_2.

3. Let $s \in \{0, 1\}$ and $n > 1$. Obtain G_3 from G_2 as follows:

   (a) For every canonical quantificational rule of G_2, which has the clauses $p(y_1)^{-s} \Rightarrow p(y_1)^s, ..., p(y_n)^{-s} \Rightarrow p(y_n)^s$ in its premises, discard the clauses $p(y_2)^{-s} \Rightarrow p(y_2)^s, ..., p(y_n)^{-s} \Rightarrow p(y_n)^s$.

   (b) For every canonical quantificational rule of G_2, which has the clauses $p(c_1)^{-s} \Rightarrow p(c_1)^s, ...p(c_n)^{-s} \Rightarrow p(c_n)^s$ in its premises, discard the clauses $p(c_2)^{-s} \Rightarrow p(c_2)^s, ..., p(c_n)^{-s} \Rightarrow p(c_n)^s$.

   (c) Replace every canonical quantificational rule S/C of G_2 which has both $p(y)^{-s} \Rightarrow p(y)^s$ and $p(c)^{-s} \Rightarrow p(c)^s$ as its premises in S, by $\{p(y)^{-s} \Rightarrow p(y)^s\}/C$.

   It is easy to see that if G_2 is coherent, then so is G_3. To show that G_3 is equivalent to G_2 and every cut-free proof in G_3 can be transformed into a cut-free proof in G_2, it suffices to show that every application of a rule R' obtained from a rule R of G_2 by one of the transformations (a) – (c), can be simulated by an application of R and vice versa. We will show the proof for (a), leaving the easy proofs for (b) and (c) to the reader.

   Let R be a rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}/\mathsf{Q}x\ p(x)^{-r} \Rightarrow \mathsf{Q}x\ p(x)^r$ and assume that both $p(y)^{-s} \Rightarrow p(y)^s$ and $p(c)^{-s} \Rightarrow p(c)^s$ are among the premises of R. Let R' be obtained from R by discarding $p(c)^{-s} \Rightarrow p(c)^s$. Obviously, an application of R can be simulated by an application of R'. An application of R' has the following form:

$$\frac{\Gamma, \Pi_1^* \Rightarrow \Delta, \Sigma_1^*, ..., \Gamma, A\{z/x\}^{-s} \Rightarrow \Delta, A\{z/x\}^s\ , ..., \Gamma, \Pi_m^* \Rightarrow \Delta, \Sigma_m^*}{\Gamma, \mathsf{Q}x\ A^{-r} \Rightarrow \Delta, \mathsf{Q}x\ A^r}$$

   where z is free for x in A and does not occur free in $\Gamma \cup \Delta \cup \{\mathsf{Q}x\ A\}$. Then the following is an application of R (the premises are listed vertically):

$$\frac{\begin{array}{c}\Gamma, \Pi_1^* \Rightarrow \Delta, \Sigma_1^* \\ \dots \\ \Gamma, A\{z/x\}^{-s} \Rightarrow \Delta, A\{z/x\}^s \\ \Gamma, A\{z/x\}^{-s} \Rightarrow \Delta, A\{z/x\}^s \\ \dots \\ \Gamma, \Pi_m^* \Rightarrow \Delta, \Sigma_m^*\end{array}}{\Gamma, \mathsf{Q}x\ A^{-r} \Rightarrow \Delta, \mathsf{Q}x\ A^r}$$

Note that in one premise of the application of R, z is an eigenvariable (taking the place of y), while in the other it is a term (taking the place of c).

4. Obtain G_4 from G_3 by replacing every canonical rule of G_3 of the form $\{p(c)^{-s} \Rightarrow p(c)^s\}/C$ by a pair of rules $\{p(y)^{-s} \Rightarrow p(y)^s\}/C$ and $\{p(c') \Rightarrow, \Rightarrow p(c'')\}/C$ where $y \in Var$ and $c', c'' \in Con$ are fresh, and C is either $\Rightarrow \mathsf{Q}x\ p(x)$ or $\mathsf{Q}x\ p(x) \Rightarrow$. It is not difficult to see that if G_3 is coherent, then so is G_4. To show that G_4 is equivalent to G_3 and a cut-free proof in G_4 can be transformed into a cut-free proof in G_3, it suffices to show that an application of $\{p(c)^{-s} \Rightarrow p(c)^s\}/C$ can be simulated by applications of $\{p(y)^{-s} \Rightarrow p(y)^s\}/C$ and $\{p(c') \Rightarrow, \Rightarrow p(c'')\}/C$, and vice versa:

   (a) An application of $\{\Rightarrow p(c)\}/C$ is of the form:

$$\frac{\Gamma \Rightarrow \Delta, A\{\mathbf{t}/x\}}{\Gamma, \mathsf{Q}z\ A^{-r} \Rightarrow \Delta, \mathsf{Q}z\ A^r}$$

   where $\mathbf{t}$ is free for x in A. Let x be a new variable (which does not occur free in $\Gamma \cup \Delta \cup \{\mathsf{Q}zA\}$). We simulate it in G_4 by applying first $\{p(c) \Rightarrow, \Rightarrow p(c')\}/C$ and then $\{\Rightarrow p(y)\}/C$ as follows:

$$\frac{\dfrac{\Gamma, A\{x/z\} \Rightarrow \Delta, A\{\mathbf{t}/x\} \quad \Gamma, A\{x/z\} \Rightarrow \Delta, A\{x/z\}}{\Gamma, \mathsf{Q}z\ A^{-r} \Rightarrow \Delta, \mathsf{Q}z\ A^r, A\{x/z\}}}{\Gamma, \mathsf{Q}z\ A^{-r} \Rightarrow \Delta, \mathsf{Q}z\ A^r}$$

   Similarly for an application of $\{p(c) \Rightarrow\}/C$.

   (b) An application of $\{p(y)^{-s} \Rightarrow p(y)^s\}/C$ is of the form:

$$\frac{\Gamma, A\{x/z\}^{-s} \Rightarrow \Delta, A\{x/z\}^s}{\Gamma, \mathsf{Q}z\ A^{-r} \Rightarrow \Delta, \mathsf{Q}z\ A^r}$$

   where x is free for z in A and does not occur free in $\Gamma \cup \Delta \cup \{\mathsf{Q}z\ A\}$, and it is also an application (in G_3) of $\{p(c)^{-s} \Rightarrow p(c)^s\}/C$.

(c) An application of $\{p(c') \Rightarrow, \Rightarrow p(c'')\}/C$ is of the form:

$$\frac{\Gamma, A\{\mathbf{t}/z\} \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A\{\mathbf{t}'/z\}}{\Gamma, \mathsf{Q}z\ A^{-r} \Rightarrow \Delta, \mathsf{Q}z\ A^{r}}$$

where $\mathbf{t}, \mathbf{t}'$ are free for z in A. We simulate it in G_3 by applying $\{p(c)^{-s} \Rightarrow p(c)^{s}\}/C$ (to the first premise if $s = 1$ and to the second if $s = 0$):

$$\frac{\Gamma, A\{\mathbf{t}/z\}^{-s} \Rightarrow \Delta, A\{\mathbf{t}/z\}^{s}}{\Gamma, \mathsf{Q}z\ A^{-r} \Rightarrow \Delta, \mathsf{Q}z\ A^{r}}$$

5. G^n is obtained from G_4 by discarding all rules which are equal to some other rule of G_4 and renaming constants so that the sets of constants in different rules are disjoint. The resulting calculus is equivalent to G_4 and if G_4 is coherent, so is G^n. It is easy to verify that G^n is indeed in normal form.

■

EXAMPLE 3.12. *Consider the calculus G consisting of the standard first-order rules for $\forall, \exists$:*

$$\{p(c_1) \Rightarrow\}/\forall x\, p(x) \Rightarrow,\ \{\Rightarrow p(y_1)\}/ \Rightarrow \forall x\, p(x),$$

$$\{\Rightarrow p(c_2)\}/ \Rightarrow \exists x\, p(x),\ \{p(y_2) \Rightarrow\}/\exists x\, p(x) \Rightarrow$$

An equivalent calculus G^n in normal form is as follows:

$$\{\{p(c_1) \Rightarrow\ ,\ \Rightarrow p(c_2)\}/\forall x\, p(x) \Rightarrow,\ \{p(y_3) \Rightarrow\}/\forall x\, p(x) \Rightarrow,\ \{\Rightarrow p(y_1)\}/ \Rightarrow \forall x\, p(x),$$

$$\{p(c_3) \Rightarrow\ ,\ \Rightarrow p(c_4)\}/ \Rightarrow \exists x\, p(x),\ \{\Rightarrow p(y_4)\}/ \Rightarrow \exists x\, p(x),\ \{p(y_2) \Rightarrow\}/\exists x\, p(x) \Rightarrow\}$$

Canonical systems, Cut-Elimination and 2Nmatrices

LEMMA 3.13. *Let G be a canonical calculus over a language L with unary quantifiers.*

1. *Suppose that if a sequent satisfying the pure-variable condition is provable in G, then it has a cut-free proof in G. Then no clause $\Gamma \Rightarrow \Delta$ such that Γ, Δ are disjoint is provable in G.*

2. *If no clause $\Gamma \Rightarrow \Delta$ such that Γ, Δ are disjoint is provable in G, then G is coherent.*

PROOF. 1. Assume that if a sequent satisfying the pure-variable condition is provable in G, then it has a cut-free proof. Let $\Gamma \Rightarrow \Delta$ be a clause, such that Γ, Δ are disjoint. Then $\Gamma \Rightarrow \Delta$ satisfies the pure-variable condition. Hence if it is provable in G, then it has a cut-free proof there. However, this is easily seen to be impossible.

2. Assume that for any two disjoint sets of atomic formulas Γ, Δ, $\Gamma \Rightarrow \Delta$ is not provable in G. Suppose by contradiction that G is not coherent. Then there exist two rules $S_1/ \Rightarrow A$ and $S_2/A \Rightarrow$, such that (i) A is either $\diamondsuit(q_1, ..., q_n)$ for some n-ary connective $\diamondsuit$ of L or $\mathsf{Q}x\ p(x)$ for some unary quantifier Q of L, and (ii) $S_1 \cup S_2$ is classically consistent. If A is $\diamondsuit(q_1, ..., q_n)$, the proof is easily adapted from [3]. Otherwise A is $\mathsf{Q}x\ p(x)$.
Suppose that no variables occur free in $S_1 \cup S_2$. $S_1 \cup S_2$ is classically consistent, so there exists some L-structure which satisfies $S_1 \cup S_2$. It is easy to see that there also exists a (classical) propositional valuation v_S, satisfying $S_1 \cup S_2$. Let $\Pi' = \{A \mid v_S[A] = 1, A \in \Gamma \cup \Delta, \Gamma \Rightarrow \Delta \in S_1 \cup S_2\}$ and $\Sigma' = \{A \mid v_S[A] = 0, A \in \Gamma \cup \Delta, \Gamma \Rightarrow \Delta \in S_1 \cup S_2\}$.
Let $B_j = \{\Pi, \Pi' \Rightarrow \Sigma, \Sigma' \mid \Pi \Rightarrow \Sigma \in S_j\}$ for $j = 1, 2$. Then B_1 and B_2 are sets of standard axioms. (Because if v_S satisfies $\Pi \Rightarrow \Sigma$, there is some $A \in \Pi$, such that $v_S[A] = 0$, or some $A \in \Delta$, such that $v_S[A] = 1$. In the former case, $A \in \Sigma'$ and in the latter case, $A \in \Pi'$.)
Obviously, the following is an application of $S_1/\mathsf{Q}x\ p(x) \Rightarrow$:

$$\frac{B_1}{\Pi', \mathsf{Q}x\ p(x) \Rightarrow \Sigma'}$$

In a similar way, by applying the second rule on B_2 we obtain $\Pi' \Rightarrow \Sigma', \mathsf{Q}x\ p(x)$. Using cut, $\Pi' \Rightarrow \Sigma'$ is provable in G, in contradiction to our assumption.
Otherwise, assume that $(p(y) \Rightarrow) \in S_1 \cup S_2$ for some $y \in Var$. Since $S_1 \cup S_2$ is classically consistent, every sequent of $S_1 \cup S_2$ is of the form $p(z) \Rightarrow$ or $p(c) \Rightarrow$. Then the following is an application of $S_1/\mathsf{Q}x\ p(x) \Rightarrow$[6], where $d \in Con$ is a new constant:

$$\frac{p(d) \Rightarrow p(d) \quad ... \quad p(d) \Rightarrow p(d)}{\mathsf{Q}x\ p(d) \Rightarrow p(d)}$$

Note that the quantification is vacuous here. Similarly, using the second rule we can derive $\Rightarrow \mathsf{Q}x\ p(d), p(d)$, and by cut $\Rightarrow p(d)$ is derivable,

[6] Let $x' \in Var$ and $c' \in Con$. Note that $p(d) = p(d)\{x'/x\} = p(d)\{c'/x\}$.

in contradiction to our assumption.

The proof for the case of $\Rightarrow p(y) \in S_1 \cup S_2$ is symmetric.

■

Now we define the 2Nmatrix induced by a coherent canonical calculus in normal form, along the lines of [3]. The intuitive idea is that every canonical propositional rule for $\Diamond$ of such calculus imposes a constraint on a set $\tilde{\Diamond}_{\mathcal{M}}[a_1, ..., a_n] \subseteq \{0,1\}$ for exactly one n-ary vector $a_1...a_n \in \{0,1\}$. For example, the rule $\{\Rightarrow q_1, \Rightarrow q_2\} \Rightarrow q_1 \wedge q_2$ dictates $\tilde{\wedge}_{\mathcal{M}}[1,1] = \{1\}$. This approach can be extended to the quantificational canonical rules as follows. A canonical quantificational rule of a calculus in normal form imposes a constraint on $\tilde{\mathsf{Q}}_{\mathcal{M}}[H]$ for exactly one (non-empty) set $H \subseteq \{0,1\}$. For example, $\{p(x) \Rightarrow\}/ \Rightarrow \mathsf{Q}y\, p(y)$ dictates $\tilde{\mathsf{Q}}_{\mathcal{M}}[\{0\}] = \{1\}$, $\{\Rightarrow p(x)\}/\mathsf{Q}y\, p(y) \Rightarrow$ dictates $\tilde{\mathsf{Q}}_{\mathcal{M}}[\{1\}] = \{0\}$ and $\{(\Rightarrow p(c))\ ,\ (p(c') \Rightarrow)\}/ \Rightarrow \mathsf{Q}y\ p(y)$ dictates $\tilde{\mathsf{Q}}_{\mathcal{M}}[\{0,1\}] = \{1\}$.

Definition 3.14. **(2Nmatrix induced by a coherent canonical calculus in normal form)** Let G^n be a coherent canonical calculus in normal form over a language L with unary quantifiers, such that the sets of constants occurring in the different rules of G^n are disjoint. Then $\mathcal{M}_{G^n}$, *the 2Nmatrix for L induced by* G^n, is defined as follows:

1. For every n-ary connective $\Diamond$ of L and every $s_1, ..., s_n, r \in \{0,1\}$:

$$\tilde{\Diamond}_{\mathcal{M}_{G^n}}(s_1, ..., s_n) = \begin{cases} \{r\} & \text{if } [\Diamond(s_1, ..., s_n) : r] \in G^n \\ \{0,1\} & \text{otherwise} \end{cases}$$

2. For every unary quantifier Q of L and every $s, r \in \{0,1\}$:

$$\tilde{\mathsf{Q}}_{\mathcal{M}_{G^n}}[\{s\}] = \begin{cases} \{r\} & \text{if } [\mathsf{Q}(s) : r] \in G^n \\ \{0,1\} & \text{otherwise} \end{cases}$$

$$\tilde{\mathsf{Q}}_{\mathcal{M}_{G^n}}[\{0,1\}] = \begin{cases} \{r\} & \text{if } [\mathsf{Q}(\mathcal{V}) : r] \in G^n \\ \{0,1\} & \text{otherwise} \end{cases}$$

Note that $\mathcal{M}_{G^n}$ is well defined since G^n is coherent and the sets of constants occurring in the different rules of G^n are disjoint (see remark at the end of the section).

Example 3.15. *Consider the first-order paraconsistent system* PLK$[(\Rightarrow \forall), (\Rightarrow \exists)]$ *from [5]. This system consists of the α-axiom and the positive fragment of classical propositional logic with the addition of the standard rules* $(\Rightarrow \neg)$, $(\Rightarrow \forall)$ *and* $(\exists \Rightarrow)$.
In our notation the rules $(\Rightarrow \neg)$, $(\Rightarrow \forall)$ *and* $(\Rightarrow \exists)$ *are formulated as follows:*

$$\{q_1 \Rightarrow\}/ \Rightarrow \neg q_1$$

$$\{\Rightarrow p(y)\}/ \Rightarrow \forall x\ p(x) \qquad \{\Rightarrow p(c)\}/ \Rightarrow \exists x\ p(x)$$

Denote PLK$[(\Rightarrow \forall), (\Rightarrow \exists)]$ *by G. After we transform G into an equivalent system G^n in normal form, the rule* $\{\Rightarrow p(c)\}/ \Rightarrow \exists x\ p(x)$ *is replaced by rules:*

$$\{\Rightarrow p(z)\}/ \Rightarrow \exists x\ p(x) \quad \{\Rightarrow p(c')\ ,\ p(c'') \Rightarrow\}/ \Rightarrow \exists x\ p(x)$$

The 2Nmatrix $\mathcal{M}_{G^n}$ induced by G^n is defined as follows:

- $\mathcal{V} = \{0,1\}$, $\mathcal{G} = \{1\}$
- $\tilde{\neg}_{\mathcal{M}_{G^n}}[0] = \{1\}, \tilde{\neg}_{\mathcal{M}_{G^n}}[1] = \{1,0\}$
- $\tilde{\wedge}_{\mathcal{M}_{G^n}}[1,1] = \{1\}, \tilde{\wedge}_{\mathcal{M}_{G^n}}[1,0] = \tilde{\wedge}_{\mathcal{M}_{G^n}}[0,1] = \tilde{\wedge}_{\mathcal{M}_{G^n}}[0,0] = \{0\}$
- $\tilde{\vee}_{\mathcal{M}_{G^n}}[0,0] = \{0\}, \tilde{\vee}_{\mathcal{M}_{G^n}}[1,0] = \tilde{\vee}_{\mathcal{M}_{G^n}}[0,1] = \tilde{\vee}_{\mathcal{M}_{G^n}}[0,0] = \{1\}$
- $\tilde{\supset}_{\mathcal{M}_{G^n}}[1,0] = \{0\}, \tilde{\supset}_{\mathcal{M}_{G^n}}[1,1] = \tilde{\supset}_{\mathcal{M}_{G^n}}[0,1] = \tilde{\supset}_{\mathcal{M}_{G^n}}[0,0] = \{1\}$
- $\tilde{\forall}_{\mathcal{M}_{G^n}}[\{1\}] = \{1\}$, $\tilde{\forall}_{\mathcal{M}_{G^n}}[\{1,0\}] = \tilde{\forall}_{\mathcal{M}_{G^n}}[\{0\}] = \{1,0\}$
- $\tilde{\exists}_{\mathcal{M}_{G^n}}[\{1\}] = \tilde{\exists}_{\mathcal{M}_{G^n}}[\{1,0\}] = \{1\}$, $\tilde{\exists}_{\mathcal{M}_{G^n}}[\{0\}] = \{1,0\}$

We will now show that G^n is sound and cut-free complete for $\mathcal{M}_{G^n}$.

Lemma 3.16. **(Soundness and cut-free completeness)** *Let G^n be a coherent canonical calculus in normal form over a language L with unary quantifiers, such that the sets of constants occurring in the different rules of G^n are disjoint. Then (i) any sequent provable in G^n is $\mathcal{M}_{G^n}$-valid, and (ii) if a sequent satisfying the pure-variable condition is $\mathcal{M}_{G^n}$-valid, then it has a cut-free proof in G^n.*

Proof.
Soundness: the proof is not hard and is left to the reader.
Completeness and Cut-Elimination:
Let $\Gamma \Rightarrow \Delta$ be a sequent satisfying the pure-variable condition. Suppose that $\Gamma \Rightarrow \Delta$ has no cut-free proof in G^n. We will show that it is not $\mathcal{M}_{G^n}$-valid. Obviously, we can limit ourselves to the language L^*, which is a subset of L,

consisting of all the constants and predicate and function symbols, occurring in $\Gamma \Rightarrow \Delta$. Let $\mathbf{T}$ be the set of all the terms in L^* which do not contain variables occurring bound in $\Gamma \Rightarrow \Delta$. It is a standard matter to show that Γ, Δ can be extended to two (possibly infinite) sets Γ', Δ' (where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$), satisfying the following properties:

1. For every $\Gamma_1 \subseteq \Gamma'$ and $\Delta_1 \subseteq \Delta'$, $\Gamma_1 \Rightarrow \Delta_1$ does not have a cut-free proof.
2. There are no $A \in \Gamma'$ and $B \in \Delta'$, such that $A \equiv_\alpha B$.
3. For every rule $[\Diamond(s_1, ..., s_n) : r]$ of G^n, if $\Diamond(A_1, ..., A_n) \in ite(r, \Delta', \Gamma')$, then for some $1 \leq i \leq n$: $A_i \in ite(s_i, \Delta', \Gamma')$.
4. For every rule $[\mathsf{Q}(s) : r]$ of G^n: if $\mathsf{Q}z\ A \in ite(r, \Delta', \Gamma')$, then there exists some $\mathbf{t} \in \mathbf{T}$, such that $A\{\mathbf{t}/z\} \in ite(s, \Delta', \Gamma')$.
5. For every rule $[\mathsf{Q}(\mathcal{V}) : r]$ of G^n: if $\mathsf{Q}z\ A \in ite(r, \Delta', \Gamma')$, then either $A\{\mathbf{t}/z\} \in \Gamma'$ for every $\mathbf{t} \in \mathbf{T}$, or $A\{\mathbf{t}/z\} \in \Delta'$ for every $\mathbf{t} \in \mathbf{T}$.

Let $S^* = \langle \mathbf{T}, I^* \rangle$ be the L^*-structure defined as follows: $I^*[p][\mathbf{t}_1, ..., \mathbf{t}_n] = 1$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$, $I^*[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$, and $I^*[c] = c$.
Let σ^* be any S^*-substitution satisfying $\sigma^*[x] = \overline{x}$ for every $x \in \mathbf{T}$. (Note that every $x \in \mathbf{T}$ is also a member of the domain and thus has an individual name referring to it in $L^*(D)$.)

LEMMA 3.17. *1.* $I^*[\sigma^*[\boldsymbol{t}]] = \boldsymbol{t}$ *for every* $\boldsymbol{t} \in \mathbf{T}$.

2. Let $A, A' \in \Gamma' \cup \Delta'$. *If* $\sigma^*[A] \sim^{S^*} \sigma^*[A']$, *then* $A \equiv_\alpha A'$.

Let v be the S^*-valuation defined as follows.

- $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I^*[p][I^*[\mathbf{t}_1], ..., I^*[\mathbf{t}_n]]$.
- $v[\Diamond(A_1, ..., A_n)] = 1$ iff one of the following conditions holds:
  1. $\tilde{\Diamond}_{\mathcal{M}_{G^n}}[v[A_1], ..., v[A_n]] = \{1\}$.
  2. $\tilde{\Diamond}_{\mathcal{M}_{G^n}}(v[A_1], ..., v[A_n]) = \{0, 1\}$ and there exists some formula $C \in \Gamma'$, such that $\Diamond(A_1, ..., A_n) \sim^{S^*} \sigma^*[C]$.
- $v[\mathsf{Q}x\ A] = 1$ iff one of the following conditions holds:
  1. $\tilde{\mathsf{Q}}_{\mathcal{M}_{G^n}}[\{v[A\{\overline{a}/x\}] \mid a \in D\}] = \{1\}$.
  2. $\tilde{\mathsf{Q}}_{\mathcal{M}_{G^n}}[\{v[A\{\overline{a}/x\}] \mid a \in D\}] = \{0, 1\}$ and there exists some formula $C \in \Gamma'$, such that $\mathsf{Q}x\ A \sim^{S^*} \sigma^*[C]$.

LEMMA 3.18. *For every $B \in \Gamma' \cup \Delta'$: if $B \in \Gamma'$, then $v[\sigma^*[B]] = 1$, and if $B \in \Delta'$, then $v[\sigma^*[B]] = 0$.*

The proof is not hard and is omitted here.

LEMMA 3.19. *Let A, A' be $L^*(D)$-sentences. If $A \sim^{S^*} A'$, then $v[A] = v[A']$.*

It is now easy to see that v is legal in $\mathcal{M}_{G^n}$. Thus we have constructed an $\mathcal{M}_{G^n}$-legal S^*-valuation v refuting $\Gamma' \Rightarrow \Delta'$. Since $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, it also refutes $\Gamma \Rightarrow \Delta$. Hence, $\Gamma \Rightarrow \Delta$ is not $\mathcal{M}_{G^n}$-valid. ■

Now we summarize the main result of the paper in the following corollary.

COROLLARY 3.20. *Let G be a canonical calculus over a language L with unary quantifiers, such that the sets of constants occurring in the different rules of G are disjoint. Then the following conditions concerning G are equivalent:*

1. *If a sequent satisfying the pure-variable condition is provable in G, then it has a cut-free proof in G.*
2. *No clause $\Gamma \Rightarrow \Delta$ such that Γ, Δ are disjoint is provable in G.*
3. *G is coherent.*
4. *G has a characteristic 2Nmatrix.*

PROOF. $1 \Rightarrow 2$ and $2 \Rightarrow 3$ follow directly from lemma 3.13. The proof of $4 \Rightarrow 2$ is trivial. To show $3 \Rightarrow 1$ and $3 \Rightarrow 4$, suppose that G is coherent. By theorem 3.11, we can construct a canonical calculus G^n in normal form, satisfying the following conditions: (i) G^n is coherent, (ii) $\vdash_G = \vdash_{G^n}$, (iii) if a sequent has a cut-free proof in G^n, then it has a cut-free proof in G, and (iv) the sets of constants in the different rules of G^n are disjoint. By lemma 3.16, G^n is sound and cut-free complete for $\mathcal{M}_{G^n}$, and by properties (ii) and (iii), so is G. ■

Remark: Note that the requirement that the sets of constants occurring in different rules of G be disjoint is crucial for the above corollary. Consider, for instance, a canonical calculus G consisting of the rules $\{p(c) \Rightarrow\ , \Rightarrow p(c')\}/ \Rightarrow \mathsf{Q}x\ p(x)$ and $\{p(c'') \Rightarrow, \Rightarrow p(c)\}/\mathsf{Q}x\ p(x) \Rightarrow$. Let G' be the non-coherent calculus obtained from G by renaming of constants: $\{p(c) \Rightarrow\ , \Rightarrow p(c')\}/ \Rightarrow \mathsf{Q}x\ p(x)$, $\{p(c'') \Rightarrow\ , \Rightarrow p(c''')\}/\mathsf{Q}x\ p(x) \Rightarrow$. Although G is coherent, neither G nor G' have a characteristic 2Nmatrix.

4. Conclusions and further research

In this paper we have extended the definition of canonical calculi, which are the most natural type of multiple conclusion Gentzen-type systems, to first-order languages and beyond. We have proposed a precise characterization of quantificational canonical rules, the well-known instances of which are the standard rules for $\forall$ and $\exists$. Moreover, we have shown that, like in the propositional case, on the level of languages with quantifiers there exists a deep connection between the possibility to eliminate cuts in a given canonical Gentzen-type system and the existence of a two-valued characteristic Nmatrix for it. We have also generalized the coherence criterion of [3, 4] for the non-triviality of canonical systems and showed that it remains constructive for languages with quantifiers. In addition to providing a better insight into the phenomenon of cut-elimination, our work also provides further evidence for the thesis that the meaning of a logical constant is given by its introduction (and "elimination") rules . We have shown that at least in the framework of multiple-conclusion consequence relations, any "reasonable" set of canonical quantificational rules (which we defined in precise terms) completely determines the semantics of the quantifier.

Some immediate directions for further research are: (a) To investigate the connection between the cut-elimination phenomenon and Nmatrices in Gentzen-type proof systems which are less restrictive than the canonical ones.

(b) To generalize the interpretation of quantifiers in Nmatrices. In particular, we intend to investigate n-ary quantifiers and such concrete important cases as Henkin quantifiers, Transitive Closure operations and other extensions.

(c) To generalize the proposed framework to arbitrary finite n-valued Nmatrices along the lines of [2] and to explore the connection of our current results on canonical systems to the work of [6, 7]. For instance, in [7] it is shown that any many-sided propositional calculus which satisfies: (i) a condition similar to coherence, i.e. certain clause sets corresponding to premises of introduction rules are refutable by resolution, and (ii) axioms can be reduced to atomic axioms, has a deterministic characteristic matrix. Our corollary 3.20 sheds some light on what happens in systems which do not satisfy condition (ii): the corresponding semantics must be generalized to non-deterministic matrices. Such is, for instance, the system PLK$[(\Rightarrow \forall), (\Rightarrow \exists)]$ from example 3.15. The exact connection between these results is still to be thoroughly investigated.

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