Combining Classical Logic, Paraconsistency and Relevance

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Abstract

We present a logic with has both a simple semantics and a cut-free Gentzen-type system on one hand, and which combines relevance logics, da Costa's paraconsistent logics, and classical logic on the other. We further show that the logic has many other nice properties, and that its language is ideal from the semantic point of view.

1 Introduction

A (propositional) logic $\mathcal{L}$ is paraconsistent with respect to a negation connective $\sim$ if whenever P and Q are two distinct atomic variables then

$$\sim P,\ P \nvdash_{\mathcal{L}} Q$$

Intuitively (and sometimes practically) the logic(s) we use should be paraconsistent (perhaps with respect to *any* unary connective!) on the ground of relevance: why should a "contradiction" concerning P imply something completely unrelated? There is no wonder that relevance logics ([1, 2, 13]) are paraconsistent with respect to their official negation. However, relevance logics have the defect that they totally reject extremely useful classical principles (like the disjunctive syllogism) without providing any indication when can these principles safely be used. This is precisely what da Costa's family of paraconsistent logics ([11, 10]) is trying to provide. However, these logics (with the exception of the 3-valued paraconsistent logic J_3) have neither convincing semantics nor decent cut-free proof systems, and their philosophical basis seems to be doubtful. In particular: relevance considerations are altogether ignored in them. There seems indeed to be little connection between this family and the family of relevance logics.

The goal of this paper is to present a logic with has both a simple semantics and a cut-free Gentzen-type system on one hand, and which combines relevance logics, da Costa's paraconsistent logics, and classical logic on the other. By "combines" we mean, first of all, that the main ideas and principles behind these logics are taken into account in our logic, and provide its basis. As a result, our logic is a combination of these 3 families also in the technical sense that for each of them it has an important fragment which belongs to that family.

Our starting point is classical logic. This, after all, is the primary logic, being both the simplest logic and the metalogic used for investigating all other logics (as is revealed by any examination of works on non-classical logics). Now classical logic is actually based on the following two principles:

(T) Whatever is not *absolutely* true is false.

(F) Whatever is not *absolutely* false is true.

These two intuitive principles might look fuzzy, but they can be translated into completely precise ones using the semantic framework of *matrices*, which is general enough for characterizing every propositional logic (by a famous theorem from [17]).

Definition 1 A matrix $\mathcal{M}$ for a propositional language $\mathcal{L}$ is a triple $\langle M, D, O\rangle$, where M is a nonempty set (of "truth values"), $\emptyset \subset D \subset M$ (D is the subset of "designated" values), and O includes an operation $\widetilde{\diamond} : M^n \rightarrow M$ for every n-ary connective $\diamond$ of $\mathcal{L}$. A *valuation* in a matrix M is a function $v : \mathcal{L} \rightarrow M$ which respects the operations, i.e.: $v(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ for every connective $\diamond$ of $\mathcal{L}$. Such a valuation v is called a *model* in $\mathcal{M}$ of a formula ψ if $v(\psi) \in D$. We say that φ *follows* in $\mathcal{M}$ from a theory T ($T \vdash_{\mathcal{M}} \varphi$) if every model in $\mathcal{M}$ of all the formulas of T is also a model of φ. $\mathcal{M}$ is a *characteristic matrix* for a logic $\mathbf{L}$ if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{M}}$, and it is *weakly characteristic* for $\mathbf{L}$ if they have the same tautologies, i.e.: for all ψ, $\vdash_{\mathbf{L}} \psi$ iff $\vdash_{\mathcal{M}} \psi$.

The conditions in Definition 1 concerning the set D imply that M contains at least two different truth values, $\top$ and $\bot$, so that $\top \in D$ while $\bot \notin D$ (note that this assumption excludes trivial "logics"). We may take these two elements as denoting absolute truth and falsehood. This leads to the following interpretations of the terms used in the formulation of (T) and (F) above:

- "φ is true" means $v(\varphi) \in D$
- "φ is absolutely true" means $v(\varphi) = \top$
- "φ is false" means $v(\varphi) \notin D$
- "φ is absolutely false" means $v(\varphi) = \bot$

With this interpretations the two classical principles reduce to:

Principle (T) : $D = \{\top\}$

Principle (F) : $D = M - \{\bot\}$

Note that together with the condition $\bot \neq \top$ (which we henceforth assume) each of these principles already implies that $\top \in D$ while $\bot \notin D$, and that whatever is absolutely true is true, and whatever is absolutely false is false. Together the two principles imply that $M = \{\top, \bot\}$, and we get the classical, bivalent semantics. $\top$ and $\bot$ may indeed be identified with the classical truth values, and we shall henceforth make this identification. Accordingly, we shall take $\bot = \sim \top$ as a necessary condition for a unary connective $\sim$ to serve as a "negation" (and usually also that $\top = \sim \bot$, so that $\sim$ behaves exactly like classical negation on the classical truth values).

After formulating the two classical principles in precise terms we immediately see that paraconsistency is in a direct conflict with Principle (T). This principle easily implies that $\{\sim P,\ P\}$ can have no model [1]. Hence every model of $\{\sim P,\ P\}$ is a model of any Q, and so $\sim P,\ P \vdash_{\mathcal{L}} Q$ in any logic whose semantics obeys Principle (T). Since most of the more famous logics (like Classical Logic, Intuitionistic Logic, Kleene's three-valued logic, Łukasiewicz' many-valued logics, other fuzzy logics, and many others) adopt this principle, all these logics are not paraconsistent with respect to their official negation.

It follows from the above discussion that any paraconsistent logic should be based on a many-valued semantics in which there exist (at least) one designated element $\top$ such that $\bot = \sim \top \notin D$, and (at least) one element I such that both I and $\sim I$ are in D (such truth values correspond to contradictory, or "paradoxical" propositions). The most economic many-valued structures with these properties are those in which there are exactly 3 truth values $\{\top, \bot, I\}$, with $D = \{\top, I\}$, and $\sim I = I$. The famous paraconsistent logic J_3 of [12, 14, 4, 20] is indeed based on such a 3-valued structure. Moreover: although J_3 rejects Principle (T), it still adheres to Principle (F). However, J_3 has the defect that it does not take relevance into consideration: any two paradoxical propositions are equivalent according to this logic. In order to avoid this, but still keep at least one of the two classical principles, we should allow for more than one paradoxical truth-value. The most natural alternative is to have a potentially infinite number of them. Paradoxical propositions that get different paradoxical truth-value should then be considered as irrelevant to each other.

What logical connectives (in addition to negation) should be used in a logic which is based on such structures? We suggest two main criteria. The main one (which we think is absolutely necessary) is *symmetry*: there should be no way to distinguish between two given paradoxical values on a *logical* basis. The other is *isolation of contradictions*: a formula may be assigned a given paradoxical value only if all its constituents are assigned that paradoxical value.

In section 2 of this paper we describe the semantics, language and consequence relation of the logic to which the above ideas lead. The main result there is that the connectives which are definable in the language of that logic are exactly those that meet the two criteria mentioned

[1] Note that for this conclusion it suffices to assume that $\sim \top \neq \top$. This condition is equivalent (in case principle (T) is respected) to the condition that $P \not\vdash \sim P$ for some P, and this is a less-than-minimal condition for any "negation"!

above. In section 3 we present proof systems for the logic. The most important among them is the hypersequential Gentzen-type system $GSRMI_m$. In the main theorem of this section we simultaneously show the strong completeness of $GSRMI_m$ for the logic, as well as the fact that the cut rule and the external contraction rule can both be eliminated from it (this fact allows for very direct proofs of valid hypersequents, and has several useful applications). In section 4 we use this system to show that relevance logic, paraconsistent logic, and classical logic can all be viewed as fragments of our logic, and discuss some further connections between these logics and ours.

The work described in this paper is a continuation of [3] and [9], and some results from these papers are reproved here (usually by new methods) in order that the reader will have the full picture. The paper is nevertheless self-contained, and its main results are new.

2 The Language and Its Semantics

Definition 2 The *pure intensional propositional language* [2] $\mathcal{IL}$ is the language $\{\sim, \otimes\}$, where $\sim$ is a unary connective, and $\otimes$ is binary [3]. The *full intensional language* $\mathcal{IL}^{\perp}$ is $\{\sim, \otimes, \perp\}$, where $\perp$ is a propositional constant.

We next describe our intended algebraic semantics for $\mathcal{IL}$ and $\mathcal{IL}^{\perp}$. For simplicity, we use the same symbols for the connectives of the languages and for their algebraic counterparts.

Definition 3

1. The structure $\mathcal{A}_\omega = \langle A_\omega, \sim, \otimes \rangle$ is defined as follows:

   (i) $A_\omega = \{\top, \perp, I_1, I_2, I_3, \dots\}$

   (ii) $\sim \top = \perp$, $\sim \perp = \top$, $\sim I_k = I_k$ $(k = 1, 2, \dots)$

   (iii) $a \otimes b = \begin{cases} \perp & a = \perp \text{ or } b = \perp \\ I_k & a = b = I_k \\ \top & \textit{otherwise} \end{cases}$

2. $\mathcal{A}_n$ $(n \geq 0)$ is the substructure of $\mathcal{A}_\omega$ which consists of $\{\top, \perp, I_1, \dots, I_n\}$.

3. A *valuation* for $\mathcal{IL}$ ($\mathcal{IL}^{\perp}$) is a function v from the set of formulas of $\mathcal{IL}$ ($\mathcal{IL}^{\perp}$) to A_ω such that ($v(\perp) = \perp$ and) $v(\sim\varphi) = \sim v(\varphi)$, $v(\varphi \otimes \psi) = v(\varphi) \otimes v(\psi)$ for all φ, ψ.

4. For $\mathcal{T}$ and φ in $\mathcal{IL}$ ($\mathcal{IL}^{\perp}$), $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi$ iff for every valuation v in $\mathcal{A}_\omega$, if $v(\psi) \neq \perp$ for all $\psi \in \mathcal{T}$ then $v(\varphi) \neq \perp$. In particular φ is *valid* in $\mathcal{A}_\omega$ ($\vdash_{\mathcal{A}_\omega} \varphi$) iff $v(\varphi) \neq \perp$ for every valuation v in $\mathcal{A}_\omega$. $\vdash_{\mathcal{A}_n}$ and validity in $\mathcal{A}_n$ are defined similarly.

[2] The terminology is from Relevance Logic. [16] uses the term "multiplicative" instead of "intensional".

[3] $\sim$ and $\otimes$ are called intensional negation and conjunction, respectively. The notation $\sim$ is from relevance logic, while $\otimes$ is taken from [16] (relevantists had used $\circ$ before).

Note $\mathcal{A}_\omega$, which was first introduced in [3], can of course be taken as a matrix which is based on Principle (F) (i.e.: its set of designated elements is $A_\omega - \{\bot\}$). $\mathcal{A}_1$ was first introduced in [21], and in that paper the set of $\mathcal{IL}$-formulas which are valid in it was axiomatized. It is known, therefore, as Sobociński 3-valued logic. $\vdash_{\mathcal{A}_1}$ is also known to be ([1]) the purely intensional fragment of the semi-relevant system RM. $\vdash_{\mathcal{A}_0}$ is, of course, just classical logic. In [3, 9] it is proved that $\vdash_{\mathcal{A}_\omega}$ is decidable, that any nontrivial logic in $\mathcal{IL}$ (or $\mathcal{IL}^\bot$) which properly extends it is identical to $\vdash_{\mathcal{A}_n}$ for some $0 \leq n < \omega$, and that each $\vdash_{\mathcal{A}_n}$ $(0 \leq n < \omega)$ is a proper extension of $\vdash_{\mathcal{A}_{n+1}}$.

The following are important connectives from relevance logic which are definable in $\mathcal{IL}$:

1. $\varphi \to \psi = \sim(\varphi \otimes \sim\psi)$
2. $\varphi \leftrightarrow \psi = (\varphi \to \psi) \otimes (\psi \to \varphi)$
3. $\varphi + \psi = \sim\varphi \to \psi$

The following properties of these connectives can easily be established:

Lemma 1 *The connective $\to$ corresponds in A_ω to the following function:*

$$a \to b = \begin{cases} \top & a = \bot \;\; or \; b = \top \\ I_k & a = b = I_k \\ \bot & otherwise \end{cases}$$

Lemma 2 *$\otimes$, $+$, and $\to$ behave on $\{\top, \bot\}$ like the classical conjunction, disjunction, and implication (respectively). Moreover: $v(\varphi + \psi) = \top$ iff either $v(\varphi) = \top$ or $v(\psi) = \top$.*

Lemma 3 *$a \leftrightarrow b \neq \bot$ iff $a = b$.*

The main goal of the rest of this section is to provide in the context of $\mathcal{A}_\omega$ a precise formulation of the symmetry and isolation conditions described in the introduction, and to show that the expressive power of $\mathcal{IL}$ and $\mathcal{IL}^\bot$ exactly corresponds to these conditions. For this we need first some notations, definitions and lemmas.

Notations:

1. Let ψ be a formula. We denote by $A(\psi)$ the set of atomic variables that occur in ψ.
2. Let $A(\psi) = \{p_1, \ldots, p_n\}$, where $p_1, \ldots, p_n$ are the first n atomic variables. We denote by g_ψ the function from A_ω^n to A_ω that corresponds to ψ (i.e., if $\vec{x} = (x_1, \ldots, x_n)$ then $g_\psi(\vec{x}) = v_{\vec{x}}(\psi)$, where $v_{\vec{x}}$ is a valuation in A_ω such that $v_{\vec{x}}(p_i) = x_i$ for $1 \leq i \leq n$).

3. Let $A(\psi) = \{p_1, \dots, p_n\}$. $S[\psi]$, the subset of A_ω^n which is *characterized* by ψ, is:

$$S[\psi] = \{(a_1, \dots, a_n) \in A_\omega^n \mid g_\psi(a_1, \dots, a_n) \neq \perp\}$$

4. Denote by I_k^n the n-tuple $(I_k, I_k, \dots, I_k)$. Let $I(n) = \{I_k^n \mid k \in \mathcal{N}\}$.

Definition 4 An n-ary operation F on A_ω (A_k) satisfies the *symmetry* condition if

$$F(h(x_1), \dots, h(x_n)) = h(F(x_1, \dots, x_n))$$

for all $x_1, \dots, x_n \in A_\omega$ (A_k)) and for every injective function h from A_ω to A_ω (A_k to A_k) such that $h(\top) = \top$ and $h(\perp) = \perp$.

Lemma 4 *If ψ is in $\mathcal{IL}^\perp$ and $A(\psi) = \{p_1, \dots, p_n\}$ then g_ψ satisfies the symmetry condition.*

Proof: Since g_ψ is obtained from the functions $\sim$, $\otimes$, the constant functions $\lambda\vec{x}.\perp$, and the projection functions (including identity) using composition, it suffices to check that all these functions satisfy the symmetry condition, and that composition preserves this property. This is easy.

Definition 5 Let F be an n-ary operation on A_ω (A_l).

1. F satisfies the *isolation* condition if for every k, $F(\vec{x}) = I_k$ only if $\vec{x} = I_k^n$.

2. F satisfies the *strong isolation* condition if for every k, $F(\vec{x}) = I_k$ iff $\vec{x} = I_k^n$.

Lemma 5

1. *If ψ is in $\mathcal{IL}^\perp$ and $A(\psi) = \{p_1, \dots, p_n\}$ then g_ψ satisfies the isolation condition.*

2. *If ψ is in $\mathcal{IL}$ and $A(\psi) = \{p_1, \dots, p_n\}$ then g_ψ satisfies the strong isolation condition.*

Proof: By induction on the structure of ψ.

Corollary 1 *If ψ is in $\mathcal{IL}^\perp$, $A(\psi) = \{p_1, \dots, p_n\}$, and $\vec{x} \notin I(n)$, then $g_\psi(\vec{x}) \in \{\top, \perp\}$.*

Corollary 2 *If ψ is in $\mathcal{IL}$, and $A(\psi) = \{p_1, \dots, p_n\}$, then $I(n) \subseteq S[\psi]$.*

Lemma 6 *Let (in $\mathcal{IL}^\perp$) $\top = \sim\perp$. If ψ is in $\mathcal{IL}^\perp$ and v is a valuation in A_ω then:*

$$v(\top \otimes \psi) = \begin{cases} \perp & v(\psi) = \perp \\ \top & v(\psi) \neq \perp \end{cases} \qquad v(\top \to \psi) = \begin{cases} \top & v(\psi) = \top \\ \perp & v(\psi) \neq \top \end{cases}$$

Lemma 7 *Let* $\top_n = (p_1 \to p_1) \otimes (p_2 \to p_2) \otimes \cdots \otimes (p_n \to p_n)$. *Then*

$$g_{\top_n}(\vec{x}) = \begin{cases} I_k & \vec{x} = I_k^n \\ \top & \vec{x} \notin I(n) \end{cases}$$

Lemma 8 *For* $1 \leq i \leq k$, *let* ψ_i *be in* $\mathcal{IL}^\perp$, *and* $A(\psi_i) = \{p_1, \ldots, p_n\}$. *Then:*

1. $\bigcap_{i=1}^k S[\psi_i] = S[\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_k]$
2. $\bigcup_{i=1}^k S[\psi_i] = S[\top\otimes\psi_1 + \top\otimes\psi_2 + \cdots + \top\otimes\psi_k]$
3. *If* ψ_i *is in* $\mathcal{IL}$ *for all* $1 \leq i \leq k$ *then* $\bigcup_{i=1}^k S[\psi_i] = S[\psi_1 + \psi_2 + \cdots + \psi_k]$

Proof: Immediate from the Definition of $\otimes$, Lemmas 2 and 6, and Corollaries 1 and 2.

Definition 6 We say that $\vec{b} = (b_1, \ldots, b_n) \in A_\omega^n$ is *similar* to $\vec{a} = (a_1, \ldots, a_n) \in A_\omega^n$ if there exists an injective function from A_ω to A_ω such that $h(\top) = \top$, $h(\perp) = \perp$, and $h(a_i) = b_i$ for $1 \leq i \leq n$.

The proof of the following two lemmas is straightforward:

Lemma 9 *Similarity of tuples is an equivalence relation.*

Lemma 10 $\vec{b}$ *is similar to* $\vec{a}$ *iff the following conditions are satisfied for all* $1 \leq i, j \leq n$*:*

- *If* $a_i = \top$ *then* $b_i = \top$
- *If* $a_i = \perp$ *then* $b_i = \perp$
- *If* $a_i \in I(1)$ *then* $b_i \in I(1)$
- *If* $a_i = a_j$ *then* $b_i = b_j$
- *If* $a_i \neq a_j$ *then* $b_i \neq b_j$

Corollary 3 *Every* $\vec{b} \in A_\omega^n$ *is similar to some* $\vec{a} \in A_n^n$.

Definition 7 A subset $C \subseteq A_\omega^n$ is *characterizable* in $\mathcal{IL}^\perp$ ($\mathcal{IL}$) if $C = S[\psi]$ for some formula ψ of $\mathcal{IL}^\perp$ ($\mathcal{IL}$) such that $A(\psi) = \{p_1, \ldots, p_n\}$.

We turn next to our two major Lemmas.

Lemma 11 $C \subseteq A_\omega^n$ *is characterizable in* $\mathcal{IL}^\perp$ *if* $\vec{b} \in C$ *whenever* $\vec{b}$ *is similar to some* $\vec{a} \in C$.

Proof: By Lemma 9 and Corollary 3, if C has this property then $C = \bigcup_{\vec{a} \in A_n^n \cap C} S^{\vec{a}}$ where $S^{\vec{a}}$ is the set of tuples which are similar to $\vec{a}$. By Lemma 8 it remains therefore to prove that $S^{\vec{a}}$ is characterizable in $\mathcal{IL}^{\perp}$ for all $\vec{a} \in A_\omega^n$. By Lemma 10, $S^{\vec{a}} = (\bigcap_{1 \leq i \leq n} S_i^{\vec{a}}) \cap (\bigcap_{1 \leq i,j \leq n} S_{i,j}^{\vec{a}})$, where

$$S_i^{\vec{a}} = \begin{cases} \{\vec{x} \in A_\omega^n \mid x_i = \top\} & a_i = \top \\ \{\vec{x} \in A_\omega^n \mid x_i = \perp\} & a_i = \perp \\ \{\vec{x} \in A_\omega^n \mid x_i \in I(1)\} & a_i \in I(1) \end{cases}$$

$$S_{i,j}^{\vec{a}} = \begin{cases} \{\vec{x} \in A_\omega^n \mid x_i = x_j\} & a_i = a_j \\ \{\vec{x} \in A_\omega^n \mid x_i \neq x_j\} & a_i \neq a_j \end{cases}$$

By Lemma 8 it remains therefore to prove that $S_i^{\vec{a}}$ and $S_{i,j}^{\vec{a}}$ are characterizable in $\mathcal{IL}^{\perp}$ for all i and j. This follows from the following equations, which easily follow from Lemmas 3, 6, and 7:[4]

$$\{\vec{x} \in A_\omega^n \mid x_i = \top\} = S[(\top_n \otimes \top) \to p_i]$$

$$\{\vec{x} \in A_\omega^n \mid x_i = \perp\} = S[(\top_n \otimes \top) \to\sim p_i]$$

$$\{\vec{x} \in A_\omega^n \mid x_i \in I(1)\} = S[(\top_n \otimes \top) \otimes p_i \otimes \sim p_i]$$

$$\{\vec{x} \in A_\omega^n \mid x_i = x_j\} = S[(\top_n \otimes \top) \otimes (p_i \leftrightarrow p_j)]$$

$$\{\vec{x} \in A_\omega^n \mid x_i \neq x_j\} = S[(\top_n \otimes \top) \to\sim (p_i \leftrightarrow p_j)]$$

Lemma 12 *A subset $C \subseteq A_\omega^n$ is characterizable in $\mathcal{IL}$ if it satisfies the following two conditions:*

- $I(n) \subseteq C$
- *If $\vec{a} \in C$ and $\vec{b}$ is similar to $\vec{a}$ then $\vec{b} \in C$.*

Proof: The proof is similar to that of Lemma 11. The main difference is that because of the extra condition, $S^{\vec{a}}$ in the equation $C = \bigcup_{\vec{a} \in A_n^n \cap C} S^{\vec{a}}$ can be taken this time to be the union of $I(n)$ and the set of tuples which are similar to $\vec{a}$. Again by Lemma 8 it remains therefore to prove that this $S^{\vec{a}}$ is characterizable in $\mathcal{IL}$ for all $\vec{a} \in A_\omega^n$. Again $S^{\vec{a}} = (\bigcap_{1 \leq i \leq n} S_i^{\vec{a}}) \cap (\bigcap_{1 \leq i,j \leq n} S_{i,j}^{\vec{a}})$, where this time:

$$S_i^{\vec{a}} = \begin{cases} I(n) \cup \{\vec{x} \in A_\omega^n \mid x_i = \top\} & a_i = \top \\ I(n) \cup \{\vec{x} \in A_\omega^n \mid x_i = \perp\} & a_i = \perp \\ \{\vec{x} \in A_\omega^n \mid x_i \in I(1)\} & a_i \in I(1) \end{cases}$$

$$S_{i,j}^{\vec{a}} = \begin{cases} I(n) \cup \{\vec{x} \in A_\omega^n \mid x_i = x_j\} & a_i = a_j \\ I(n) \cup \{\vec{x} \in A_\omega^n \mid x_i \neq x_j\} & a_i \neq a_j \end{cases}$$

[4]The reason for using $\top_n$ in these equations is to make sure that we use only formulas ψ s.t. $A(\psi) = \{p_1, \ldots, p_n\}$.

The fact that these $S_i^{\vec{a}}$ and $S_{i,j}^{\vec{a}}$ are characterizable in $\mathcal{IL}$ for all i and j follows from the following easily established equations:

$$I(n) \cup \{\vec{x} \in A_\omega^n \mid x_i = \top\} = S[\top_n \to p_i]$$

$$I(n) \cup \{\vec{x} \in A_\omega^n \mid x_i = \bot\} = S[\top_n \to\sim p_i]$$

$$\{\vec{x} \in A_\omega^n \mid x_i \in I(1)\} = S[\top_n \otimes p_i \otimes \sim p_i]$$

$$I(n) \cup \{\vec{x} \in A_\omega^n \mid x_i = x_j\} = S[\top_n \otimes (p_i \leftrightarrow p_j)]$$

$$I(n) \cup \{\vec{x} \in A_\omega^n \mid x_i \neq x_j\} = S[\top_n \to\sim (p_i \leftrightarrow p_j)]$$

Note It is easy to see that the converse implications in Lemmas 11 and 12 are also true.

Theorem 1

1. *An n-ary operation F on A_ω (A_k) is definable in $\mathcal{IL}$ by a formula ψ in the distinct propositional variables $p_1, \dots, p_n$ (i.e., $F = g_\psi$) iff it satisfies both the symmetry condition and the strong isolation condition.*

2. *An n-ary operation F on A_ω (A_k) is definable in $\mathcal{IL}^\perp$ by a formula in the distinct propositional variables $p_1, \dots, p_n$ iff it satisfies both the symmetry condition and the isolation condition.*

Proof: The necessity parts have been shown in Lemmas 4 and 5. For the converse assume first that F satisfies the conditions of symmetry and strong isolation. It follows that the set $E = \{\vec{x} \in A_\omega^n \mid F(\vec{x}) \neq \bot\}$ satisfies the two conditions from Lemma 12. Hence $E = S[\psi_F]$ for some formula ψ_F in $\mathcal{IL}$. We show now that $F = g_{\psi_F}$. We consider 3 cases:

- If $F(\vec{x}) = I_k$ for some k then (by isolation) $\vec{x} = I_k^n$, and since ψ_F is in $\mathcal{IL}$ it follows that $g_{\psi_F}(\vec{x}) = I_k = F(\vec{x})$.

- If $F(\vec{x}) = \bot$ then $\vec{x} \notin E$. Hence $\vec{x} \notin S[\psi_F]$, and so $g_{\psi_F}(\vec{x}) = \bot = F(\vec{x})$

- If $F(\vec{x}) = \top$ then $\vec{x} \in E = S[\psi_F]$, and (by strong isolation) $\vec{x} \notin I(n)$. Hence $g_{\psi_F}(\vec{x}) \neq \bot$, and (by isolation) $g_{\psi_F}(\vec{x}) \notin I(1)$. It follows that $g_{\psi_F}(\vec{x}) = \top = F(\vec{x})$.

Assume now that F satisfies symmetry and isolation. If $F(\vec{x}) \notin \{\top, \bot\}$ for some $\vec{x} \in A_\omega^n$ then these two conditions together imply that F actually satisfies strong isolation as well. Hence F is definable in this case by a formula in $\mathcal{IL}$ (by what we have just proved). Assume therefore that $F(\vec{x}) \in \{\top, \bot\}$ for all $\vec{x} \in A_\omega^n$. Since F satisfies the symmetry condition, Lemma 11 entails that $E = S[\psi_F]$ for some formula ψ_F in $\mathcal{IL}^\perp$, where again $E = \{\vec{x} \in A_\omega^n \mid F(\vec{x}) \neq \bot\}$. It is easy now to see that our assumptions on F imply that $F = g_{\top \otimes \psi_F}$.

Corollary 4 *An n-ary operation F on A_1 is definable in $\mathcal{IL}^\perp$ ($\mathcal{IL}$) by a formula ψ in the distinct propositional variables $p_1, \ldots, p_n$ iff it satisfies the (strong) isolation condition.*

Proof: The symmetry condition is trivially true for any operation on A_1.

Of the two conditions we have imposed on the connectives of our languages the more fundamental one is no doubt the symmetry condition. This condition seems to us absolutely essential for any *logical* language which is based on A_ω. It is interesting therefore to note that a finite set of connectives which is functionally complete for the set of operations which satisfy this condition can be obtained from $\mathcal{IL}^\perp$ by adding one extra binary connective which again is closely related to one which is used in relevance logic.

Definition 8 The partial order $\leq_r$ is defined on A_ω by: $\perp \leq_r I_k \leq_r \top$.

Lemma 13 *The structure $\langle A_\omega, \leq_r \rangle$ is a lattice. moreover: $a \to b \neq \perp$ iff $a \leq_r b$.*

Definition 9 $a \vee b = sup_{\leq_r}(a, b) \qquad a \wedge b = inf_{\leq_r}(a, b)$

It is easy to see that $\vee$ and $\wedge$ are connected by De Morgan's rules. Thus $a \wedge b = \sim (\sim a \vee \sim b)$. Hence it suffices to add just one of them to the language.

Note: $\otimes$ itself can be defined as $inf_{\leq_o}(a, b)$, where $\perp \leq_o \top \leq_o I_k$.

Theorem 2 *An n-ary operation F on A_ω is definable in $\{\sim, \otimes, \perp, \vee\}$ by a formula ψ in the distinct propositional variables $p_1, \ldots, p_n$ iff it satisfies the symmetry condition.*[5]

Proof: It is easy to see that every operation which is definable in $\{\sim, \otimes, \perp, \vee\}$ satisfies the symmetry condition. For the converse, assume that the n-ary operation F satisfies this condition. This implies that the sets $\{\vec{x} \in A_\omega^n \mid F(\vec{x}) = \top\}$, $\{\vec{x} \in A_\omega^n \mid F(\vec{x}) \in I(1)\}$, and $\{\vec{x} \in A_\omega^n \mid \vec{x}$ is similar to $\vec{a}\}$ (where $\vec{a} \in A_\omega^n$) all satisfy the condition from Lemma 11, and so they are characterizable in $\mathcal{IL}^\perp$ by formulas $\psi_\top$, ψ_I and $\psi_{\vec{a}}$ (respectively). By moving if necessary from ψ to $\top \otimes \psi$ we may assume that all these formulas take values only in $\{\top, \perp\}$. The symmetry condition entails also that if $F(\vec{a}) = I_k$ then $I_k = a_{i(\vec{a})}$ for some $1 \leq i(\vec{a}) \leq n$ [6], and that $F(\vec{x}) = x_{i(\vec{a})} = x_{i(\vec{x})}$ for every $\vec{x}$ which is similar to $\vec{a}$. This entails that $F = g(\psi_F)$ where

$$\psi_F = \psi_\top \vee (\psi_I \wedge \vee_{\vec{a} \in A_n^n \cap \{\vec{x} \in A_\omega^n \mid F(\vec{x}) \in I(1)\}} (\psi_{\vec{a}} \wedge p_{i(\vec{a})}))$$

Indeed, if $F(\vec{x}) = \top$ then $g_{\psi_\top}(\vec{x}) = \top$, and so $g_{\psi_F}(\vec{x}) = \top$. If $F(\vec{x}) = \perp$ then $g_{\psi_\top}(\vec{x}) = \perp$ and $g_{\psi_I}(\vec{x}) = \perp$, and so $g_{\psi_F}(\vec{x}) = \perp$. Finally, if $F(\vec{x}) = x_{i(\vec{x})} \in I(1)$ then $g_{\psi_\top}(\vec{x}) = \perp$, $g_{\psi_I}(\vec{x}) = \top$,

[5] Note that this theorem is valid only in the infinite case, but *not* in A_k for $k < \omega$!

[6] This is the step in the proof which fails for A_k in case k is finite!

$g_{\psi_{\vec{a}} \wedge p_{i(\vec{a})}}(\vec{x}) = \perp$ for every $\vec{a} \in A_n^n \cap \{\vec{x} \in A_\omega^n \mid F(\vec{x}) \in I(1)\}$ which is not similar to $\vec{x}$, while $g_{\psi_{\vec{a}} \wedge p_{i(\vec{a})}}(\vec{x}) = x_{i(\vec{a})} = x_{i(\vec{x})}$ for every $\vec{a} \in A_n^n \cap \{\vec{x} \in A_\omega^n \mid F(\vec{x}) \in I(1)\}$ which *is* similar to $\vec{x}$. It follows that $g_{\psi_F}(\vec{x}) = x_{i(\vec{x})} = F(\vec{x})$ in this case as well.

Note Using where necessary $\top_n$ instead of $\top$, one can prove in a similar way that the n-ary operations F which are definable in $\{\sim, \otimes, \vee\}$ by a formula ψ such that $A(\psi) = \{p_1, \ldots, p_n\}$, are precisely those which satisfy the symmetry condition as well as the condition $F(I_k^n) = I_k$.

Similar proofs can be used to show that an operation F on A_1 is definable in $\{\sim, \otimes, \perp, \vee\}$ iff it is *classically closed* (i.e.: if $\{x_1, , \ldots, , x_n\} \subseteq \{\top, \perp\}$, then $F(x_1, , \ldots, , x_n) \in \{\top, \perp\}$), and that such F is definable in $\{\sim, \otimes, \vee\}$ iff it is both classically closed and *free* (i.e.: $F(I^n) = I$).

3 Corresponding Proof Systems

3.1 Gentzen-type Systems

We use simplified versions of the Gentzen-type calculi introduced in [3] and [9]. Unlike the systems there, we employ here sequents of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite *sets* (rather than sequences or multisets) of formulas (so that the structural rules of contraction, its converse, and permutation are all built in). As usual, we write Γ, Δ and Γ, φ instead of $\Gamma \cup \Delta$ and $\Gamma \cup \{\varphi\}$ (respectively). We use P, Q, p, q as metavariables for propositional variables (i.e.: atomic formulas other than $\perp$), φ, ψ, A, B, C as variables for arbitrary formulas, and s as a variable for sequents.

Our main system use hypersequents as the main data structure. We start however with the following ordinary sequential calculus from [3], on which our main system is based.

THE SYSTEM $GRMI_m$ [7]

Axioms:

$$P \Rightarrow P \qquad\qquad \perp, \Gamma \Rightarrow \Delta$$

Logical rules:

$$(\neg \Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\quad (\Rightarrow \neg)$$

$$(\otimes \Rightarrow)\quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \otimes \psi \Rightarrow \Delta} \quad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \Gamma_2 \Rightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \otimes \psi}\quad (\Rightarrow \otimes)$$

Note In the axiom for $\perp$ we may (and will) assume that Γ and Δ are sets of *atomic* formulas.

[7] This system was called $GRMI_m^\perp$ in [9], while the name $GRMI_m$ was used there for the fragment without $\perp$.

THE SYSTEM $\boldsymbol{GRM}_{\mathbf{m}}$ is defined like $GRMI_m$, but its axioms are the sequents of the form $\Gamma \Rightarrow \Gamma$ where Γ is a finite set of atomic formulas.

Notes

1. It is important to note that all the rules of this system are *multiplicative* (or *pure*, in the terminology of [6]). This means that from any correct application of a rule one can get another correct application of that rule by adding arbitrary finite sets to one side of the premises (*not* necessarily the same set to each premise!) and the union of these sets to the same side of the conclusion. In other words: the rules are context-free.

2. It is easy to show that both $GRMI_m$ and GRM_m are closed under substitutions. Hence we could have taken the axioms of $GRMI_m$ to be $\varphi \Rightarrow \varphi$ for every φ, and $\bot, \Gamma \Rightarrow \Delta$ for every finite sets Γ, Δ of *arbitrary* formulas (and in the case of GRM_m also $\Gamma \Rightarrow \Gamma$ for every Γ).

3. It is easy to see that the derived rules for $\rightarrow$ and $+$ in both systems are the standard multiplicative versions of the classical rules for implication and disjunction (respectively). Thus the rules for $+$ are just the duals of the rules for $\otimes$.

4. Another rule that was taken in [3] as primitive is the cut rule. It is easy however to use Gentzen's method from [15] to show that it is eliminable in both systems (See [3]). Below we shall present a new, semantic proof of this fact (see Corollary 8).

Notation $At(E)$ denotes the sets of atomic formulas (i.e.: atomic variables or $\bot$) which occur in E (here E can be a formula, a sequent, or a hypersequent).

Lemma 14 *$GRMI_m$ and GRM_m are closed under the* strong expansion *rule: If $\Gamma \Rightarrow \Delta$ is provable and $At(\varphi) \subseteq At(\Gamma \Rightarrow \Delta)$ then also $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi$ are provable.*

Proof: By induction on the complexity of φ. The base case (where φ is atomic) is done by an inner induction on the length of the proof of $\Gamma \Rightarrow \Delta$. The base case of this inner induction uses the special form of the axioms of $GRMI_m$, while both induction steps (that of the inner induction and that of the main one) rely on the multiplicativity of the rules.

Lemma 15

1. *$GRMI_m$ is closed under the* strong mingle *rule: If $At(\Gamma_1 \Rightarrow \Delta_1) \cap At(\Gamma_2 \Rightarrow \Delta_2) \neq \emptyset$, and both $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ are provable in $GRMI_m$, then so is $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.*

2. *GRM_m is closed under the* mix *(or combining) rule: If both $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ are provable in GRM_m, then so is $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.*

Proof: By induction on the sum of the lengths of the proofs of $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ (again the special axioms of $GRMI_m$ and GRM_m are used for the base case, while the multiplicativity of the rules is used for the induction step).

In [3] (and in Corollary 8 below) it is proved that $GRMI_m$ is *weakly* sound and complete with respect to $\mathcal{A}_\omega$, in the sense that $\vdash_{\mathcal{A}_\omega} \varphi$ iff $\vdash_{GRMI_m} \Rightarrow \varphi$. The same is true for GRM_m with respect to $\mathcal{A}_1$. Neither system is strongly complete, though (see [9]). Thus $\varphi \otimes \psi \vdash_{\mathcal{A}_\omega} \varphi$, but $\not\vdash_{GRM_m} \varphi \otimes \psi \Rightarrow \varphi$. In order to get strong completeness we need (like in [9]) to use calculi of hypersequents. A *hypersequent* is a finite *multiset* of ordinary sequents. The elements of this multiset are called its *components*. We denote by $s_1 \mid \cdots \mid s_n$ the hypersequent whose components are $s_1, \ldots, s_n$, and use G a a variable for (possibly empty) hypersequents [8].

Definition 10 The n-part of a hypersequential calculus or a logic is the fragment in which only hypersequents with at most n components are allowed.

THE SYSTEM $GSRMI_m$

Axioms:

$$P \Rightarrow P$$

Logical rules:

$$(\neg \Rightarrow)\quad \frac{G \mid \Gamma \Rightarrow \Delta, \varphi}{G \mid \neg\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{G \mid \varphi, \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta, \neg\varphi}\quad (\Rightarrow \neg)$$

$$(\otimes \Rightarrow)\quad \frac{G \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{G \mid \Gamma, \varphi \otimes \psi \Rightarrow \Delta} \qquad \frac{G \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad G \mid \Gamma_2 \Rightarrow \Delta_2, \psi}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \otimes \psi}\quad (\Rightarrow \otimes)$$

Structural rules:

$$\frac{G \mid s \mid s}{G \mid s}$$

$$\frac{G \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \qquad G \mid \varphi, \Gamma_2 \Rightarrow \Delta_2}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'}$$

[8] Hypersequents were first introduced by Pottinger in [19], and independently in [5]. Related structures were used before by Mints (see [18]).

(external contraction, cut, and strong splitting, respectively).

THE SYSTEM $\boldsymbol{GSRM_m}$: Similar to $GSRMI_m$, but with axioms like in GRM_m.

Note Unlike in [9], we use here the externally *additive* versions of the rules which have more than one premise (this means that both premises have the same inactive side-hypersequent G)[9]. This is equivalent to the externally multiplicative versions of [9], because of the presence of external contraction and external weakening (the latter, which allows to infer $G \mid s$ from G, is a special case of strong splitting). We shall show that this formulation of the rules makes the problematic rule of external contraction superfluous.

The semantics of these hypersequential calculi is given in the next definition.

Definition 11

1. A valuation v is a *model* in $\mathcal{A}_\omega$ of a sequent $\Gamma \Rightarrow \Delta$ if either $v(\varphi) = \perp$ for some $\varphi \in \Gamma$, or $v(\psi) = \top$ for some $\psi \in \Delta$, or $\Gamma \Rightarrow \Delta$ is not empty and there exists k such that $v(\varphi) = I_k$ for all $\varphi \in \Gamma \cup \Delta$, or $\Gamma \Rightarrow \Delta$ is empty and there exists k such that $v(P) = I_k$ for some atomic P (i.e.: v is a model of the empty sequent iff it is not a classical valuation).

2. A valuation v is a *model* of a hypersequent G in $\mathcal{A}_\omega$ ($v \models_{\mathcal{A}_\omega} G$) if v is a model in $\mathcal{A}_\omega$ of at least one of the components of G.

3. A hypersequent G is valid in $\mathcal{A}_\omega$ ($\vdash^h_{\mathcal{A}_\omega} G$) if every valuation in $\mathcal{A}_\omega$ is a model of G.

4. The concepts of model and validity in $\mathcal{A}_1$ are defined similarly.

Note It is easy to see that $\Rightarrow \varphi$ is valid in $\mathcal{A}_\omega$ ($\mathcal{A}_1$) according to Definition 11 iff φ is.

Theorem 3 (Soundness Theorem)

1. *The axioms of $GSRMI_m$ ($GSRM_m$) are valid in $\mathcal{A}_\omega$ ($\mathcal{A}_1$), and all its rules are* truth *preserving: every model of all the premises of a rule is also a model of its conclusion.*

2. *If $\vdash_{GSRMI_m} G$ then $\vdash^h_{\mathcal{A}_\omega} G$. If $\vdash_{GSRM_m} G$ then $\vdash^h_{\mathcal{A}_1} G$.*

Proof: The second part is immediate from the first, while the proof of the first is straightforward. We only note that the cut rule is nontrivially sound here even in case the resulting component is empty. Indeed, v can be a model of both $\Rightarrow \varphi$ and $\varphi \Rightarrow$ only if $v(\varphi) = I_k$ for some k. Such v is not a classical valuation, and so it is a model of $\Rightarrow$ as well.

[9]Note that *internally* cut and $(\Rightarrow \otimes)$ still have a multiplicative form!

3.2 Completeness and Cut Elimination

The main result of this section is the following:

Theorem 4 *A hypersequent G is valid in $\mathcal{A}_\omega$ iff G has a proof in $GSRMI_m$ in which the cut rule and the external contraction rule are not used.*

This subsection is mainly devoted to a proof of this theorem. For convenience, in the rest of it $\vdash G$ means that G has a a proof in $GSRMI_m$ in which cut and external contraction are not used.

Definition 12 A hypersequent $\Gamma'_1 \Rightarrow \Delta'_1 \mid \ldots \mid \Gamma'_n \Rightarrow \Delta'_n$ *relevantly extends* the hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ if for all $1 \leq i \leq n$ we have that $\Gamma_i \subseteq \Gamma'_i$, $\Delta_i \subseteq \Delta'_i$, and every formula in $\Gamma'_i \Rightarrow \Delta'_i$ is a subformula of some formula in $\Gamma_i \Rightarrow \Delta_i$.

Lemma 16 *Relevant extension is a transitive relation: if G_1 relevantly extends G_2, and G_2 relevantly extends G_3, then G_1 relevantly extends G_3.*

Lemma 17 *A model of a hypersequent G is also a model of every relevant extension of G.*

Proof: Let v be a model of G and let G' be a relevant extension of G. Then v is a model of some component $\Gamma_i \Rightarrow \Delta_i$ of G. If $v(\psi) = \bot$ for some $\psi \in \Gamma_i$, or $v(\psi) = \top$ for some $\psi \in \Delta_i$, then the same is true for the corresponding component $\Gamma'_i \Rightarrow \Delta'_i$ of G'. If $v(\psi) = I_k$ for all formulas of $\Gamma_i \Rightarrow \Delta_i$, or if $\Gamma_i \Rightarrow \Delta_i$ is empty, then the same is again true for $\Gamma'_i \Rightarrow \Delta'_i$, since it consists only of subformulas of formulas in $\Gamma_i \Rightarrow \Delta_i$. In either case v is also a model of $\Gamma'_i \Rightarrow \Delta'_i$ and so of G'.

Notation Let $\Gamma_i \Rightarrow \Delta_i$ be a component of the hypersequent G. Denote by G_i the hypersequent which is obtained from G by deleting $\Gamma_i \Rightarrow \Delta_i$ (and so $G = \Gamma_i \Rightarrow \Delta_i \mid G_i$ up to the order of the components. Note also that G_i may be empty).

Definition 13 Let G be a hypersequent such that $\not\vdash G$. G is called *saturated* if every component $\Gamma_i \Rightarrow \Delta_i$ of G satisfies the following conditions:

(i) If $\neg\varphi \in \Gamma_i$ then $\varphi \in \Delta_i$

(ii) If $\neg\varphi \in \Delta_i$ then $\varphi \in \Gamma_i$

(iii) If $\varphi \otimes \psi \in \Gamma_i$ then $\varphi \in \Gamma_i$ and $\psi \in \Gamma_i$

(iv) If $\varphi \otimes \psi \in \Delta_i$ and $\not\vdash \Gamma_i \Rightarrow \Delta_i, \varphi \mid G_i$ then $\varphi \in \Delta_i$

(v) If $\varphi \otimes \psi \in \Delta_i$ and $\not\vdash \Gamma \Rightarrow \Delta_i, \psi \mid G_i$ then $\psi \in \Delta_i$.

Lemma 18 *If $\not\vdash G$ then G has an unprovable, saturated relevant extension.*

Proof: If $\not\vdash G$ and G is not saturated then it is possible to properly and relevantly extend G without making the new hypersequent provable (this is obvious and standard if one of the conditions (i)–(iii) is violated by some component $\Gamma \Rightarrow \Delta$ of G, and is trivial in the special cases (iv)–(v)). Since G has only finitely many subformulas, this process must stop by lemma 16 with a saturated sequent which relevantly extends G.

Lemma 19 *Every unprovable saturated hypersequent has a countermodel in $\mathcal{A}_\omega$.*

Proof: Let $G = \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ be an unprovable saturated sequent. Define:

$$\Gamma = \bigcup_{i=1}^{n} \Gamma_i \quad \Delta = \bigcup_{i=1}^{n} \Delta_i$$

$$I(G) = \{p \in At(G) \mid p \in \Gamma \cap \Delta\}$$

$$R = \{\langle p, q\rangle \in I(G)^2 \mid \exists \Gamma' \subseteq \Gamma \exists \Delta' \subseteq \Delta. \vdash_{GRMI_m} \Gamma' \Rightarrow \Delta' \text{ and } \{p,q\} \subseteq At(\Gamma' \Rightarrow \Delta')\}$$

We first show that R is an equivalence relation. That R is reflexive follows immediately from the definition of $I(G)$, and the symmetry of R is trivial. It remains to show that R is transitive. So assume that pRq and qRr. Then there exist $\Gamma', \Delta', \Gamma'', \Delta''$ such that $\Gamma', \Gamma' \subseteq \Gamma$, $\Delta', \Delta'' \subseteq \Delta$, $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta'$, $\vdash_{GRMI_m} \Gamma'' \Rightarrow \Delta''$, $\{p,q\} \subseteq At(\Gamma' \Rightarrow \Delta')$, and $\{q,r\} \subseteq At(\Gamma'' \Rightarrow \Delta'')$. Since q belongs to both $At(\Gamma' \Rightarrow \Delta')$ and $At(\Gamma'' \Rightarrow \Delta'')$, Lemma 15 entails that $\vdash_{GRMI_m} \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''$. But $\{p,r\} \subseteq \{p,q,r\} \subseteq At(\Gamma', \Gamma'' \Rightarrow \Delta', \Delta'')$. Hence pRr.

Let $C_1, \ldots, C_\ell$ be the equivalence classes of R (in some order.) Obviously, ℓ has at most the cardinality of $At(G)$. From the proof of the transitivity of R it also easily follows that for every $1 \leq i \leq \ell$ there exist $\Gamma_i \subseteq \Gamma, \Delta_i \subseteq \Delta$ such that $\vdash_{GRMI_m} \Gamma_i \Rightarrow \Delta_i$ and $C_i \subseteq At(\Gamma_i \Rightarrow \Delta_i)$.

We now define a countermodel v of G in $\mathcal{A}_\ell$ (and so in $\mathcal{A}_\omega$) as follows:

$$v(p) = \begin{cases} I_i & p \in C_i \\ \top & p \in \Gamma, p \notin \Delta \\ \bot & p \notin \Gamma \end{cases}$$

To show that v indeed refutes G, we first show by induction on the complexity of φ that if $\varphi \in \Gamma$ then $v(\varphi) \neq \bot$, and if $\varphi \in \Delta$ then $v(\varphi) \neq \top$. This is obvious in case φ is atomic (including the case $\varphi = \bot$, by the special axiom for $\bot$ and the fact that G is unprovable). In case $\varphi = \neg\psi$ the claim follows easily from the induction hypothesis and conditions (i)–(ii) in the definition of a saturated sequent (Definition 13). If $\varphi = \psi_1 \otimes \psi_2$ and $\varphi \in \Gamma$ then the claim follows from the induction hypothesis concerning ψ_1 and ψ_2 and condition (iii) of Definition 13. Finally assume that $\varphi = \psi_1 \otimes \psi_2$ and $\varphi \in \Delta$. So $\varphi \in \Delta_i$ for some i. Had both $\Gamma_i \Rightarrow \Delta_i, \psi_1 \mid G_i$ and $\Gamma_i \Rightarrow \Delta_i, \psi_2 \mid G_i$

been provable, so would have been G (using our externally additive version of $(\Rightarrow \otimes)$, and the fact that $\varphi \in \Delta_i$). Hence one of these sequents is unprovable. Assume, e.g., that $\not\vdash \Gamma_i \Rightarrow \Delta_i, \psi_1 \mid G_i$. Then $\psi_1 \in \Delta_i$ by condition (iv) of Definition 13. Hence $v(\psi_1) \neq \top$ by induction hypothesis. If $v(\psi_1) = \bot$ then $v(\varphi) = \bot \neq \top$. Assume, therefore, that $v(\psi_1) = I_k$ for some k. Then $\bot \notin At(\psi_1)$, and $v(P) = I_k$ for every $P \in At(\psi_1)$. Hence $At(\psi_1) \subseteq C_k$. By the observation above concerning C_k there exist $\Gamma'_j \subseteq \Gamma_j$, $\Delta'_j \subseteq \Delta_j$ $(j = 1, \ldots, n)$ such that $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta'$ and $At(\psi_1) \subseteq C_k \subseteq At(\Gamma' \Rightarrow \Delta')$, where $\Gamma' = \bigcup_{j=1}^{n} \Gamma'_j, \Delta' = \bigcup_{j=1}^{n} \Delta'_j$. Hence $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta', \psi_1$ by Lemma 14. Using the strong splitting rule of $GSRMI_m$, this implies that $\vdash \Gamma'_i \Rightarrow \Delta'_i, \psi_1 \mid G_i$. It is not possible therefore that $\vdash \Gamma_i \Rightarrow \Delta_i, \psi_2 \mid G_i$, since otherwise we would have (using again $(\Rightarrow \otimes)$, and the facts that $\Gamma'_i \subseteq \Gamma_i$, $\Delta'_i \subseteq \Delta_i$, and $\varphi \in \Delta_i$) that $\vdash G$. It follows by condition (v) of Definition 13 that $\psi_2 \in \Delta_i$, and so $v(\psi_2) \neq \top$ by induction hypothesis. If $v(\psi_2) = \bot$ then again $v(\varphi) = \bot \neq \top$. Assume therefore that $v(\psi_2) = I_m$ for some m. Then again $At(\psi_2) \subseteq C_m$, and there exist $\Gamma''_j \subseteq \Gamma_j$, $\Delta''_j \subseteq \Delta_j$ $(j = 1, \ldots, n)$ such that $\vdash_{GRMI_m} \Gamma'' \Rightarrow \Delta''$ and $At(\psi_2) \subseteq C_m \subseteq At(\Gamma'' \Rightarrow \Delta'')$, where $\Gamma'' = \bigcup_{j=1}^{n} \Gamma''_j, \Delta'' = \bigcup_{j=1}^{n} \Delta''_j$. Hence $\vdash_{GRMI_m} \Gamma'' \Rightarrow \Delta'', \psi_2$ by Lemma 14. This and the fact that $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta', \psi_1$ entail that $\vdash_{GRMI_m} \Gamma', \Gamma'' \Rightarrow \Delta', \Delta'', \varphi$. Since $\varphi \in \Delta$, this fact entails that pRq for every $p, q \in At(\varphi)(= At(\psi_1) \cup At(\psi_2))$. It follows that $C_k = C_m$ and so $I_k = I_m$ and $v(\varphi) = I_k \otimes I_k = I_k \neq \bot$.

Next we observe that if $p \in I(G)$ then $G \mid\Rightarrow$ is derivable from the axiom $p \Rightarrow p$ using strong splittings. It follows that if the empty sequent is a component of G than $I(G)$ is empty, and so $v(p) \in \{\top, \bot\}$ for all p. Hence v refutes the empty sequent in case it is a component of G.

To show that v is a countermodel of G it remains now only to eliminate the possibility that there exists $1 \leq i \leq n$ and $1 \leq k \leq \ell$ such that $v(\varphi) = I_k$ for all $\varphi \in \Gamma_i \cup \Delta_i$. Well, had there been such i and k we would have that $v(P) = I_k$ for all $P \in At(\Gamma_i \Rightarrow \Delta_i)$, and so $At(\Gamma_i \Rightarrow \Delta_i) \subseteq C_k$ (note that $\bot \notin At(\Gamma_i \Rightarrow \Delta_i)$ in such a case!). Hence there would have been $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ such that $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta'$ and $At(\Gamma_i \Rightarrow \Delta_i) \subseteq C_k \subseteq At(\Gamma' \Rightarrow \Delta')$. Lemma 14 would have implied then that $\vdash_{GRMI_m} \Gamma_i, \Gamma' \Rightarrow \Delta_i, \Delta'$. From this it is possible to derive G using strong splittings (Since $\Gamma_i \Rightarrow \Delta_i$ is a component of G and $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$). A contradiction.

Proof of Theorem 4: The "if" part is Theorem 3. The "only if" part is immediate from Lemmas 17, 18, and 19.

Theorem 5 *A hypersequent G is valid in $\mathcal{A}_1$ iff G has a proof in $GSRM_m$ in which the cut rule and the external contraction rule are not used.*

Proof: The proof is similar to that of $GSRMI_m$. The main difference is that the form of the axioms of GRM_m implies that the equivalence relation R used in the proof of lemma 18 has at most one equivalence class (If p and q are in $I(G)$ then $\{p,q\} \subseteq \Gamma$, $\{p,q\} \subseteq \Delta$ and $\vdash_{GRM_m} p,q \Rightarrow p,q$). Hence $\ell \leq 1$ and the countermodel we get is actually in A_1.

Note It is not necessary, of course, to introduce R at all if one proves the completeness of $GSRM_m$ directly, and the proof is therefore simpler than in the case of $GSRMI_m$.

Corollary 5 *The cut elimination theorem is valid for $GSRMI_m$ and $GSRM_m$. Moreover: if a hypersequent is provable in either of these systems then it has a proof there in which the cut rule and the external contraction rule are not used.*

Proof: This follows from Theorems 3, 4, and 5.

Corollary 6 *The n-part (Definition 10) of $\vdash^h_{\mathcal{A}_\omega}$ ($\vdash^h_{\mathcal{A}_1}$) is identical to the (external contraction and cut free) n-part of $GSRMI_m$ ($GSRM_m$).*

A close examination of the proof of the completeness theorem reveals that this Corollary can be strengthened as follows:

Corollary 7 *Let $GSRMI_m(n)$ ($GSRM_m(n)$) be the system for hypersequents with n components which has as rules the logical rules of $GSRMI_m$ (not including cut!) and as axioms the hypersequents with n components which can be derived from a theorem of $GSRMI_m$ ($GSRM_m$) using strong splittings. Then a hypersequent G with n components is valid in $\mathcal{A}_\omega$ ($\mathcal{A}_1$) iff it has a proof in $GSRMI_m(n)$ ($GSRM_m(n)$).*

Corollary 8 *A sequent is valid in $\mathcal{A}_\omega$ ($\mathcal{A}_1$) iff it has a (cut-free) proof in $GRMI_m$ (GRM_m). Hence the cut rule is admissible in $GRMI_m$ and GRM_m.*

Corollary 9 *$GSRMI_m$ ($GSRM_m$) is a conservative extension of $GRMI_m$ (GRM_m).*

Note The cut-elimination part of Corollary 5 was first stated (with a hint for a very complicated syntactical proof) in [9]. This is the first time it is given a real (and much simpler) proof. That external contraction can also be eliminated is a new result. Corollaries 9 and 8 were first proved in [9] and [3] (respectively).

3.3 Compactness and Characterizations of the Consequence Relations

We turn now to a proof-theoretical characterization of $\vdash_{\mathcal{A}_\omega}$ and $\vdash_{\mathcal{A}_1}$ using our hypersequential calculi (this, recall, was the main motivation for introducing these calculi, because for characterizing the logically valid formulas the purely sequential fragments suffice).

The most standard way of using a sequential calculus G for defining a (Tarskian) consequence relation is to let $\mathcal{T} \vdash_G \varphi$ iff there exists a finite $\Gamma \subseteq \mathcal{T}$ such that $\vdash_G \Gamma \Rightarrow \varphi$. This method is not applicable here, because $p \otimes q \vdash_{\mathcal{A}_\omega} p$, but neither $p \otimes q \Rightarrow p$ nor $\Rightarrow p$ is provable even in $GSRM_m$ (since both are not valid in $\mathcal{A}_1$). Another common way to use G for this purpose is to let $\mathcal{T} \vdash^e_G \varphi$ if the sequent $\Rightarrow \varphi$ is derivable in G from the set of sequents $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\}$. This method *does* work for $GSRMI_m$ and $GSRM_m$, since $\vdash^e_{GSRMI_m} = \vdash_{\mathcal{A}_\omega}$ and $\vdash^e_{GSRM_m} = \vdash_{\mathcal{A}_1}$ (see Corollary 13 below). It is not very useful, though, so a better one should be sought. Now in classical logic $\Rightarrow \varphi$ is derivable from $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\}$ iff the set $\{\varphi \Rightarrow\} \cup \{\Rightarrow \psi \mid \psi \in \mathcal{T}\}$ is not satisfiable, i.e.: there is no valuation v which assigns $\top$ to all sequents of this set. This characterization is based on the role of $\top$ (and of principle (T)). An equivalent characterization, based on the role of $\bot$ (which is more natural here), is that there is no valuation v which assigns $\bot$ to all elements of $\{\Rightarrow \varphi\} \cup \{\psi \Rightarrow \mid \psi \in \mathcal{T}\}$. This formulation relies, however, on the assumption that $\psi \Rightarrow$ can be true only if ψ is not true. This is a variant of principle (T) which fails in our framework. This can be remedied by using instead the (classically equivalent) condition that there is no valuation v which assigns $\bot$ to all elements of $\{\Rightarrow \varphi\} \cup \{\psi \Rightarrow \varphi \mid \psi \in \mathcal{T}\}$. This characterization *can* be used in the case of $\vdash_{\mathcal{A}_\omega}$ and $\vdash_{\mathcal{A}_1}$. This line of thought leads to the following definitions and propositions:

Definition 14 A set S of sequents is called *negatively satisfiable* (*n-satisfiable* in short) in $\mathcal{A}_\omega$ ($\mathcal{A}_1$) if there exists a valuation v there which is not a model of any element of S.

Proposition 1 *A finite set* $\{s_1, \ldots, s_n\}$ *of sequents is n-satisfiable in* $\mathcal{A}_\omega$ ($\mathcal{A}_1$) *iff the hypersequent* $s_1 \mid \ldots \mid s_n$ *is not valid there.*

Proposition 2 *For a theory* $\mathcal{T}$ *and a formula* φ *let* $S_{\mathcal{T},\varphi} = \{\Rightarrow \varphi\} \cup \{\psi \Rightarrow \varphi \mid \psi \in \mathcal{T}\}$. *Then* $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi$ ($\mathcal{T} \vdash_{\mathcal{A}_1} \varphi$) *iff* $S_{\mathcal{T},\varphi}$ *is not n-satisfiable in* $\mathcal{A}_\omega$ ($\mathcal{A}_1$).

The two propositions easily follow from the relevant definitions. Together they yield the following characterization of $\vdash_{\mathcal{A}_\omega}$ ($\vdash_{\mathcal{A}_1}$) in the case of finite theories:

Proposition 3 $\{\psi_1, \ldots, \psi_n\} \vdash_{\mathcal{A}_\omega} \varphi$ *iff the hypersequent* $\Rightarrow \varphi \mid \psi_1 \Rightarrow \varphi \mid \ldots \mid \psi_n \Rightarrow \varphi$ *is valid in* $\mathcal{A}_\omega$. *A similar result holds for* $\mathcal{A}_1$.

Another, equivalent characterization is:

Corollary 10 *If* Γ *is finite then* $\Gamma \vdash_{\mathcal{A}_\omega} \varphi$ ($\Gamma \vdash_{\mathcal{A}_1} \varphi$) *iff* $\Gamma \Rightarrow \varphi \mid \Rightarrow \varphi$ *is valid in* $\mathcal{A}_\omega$ ($\mathcal{A}_1$).

Proof: It is easy to see that $\Rightarrow \varphi \mid \psi_1 \Rightarrow \varphi \mid \ldots \mid \psi_n \Rightarrow \varphi$ is valid in $\mathcal{A}_\omega$ iff $\psi_1, \ldots, \psi_n \Rightarrow \varphi \mid \Rightarrow \varphi$ is valid there. The same applies to $\mathcal{A}_1$.

In order to generalize these characterizations to arbitrary theories we need the following

Theorem 6 (Compactness Theorem) *Let S be a set of sequents such that every finite subset of S is n-satisfiable in $\mathcal{A}_\omega$ ($\mathcal{A}_1$). Then S itself is n-satisfiable there.*

Proof: We do here the case of $\mathcal{A}_\omega$. We may assume without a loss in generality that S is a maximal set of sequents with the property that every finite subset of it is n-satisfiable. Hence by Proposition 1 a sequent s is not in S iff there exist $s_1, \dots, s_k \in S$ such that $\vdash^h_{\mathcal{A}_\omega} s_1 \mid \dots \mid s_k \mid s$ (while there exist no $s_1, \dots, s_k \in S$ such that $\vdash^h_{\mathcal{A}_\omega} s_1 \mid \dots \mid s_k$).

Let $S = \{\Gamma_\alpha \Rightarrow \Delta_\alpha \mid \alpha \in I\}$. The construction of a valuation v which is a countermodel in A_ω of all the sequents of S is similar to that in the proof of Lemma 19. We define:

$$\Gamma = \bigcup_{\alpha \in I} \Gamma_\alpha \quad \Delta = \bigcup_{\alpha \in I} \Delta_\alpha$$

$$I(S) = \{p \in At(S) \mid p \in \Gamma \cap \Delta\}$$

$$R = \{\langle p, q\rangle \in I(S)^2 \mid \exists \Gamma' \subseteq \Gamma \exists \Delta' \subseteq \Delta. \vdash_{GRMI_m} \Gamma' \Rightarrow \Delta' \text{ and } \{p, q\} \subseteq At(\Gamma' \Rightarrow \Delta')\}$$

Again R is an equivalence relation. Let $C_1, C_2, \dots$ be the equivalence classes of R in some order (the set of equivalence classes may be finite or countable). Again it can easily be proved that if C'_i is a finite subset of C_i then there exist $\Gamma_i \subseteq \Gamma, \Delta_i \subseteq \Delta$ such that $\vdash_{GRMI_m} \Gamma_i \Rightarrow \Delta_i$ and $C'_i \subseteq At(\Gamma_i \Rightarrow \Delta_i)$.

We define now our countermodel v exactly as in the proof of Lemma 19, and again first show by induction on the complexity of φ that if $\varphi \in \Gamma$ then $v(\varphi) \neq \bot$, and if $\varphi \in \Delta$ then $v(\varphi) \neq \top$. As before, this is obvious in case φ is atomic. Assume next that $\varphi = \neg\psi$ and $\varphi \in \Gamma$. Then there exists $\alpha \in I$ s.t. $\varphi \in \Gamma_\alpha$. Assume that $\Gamma_\alpha \Rightarrow \psi, \Delta_\alpha$ is not in S. Then there exist $s_1, \dots, s_k \in S$ such that $\vdash^h_{\mathcal{A}_\omega} s_1 \mid \dots \mid s_k \mid \Gamma_\alpha \Rightarrow \psi, \Delta_\alpha$. Since $\neg\psi \in \Gamma_\alpha$, this entails that $\vdash^h_{\mathcal{A}_\omega} s_1 \mid \dots \mid s_k \mid \Gamma_\alpha \Rightarrow \Delta_\alpha$. This contradicts the basic property of S. It follows that $\psi \in \Delta$, and so $v(\psi) \neq \top$ by the induction hypothesis. Hence $v(\varphi) \neq \bot$. The cases where $\varphi = \neg\psi$ and $\varphi \in \Delta$ and where $\varphi = \psi_1 \otimes \psi_2$ and $\varphi \in \Gamma$ are similarly handled. Finally assume that $\varphi = \psi_1 \otimes \psi_2$ and $\varphi \in \Delta$. So $\varphi \in \Delta_\alpha$ for some $\alpha \in I$. It is impossible that both $\Gamma_\alpha \Rightarrow \Delta_\alpha, \psi_1$ and $\Gamma_\alpha \Rightarrow \Delta_\alpha, \psi_2$ are not in S, since in such a case there would exist $s_1, \dots, s_k \in S$ such that $\vdash^h_{\mathcal{A}_\omega} s_1 \mid \dots \mid s_k \mid \Gamma_\alpha \Rightarrow \Delta_\alpha, \psi_1$ and $\vdash^h_{\mathcal{A}_\omega} s_1 \mid \dots \mid s_k \mid \Gamma_\alpha \Rightarrow \Delta_\alpha, \psi_2$, and this implies that $\vdash^h_{\mathcal{A}_\omega} s_1 \mid \dots \mid s_k \mid \Gamma_\alpha \Rightarrow \Delta_\alpha$ (since $\varphi = \psi_1 \otimes \psi_2 \in \Delta_\alpha$). A contradiction. Assume, accordingly, that $\Gamma_\alpha \Rightarrow \Delta_\alpha, \psi_1$ (say) is in S. Then $\psi_1 \in \Delta$, and so $v(\psi_1) \neq \top$ by induction hypothesis. If $v(\psi_1) = \bot$ then $v(\varphi) = \bot \neq \top$. Assume, therefore, that $v(\psi_1) = I_k$ for some k. Then $\bot \notin At(\psi_1)$, and $v(P) = I_k$ for every $P \in At(\psi_1)$. Hence $At(\psi_1) \subseteq C_k$, and so there exist $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ such that $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta'$ and $At(\psi_1) \subseteq At(\Gamma' \Rightarrow \Delta')$. Hence $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta', \psi_1$ by Lemma 14. By the soundness of strong splitting, this implies that there exist $s'_1, \dots, s'_k \in S$ such that $\vdash^h_{\mathcal{A}_\omega} s'_1 \mid \dots \mid s'_k \mid \Rightarrow \psi_1$. It is impossible therefore that $\Gamma_\alpha \Rightarrow \Delta_\alpha, \psi_2$ is not in S, since in such a case there would exist $s''_1, \dots, s''_l \in S$ such that $\vdash^h_{\mathcal{A}_\omega} s''_1 \mid \dots \mid s''_l \mid \Gamma_\alpha \Rightarrow \Delta_\alpha, \psi_2$, and together with $\vdash^h_{\mathcal{A}_\omega} s'_1 \mid \ldots \mid s'_k \mid \Rightarrow \psi_1$ we would get that $\vdash^h_{\mathcal{A}_\omega} s'_1 \mid \ldots \mid s'_k \mid s''_1 \mid \ldots \mid s''_l \mid \Gamma_\alpha \Rightarrow \Delta_\alpha$ (since $\varphi = \psi_1 \otimes \psi_2 \in \Delta_\alpha$). A contradiction. It follows that $\psi_2 \in \Delta$, and so $v(\psi_2) \neq \top$ by induction hypothesis. If $v(\psi_2) = \bot$ then again $v(\varphi) = \bot \neq \top$. Assume therefore that $v(\psi_2) = I_m$ for some m. Then again $At(\psi_2) \subseteq C_m$, and there exist $\Gamma'' \subseteq \Gamma$, $\Delta'' \subseteq \Delta$ such that $\vdash_{GRMI_m} \Gamma'' \Rightarrow \Delta''$ and $At(\psi_2) \subseteq At(\Gamma'' \Rightarrow \Delta'')$, Hence $\vdash_{GRMI_m} \Gamma'' \Rightarrow \Delta'', \psi_2$ by Lemma 14. This and the fact that $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta', \psi_1$ entail that $\vdash_{GRMI_m} \Gamma', \Gamma'' \Rightarrow \Delta', \Delta'', \varphi$. Since $\varphi \in \Delta$, this fact entails that pRq for every $p, q \in At(\varphi)$. It follows that $C_k = C_m$ and so $I_k = I_m$ and $v(\varphi) = I_k \otimes I_k = I_k \neq \bot$.

Assume now that the empty sequent is in S. Than $I(S)$ is empty, since if $p \in I(S)$ then $p \in \Gamma_{\alpha_1} \cap \Delta_{\alpha_2}$ for some $\alpha_1, \alpha_2 \in I$, and so $\{\Rightarrow, \Gamma_{\alpha_1} \Rightarrow \Delta_{\alpha_1}, \Gamma_{\alpha_2} \Rightarrow \Delta_{\alpha_2}\}$ is not n-satisfiable. It follows that v refutes the empty sequent in case it is in S.

To show that v is a countermodel of all the sequents in S it remains now only to eliminate the possibility that there exists $\alpha \in I$ and k such that $v(\varphi) = I_k$ for all $\varphi \in \Gamma_\alpha \cup \Delta_\alpha$. Well, had there been such α and k we would have that $v(P) = I_k$ for all $P \in At(\Gamma_\alpha \Rightarrow \Delta_\alpha)$, and so $At(\Gamma_\alpha \Rightarrow \Delta_\alpha) \subseteq C_k$. Hence there would have been $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ such that $\vdash_{GRMI_m} \Gamma' \Rightarrow \Delta'$ and $At(\Gamma_\alpha \Rightarrow \Delta_\alpha) \subseteq At(\Gamma' \Rightarrow \Delta')$. Lemma 14 would have implied then that $\vdash_{GRMI_m} \Gamma_\alpha, \Gamma' \Rightarrow \Delta_\alpha, \Delta'$. By the soundness of strong splittings this entails that $\vdash^h_{\mathcal{A}_\omega} s_1 \mid \ldots \mid s_k \mid \Gamma_\alpha \Rightarrow \Delta_\alpha$, for some $s_1, \ldots, s_k \in S$. A contradiction (Since $\Gamma_\alpha \Rightarrow \Delta_\alpha$ is also in S).

Corollary 11 $\vdash_{\mathcal{A}_\omega}$ *is finitary:* $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi$ *iff there exists a finite subset* Γ *of* $\mathcal{T}$ *such that* $\Gamma \vdash_{\mathcal{A}_\omega} \varphi$. *The same is true for* $\vdash_{\mathcal{A}_1}$.

Proof: Immediate from Theorem 6 and Proposition 2.

The following theorem and its corollary provide our best syntactic characterization of $\vdash_{\mathcal{A}_\omega}$ ($\vdash_{\mathcal{A}_1}$) in terms of $GSRMI_m$ ($GSRM_m$):

Theorem 7 $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi$ *(*$\mathcal{T} \vdash_{\mathcal{A}_1} \varphi$*) iff there exists a finite subset* Γ *of* $\mathcal{T}$ *such that* $\Gamma \Rightarrow \varphi \mid \Rightarrow \varphi$ *is provable in* $GSRMI_m$ *(*$GSRM_m$*) without using cut or external contraction.*

Proof: This follows from Corollary 11, Corollary 10, and Theorems 4 and 5.

Corollary 12

1. *A formula* φ *is valid in* $\mathcal{A}_\omega$ *iff* $\Rightarrow \varphi$ *has a cut-free proof in the 1-part of* $GSRMI_m$ *(which is just* $GRMI_m$*). Similar relations hold between* $\vdash_{\mathcal{A}_1}$ *and the 1-part of* $GSRM_m$ *(*$GSRM_m$*).*

2. $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi$ *(*$\mathcal{T} \vdash_{\mathcal{A}_1} \varphi$*) iff there exists a finite subset* Γ *of* $\mathcal{T}$ *such that* $\Gamma \Rightarrow \varphi \mid \Rightarrow \varphi$ *has a cut-free (and external contraction-free) proof in the 2-part of* $GSRMI_m$ *(*$GSRM_m$*).*

Proof: Immediate from Theorem 7 and Corollary 6.

Another important characterization of $\vdash_{\mathcal{A}_\omega}$ ($\vdash_{\mathcal{A}_1}$) in terms of $GSRMI_m$ ($GSRM_m$) is given in the following corollary:

Corollary 13 *$\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi$ ($\mathcal{T} \vdash_{\mathcal{A}_1} \varphi$) iff the sequent $\Rightarrow \varphi$ is derivable in the 2-part of $GSRMI_m$ ($GSRM_m$) from the set of sequents $\{\Rightarrow \psi \mid \psi \in \mathcal{T}\}$.*

Proof: The "if" part follows from the soundness of the rules of $GSRMI_m$ ($GSRM_m$). The "only if" part follows from Corollary 12, since $\Rightarrow \varphi$ is derivable from $\Gamma \Rightarrow \varphi \mid\Rightarrow \varphi$ and the set $\{\Rightarrow \psi \mid \psi \in \Gamma\}$ using n cuts followed by an external contraction.

Both of the two last corollaries mean that for characterizing the consequence relation induced by $\mathcal{A}_\omega$ ($\mathcal{A}_1$) only the 2-part of $GSRMI_m$ ($GSRM_m$) is needed!

Note Proposition 3, Corollary 10, Corollary 11, and the first part of Corollary 12 have already been proved in [9]. A weak version of Theorem 7, in which only the possibility of eliminating cuts is mentioned, was also claimed (without a proof) in [9]. All other results of this subsection are new.

3.4 Hilbert-type Systems

For the sake of completeness, we present now the Hilbert-type systems for our logics given in [3, 9] (with new proofs of their completeness). For this it is preferable to take $\rightarrow$ as primitive.

THE SYSTEM $HRMI_m$

Axioms

(I)	$A \rightarrow A$	(Identity)
(T)	$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$	(Transitivity)
(P)	$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$	(Permutation)
(R1)	$(A \rightarrow (B \rightarrow C)) \rightarrow (A \otimes B \rightarrow C)$	(Residuation)
(R2)	$(A \otimes B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$	
(C)	$A \rightarrow A \otimes A$	(Contraction)
(M)	$A \otimes A \rightarrow A$	(Mingle)
(N1)	$(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$	(Contraposition)
(N2)	$\sim\sim A \rightarrow A$	(Double Negation)
(F)	$\bot \rightarrow A$	(Falsehood)

Rule of inference

$$\frac{A \qquad A \rightarrow B}{B}$$

Note Instead of leaving $\otimes$ as primitive, we could have defined it by $\varphi \otimes \psi =_{Df} \sim (\varphi \rightarrow \sim \psi)$ (This equivalence is a theorem of the version we have preferred here). Axiom (F) can and should be deleted if we are interested only in the pure intensional language $\mathcal{IL}$.

Proposition 4

1. $\vdash_{GRMI_m} \Rightarrow \varphi$ *iff* $\vdash_{HRMI_m} \varphi$. *More generally:*

$$\vdash_{GRMI_m} \varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k \quad \text{iff} \quad \vdash_{HRMI_m} \sim \varphi_1 + \sim \varphi_2 + \cdots + \sim \varphi_n + \psi_1 + \cdots + \psi_k$$

2. $\mathcal{T} \vdash_{HRMI_m} \varphi$ *iff there exists a finite subset* Γ *of* $\mathcal{T}$ *such that* $\vdash_{GRMI_m} \Gamma \Rightarrow \varphi$.

Proof: The proof of the first part of Proposition 4 with the help of the admissible cut rule is standard. The second part follows from the first using the relevant deduction theorem for $HRMI_m$ (see Proposition 5).

Definition 15 $HSRMI_m$ is the system which is obtained from $HRMI_m$ by adding $\frac{A \otimes B}{A}$ as an extra rule of inference.

Theorem 8 $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi$ *iff* $\mathcal{T} \vdash_{HSRMI_m} \varphi$.

Proof: By Proposition 4 all axioms of $HSRMI_m$ are valid in $\mathcal{A}_\omega$, and it is easy to check that its two rules of inference are also sound with respect to $\mathcal{A}_\omega$. Hence the "if" part. For the converse, assume $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi$. Then by Theorem 7 there exists a finite subset $\Gamma = \{\psi_1, \ldots, \psi_n\}$ of $\mathcal{T}$ such that $\Gamma \Rightarrow \varphi \mid \Rightarrow \varphi$ is provable in $GSRMI_m$. Now from each of the two components of this hypersequent it is easy to derive (using the provability of $\psi_i \Rightarrow \psi_i$ and n applications of $(\Rightarrow \otimes)$) the sequent $\Gamma \Rightarrow \varphi \otimes \psi$, where $\psi = \psi_1 \otimes \cdots \otimes \psi_n$. Hence this sequent is provable in $GSRMI_m$ (using an external contraction), and so also in $GRMI_m$ (by Corollary 9). It follows by Proposition 4 that $\mathcal{T} \vdash_{HRMI_m} \varphi \otimes \psi$, and so $\mathcal{T} \vdash_{HSRMI_m} \varphi$ (using a single application of the extra rule of $HSRMI_m$).

Definition 16 HRM_m is the system which is obtained from $HRMI_m$ by adding to it the axiom $\sim (A \rightarrow A) \rightarrow (B \rightarrow B)$. $HSRM_m$ is the system which is obtained from HRM_m by adding $\frac{A \otimes B}{A}$ as an extra rule of inference.

Theorem 9 $\mathcal{T} \vdash_{\mathcal{A}_1} \varphi$ *iff* $\mathcal{T} \vdash_{HRM_m} \varphi$.

Proof: Similar to that of Theorem 8.

4 Relations with Other Logics

4.1 Relations with Relevance Logic

Our logic is based on the purely intensional fragment of relevance logic [10]. Indeed, the basic language $\mathcal{IL}$ is exactly that of the "intensional" fragment of the relevant logic R [11], and the 1-part of the $\mathcal{IL}$-fragment of our logic strictly belongs to the family of intensional relevance logics. In fact, it is a very close relative of $R_{\stackrel{\sim}{\rightarrow}}$ (which to our opinion is superior to $R_{\stackrel{\sim}{\rightarrow}}$).

The connections (and the differences) are the following:

1. $GRMI_{\stackrel{\sim}{\rightarrow}}$, the $\mathcal{IL}$-fragment of $GRMI_m$, is almost identical to $GR_{\stackrel{\sim}{\rightarrow}}$, the calculus given for $R_{\stackrel{\sim}{\rightarrow}}$ in [13]. Indeed, both systems are obtained from the classical Gentzen-type system by deleting its structural rule of weakening (and using the multiplicative versions of its other rules). The main difference is that in $GR_{\stackrel{\sim}{\rightarrow}}$ the two sides of a sequent basically consist of *multisets* of formulas rather than of sets (as in $GRMI_{\stackrel{\sim}{\rightarrow}}$), and the structural rule of contraction (but not its converse!) is therefore added (Note that the constant $\perp$ has also been considered and used in the relevance literature, as well as in Linear Logic, and the axiom used for it in $GRMI_m$ is practically the same as the one used for it in [16]).

2. $HRMI_{\stackrel{\sim}{\rightarrow}}$, the $\mathcal{IL}$-fragment of $HRMI_m$ is obtained from $HR_{\stackrel{\sim}{\rightarrow}}$ (the Hilbert-type systems which *defines* $R_{\stackrel{\sim}{\rightarrow}}$) by adding to it Axiom (M) (subsection 3.4). This addition turns the intensional conjunction $\otimes$ into a fully idempotent connective.

3. In both $HR_{\stackrel{\sim}{\rightarrow}}$ and $HRMI_{\stackrel{\sim}{\rightarrow}}$ the implication connective $\rightarrow$ enjoys some relevant deduction theorems. These theorems are reviewed in the next proposition (the proofs are standard - see e.g. [8]).

Proposition 5

1. *For $L = HR_{\stackrel{\sim}{\rightarrow}}$, $HRMI_{\stackrel{\sim}{\rightarrow}}$ and $HRM_{\stackrel{\sim}{\rightarrow}}$ (or any other extension of $HR_{\stackrel{\sim}{\rightarrow}}$ by axiom schemes) $\mathcal{T}, A \vdash_L B$ iff either $\mathcal{T} \vdash_L B$ or $\mathcal{T} \vdash_L A \rightarrow B$.*

2. *$\vdash_{HR_{\stackrel{\sim}{\rightarrow}}} \varphi_1 \otimes \cdots \otimes \varphi_n \rightarrow \psi$ iff there is a proof in $HR_{\stackrel{\sim}{\rightarrow}}$ of ψ from the multiset $[\varphi_1, \ldots, \varphi_n]$ in which each element of this* multiset *is used at least once (the same applies to any extension of $HR_{\stackrel{\sim}{\rightarrow}}$ by axiom schemes).*

3. *$\vdash_{HRMI_{\stackrel{\sim}{\rightarrow}}} \varphi_1 \otimes \cdots \otimes \varphi_n \rightarrow \psi$ iff there is a proof in $HRMI_{\stackrel{\sim}{\rightarrow}}$ of ψ from the set $\{\varphi_1, \ldots, \varphi_n\}$ in which each element of this* set *is used at least once (the same applies to any extension of $HRMI_{\stackrel{\sim}{\rightarrow}}$ by axiom schemes, e.g.: $HRMI_m$ or HRM_m).*

[10] In [7] we have argued in length that this is the only fragment that is well motivated.

[11] This fragment is denoted by $R_{\stackrel{\sim}{\rightarrow}}$ in the relevant literature, like [1, 2, 13]). Note that the relevantists prefer to take the relevant implication $\rightarrow$ as primitive rather than $\otimes$, which is preferred here for purely technical reasons.

These deduction theorems are no longer valid when we pass from the logic induced by the 1-part of $\vdash_{\mathcal{A}_\omega}$ to the full logic (as the example of $p \otimes q \vdash_{\mathcal{A}_\omega} p$ shows). Instead we have there the following relevant deduction theorem:

Theorem 10 *Call a sentence* φ fully relevant *to a sentence* ψ *if* $At(\varphi) \subseteq At(\psi)$. *If* φ *is fully relevant to* ψ *then* $\mathcal{T}, \varphi \vdash_{\mathcal{A}_\omega} \psi$ *iff* $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi \to \psi$ *(the same is true for any* $\mathcal{A}_n$*).*

Proof: Assume that $\mathcal{T}, \varphi \vdash_{\mathcal{A}_\omega} \psi$, and that $v(\phi) \neq \perp$ for all $\phi \in \mathcal{T}$. If $v(\varphi) = \perp$ then $v(\varphi \to \psi) = \top$. Otherwise $v(\psi) \neq \perp$ (since $\mathcal{T}, \varphi \vdash_{\mathcal{A}_\omega} \psi$). If $v(\psi) = \top$ then again $v(\varphi \to \psi) = \top$. If $v(\psi) = I_k$ for some k then also $v(\varphi) = I_k$ (by the Isolation property and the fact that $At(\varphi) \subseteq At(\psi)$), and so $v(\varphi \to \psi) = I_k \neq \perp$ in this case as well.

The converse is easy and is left to the reader.

We note finally that in [3] it was shown that the implication $\to$ has in the $\mathcal{IL}$-fragment of our logic the crucial *variable-sharing property*, which is the main characteristic of relevance logics:

Theorem 11 *If* $\vdash_{\mathcal{A}_\omega} \varphi \to \psi$ *(where* φ *and* ψ *are in* $\mathcal{IL}$*) than* φ *and* ψ *share a variable.*

Proof: This follows from the fact that $I_1 \to I_2 = \perp$: If φ and ψ share no variable we can assign I_1 to all the atomic formulas of φ and I_2 to all the atomic formulas of ψ. We shall get that $v(\varphi) = I_1$, $v(\psi) = I_2$, and $v(\varphi \to \psi) = \perp$.

Note Using Theorem 1, a similar proof can show that for *every* binary relation $*$ definable in $\mathcal{IL}$, either $*$ or its complement $\bar{*}$ (defined by $\varphi \bar{*} \psi = \sim (\varphi * \psi)$) has in $\vdash_{\mathcal{A}_\omega}$ (and so also in $R_{\underset{\to}{\sim}}$) the variable-sharing property (Whether it is $\bar{*}$ or $*$ depends on whether $I_1 * I_2 = \top$ or $I_1 * I_2 = \perp$). We note also that in the full language one should add to the the claim made in Theorem 11 two other possibilities: that φ is equivalent to $\perp$, and that ψ is equivalent to $\top$. The proof is similar.

4.2 Relations with Classical Logic

We next show that classical logic (CL) can be identified with a special fragment of our logic. For this we associate with each n-ary operation on $\mathcal{A}_\omega$ definable in $\mathcal{IL}^\perp$ (or just $\mathcal{IL}$) its restriction to $\{\top, \perp\}^n$. In particular: we associate classical conjunction with $\otimes$, and classical negation with $\sim$ (and so classical implication is associated with $\to$, classical disjunction with $+$, etc.). The following theorem should be understood accordingly.

Theorem 12

1. *A sequent* s *is classically valid iff* $s \mid\Rightarrow$ *is valid in* $\mathcal{A}_\omega$ *(iff it is provable in* $GSRMI_m$ *without using cut or external contraction).*

2. *$\mathcal{T} \vdash_{CL} \varphi$ iff there exists a finite subset Γ of $\mathcal{T}$ such that $\Gamma \Rightarrow \varphi \mid\Rightarrow$ has a cut-free (and external contraction-free) proof in the 2-part of $GSRMI_m$.*

Proof: The first part immediately follows from the semantic definitions (especially the definition of a model of the empty sequent). The second part follows from the first, Corollary 6, and well-known properties of $\vdash_{CL}$.

Notes

1. The second part of Theorem 12 should be compared with the second part of Corollary 12!

2. Theorem 12 remains valid if we restrict ourselves to the $\mathcal{IL}$-fragment of $GSRMI_m$.

3. Theorem 12 is easily seen to imply both the completeness of the usual Gentzen-type system for classical propositional logic, as well as the cut elimination theorem for it. In fact, a splitting in a cut-free and external contraction-free proof in $GSRMI_m$ of $s \mid\Rightarrow$ amounts to a generalized form of weakening applied to some subsequent of s (in which more than one formula is added). A proof of $s \mid\Rightarrow$ in $GSRMI_m$ exactly simulates therefore a special type of a cut-free proof of s in a Gentzen-type system for CL.

We next show that CL can be taken as a sublogic of $\vdash_{\mathcal{A}_\omega}$ also from another point of view: it can be interpreted in $\vdash_{\mathcal{A}_\omega}$.

Definition 17 $\varphi \supset \psi =_{Df} \varphi \rightarrow \varphi \otimes \psi \quad \varphi \bar{\vee} \psi =_{Df} (\varphi \supset \psi) \supset \psi$

Theorem 13 [9] *The $\{\otimes, \bar{\vee}, \supset, \bot\}$-fragment of $\vdash_{\mathcal{A}_\omega}$ is identical to classical logic (where these connectives are respectively taken as the interpretations of classical conjunction, disjunction, implication, and falsehood).*

Proof: It can easily be checked that $a \otimes b$ is true (i.e. $a \otimes b \neq \bot$) iff both a and b are true, $a \bar{\vee} b$ is true iff either a is true or b is true, $a \supset b$ is true iff either a is not true or b is true, and $\bot$ is of course never true.

Notes

1. It follows from this theorem that the $\mathcal{IL}$-fragment of our logic is sufficient for interpreting *positive* classical logic. It is easy however to see that the strong isolation condition (see Lemma 5) entails that no faithful interpretation of classical negation is available in it.

2. $\supset$ is in fact an implication connective for $\vdash_{\mathcal{A}_\omega}$ as a whole, since it is easy to see that the following deduction theorem from [9] obtains for it: $\mathcal{T}, \varphi \vdash_{\mathcal{A}_\omega} \psi$ iff $\mathcal{T} \vdash_{\mathcal{A}_\omega} \varphi \supset \psi$.

3. The interpretation of classical negation is defined of course in the language $\{\otimes, \bar{\vee}, \supset, \perp\}$ as $\neg\varphi =_{Df} \varphi \supset \perp$ (which is equivalent to $\varphi \to \perp$). It is easy to see that $\neg a$ is true iff a is not true. Note also that the languages $\{\otimes, \bar{\vee}, \supset, \perp\}$ and $\{\otimes, \bar{\vee}, \supset, \neg\}$ are equivalent, since $\perp$ is equivalent in $\mathcal{A}_\omega$ to $\neg\varphi \otimes \varphi$, where φ is arbitrary.

The following propositions provide two other connections between our logic and classical logic:

Theorem 14 *(***Substitution of classical equivalents***) Let A and B be two classically equivalent formulas in the language $\{\otimes, \bar{\vee}, \supset, \perp\}$ such that $At(A) = At(B)$. Let the formula ψ be obtained from the formula φ of $\mathcal{IL}^{\perp}$ by replacing some occurrences of A in φ by B. Then $\vdash_{\mathcal{A}_\omega} \varphi \leftrightarrow \psi$.*

Proof: Let v be a valuation in $\mathcal{A}_\omega$. By Theorem 13 $A \supset B$ and $B \supset A$ are both valid in $\mathcal{A}_\omega$. This implies that $v(A) = v(B)$ in case both $v(A)$ and $v(B)$ are in $\{\top, \perp\}$. On the other hand the fact that $At(A) = At(B)$ entails that $v(A) = I_k$ for some k iff $v(B) = I_k$ (note that this is impossible if $\perp \in At(A) = At(B)$). It follows that $v(A) = v(B)$ in all cases, and so $v(\varphi) = v(\psi)$ for all v. Hence $\varphi \leftrightarrow \psi$ is valid in $\mathcal{A}_\omega$.

Proposition 6 *Let ψ be a formula of $\mathcal{IL}$ in which exactly one atomic formula P occurs. If ψ is a classical tautology (with $\sim$ and $\otimes$ interpreted as negation and conjunction, respectively), then ψ is valid in $\mathcal{A}_\omega$.*

Proof: Let v be a valuation in $\mathcal{A}_\omega$. If $v(P) = I_k$ for some k then $v(\psi) = I_k$ by Theorem 1. If $v(P) \in \{\top, \perp\}$ then v behaves with respect to ψ like a classical valuation, and so $v(\psi) = \top$ (since ψ is a classical tautology). In either case $v(\psi) \neq \perp$.

4.3 Relations with Paraconsistent Logics

We finally turn to investigate the relations between our logic and da Costa's family of paraconsistent logics. We start with the trivial observation that with respect to $\sim$ even $\vdash_{\mathcal{A}_1}$ is paraconsistent:

Proposition 7 $\sim P, P \not\vdash_{\mathcal{A}_1} Q$ *if P and Q are distinct atomic formulas.*

This proposition is true in fact not only for $\sim$, but for any other "negation" that one might try to define in $\mathcal{IL}$. Moreover: we have the following stronger result:

Proposition 8 *No finite theory Γ in $\mathcal{IL}$ is trivial in $\vdash_{\mathcal{A}_1}$: if Q does not occur in Γ then $\Gamma \not\vdash_{\mathcal{A}_1} Q$.*

Proof: Let v be the valuation which assigns $\perp$ to Q and I_1 to any other atomic variable. Then by Theorem 1 v is a model of Γ which is not a model of Q.

The value of $\vdash_{\mathcal{A}_\omega}$ (and $\vdash_{\mathcal{A}_1}$) as a paraconsistent logic is due not only to these negative results, but also to the following positive one:

Proposition 9 *A hypersequent G is valid in $\mathcal{A}_\omega$ ($\mathcal{A}_1$) under the assumption that $\varphi_1, \ldots, \varphi_n$ are all "normal" (i.e.: G is true for any v which assigns to $\varphi_1, \ldots, \varphi_n$ truth value in $\{\top, \bot\}$) iff $G \mid\Rightarrow \varphi_1 \otimes \sim \varphi_1 \mid \cdots \mid\Rightarrow \varphi_n \otimes \sim \varphi_n$ is valid in $\mathcal{A}_\omega$.*

This proposition (the easy proof of which we leave to the reader) is important because it makes it possible to use within $\mathcal{A}_\omega$ default assumptions of the form "the formula ψ is normal" (and classical logic when it seems safe) by applying nonmonotonically the rule: from $G \mid\Rightarrow \varphi \otimes {\sim}\varphi$ infer G.

We next show that the $\mathcal{IL}$-fragment of $\vdash_{\mathcal{A}_\omega}$ is a proper extension of da Costa's basic paraconsistent logic C_ω ([11]). Identifying $\sim$, $\otimes$, $\bar{\vee}$, and $\supset$ (respectively) with the negation, conjunction, disjunction, and implication of C_ω we have:

Proposition 10 *The $\{\sim, \otimes, \bar{\vee}, \supset\}$-fragment of $\vdash_{\mathcal{A}_\omega}$ is a proper extension of C_ω. This remains true if we replace $\otimes$ by any other binary connective & such that $a\&b = \bot$ iff either $a = \bot$ or $b = \bot$.*

Proof: C_ω is obtained from positive classical logic by adding to it as axioms excluded middle (${\sim}\psi\bar{\vee}\psi$) and double negation elimination (${\sim\sim}\psi \supset \psi$). Hence Theorem 13 and Proposition 6 imply that $\vdash_{\mathcal{A}_\omega}$ is an extension of C_ω. Proposition 6 implies also that $\vdash_{\mathcal{A}_\omega} \psi \supset {\sim\sim}\psi$. Since this schema is not derivable in C_ω, $\vdash_{\mathcal{A}_\omega}$ is in fact a proper extension of C_ω.

C_ω is the weakest logic in the sequence $\{C_i\}_{i=1}^{\omega}$ of da Costa's paraconsistent logics. We turn now to compare the $\{\sim, \otimes, \bar{\vee}, \supset\}$-fragment of $\vdash_{\mathcal{A}_\omega}$ (which we denote by $\vdash^{pac}_{\mathcal{A}_\omega}$ in what follows) with the strongest logic in this sequence: C_1. We first note three big advantages that $\vdash^{pac}_{\mathcal{A}_\omega}$ has over C_1:

- C_1 is paraconsistent only with respect to $\sim$, but it is possible to define within it another negation with respect to which it is not paraconsistent. By Proposition 8 this is impossible in $\vdash^{pac}_{\mathcal{A}_\omega}$, and so this logic is *absolutely paraconsistent*.
- While the choice of the basic axioms concerning $\sim$ in C_1 seems quite arbitrary (why is ${\sim\sim}\psi \supset \psi$ accepted while its converse is not?), Proposition 6 implies that in $\vdash^{pac}_{\mathcal{A}_\omega}$ *all* classical tautologies of this type are accepted.
- Equivalence in C_1 is never a real equivalence, since substitution of equivalents always fails in the context of negation. Thus even $\sim(\varphi\bar{\vee}\psi)$ and $\sim(\psi\bar{\vee}\varphi)$ are not equivalent in C_1 (although both $(\varphi\bar{\vee}\psi) \supset (\psi\bar{\vee}\varphi)$ and its converse are theorems). By Theorem 14, in contrast, A can be substituted for B whenever A and B are (instances of) positive equivalent formulas of $\vdash^{pac}_{\mathcal{A}_\omega}$ having the same atomic variables. The same is true (by Proposition 6) if A and B are (instances of) classically equivalent formulas s.t. $At(A) = At(B) = \{P\}$ (thus in $\vdash^{pac}_{\mathcal{A}_\omega}$ the formula φ can always be substituted for ${\sim\sim}\,\varphi$, and vice versa).

On the other hand C_1 has the advantage that it is possible to express within its language the assumption that a given formula should be taken as "normal" (i.e. its truth value should be in $\{\top, \bot\}$). This normality of a formula φ is defined by $\varphi^\circ =_{Df} \sim(\sim\varphi \& \varphi)$ (where & is the official conjunction of the system). This makes it possible to use full classical logic within C_1 under the assumption that certain formulas are normal. Thus the most characteristic axiom of this system allows to infer $\sim\varphi$ from $\varphi \supset \psi$, $\varphi \supset\sim \psi$, and ψ°. Proposition 9 provides some substitute for this property of C_1 in the case of $\vdash^{pac}_{\mathcal{A}_\omega}$, but normality is not expressible within the language of this logic. It is easy in fact to show (using Theorem 1) that one cannot define in $\mathcal{IL}$ a unary connective $\diamond$ such that $\diamond(P), Q \supset P, Q \supset\sim P \vdash_{\mathcal{A}_1} \sim Q$. This shortcoming(?) can be remedied in the full language $\mathcal{IL}^\perp$ at the cost of losing the first advantage of $\vdash^{pac}_{\mathcal{A}_\omega}$ over C_1 which we have noted above.

Definition 18 $\varphi \& \psi =_{Df} \varphi \otimes \psi \otimes \top$

Note that $v(\varphi \& \psi)$ is $\bot$ iff either $v(\varphi) = \bot$ or $v(\psi) = \bot$, and $v(\varphi \& \psi) = \top$ otherwise. Hence if we define $\varphi^\circ =_{Df} \sim(\sim\varphi \& \varphi)$ then $v(\varphi^\circ) = \top$ if $v(\varphi) \in \{\top, \bot\}$, and $v(\varphi^\circ) = \bot$ otherwise. Using this fact, Proposition 10 and some routine checks it is easy to prove:

Theorem 15 *The* $\{\sim, \&, \bar{\vee}, \supset\}$*-fragment of* $\vdash_{\mathcal{A}_\omega}$ *is a proper extension of* C_1.

Note The $\{\sim, \&, \bar{\vee}, \supset\}$-fragment of $\vdash_{\mathcal{A}_\omega}$ is a paraconsistent logic which still has over C_1 the second and third advantages noted above. In addition it is sensitive to relevance considerations, and it has both simple semantics and a nice proof system. These properties make it (so we believe) superior to C_1 or to any other paraconsistent logic which has been suggested in the literature.

A final note: $\mathcal{A}_1$ can be taken as the restriction of J_3 (the strongest logic produced by da Costa's school), to the language described in Corollary 4. Indeed the two structures have the same (designated) truth-values, and the connectives of $\mathcal{IL}^\perp$ are definable in J_3 (see [4]).

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