Non-deterministic Semantics for Logics with a Consistency Operator

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Abstract

In order to handle inconsistent knowledge bases in a reasonable way, one needs a logic which allows nontrivial inconsistent theories. Logics of this sort are called *paraconsistent*. One of the oldest and best known approaches to the problem of designing useful paraconsistent logics is da Costa's approach, which seeks to allow the use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. Da Costa's approach has led to the family of logics of formal (in)consistency (LFIs). In this paper we provide in a modular way simple non-deterministic semantics for 64 of the most important logics from this family. Our semantics is 3-valued for some of the systems, and infinite-valued for the others. We prove that these results cannot be improved: neither of the systems with a three-valued non-deterministic semantics has either a finite characteristic ordinary matrix or a two-valued characteristic non-deterministic matrix, and neither of the other systems we investigate has a finite characteristic non-deterministic matrix. Still, our semantics provides decision procedures for all the systems investigated, as well as easy proofs of important proof-theoretical properties of them.

1 Introduction

One of the most difficult problems concerning reasoning with uncertainty is that of contradictory data. The following quotation from [17] describes the problem as follows:

> "It is a fact of life that large knowledge bases are inherently inconsistent, in the same way large programs are inherently buggy. Moreover, within a con-

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ventional logic, the inconsistency of a knowledge base has the catastrophic consequence that *everything* is derivable from the knowledge base."

It follows that in order to handle inconsistent knowledge bases in a reasonable way, one needs an *unconventional* logic: a logic which allows nontrivial inconsistent theories. Logics of this sort are called *paraconsistent*.

There are several approaches to the problem of designing a useful paraconsistent logic (see e.g. [9,7,14]). One of the oldest and best known is da Costa's approach ([15]), which seeks to allow the use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. Da Costa's approach has led to the family of LFIs (Logics of Formal (In)consistency — see [13]). This family is based on two main ideas. The first is that propositions should be divided into two sorts: the "normal" (or consistent) propositions, and the "abnormal" (or inconsistent) ones. Classical logic can (and should) be applied freely to normal propositions, but not to abnormal ones. The second idea is to formally introduce this classification into the language. When this is done by employing a special (primitive or defined) unary connective $\circ$ (where the intuitive meaning of $\circ\varphi$ is : "φ is consistent") we get a special type of LFIs: the C-systems ([12]).

The class of C-systems is the most important and useful subclass of the class of logics of formal (in)consistency. So far, the main shortcoming of this class has been the lack of a corresponding intuitive semantics, which would be easy to use and would provide real insight into these logics. [1] In this paper we remedy this by providing simple, modular *non-deterministic* semantics for the 64 most basic C-systems. Our semantics is based on the use of non-deterministic matrices (Nmatrices). These are multi-valued structures (introduced in [1–3]) where the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. Although applicable to a much larger family of logics, the semantics of finite Nmatrices has all the advantages that the semantics of ordinary finite-valued semantics provides. In particular:

(1) The semantics of finite Nmatrices is *effective* (see Proposition 2 below). Hence a logic with a finite characteristic Nmatrix is necessarily decidable.
(2) A logic with a finite characteristic Nmatrix is finitary (i.e.: the compactness theorem obtains for it – see [3]).
(3) There is a well-known uniform method ([18,8]) for constructive cut-free calculus of n-sequents for any logic which has an n-valued characteristic matrix. This method can easily be extended to logics which have a finite characteristic Nmatrix (see [4]).

[1] The bivaluations semantics and the possible translations semantics described in [12,13] are not satisfactory from these points of view. See e.g. footnote 5 below.

Nmatrices seem to be particularly useful for reasoning on uncertainty because uncertainty concerning the truth-value assigned to a formula is their most crucial feature. This potential in this area is demonstrated in this paper by applying them for the special case of paraconsistent reasoning.

The semantics we provide in this paper is 3-valued for some of the systems, and infinite-valued for the others. We also prove that our results cannot be improved: neither of the simpler systems has either a finite characteristic ordinary matrix or a two-valued characteristic Nmatrix, and neither of the other systems has a finite characteristic Nmatrix. Still, our semantics leads to easy decision procedures for all the systems we investigate. Moreover: as a demonstration of the power of our semantics, we use it to show a very important proof-theoretical property: in most of the systems we consider two formulas are logically indistinguishable iff they are identical, and all of the systems have this property with respect to formulas without $\circ$.

2 Preliminaries

2.1 A Taxonomy of C-Systems

Let $\mathcal{L}_{\mathrm{cl}}^{+} = \{\wedge, \vee, \supset\}$, $\mathcal{L}_{\mathrm{cl}} = \{\wedge, \vee, \supset, \neg\}$, and $\mathcal{L}_{\mathrm{C}} = \{\wedge, \vee, \supset, \neg, \circ\}$.

Definition 1 Let $\mathbf{HCL}^{+}$ be some standard Hilbert-type system which has MP as the only inference rule, and is sound and strongly complete for the $\mathcal{L}_{\mathrm{cl}}^{+}$-fragment of CPL (classical propositional logic). The logic $\mathbf{B}$ [2] is the logic in $\mathcal{L}_{\mathrm{C}}$ which is obtained from $\mathbf{HCL}^{+}$ by adding the schemata:

(t) $\neg\varphi \vee \varphi$
(p) $\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi)$

Definition 2 Let Ax be the set consisting of the following axioms:

(c) $\neg\neg\varphi \supset \varphi$
(e) $\varphi \supset \neg\neg\varphi$
(i) $\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)$
(a) $(\circ\varphi \wedge \circ\psi) \supset (\circ(\varphi \wedge \psi) \wedge \circ(\varphi \vee \psi) \wedge \circ(\varphi \supset \psi))$
(o) $(\circ\varphi \vee \circ\psi) \supset (\circ(\varphi \wedge \psi) \wedge \circ(\varphi \vee \psi) \wedge \circ(\varphi \supset \psi))$
(l) $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$
(d) $\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$
(b) $(\neg(\varphi \wedge \neg\varphi) \vee \neg(\neg\varphi \wedge \varphi)) \supset \circ\varphi$

For $X \subseteq Ax$, $\mathbf{B}[X]$ is the system obtained by adding the axioms in X to $\mathbf{B}$.

[2] The logic $\mathbf{B}$ is called mbC in [13].

Notation: We'll usually denote $\mathbf{B}[X]$ by $\mathbf{B}s$, where s is a string consisting of the names of the axioms in X (thus we'll write **Biel** rather than $\mathbf{B}[\{(\mathbf{i}),(\mathbf{e}),(\mathbf{l})\}]$. [3]

Note. It is easy to see that the converse of **(i)** (i.e. $\varphi \wedge \neg\varphi \supset \neg\circ\varphi$) and the converse of **(l)** (i.e. $\circ\varphi \supset \neg(\varphi \wedge \neg\varphi)$) are theorems of **B**. Together the four implications intuitively mean that $\circ\varphi$ and $\neg(\varphi\wedge\neg\varphi)$ "have the same meaning". On the other hand **(d)** and its converse (together with **(i)**) intuitively mean that $\circ\varphi$ and $\neg(\neg\varphi \wedge \varphi)$ "have the same meaning". At this point it should be emphasized that **(d)** and **(l)** are *not* equivalent in the present context (in fact, this well-known fact easily follows from the results of this paper).

2.2 Non-deterministic Matrices

Our main semantic tool in what follows will be the following generalization of the concept of a matrix given in [1–3]: [4]

Definition 3

(1) A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$, where:
   (a) $\mathcal{V}$ is a non-empty set of *truth values.*
   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.
   (c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.
   We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{V}$.

(2) Let $\mathcal{W}$ be the set of formulas of $\mathcal{L}$. A *(legal) valuation* in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{V}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{W}$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

(3) A valuation v in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model* in $\mathcal{M}$ of a set Γ of formulas (notation: $v \models^{\mathcal{M}} \Gamma$) if it satisfies every formula in Γ.

(4) $\vdash_{\mathcal{M}}$, the consequence relation induced by the Nmatrix $\mathcal{M}$, is defined by: $T \vdash_{\mathcal{M}} \varphi$ if for every v such that $v \models^{\mathcal{M}} T$, also $v \models^{\mathcal{M}} \varphi$.

(5) A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}}\rangle$ is *sound* for an Nmatrix $\mathcal{M}$ (where $\mathcal{L}$ is the language of $\mathcal{M}$) if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{M}}$. $\mathbf{L}$ is *complete* for $\mathcal{M}$ if $\vdash_{\mathbf{L}} \supseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is *characteristic*

[3] In the literature on LFIs one usually writes $\mathbf{C}s$ instead of our $\mathbf{Bc}s$ When X includes the axiom **(c)**. Thus what we denote by **Bcial** is called **Cila** in [12,13].

[4] A special two-valued case of this definition was essentially introduced in [6]. Another particular case of the same idea, using a similar name, was used in [11]. It should also be noted that Carnielli's "possible-translations semantics" (see [10]) was originally called "non-deterministic semantics".

for **L** if **L** is both sound and complete for it (i.e.: if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{M}}$). $\mathcal{M}$ is *weakly-characteristic* for **L** if for every formula φ of $\mathcal{L}$, $\vdash_{\mathbf{L}} \varphi$ iff $\vdash_{\mathcal{M}} \varphi$.

Note: We shall identify an ordinary (deterministic) matrix with an Nmatrix whose functions in $\mathcal{O}$ always return singletons.

Definition 4 Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$. $\mathcal{M}_2$ is called a *refinement* of $\mathcal{M}_1$ if $\mathcal{V}_2 \subseteq \mathcal{V}_1$, $\mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{V}_2$, and $\tilde{\diamond}_{\mathcal{M}_2}(\vec{x}) \subseteq \tilde{\diamond}_{\mathcal{M}_1}(\vec{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{V}_2^n$.

The following proposition can easily be proved:

Proposition 1 *If $\mathcal{M}_2$ is a refinement of $\mathcal{M}_1$ then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$. Hence if* **L** *is sound for $\mathcal{M}_1$ then* **L** *is also sound for $\mathcal{M}_2$.*

Perhaps the most important property of the semantics of Nmatrices is its being *effective* in the sense that for determining whether $T \vdash_{\mathcal{M}} \varphi$ (where $\mathcal{M}$ is an Nmatrix) it always suffices to check only *partial* valuations, defined only on subformulas of $T \cup \{\varphi\}$. This is due to the following obvious proposition: [5]

Proposition 2 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$, and let $\mathcal{W}$ be the set of formulas of $\mathcal{L}$. Assume that $\mathcal{W}'$ is a subset of $\mathcal{W}$ which is closed under subformulas, and that $v' : \mathcal{W}' \to \mathcal{V}$ is a partial valuation that respects $\mathcal{M}$ (i.e., if $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{W}'$, then $v'(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v'(\psi_1), \ldots, v'(\psi_n))$). Then v' can be extended to a (full legal) valuation in $\mathcal{M}$.*

Corollary 1 *If $\mathcal{M}$ is a finite Nmatrix then $\vdash_{\mathcal{M}}$ is decidable.*

3 The Systems with finite-valued Nmatrices

3.1 The basic system **B**

Definition 5

• A basic **B**-Nmatrix is an Nmatrix for the language $\mathcal{L}_C$ such that:
(1) $\mathcal{V} = \mathcal{T} \uplus \mathcal{I} \uplus \mathcal{F}$, where $\mathcal{T}$, $\mathcal{I}$, and $\mathcal{F}$ are disjoint nonempty sets.
(2) $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$

[5] No general similar theorem is available for the semantics of bivaluations or for possible translations semantics described in [12,13]. Hence a corresponding proposition should be proved from scratch for any useful instance of these types of semantics.

(3) $\mathcal{O}$ is defined by:

$$a\tilde{\vee}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a\tilde{\supset}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a\tilde{\wedge}b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\tilde{\neg}a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\tilde{\circ}a = \begin{cases} \mathcal{V} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

- A **B**-Nmatrix is an Nmatrix for $\mathcal{L}_C$ which is a refinement of some basic **B**-Nmatrix.
- $\mathcal{M}_B$ is the 3-valued basic **B**-Nmatrix in which $\mathcal{T} = \{t\}$, $\mathcal{F} = \{f\}$, and $\mathcal{I} = \{I\}$.

It is easy to see that the non-deterministic truth tables corresponding to the operations in $\mathcal{M}_B$ are:

$\tilde{\vee}$	**f**	**I**	**t**
f	$\{f\}$	$\{I,t\}$	$\{I,t\}$
I	$\{I,t\}$	$\{I,t\}$	$\{I,t\}$
t	$\{I,t\}$	$\{I,t\}$	$\{I,t\}$

$\tilde{\wedge}$	**f**	**I**	**t**
f	$\{f\}$	$\{f\}$	$\{f\}$
I	$\{f\}$	$\{I,t\}$	$\{I,t\}$
t	$\{f\}$	$\{I,t\}$	$\{I,t\}$

$\tilde{\supset}$	**f**	**I**	**t**
f	$\{I,t\}$	$\{I,t\}$	$\{I,t\}$
I	$\{f\}$	$\{I,t\}$	$\{I,t\}$
t	$\{f\}$	$\{I,t\}$	$\{I,t\}$

$\tilde{\neg}$	**f**	**I**	**t**
	$\{I,t\}$	$\{I,t\}$	$\{f\}$

$\tilde{\circ}$	**f**	**I**	**t**
	$\{t,I,f\}$	$\{f\}$	$\{t,I,f\}$

Theorem 1 **B** *is sound for any* **B**-*Nmatrix.*

Proof: It is straightforward to check that **B** is sound for any basic **B**-Nmatrix. Hence the theorem follows from Proposition 1. □

Theorem 2 *$\mathcal{M}_B$ is a characteristic Nmatrix for* **B**.[6]

Proof: Soundness follows from Theorem 1. For completeness, assume that $\mathbf{T}$ is a theory and φ_0 a sentence such that $\mathbf{T} \nvdash_{\mathbf{B}} \varphi_0$. We construct a model of $\mathbf{T}$ in $\mathcal{M}_B$ which is not a model of φ_0. For this extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \nvdash_{\mathbf{B}} \varphi_0$. $\mathbf{T}^*$ has the following properties:

(1) $\psi \notin \mathbf{T}^*$ iff $\psi \supset \varphi_0 \in \mathbf{T}^*$.
(2) If $\psi \notin \mathbf{T}^*$ then $\psi \supset \varphi \in \mathbf{T}^*$ for every sentence φ of $\mathcal{L}_C$.
(3) $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.
(4) $\varphi \wedge \psi \in \mathbf{T}^*$ iff both $\varphi \in \mathbf{T}^*$ and $\psi \in \mathbf{T}^*$.
(5) $\varphi \supset \psi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.
(6) For every sentence φ of $\mathcal{L}_C$, either $\varphi \in \mathbf{T}^*$ or $\neg\varphi \in \mathbf{T}^*$.
(7) If both $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$ then $\circ\varphi \notin \mathbf{T}^*$.

The proofs of Properties 1–5 are exactly as in the case of $\mathbf{HCL}^+$: Property 1 follows from the deduction theorem (which is obviously valid for **B**) and the maximality of $\mathbf{T}^*$. Property 2 is proved first for φ_0 using 1 and the tautology $((\varphi_0 \supset \varphi) \supset \varphi_0) \supset \varphi_0$. It then follows for all $\psi \notin \mathbf{T}^*$ by 1. Properties 3–5 are easy corollaries of 1, 2, and the maximality of $\mathbf{T}^*$. Finally, Property 6 is immediate from Property 3 and Axiom **(t)**, and Property 7 follows from Axiom (**p**).

Define now a valuation v in $\mathcal{M}_B$ as follows:

$$v(\psi) = \begin{cases} f & \psi \notin \mathbf{T}^* \\ t & \neg\psi \notin \mathbf{T}^* \\ I & \psi \in \mathbf{T}^*, \neg\psi \in \mathbf{T}^* \end{cases}$$

Note that by property 6, v is well defined, and $v(\psi) \in \mathcal{D} = \{I, t\}$ iff $\psi \in \mathbf{T}^*$. We use this to prove that v is a legal valuation, i.e.: it respects the interpretations of the connectives in $\mathcal{M}_B$. That this is the case for the positive connectives easily follows from Properties 3–5 of $\mathbf{T}^*$. We prove next the cases of $\neg$ and $\circ$:

- Assume $v(\psi) = f$. Then $\psi \notin \mathbf{T}^*$. Hence $\neg\psi \in \mathbf{T}^*$ by Property 6 of $\mathbf{T}^*$. Thus $v(\neg\psi) \in \{I, t\}$.
- Assume $v(\psi) = t$. By definition, this implies $\neg\psi \notin \mathbf{T}^*$, whence $v(\neg\psi) = f$.

[6] This Theorem and the next one were first proved (using a different argument) in [5]. What is new here concerning them are the proofs, and their various applications (like Theorems 4-6 below). These results and their current proofs provide also the necessary basis for the infinite semantics described in the main part of this paper: Section 4.

- Assume $v(\psi) = I$. By definition, this implies $\psi \in \mathbf{T}^*$ and $\neg\psi \in \mathbf{T}^*$. The latter implies $v(\neg\psi) \in \{I, t\}$. Together with the former it also implies that $\circ\psi \notin \mathbf{T}^*$, by Property 7 of $\mathbf{T}^*$. Hence $v(\circ\psi) = f$.

Since $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$, $v(\psi) \in \mathcal{D}$ for every $\psi \in \mathbf{T}$, while $v(\varphi_0) \notin \mathcal{D}$. Hence v is a model of $\mathbf{T}$ which is not a model of $v(\varphi_0)$. □

3.2 Handling (i)*,* (c)*,* (e)*,* (a)*, and* (o)

Definition 6 The conditions on **B**-Nmatrices which correspond to the axioms **(i)**, **(c)**, **(e)**, **(a)** and **(o)** are:

Cond(**i**): $a \in \mathcal{T} \cup \mathcal{F} \Rightarrow \tilde{\circ}a \subseteq \mathcal{T}$
Cond(**c**): $a \in \mathcal{F} \Rightarrow \tilde{\neg}a \subseteq \mathcal{T}$
Cond(**e**): $a \in \mathcal{I} \Rightarrow \tilde{\neg}a \subseteq \mathcal{I}$
Cond(**a**): $a \in \mathcal{T} \cup \mathcal{F}$ and $b \in \mathcal{T} \cup \mathcal{F} \Rightarrow a\tilde{\sharp}b \subseteq \mathcal{T} \cup \mathcal{F}$ $(\sharp \in \{\vee, \wedge, \supset\})$
Cond(**o**): $a \in \mathcal{T} \cup \mathcal{F}$ or $b \in \mathcal{T} \cup \mathcal{F} \Rightarrow a\tilde{\sharp}b \subseteq \mathcal{T} \cup \mathcal{F}$ $(\sharp \in \{\vee, \wedge, \supset\})$

Definition 7 Let $S = \{\mathbf{i}, \mathbf{c}, \mathbf{e}, \mathbf{ci}, \mathbf{ie}, \mathbf{ce}, \mathbf{cie}, \mathbf{ia}, \mathbf{cia}, \mathbf{iae}, \mathbf{ciae}, \mathbf{io}, \mathbf{cio}, \mathbf{ioe}, \mathbf{cioe}\}$

- For $s \in S$, a **B**s-Nmatrix is a **B**-Nmatrix which satisfies *Cond*(x) for every x which occurs in s.
- $\mathcal{M}_{Bs}$ is the unique **B**s-Nmatrix in which $\mathcal{T} = \{t\}$, $\mathcal{F} = \{f\}$, and $\mathcal{I} = \{I\}$.

Thus if **i** occurs in s then in $\mathcal{M}_{Bs}$ the truth table corresponding to $\circ$ is:

$\tilde{\circ}$	**f**	**I**	**t**
	$\{t\}$	$\{f\}$	$\{t\}$

In $\mathcal{M}_{Bc}$, $\mathcal{M}_{Bci}$, $\mathcal{M}_{Bcia}$, and $\mathcal{M}_{Bcio}$ the truth table corresponding to $\neg$ is:

$\tilde{\neg}$	**f**	**I**	**t**
	$\{t\}$	$\{I, t\}$	$\{f\}$

In $\mathcal{M}_{Be}$, $\mathcal{M}_{Bie}$, $\mathcal{M}_{Biae}$, and $\mathcal{M}_{Bioe}$ the truth table corresponding to $\neg$ is:

$\tilde{\neg}$	**f**	**I**	**t**
	$\{I, t\}$	$\{I\}$	$\{f\}$

In $\mathcal{M}_{Bce}$, $\mathcal{M}_{Bcie}$, $\mathcal{M}_{Bciae}$, and $\mathcal{M}_{Bcioe}$ the truth table corresponding to $\neg$ is:

$\tilde{\neg}$	**f**	**I**	**t**
	$\{t\}$	$\{I\}$	$\{f\}$

If **a** occurs in s then in $\mathcal{M}_{Bs}$ the tables corresponding to $\{\vee, \wedge, \supset\}$ are:

$\tilde{\vee}$	**f**	**I**	**t**
f	$\{f\}$	$\{I,t\}$	$\{t\}$
I	$\{I,t\}$	$\{I,t\}$	$\{I,t\}$
t	$\{t\}$	$\{I,t\}$	$\{t\}$

$\tilde{\wedge}$	**f**	**I**	**t**
f	$\{f\}$	$\{f\}$	$\{f\}$
I	$\{f\}$	$\{I,t\}$	$\{I,t\}$
t	$\{f\}$	$\{I,t\}$	$\{t\}$

$\tilde{\supset}$	**f**	**I**	**t**
f	$\{t\}$	$\{I,t\}$	$\{t\}$
I	$\{f\}$	$\{I,t\}$	$\{I,t\}$
t	$\{f\}$	$\{I,t\}$	$\{t\}$

If **o** occurs in s then in $\mathcal{M}_{Bs}$ the tables corresponding to $\{\vee, \wedge, \supset\}$ are:

$\tilde{\vee}$	**f**	**I**	**t**
f	$\{f\}$	$\{t\}$	$\{t\}$
I	$\{t\}$	$\{I,t\}$	$\{t\}$
t	$\{t\}$	$\{t\}$	$\{t\}$

$\tilde{\wedge}$	**f**	**I**	**t**
f	$\{f\}$	$\{f\}$	$\{f\}$
I	$\{f\}$	$\{I,t\}$	$\{t\}$
t	$\{f\}$	$\{t\}$	$\{t\}$

$\tilde{\supset}$	**f**	**I**	**t**
f	$\{t\}$	$\{t\}$	$\{t\}$
I	$\{f\}$	$\{I,t\}$	$\{t\}$
t	$\{f\}$	$\{t\}$	$\{t\}$

Theorem 3 *For $s \in S$:*

- **B**s *is sound for any* **B**s*-Nmatrix.*
- $\mathcal{M}_{Bs}$ *is a characteristic Nmatrix for* **B**s*.*

Proof: To show the first part it suffices by Theorem 1 to check that for every $\mathbf{x} \in \{\mathbf{c}, \mathbf{i}, \mathbf{e}\}$, the validity of schema ($\mathbf{x}$) (in a given **B**-Nmatrix) follows from Cond($\mathbf{x}$), and that or every $\mathbf{x} \in \{\mathbf{a}, \mathbf{o}\}$, the validity of schema ($\mathbf{x}$) follows from Cond($\mathbf{x}$) together with Cond($\mathbf{i}$). This is easy.

The proof of the second part is very similar to the proof of Theorem 2, using **B**s instead of **B**. We only have to show that the extra conditions imposed by the extra axioms of **B**s on the valuation v defined in that proof are respected:

- Suppose that (**i**) is an axiom of **B**s, and assume $v(\varphi) \in \{t, f\}$. Then either $\varphi \notin \mathbf{T}^*$, or $\neg\varphi \notin \mathbf{T}^*$. Hence (by schema (**i**)) $\neg\circ\varphi \notin \mathbf{T}^*$ By definition of v, $v(\circ\varphi) = t$, as required.
- Suppose that both (**i**) and (**a**) are axioms of **B**s, and that $v(\varphi) \in \{t, f\}$, $v(\psi) \in \{t, f\}$. Then by schemata (**i**) and (**t**), $\circ(\varphi)$ and $\circ(\psi)$ are in $\mathbf{T}^*$, and so $\circ(\varphi\sharp\psi) \in \mathbf{T}^*$ by Schema **(a)**. Hence (by Schema **(p)**) either $\varphi\sharp\psi \notin \mathbf{T}^*$, or $\neg(\varphi\sharp\psi) \notin \mathbf{T}^*$. By definition of v, this yields $v(\varphi\sharp\psi) \in \{t, f\}$, as required.
- Suppose that both (**i**) and (**o**) are axioms of **B**s, and that either $v(\varphi)$ or $v(\psi)$ is in $\{t, f\}$. Then by schemata (**i**) and (**t**), either $\circ(\varphi)$ or $\circ(\psi)$ is in $\mathbf{T}^*$. Hence $\circ(\varphi\sharp\psi) \in \mathbf{T}^*$ by Schema **(o)**. This again implies that $v(\varphi\sharp\psi) \in \{t, f\}$.
- Suppose that (**c**) is an axiom of **B**s, and $v(\varphi) = f$. Then $\varphi \notin \mathbf{T}^*$. Hence $\neg\neg\varphi \notin \mathbf{T}^*$ by (**c**). By definition of v, this implies that $v(\neg\varphi) = t$, as required.

- Suppose that (**e**) is an axiom of **B**s, and $v(\varphi) = I$. Then $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$. Hence $\neg\varphi \in \mathbf{T}^*$, and by (**e**) also $\neg\neg\varphi \in \mathbf{T}^*$. By definition of v, this implies that $v(\neg\varphi) = I$, as required. □

Corollary 2 *All the 16 systems investigated in this section are decidable, and they are all different from each other.*

Note. Since (**o**) obviously entails (**a**) (using positive classical logic), no element of S includes both **a** and **o**. We also do not consider here logics that include the schema (**a**), or the schema (**o**), but do not include schema (**i**). In [5] it is shown that a completely modular semantics, in which (**a**) and (**o**) are treated independently of (**i**), can be given if we use Nmatrices with 5 truth-values rather than just 3. It is worth noting also that in that paper the schema (**i**) was naturally split into two independent axioms, (**a**) and (**o**) were split into three axioms, and completely modular semantics was given to each of the resulting axioms. This can easily be done also within the present 3-valued framework.

Examples:

(1) $\vdash_{Bia} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$
(2) $\nvdash_{Bcie} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$
(3) Let $\neg^0\psi = \psi$, $\neg^{n+1}\psi = \neg(\neg^n\psi)$. Then for all n and φ, $\vdash_{Bci} \circ\neg^n\circ\varphi$.

To show the first part, let v be a legal valuation in $\mathcal{M}_{Bia}$. If $v(\neg\varphi \vee \neg\psi) \neq f$ then $v(\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)) \in \{t, I\}$. Assume that $v(\neg\varphi \vee \neg\psi) = f$. Then $v(\neg\varphi) = f$ and $v(\neg\psi) = f$. Hence $v(\varphi) = v(\psi) = t$, and so $v(\varphi \wedge \psi) = t$. Thus $v(\neg(\varphi \wedge \psi)) = f$, implying that $v(\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)) = t$ in this case.

For the second part, it suffices by Theorem 3 and Proposition 2 to provide the following refutation in $\mathcal{M}_{Bcie}$: $v(\varphi) = t$, $v(\psi) = t$, $v(\neg\varphi) = f$, $v(\neg\psi) = f$, $v(\varphi\wedge\psi) = I$, $v(\neg(\varphi\wedge\psi)) = I$, $v(\neg\varphi\vee\neg\psi) = f$, $v(\neg(\varphi\wedge\psi) \supset (\neg\varphi\vee\neg\psi)) = f$.

For the third part, note that if v is a valuation in $\mathcal{M}_{Bci}$ then $v(\circ\psi) \in \{t, f\}$ for all ψ, and if $v(\psi) \in \{t, f\}$ then also $v(\neg\psi) \in \{t, f\}$. It follows that for all n and φ, $v(\neg^n\circ\varphi) \in \{t, f\}$, and so $v(\circ\neg^n\circ\varphi) = t$.

Now we show that the semantics given in this section to the various systems cannot be simplified.

Theorem 4 *Let $\mathcal{L}$ be either $\mathcal{L}_{cl}$ or $\mathcal{L}_C$. Then no logic in $\mathcal{L}$ which lies between positive classical logic and* **Bcioe** *can have a finite characteristic matrix.* [7]

[7] Many particular cases of this theorem have been proved by J. Marcos and others (see [13]). However, the cases of **Bcioe** and **Bcio** were explicitly left open in [13].

Proof: It is easy to see that if N is a unary connective definable in $\mathcal{L}$ and $\mathcal{M}$ is an n-valued *deterministic* matrix then for every valuation v in $\mathcal{M}$ and for every ψ, $v(N^n\psi) \in \{v(\psi), v(N\psi), \ldots, v(N^{n-1}\psi)\}$. Since for every $k < n$, $p \wedge Np \wedge NNp \wedge ...N^{n-1}p \supset N^k p$ is valid in positive classical logic, it follows that if $\mathcal{M}$ is an n-valued deterministic matrix for which positive classical logic is sound, then $\psi_n = p \wedge Np \wedge NNp \wedge ...N^{n-1}p \supset N^n p$ should be valid in $\mathcal{M}$ (where p is atomic). Hence this formula is a theorem in every extension of positive classical logic which has an n-valued characteristic matrix. However, if we take $N\varphi$ to be $\neg(\varphi \wedge \varphi)$ we can refute ψ_n in $\mathcal{M}_{Bcioe}$ by letting $v(p) = v(Np) = \ldots = v(N^{n-1}p) = I, v(N^{n-1}p \wedge N^{n-1}p) = t$, and $v(N^n p) = f$. Hence ψ_n is not provable in **Bcioe** (or in any weaker system). $\square$

Theorem 5 *No logic* **L** *between* **B** *and* **Bcioe** *can have a weakly characteristic two-valued Nmatrix.*

Proof: Let $\mathcal{M}$ be a two-valued Nmatrix which is weakly sound for **L**. It is easy to see that the interpretations of the positive connectives in $\mathcal{M}$ should be identical to the classical ones. The validity of the law of excluded middle forces therefore the condition $\tilde{\neg} f = \{t\}$. Hence there are two cases to consider:

- If $\tilde{\neg} t = \{f\}$ then $\neg p \supset \neg(p \wedge p)$ is valid in $\mathcal{M}$. Since this formula is not a theorem of **Bcioe** (it can easily be refuted in $\mathcal{M}_{Bcioe}$), $\mathcal{M}$ is not weakly characteristic for **L**.
- If $t \in \tilde{\neg} t$ then it cannot be the case that also $t \in \tilde{\circ} t$, since otherwise (**p**) would not be valid in $\mathcal{M}$. It follows that $\tilde{\circ} t = \{f\}$, and so $p \wedge \circ p \supset q$ is valid in $\mathcal{M}$. Since this formula is not valid in $\mathcal{M}_{Bcioe}$, $\mathcal{M}$ is not weakly characteristic for **L** in this case too. $\square$

We end this section with a proof of a very important property of the systems investigated here.

Definition 8 Let **L** be a logic which includes the positive classical logic. Two formulas A and B are called *logically indistinguishable* in **L** if $\varphi(A) \vdash_{\mathbf{L}} \varphi(B)$ and $\varphi(B) \vdash_{\mathbf{L}} \varphi(A)$ for every formula $\varphi(p)$ in the language of **L**.

Theorem 6 *Let* **L** *be a logic in a language which includes* $\{\neg, \wedge, \vee, \supset\}$.

(1) If **Bciae** *is an extension of* **L** *then two formulas are logically indistinguishable in* **L** *iff they are identical.*
(2) If **Bioe** *is an extension of* **L** *then two formulas are logically indistinguishable in* **L** *iff they are identical.*
(3) If **Bcioe** *is an extension of* **L** *then two formulas in* $\{\neg, \wedge, \vee, \supset\}$ *are logically indistinguishable in* **L** *iff they are identical.*

Proof: For the first part, assume that A and B are two different formulas of **L**, and let q be any atomic formula not occurring in either A or B. Let v

be some legal partial valuation in $\mathcal{M}_{Bciae}$ such that $v(\psi)$ is defined iff ψ is a subformula of either $A \supset A$ or $B \supset B$. Extend v to a larger legal partial valuation by letting $v(q) = I$, $v(q \supset (B \supset B)) = I$, $v(\neg(q \supset (B \supset B))) = I$, $v(q \supset (A \supset A)) = t$, and $v(\neg(q \supset (A \supset A))) = f$ (this is possible since both $v(A \supset A)$ and $v(B \supset B)$ are in $\mathcal{D}$). By Proposition 2 it follows that $\neg(q \supset (B \supset B)) \nvdash_{\mathbf{L}} \neg(q \supset (A \supset A))$. Hence A and B are not logically indistinguishable in **L**.

For the second part, assume again that A and B are two different formulas of **L**, and assume without a loss in generality that $A \supset A$ is not a subformula of $B \supset B$. It is easy to see that there exists a legal partial valuation v in $\mathcal{M}_{Bioe}$ such that $v(\psi)$ is defined iff ψ is a subformula of $\neg\neg\neg(B \supset B)$, and $v(B \supset B) = t$, $v(\neg(B \supset B)) = f$, $v(\neg\neg(B \supset B)) = I$, $v(\neg\neg\neg(B \supset B)) = I$. Extend v to a partial valuation defined also for all subformulas of $\neg\neg\neg(A \supset A)$ such that $v(A \supset A) = t$, $v(\neg(A \supset A)) = f$, $v(\neg\neg(A \supset A)) = t$, $v(\neg\neg\neg(A \supset A)) = f$. By Proposition 2 it follows that $\neg\neg\neg(B \supset B) \nvdash_{\mathbf{L}} \neg\neg\neg(A \supset A)$. Hence A and B are not logically indistinguishable in **L**.

The proof of the third part is similar to that of the first. However, this time we use $\mathcal{M}_{Bcioe}$ instead of $\mathcal{M}_{Bciae}$, and we start with a partial valuation v such that $v(\psi) = I$ for every subformula of either $A \supset A$ or $B \supset B$ (such v is legal since neither A nor B contains $\circ$). $\square$

Note. Extensions of **Bcio** do not have the property described in the first two parts of the last theorem (and so its third part cannot be improved). Thus by using $\mathcal{M}_{Bcio}$ it is easy to show that $\circ(A \supset A)$ and $\circ(B \supset B)$ are logically indistinguishable in **Bcio** for any two formulas A and B.

4 The Systems with Infinite-valued Nmatrices

We turn now to to the extensions of the systems handled in the previous section by the schemas **(l)**,**(d)** and their combination **(b)**.

Definition 9 Let $\mathcal{T} = \{t_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{I} = \{I_i^j \mid i \geq 0, j \geq 0\}$, $\mathcal{F} = \{f\}$. Define the following Nmatrices for the language $\mathcal{L}_C$:

$\mathcal{M}_{Bl}$**:** This is the Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where:
(1) $\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}$
(2) $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$
(3) $\mathcal{O}$ is defined by:

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a\tilde{\supset}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a\tilde{\wedge}b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\tilde{\neg}a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{if } a \in \mathcal{F} \\ \{I_i^{j+1}, t_i^{j+1}\} & \text{if } a = I_i^j \end{cases}$$

$$\tilde{\circ}a = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

$\mathcal{M}_{Bd}$**:** This is defined like $\mathcal{M}_{Bl}$, except that $\tilde{\wedge}$ is defined as follows:

$$a\tilde{\wedge}b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if if } b = I_i^j \text{ and } a \in \{I_i^{j+1}, t_i^{j+1}\} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$\mathcal{M}_{Bb}$**:** This is defined like $\mathcal{M}_{Bl}$, except that $\tilde{\wedge}$ is defined as follows:

$$a\tilde{\wedge}b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & (\text{if } a = I_i^j \text{ and } b \in \{I_i^{j+1}, t_i^{j+1}\}) \text{ or } (b = I_i^j \text{ and } a \in \{I_i^{j+1}, t_i^{j+1}\}) \\ \mathcal{D} & \text{otherwise} \end{cases}$$

Theorem 7 *For $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$, $\mathcal{M}_{By}$ is a characteristic Nmatrix for* **By**.

Proof: We do the proof in the case of **Bl**, leaving the others to the reader.

It is easy to check that $\mathcal{M}_{Bl}$ is a **B**-Nmatrix. To prove soundness, we need therefore to show only that (**l**) is valid in it. So let v be an $\mathcal{M}_{Bl}$-legal valuation, and let φ be a sentence such that $v(\circ\varphi) \notin \mathcal{D}$. Then $v(\varphi) = I_i^j$ for some i, j. Hence $v(\neg\varphi) \in \{I_i^{j+1}, t_i^{j+1}\}$, and so $v(\varphi \wedge \neg\varphi) \in \mathcal{T}$, and $v(\neg(\varphi \wedge \neg\varphi)) = f$. Hence $v(\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi) \in \mathcal{D}$, as required.

To prove completeness, assume that $\mathbf{T} \not\vdash_{Bl} \varphi_0$. Extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \not\vdash_{Bl} \varphi_0$. Then $\mathbf{T}^*$ has the Properties 1-8 from the proof of Theorem 2. Let $\lambda i.\alpha_i$ be an enumeration of all the formulas of $\mathcal{L}_C$ that do not begin with $\neg$. Then for any formula ψ of $\mathcal{L}_C$ there are unique $n(\psi), k(\psi)$ such that $\psi = \neg_{k(\psi)}\alpha_{n(\psi)}$, where $\neg_k\theta$ is θ preceded by k negation symbols. Define a valuation v in $\mathcal{M}_{Bl}$ as follows:

$$v(\psi) = \begin{cases} f & \psi \notin \mathbf{T}^* \\ t^{k(\psi)}_{n(\psi)} & \neg\psi \notin \mathbf{T}^* \\ I^{k(\psi)}_{n(\psi)} & \psi \in \mathbf{T}^*, \neg\psi \in \mathbf{T}^* \end{cases}$$

Now we show that v is $\mathcal{M}_{Bl}$-legal. The proofs that it is well defined and respects the operations corresponding to $\vee$ and $\supset$ are like in the proof of Theorem 2. We consider next the cases of $\circ$, $\neg$ and $\wedge$.

$\circ$**:** That $v(\circ\psi) = f$ in case $v(\psi) \in \mathcal{I}$ is shown as in the proof of Theorem 2. Assume next that $v(\psi) \in \mathcal{T} \cup \mathcal{F}$. Then either $\psi \notin \mathbf{T}^*$, or $\neg\psi \notin \mathbf{T}^*$. It follows that $\psi \wedge \neg\psi \notin \mathbf{T}^*$, and so $\neg(\psi \wedge \neg\psi) \in \mathbf{T}^*$ (by Property 6). Hence $\circ\psi \in \mathbf{T}^*$ by **(l)**, and so $v(\circ\psi) \in \mathcal{D}$.

$\neg$**:** The proofs that $v(\psi) = f$ implies $v(\neg\psi) \in \mathcal{D}$ and that $v(\psi) \in \mathcal{T}$ implies $v(\neg\psi) = f$ are like in the proof of Theorem 2. Assume next that $v(\psi) = I^k_n$. Then both ψ and $\neg\psi$ are in $\mathbf{T}^*$, and $\psi = \neg_k\alpha_n$. Thus $\neg\psi \in \mathbf{T}^*$, and $\neg\psi = \neg_{k+1}\alpha_n$. It follows by definition of v that $v(\neg\psi)$ is either I^{k+1}_n or t^{k+1}_n (depending whether $\neg\neg\psi$ is in $\mathbf{T}^*$ or not).

$\wedge$**:** The proofs that if $v(\psi_1) = f$ or $v(\psi_2) = f$ then $v(\psi_1 \wedge \psi_2) = f$, and that $v(\psi_1 \wedge \psi_2) \in \mathcal{D}$ otherwise, are like in the proof of Theorem 2. Assume next that $v(\psi_1) = I^k_n$ and $v(\psi_2) \in \{I^{k+1}_n, t^{k+1}_n\}$. Then both ψ_1 and ψ_2 are in $\mathbf{T}^*$, and $\psi_1 = \neg_k\alpha_n$, $\psi_2 = \neg_{k+1}\alpha_n$. It follows that $\psi_2 = \neg\psi_1$, and that $\psi_1 \wedge \neg\psi_1 \in \mathbf{T}^*$. This entails (by Property 8) that $\circ\psi_1 \notin \mathbf{T}^*$. Hence schema **(l)** implies that $\neg(\psi_1 \wedge \neg\psi_1) \notin \mathbf{T}^*$, and so $\neg(\psi_1 \wedge \psi_2) \notin \mathbf{T}^*$. Thus $v(\psi_1 \wedge \psi_2) \in \mathcal{T}$.

Obviously, $v(\psi) \in \mathcal{D}$ for every $\psi \in \mathbf{T}$, while $v(\varphi_0) = f$. Hence $\mathbf{T} \not\vdash_{\mathcal{M}_{Bl}} \varphi_0$. $\square$

Definition 10 For $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$ and $s \in S$ (see Definition 7), $\mathcal{M}_{Bsy}$ is the weakest refinement of $\mathcal{M}_{By}$ which satisfies $Cond(x)$ for every x which occurs in s. In other words, $\mathcal{M}_{Bsy}$ is obtained from $\mathcal{M}_{By}$ through the following modifications:

(1) If **i** occurs in s then: $a \in \mathcal{F} \cup \mathcal{T} \Rightarrow \tilde{\circ}(a) = \mathcal{T}$
(2) If **c** occurs in s then: $\tilde{\neg} f = \mathcal{T}$
(3) If **e** occurs in s then: $\tilde{\neg}(I^j_i) = \{I^{j+1}_i\}$
(4) If **a** occurs in s then:
- $a \in \mathcal{T}$ and $b \in \mathcal{T} \Rightarrow a \tilde{\wedge} b = \mathcal{T}$
- $a \in \mathcal{T}, b \notin \mathcal{I}$ or $b \in \mathcal{T}, a \notin \mathcal{I} \Rightarrow a \tilde{\vee} b = \mathcal{T}$
- $a \in \mathcal{F}, b \notin \mathcal{I}$ or $b \in \mathcal{T}, a \notin \mathcal{I} \Rightarrow a \tilde{\supset} b = \mathcal{T}$

(5) If **o** occurs in s then:
- $a \in \mathcal{T}, b \notin \mathcal{F}$ or $b \in \mathcal{T}, a \notin \mathcal{F} \Rightarrow a \tilde{\wedge} b = \mathcal{T}$
- $a \in \mathcal{T}$ or $b \in \mathcal{T} \Rightarrow a \tilde{\vee} b = \mathcal{T}$
- $a \in \mathcal{F}$ or $b \in \mathcal{T} \Rightarrow a \tilde{\supset} b = \mathcal{T}$

Theorem 8 *For $y \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}$ and $s \in S$, $\mathcal{M}_{Bsy}$ is a characteristic Nmatrix for $\mathbf{B}sy$.*

Proof: It is easy to check that $\mathcal{M}_{Bsy}$ is both a $\mathbf{B}s$-Nmatrix, and a refinement of $\mathcal{M}_{By}$. Hence the soundness of $\mathbf{B}sy$ for $\mathcal{M}_{Bsy}$ follows from the first part of Theorem 3, Theorem 7, and Proposition 1. The proof of its completeness is a straightforward adaptation of the completeness proof of $\mathcal{M}_{By}$, similar to what was done in the proof of the second part of Theorem 3. □

Corollary 3 *All the 64 systems considered in this paper are different from each other.*

Proof: Let $\mathbf{L}$ be one of these logics. It can easily be checked that any axiom from Ax (Definition 2) which does not directly follows by classical positive logic from an official axiom of L, is not valid in $\mathcal{M}_L$. □

Definition 11 $\mathcal{L}DB = \{By \mid y \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}\} \cup \{Bsy \mid s \in S, y \in \{\mathbf{l}, \mathbf{d}, \mathbf{b}\}\}$

Corollary 4 *Every logic $\mathbf{L} \in \mathcal{L}DB$ is decidable.*

Proof: Theorem 8 and its proof easily imply that to check whether a given formula φ is provable in $\mathbf{L}$, it suffices to check all legal partial valuations v in $\mathcal{M}_L$ which assign to subformulas of φ values in

$$\{f\} \cup \{t_i^j \mid 0 \leq i \leq n^*(\varphi), 0 \leq j \leq k^*(\varphi)\} \cup \{I_i^j \mid 0 \leq i \leq n^*(\varphi), 0 \leq j \leq k^*(\varphi)\}$$

where $n^*(\varphi)$ is the number of subformulas of φ which do not begin with $\neg$, and $k^*(\varphi)$ is the maximal number of consecutive negation symbols occurring within φ. This is a finite process. □

The proof of Corollary 4 indicates that simpler infinite Nmatrices would be sufficient for characterizing the sets of provable formulas considered there.

Definition 12 For $\mathbf{L} \in \mathcal{L}DB$, let $\mathcal{M}_L^{\text{weak}}$ be the simplest refinement of $\mathcal{M}_L$ in which the set of truth values is $\{f\} \cup \{t_0^j \mid 0 \leq j\} \cup \{I_0^j \mid 0 \leq j)\}$.

Theorem 9 *For $\mathbf{L} \in \mathcal{L}DB$:*

(1) $\mathcal{M}_L^{\text{weak}}$ is weakly-characteristic for $\mathbf{L}$.
(2) $\mathcal{M}_L^{\text{weak}}$ is not characteristic for $\mathbf{L}$.

Proof:

(1) Since $\mathcal{M}_L^{\text{weak}}$ is a refinement of $\mathcal{M}_L$, $\mathbf{L}$ is sound for $\mathcal{M}_L^{\text{weak}}$. For completeness, assume $\not\vdash_{\mathbf{L}} \varphi$. Then φ has a refuting valuation in $\mathcal{M}_L$ of the type

described in the proof of Corollary 4. Obtain from it a a refuting valuation in $\mathcal{M}_L^{\text{weak}}$ by using $t_0^{(k^*(\varphi)+1)i+j}$ and $I_0^{(k^*(\varphi)+1)i+j}$ instead of t_i^j and I_i^j, respectively.

(2) Let p, q and r be three distinct propositional variables. Define:

$$N(p) = \{\neg_k p \mid k \geq 0\} \quad C(p,q) = \{\neg(\varphi \wedge \psi) \mid \varphi \in N(p), \psi \in N(q)\}$$

Let $\mathbf{T} = N(p) \cup N(q) \cup C(p,q) \cup C(q,p)$. It is easy to construct a model of $\mathbf{T}$ in $\mathcal{M}_L$ which is not a model of r. Hence $\mathbf{T} \nvdash_{\mathbf{L}} r$. We show that in contrast, $\mathbf{T} \vdash_{\mathcal{M}_L^{\text{weak}}} r$. Assume e.g. that **(l)** is an axiom of **L**, and let v be a model in $\mathcal{M}_L^{\text{weak}}$ of $N(p) \cup N(q)$. Then in v all formulas in $N(p) \cup N(q)$ should get a value in $\{I_0^j \mid 0 \leq j)\}$. It follows that for some m, either $v(p) = v(\neg_m q)$ or $v(q) = v(\neg_m p)$. In the first case $v(\neg(p \wedge \neg_{m+1} q)) = f$, while in the second $v(\neg(q \wedge \neg_{m+1} p)) = f$. Hence $\mathbf{T}$ has no model in $\mathcal{M}_L^{\text{weak}}$, and so indeed $\mathbf{T} \vdash_{\mathcal{M}_L^{\text{weak}}} r$. □

We next extend Theorem 6 to the family $\mathcal{L}DC$:

Theorem 10 *Let* **L** *be a logic in a language which includes* $\{\neg, \wedge, \vee, \supset\}$.

(1) If **Bciaeb** *is an extension of* **L** *then two formulas are logically indistinguishable in* **L** *iff they are identical.*
(2) If **Bioeb** *is an extension of* **L** *then two formulas are logically indistinguishable in* **L** *iff they are identical.*
(3) If **Bcioeb** *is an extension of* **L** *then two formulas in* $\{\neg, \wedge, \vee, \supset\}$ *are logically indistinguishable in* **L** *iff they are identical.*

Proof: The proof is similar to that of Theorem 6. □

Corollary 5 *da Costa's system C_1 ([15]) is decidable, and two formulas are logically indistinguishable in it iff they are identical.* [8]

Proof: In [12,13] it is shown that C_1 and **Bcial** (which is called there **Cila**) are equivalent in the sense that **Bcial** is a conservative extension of C_1, and it is also interpretable in C_1. Hence the corollary follows from Corollary 4 and Theorem 10 (both applied to **Bcial**). □

We end the paper by showing that the infinite-valued semantics given in this section cannot be replaced by a finite-valued one.

Theorem 11 *No logic between* **Bl** *or* **Bd** *and* **Bcioeb** *can have a finite characteristic Nmatrix.*

[8] The decidability of C_1, as well as most of the other decidability results of this paper, are not new. See [13] for references.

Proof: Assume e.g. that **L** is a logic between **Bl** and **Bcioeb**, and assume for contradiction that $\mathcal{M} = \langle \mathcal{V}_M, \mathcal{D}_M, \mathcal{O}_M \rangle$ is an n-valued characteristic Nmatrix for **L** (n finite). Define **T** to be the union of the following three sets:

$$\{p_i \mid 1 \leq i \leq n+1\}$$
$$\{\neg p_i \mid 1 \leq i \leq n+1\}$$
$$\{\neg(p_i \wedge \neg p_j) \mid i \neq j, 1 \leq i \leq n+1, 1 \leq j \leq n+1\}$$

Define a valuation v in $\mathcal{M}_{Bcioeb}$ by $v(p_{n+2}) = f$, $v(p_i) = I_i^0$, $v(\neg p_i) = I_i^1$, $v(p_i \wedge \neg p_j) = I_0^0$, and $v(\neg(p_i \wedge \neg p_j)) = I_0^1$ for $1 \leq i \leq n+1, 1 \leq j \leq n+1, i \neq j$. It is easy to check that v is a (partial) legal valuation in $\mathcal{M}_{Bcioeb}$. Obviously, v is a model of **T** which is not a model of p_{n+2}. Hence $\mathbf{T} \nvdash_{\mathbf{Bcioeb}} p_{i+2}$, and so $\mathbf{T} \nvdash_{\mathbf{L}} p_{i+2}$. Since $\mathcal{M}$ is characteristic for **L**, there is a valuation v_0 in $\mathcal{M}$ which is a model of **T** but not a model of p_{n+2}. By the pigeonhole principle, there are $1 \leq i_0 < j_0 \leq n+1$ such that $v_0(p_{i_0}) = v_0(p_{j_0})$. Define now a new valuation v_1 by $v_1(p_{n+2}) = v_0(p_{n+2})$, $v_1(p_{i_0}) = v_0(p_{i_0})$, $v_1(\neg p_{i_0}) = v_0(\neg p_{j_0})$, $v_1(p_{i_0} \wedge \neg p_{i_0}) = v_0(p_{i_0} \wedge \neg p_{j_0})$, and $v_1(\neg(p_{i_0} \wedge \neg p_{i_0})) = v_0(\neg(p_{i_0} \wedge \neg p_{j_0}))$. Since v_0 is legal in $\mathcal{M}$, and $v_0(p_{i_0}) = v_0(p_{j_0})$, v_1 is also legal in $\mathcal{M}$. Now v_0 is a model of **T**, and so it is a model of $\{p_{i_0}, \neg p_{j_0}, \neg(p_{i_0} \wedge \neg p_{j_0})\}$. Hence v_1 is a model of $\{p_{i_0}, \neg p_{i_0}, \neg(p_{i_0} \wedge \neg p_{i_0})\}$. On the other hand, v_1 is not a model of p_{n+2} (because v_0 is not a model of p_{n+2}). Since **L** is sound for $\mathcal{M}$, it follows that $\{p_{i_0}, \neg p_{i_0}, \neg(p_{i_0} \wedge \neg p_{i_0})\} \nvdash_{\mathbf{L}} p_{n+2}$. This contradicts the fact that $\{p_{i_0}, \neg p_{i_0}, \neg(p_{i_0} \wedge \neg p_{i_0})\} \vdash_{\mathbf{Bl}} p_{n+2}$, for **L** is an extension of **Bl**. □

Corollary 6 *da Costa C_1 has no finite characteristic Nmatrix.*

Proof: This follows from Theorem 11, and the fact that **Bcial** is interpretable in C_1. □

Note. The lack of finite characteristic *ordinary* matrices for some C-systems has been known before (see [13]). However, what is proved in Theorem 11 is a much stronger result!

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