On The Expressive Power of Three-Valued and Four-Valued Languages

Arnon Avron
School of Mathematical Sciences
Sackler Faculty of Exact Sciences
Tel Aviv University
Tel Aviv 69978, Israel

Abstract

We investigate the expressive power relative to three-valued and four-valued logics of various subsets of the set of connectives which are used in the bilattices-based logics. Our study of a language is done in two stages. In the first stage the ability of the language to characterize sets of tuples of truth-values is determined. In the second stage the results of the first are used to determine its power to represent operations. Special attention is given to the role of monotonicity, closure and freedom properties in classifying languages, as well as to maximality properties (for example: we prove that by adding any nonmonotonic connective to the set of four-valued monotonic connectives, we get a functionally complete set).

1 Introduction

In [Be77a, Be77b] Belnap introduced a logic intended to deal in a useful way with inconsistent and incomplete information. This logic is based on four truth values: the classical ones, denoted here by t and f, and two new ones: $\perp$, that intuitively denotes lack of information (no knowledge), and $\top$, that indicates inconsistency ("over"-knowledge).

The truth values of Belnap's logic have two natural orderings: One, $\leq_t$, intuitively reflects differences in the "measure of truth" that every value represents. According to this order, f is the minimal element, t is the maximal one, and $\perp, \top$ are two intermediate values that are incomparable. $(\{t, f, \top, \perp\}, \leq_t)$ is a distributive lattice with an order reversing involution $\neg$, for which $\neg\top = \top$ and $\neg\perp = \perp$. We shall denote the meet and the join of this lattice by $\wedge$ and $\vee$, respectively.

The other partial order, $\leq_k$, is understood (again, intuitively) as reflecting differences in the amount of *knowledge* or *information* that each truth value exhibits. Again, $(\{t, f, \top, \perp\}, \leq_k)$ is a distributive lattice where $\perp$ is its minimal element, $\top$ – the maximal element, and t, f are incomparable. Following Fitting [Fi90] we shall denote the meet and the join of the $\leq_k$-lattice by

$\otimes$ and $\oplus$, respectively.

The two lattice orderings are closely related. The knowledge operators $\otimes$ and $\oplus$ are monotonic w.r.t. the truth ordering $\leq_t$, and the truth operators $\wedge$, $\vee$, and $\neg$ (as well, of course, as $\otimes$ and $\oplus$) are monotonic w.r.t. $\leq_k$. Further, all the 12 distributive laws hold, and so are De-Morgan's laws. The structure that consists of these four elements and the five basic operators $(\wedge, \vee, \neg, \otimes, \oplus)$ is usually called *FOUR*. A double Hasse diagram of *FOUR* is given in Figure 1.

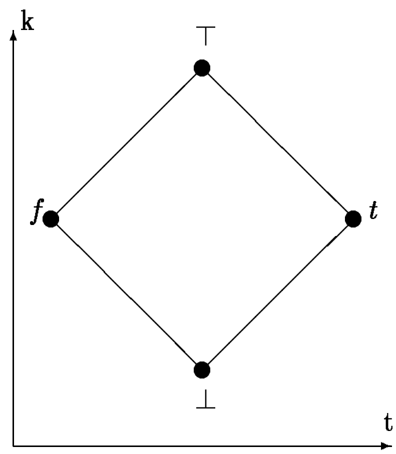

Figure 1: *FOUR*

The next step in using *FOUR* for reasoning is to choose its set of *designated* elements. The obvious choice is $\mathcal{D} = \{t, \top\}$, since both values intuitively represent formulae known to be true. The set $\mathcal{D}$ has the property that $a \wedge b \in \mathcal{D}$ (or $a \otimes b \in \mathcal{D}$) iff both a and b are in $\mathcal{D}$, while $a \vee b \in \mathcal{D}$ (or $a \oplus b \in \mathcal{D}$) iff either a or b is in $\mathcal{D}$. From this point the various semantic notions and the associated consequence relation are defined on *FOUR* as in every many-valued logic.

The algebraic structure of *FOUR* has been generalized by Ginsberg [Gi88] to the general concept of a *bilattice*. He proposed Bilattices as a basis for a general framework for many applications. Bilattices were further investigated by Fitting, who used them for extending some well known logics (like Kleene 3-valued logics) and for logic programming (see, e.g., [Fi90, Fi91, Fi94]). In [AA96] the set $\mathcal{D}$ is also generalized to what is called there *a bifilter*, and bilattices-based logics are introduced. It turned out, however, that from a logical point of view, *FOUR* has among bilattices the same role that the two-valued Boolean algebra has among Boolean algebras. It is therefore a particularly important bilattice (and so far it has also been the most useful one in applications).

It can be shown that in the language of $\{\neg, \wedge, \vee, \otimes, \oplus, t, f, \top, \bot\}$ no implication connective can be defined for which both MP and the deduction theorem obtain. In [AA96, AA98] the following implication, which does have these properties in all bilattice-based logics, has been used:

Definition 1.1 [Av91, AA96]

$$a \supset b \ = \begin{cases} b & \textit{if } a \in \mathcal{D} \\ t & \textit{if } a \notin \mathcal{D} \end{cases}$$

Another connective which has been found useful in the literature on bilattices is Fitting's conflation, $\perp$, which is the $\leq_k$-dual of negation. Not all bilattices possess a conflation, but *FOUR* does: $\perp\perp = \top, \perp\top = \perp, \perp t = t, \perp f = f$.

The purpose of this paper is to explore the expressive power relative to *FOUR* and its three-valued counterpart of various subsets of $\mathcal{L}_{all}$, the set of connectives which are used in the bilattices-based logics (i.e.: $\mathcal{L}_{all} = \{\neg, \wedge, \vee, \otimes, \oplus, t, f, \top, \perp, \supset, \perp\}$). Since negation will be included in every subset we consider, $\wedge$ and $\vee$ will always be interdefinable (using De-Morgan laws), and the same applies to t and f. Hence in what follows we will investigate, in fact, the following set of connectives (which we also call $\mathcal{L}_{all}$): $\{\neg, \wedge, \otimes, \oplus, f, \top, \perp, \supset, \perp\}$. Other important connections in *FOUR* between elements of $\mathcal{L}_{all}$ are:

(i) $f = \top \wedge \perp$

(ii) $\perp = f \otimes \neg f$

(iii) $\top = f \oplus \neg f$

(iv) $a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b)$

(v) $a \otimes b = (a \wedge \perp) \vee (b \wedge \perp) \vee (a \wedge b)$

(vi) $f = \neg(\perp \supset \perp)$

(vii) $\top = (a \supset a) \oplus \neg(a \supset a)$

(viii) $f = \perp\neg a \wedge a$

(ix) $\top = \perp\perp = \perp\neg a \oplus a$ $\quad\quad$ $\perp = \perp\top = \perp\neg a \otimes a$

(x) $a \oplus b = \perp(\perp a \otimes \perp b)$ $\quad\quad$ $a \otimes b = \perp(\perp a \oplus \perp b)$

Following [Th92], we shall take $\{\neg, \wedge\}$ as the "hard core" of the language, and consider only sets of connectives which contain it.

Our investigations of the expressive power of the various languages will be done from two different points of view, which are equivalent in two-valued logic, but not in general. One is the ability to characterize sets of tuples of truth values. The other is the ability to represent operations (of arbitrary arity) on the set of truth-values [1]. We provide now the precise definitions in the case of $FOUR$. Obviously, similar definitions can be made for any other many-valued logic. First we need some notations (which can easily be generalized as well):

1. Let ψ be a formula. We denote by $A(\psi)$ the set of atomic formulas that occur in ψ.
2. Let $A(\psi) \subseteq \{p_1, \ldots, p_n\}$. we denote by F^n_ψ the function from $FOUR^n$ to $FOUR$ that corresponds to ψ (i.e., the n-ary truth table which corresponds to ψ).
3. Let $A(\psi) \subseteq \{p_1, \ldots, p_n\}$. S^n_ψ, the subset of $FOUR^n$ which is *characterized* by ψ, is:

$$S^n_\psi = \{(a_1, \ldots, a_n) \in FOUR^n \mid F^n_\psi(a_1, \ldots, a_n) \in \mathcal{D}\}$$

Definition 1.2 A subset $C \subseteq FOUR^n$ is *characterizable* in a language iff there exists a formula ψ of that language such that $C = S^n_\psi$.

Definition 1.3 We say that a formula φ such that $A(\varphi) \subseteq \{p_1, \ldots, p_n\}$ *represents* an operator $g : FOUR^n \to FOUR$ iff $F^n_\varphi = g$.

Obviously, the characterization power of a language depends on the choice of the set of designated values. Its representation power, in contrast, has nothing to do with this choice, and depends only on the interpretations of the connectives. It is somewhat surprising, therefore, that our results and proofs concerning representability heavily depend on those concerning characterizability. Indeed, the main innovation of this work is perhaps the separation of the study of the expressive power of a language into two stages, which corresponds to these two points of view. In the first stage the characterization power of the language is determined. In the second the results of the first stage are used to determine its representation power.

[1] This is similar to the two different roles that automata and other machines have in computability theory: They are investigated both as acceptors of languages, and as input-output devices.

Our investigations of the representation power of a given set of connectives concentrate on two central problems. One is maximality: is the set functionally complete, and if not— is it maximally so (which means that by adding any connective which is not definable from it we get a functionally complete set). The other problem is to find a set of properties which characterizes the given set (i.e. properties that all connectives which are definable in that set have, and only these connectives have all of these properties). Now in previous works on this subject in the context of partial logic ([Bl86, vB88, La88, Th92]), three particularly important properties were identified: monotonicity (also called "persistence"), closure, and freedom. Since this paper is a continuation (and in certain cases a completion) of those works, it is no wonder that these properties play a key role here as well. In the case of closure and freedom we have however to generalize somewhat the previous definitions.

Definition 1.4 Let $\langle V, \leq \rangle$ be a poset. An operation $H : V^n \to V$ is called *monotonic* (relative to $\leq$) if $H(\vec{y}) \geq H(\vec{x})$ whenever $\vec{y} \geq \vec{x}$ (we say that $\vec{y} = (y_1, \ldots, y_n) \geq \vec{x} = (x_1, \ldots, x_n)$ iff $y_i \geq x_i$ for all $1 \leq i \leq n$).
In the context of $FOUR$ monotonicity is taken to be relative to the $\leq_k$ partial order.

Definition 1.5 Let V be a set which contains $\{t, f\}$, and let $\{t, f\} \subseteq S \subset V$. An operation $H : V^n \to V$ is called S*-closed* if

$$\forall 1 \leq i \leq n \; x_i \in S \Longrightarrow H(\vec{x}) \in S$$

Definition 1.6 Let V be a set which contains $\{t, f\}$, and let $a \in V \perp \{t, f\}$. An operation $H : V^n \to V$ is called a*-free* if

$$H(a, , \ldots, a) = a$$

A final remark: many of the results below are new (to the best of our knowledge), but many others are not. We have tried to give appropriate credits whenever possible. For the sake of completeness, and in order to demonstrate the unifying power of our method, we provide proofs to all results, not only to the new ones.

2 Expressive Power in Three-Valued Logic

We start with an examination of the three-valued case. For this we use the substructure of $FOUR$ with consists of $\{t, f, \top\}$. Let us call this substructure $THREE$. Using $THREE$ (rather than $\{t, f, \bot\}$) means that we take both t and $\top$ as designated, instead of just t. It means also that the connective $\supset$ we use is the implication connective of the paraconsistent logic J_3 (see [DO85, Av86, Av91, Ro89, Ep95]), which is defined by: $a \supset b$ is t if $a = f$, b otherwise. Our choice does not affect the definitions of the other connectives in which we are interested (only the notations we pick for some of them), and it is immaterial from the point of view of representability of truth-functions. It is also irrelevant as far as monotonicity, closure and freedom of connectives are concerned (Note that a three-valued function is monotonic according to $\{t, f, \top\}$ iff it is monotonic according to $\{t, f, \bot\}$!). It *is* relevant, however, to the question what sets can be characterized by what sets of connectives.

Of the nine connectives on which we concentrate in this paper, three ($\bot, \bot$ and $\otimes$) are here meaningless [2]. Moreover: $\top$ and $\oplus$ are interdefinable, using equation (iv) from the introduction and the fact that in $THREE$ $\top = \neg a \oplus a$. Accordingly, we concentrate in this section on the following five connectives: $\neg, \wedge, \supset, f, \top$.

Since in $THREE$ we have only one extra truth-value (in addition to t and f), we shall simply write in this section "free" instead of "$\top$-free", and "closed" instead of "$\{t, f\}$-closed" [3].

2.1 Characterization of subsets of $THREE^n$

Lemma 2.1 In $FOUR$ and in $THREE$ we have:

1. $\bigcup_{i=1}^{k} S^n_{\psi_i} = S^n_{\psi_1 \vee \psi_2 \vee \cdots \vee \psi_k}$

2. $\bigcap_{i=1}^{k} S^n_{\psi_i} = S_{\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_k}$

Proof: This easily follows from the fact that $a \wedge b \in \mathcal{D}$ iff $a \in \mathcal{D}$ and $b \in \mathcal{D}$, while $a \vee b \in \mathcal{D}$ iff $a \in \mathcal{D}$ or $b \in \mathcal{D}$.

[2] Concerning $\bot$ and $\otimes$ it is more accurate to say that there is no difference between $\bot$ and $\top$ and between $\otimes$ and $\oplus$.

[3] What we call here "closed" is called "classically closed" in [Th92], "pure" in [He83] and "deterministic" in [La88].

Theorem 2.2 A subset S of $THREE^n$ is characterizable by some formula in the language of $\{\neg, \wedge, \supset\}$ (or $\{\neg, \wedge, \supset, \top\}$) iff $\vec{\top} = (\top, \top, \ldots, \top) \in S$.

Proof: If ψ is any formula in the language of $\{\neg, \wedge, \supset, \top\}$ s.t. $A(\psi) \subseteq \{p_1, \ldots, p_n\}$ and $\nu(p_1) = \nu(p_2) = \ldots = \nu(p_n) = \top$, then $\nu(\psi) = \top$. Hence the condition is necessary. For the converse we use the following formula:

$$f_n = p_1 \wedge \neg p_1 \wedge p_2 \wedge \neg p_2 \wedge \ldots \wedge p_n \wedge \neg p_n$$

Obviously, f_n has the following property:

$$\nu(f_n) = \begin{cases} \top & \forall 1 \leq i \leq n \ \nu(p_i) = \top \\ f & \text{otherwise} \end{cases}$$

Let $\vec{a} = (a_1, \ldots, a_n) \in THREE^n$. Define, for every $1 \leq i \leq n$,

$$\psi_i^{\vec{a}} = \begin{cases} p_i \ \wedge \ \neg p_i & \text{if } a_i = \top \\ \neg p_i \supset f_n & \text{if } a_i = t \\ p_i \supset f_n & \text{if } a_i = f \end{cases}$$

Using the observation above concerning f_n , it is easy to see that $\psi_1^{\vec{a}} \bar{\wedge} \psi_2^{\vec{a}} \bar{\wedge} \ldots \bar{\wedge} \psi_n^{\vec{a}}$ characterizes $\{\vec{\top}, \vec{a}\}$. This and the first part of Lemma 2.1 entail the theorem.

Theorem 2.3 Every subset of $THREE^n$ is characterizable in the language of $\{\neg, \wedge, \supset, f\}$

Proof: All we need to change in the proof of Theorem 2.2 is to use f instead of f_n in the definition of $\psi_i^{\vec{a}}$. After this change the conjunction of the new $\psi_i^{\vec{a}}$'s characterizes $\{\vec{a}\}$ and not $\{\vec{\top}, \vec{a}\}$. This suffices (using $\vee$) for the characterization of every nonempty set. The empty set itself is characterized by f.

Theorems 2.2 and 2.3 can be strengthened as follow:

Theorem 2.4 A subset S of $THREE^n$ is characterizable by some formula in the language of $\{\neg, \supset\}$ iff $\vec{\top} \in S$, while every subset of $THREE^n$ is characterizable in the language of $\{\neg, \supset, f\}$.

Proof: We need to find substitutes for $\wedge$ and $\vee$ in the proofs of theorems 2.2 and 2.3. For this define: $p \bar{\wedge} q = \neg(p \supset \neg q)$, $p \bar{\vee} q = (p \supset q) \supset q$. The following properties are easily verified:

1. $\bar{\wedge}$ is associative. Moreover,

$$\nu(\psi_1 \bar{\wedge} \psi_2 \bar{\wedge} \ldots \bar{\wedge} \psi_n) = \begin{cases} f & \exists 1 \leq i \leq n-1 \ \nu(\psi_i) \notin \mathcal{D} \\ \nu(\psi_n) & \forall 1 \leq i \leq n-1 \ \nu(\psi_i) \in \mathcal{D} \end{cases}$$

2. $\nu(\psi_1 \bar{\wedge} \psi_2 \bar{\wedge} \ldots \bar{\wedge} \psi_n) \in \mathcal{D}$ iff $\forall 1 \leq i \leq n \ \nu(\psi_i) \in \mathcal{D}$.

3. $\bar{\vee}$ is associative. Moreover,

$$\nu(\psi_1 \bar{\vee} \psi_2 \bar{\vee} \ldots \bar{\vee} \psi_n) = \begin{cases} \nu(\psi_n) & \forall 1 \leq i \leq n-1 \ \nu(\psi_i) \notin \mathcal{D} \text{ or } \nu(\psi_n) = \top \\ t & \text{otherwise} \end{cases}$$

4. $\nu(\psi_1 \bar{\vee} \psi_2 \bar{\vee} \ldots \bar{\vee} \psi_n) \in \mathcal{D}$ iff $\exists 1 \leq i \leq n \ \nu(\psi_i) \in \mathcal{D}$.

Using these facts it is easy to see that f_n is equivalent to $p_1 \bar{\wedge} \neg p_1 \bar{\wedge} p_2 \bar{\wedge} \neg p_2 \bar{\wedge} \ldots \bar{\wedge} p_n \bar{\wedge} \neg p_n$, and

$$(i) \ S^n_{\psi_1 \bar{\wedge} \ldots \bar{\wedge} \psi_m} = S^n_{\psi_1} \cap \ldots \cap S^n_{\psi_m} \qquad (ii) \ S^n_{\psi_1 \bar{\vee} \ldots \bar{\vee} \psi_m} = S^n_{\psi_1} \cup \ldots \cup S^n_{\psi_m}$$

From this point we proceed as in in the proofs of 2.2 and 2.3, using $\bar{\wedge}$ and $\bar{\vee}$ instead of $\wedge$ and $\vee$.

We turn now to the languages without $\supset$.

Definition 2.5 Let $\langle V, \leq \rangle$ be a poset. A set $\mathcal{S} \subseteq V^n$ is called a *cone* in $\langle V^n, \leq \rangle$ if $\vec{y} \in \mathcal{S}$ whenever $\vec{y} \geq \vec{x}$ and $\vec{x} \in \mathcal{S}$. If $S = V^n$ then the cone is called *trivial.*

Note Obviously, a cone S in $THREE^n$ is nonempty iff $\vec{\top} = (\top, \ldots, \top) \in \mathcal{S}$.

Definition 2.6 Let $\langle V, \leq \rangle$ be a poset, and let $\mathcal{S}$ be a cone in $\langle V^n, \leq \rangle$. An element $\vec{x} \in \mathcal{S}$ is called a *stable* element of $\mathcal{S}$ if $\{\vec{y} \in V^n | \vec{y} \leq \vec{x}\} \subseteq \mathcal{S}$.

Theorem 2.7 Any subset of $THREE^n$ which can be characterized by some formula in $\{\neg, \wedge, f, \top\}$ is a cone. Conversely, every cone $\mathcal{C}$ in $THREE^n$ can be characterized by a formula ψ_C in $\{\neg, \wedge, f\}$, so that if $\vec{x}$ is a stable element of $\mathcal{C}$ then $F^n_{\psi_C}(\vec{x}) = t$.

Proof: The first part is immediate from the fact that $\neg, \wedge, f, \top$ correspond all to monotonic operations. For the converse, we define for every $\vec{a} \in \mathcal{C}$ and every $1 \leq i \leq n$ a formula $\psi^i_{\vec{a}}$ as follows:

If $\vec{a}$ is not a stable element of $\mathcal{C}$ then

$$\psi^i_{\vec{a}} = \begin{cases} p_i \wedge \neg p_i & a_i = \top \\ p_i & a_i = t \\ \neg p_i & a_i = f \end{cases}$$

If $\vec{a}$ *is* a stable element of $\mathcal{C}$ then

$$\psi^i_{\vec{a}} = \begin{cases} t & a_i = \top \\ p_i & a_i = t \\ \neg p_i & a_i = f \end{cases}$$

Let $\psi_{\vec{a}}$ be $\psi^1_{\vec{a}} \wedge \psi^2_{\vec{a}} \wedge \cdots \wedge \psi^n_{\vec{a}}$. It is easy to see that $\vec{a} \in S_{\psi_{\vec{a}}}$ for every $\vec{a}$, and that $F^n_{\psi_{\vec{a}}}(\vec{a}) = t$ in case $\vec{a}$ is a stable element of $\mathcal{C}$. We show now that $S^n_{\psi_{\vec{a}}} \subseteq \mathcal{C}$. This is obvious in case $\vec{a}$ is an element of $\mathcal{C}$ which is not stable (because $\mathcal{C}$ is a cone). Assume that $\vec{a}$ is a stable element of $\mathcal{C}$, and that $\vec{x} \in S^n_{\psi_{\vec{a}}}$. Then $x_i \geq a_i$ for every i such that $a_i \neq \top$. Define:

$$c_i = \begin{cases} a_i & a_i \neq \top \\ x_i & a_i = \top \end{cases}$$

Then $\vec{c} \leq \vec{a}$, and so $\vec{c} \in \mathcal{C}$ (since $\vec{a}$ is stable in $\mathcal{C}$). But $\vec{c} \leq \vec{x}$ also, and so $\vec{x} \in \mathcal{C}$ (because $\mathcal{C}$ is a cone).

Define now ψ_C to be $\bigvee_{\vec{a} \in \mathcal{C}} \psi_{\vec{a}}$ in case $\mathcal{C}$ is not empty, f otherwise. Given what we have shown, it is obvious that ψ_C has the required properties.

Theorem 2.8 Any subset of $THREE^n$ which can be characterized by some formula in $\{\neg, \wedge, \top\}$ is a nonempty cone. Conversely, every nonempty cone $\mathcal{C}$ in $THREE^n$ can be characterized by a formula ψ_C in $\{\neg, \wedge\}$, so that if $\vec{x}$ is a stable element of $\mathcal{C}$ other than $\vec{\top}$ then $\psi^n_C(\vec{x}) = t$.

Proof: If ϕ is a formula in $\{\neg, \wedge, \top\}$ then $\vec{\top} \in S^n_{\phi}$. This entails the first part. For the second part, note that the propositional constant f was used in the previous proof twice. It was used at the end, for characterizing the empty cone. This is not needed here. It was also used in the definition of $\psi^i_{\vec{a}}$ in case $\vec{a}$ is stable. But unless $\vec{a} = \vec{\top}$, the conjuncts t can be deleted from this definition. It remains to check the case when $\vec{\top}$ is a stable element of $\mathcal{C}$. This happens iff $\mathcal{C}$ is $THREE^n$, and in this case $\neg f_n$ (see the proof of Theorem 2.2) is a formula as required.

2.2 Representation of operations on $THREE^n$

Theorem 2.9 The language $\{\neg, \wedge, \supset, f, \top\}$ is functionally complete for $THREE$ (i.e.: every function from $THREE^n$ to $THREE$ is representable by some formula in this language).

Proof: Let $g : THREE^n \to THREE$. By Theorem 2.3, every subset of $THREE^n$ is characterizable in the language. Let, accordingly, ψ^g_f and $\psi^g_\top$ characterize $g^{-1}(\{f\})$ and $g^{-1}(\{\top\})$ respectively. Define: $\Psi^g = (\psi^g_f \supset f) \wedge (\psi^g_\top \supset \top)$. It is easy to verify that Ψ^g represents g.

Theorem 2.10 A 3-valued operation g is representable in the language $\{\neg, \wedge, \supset, f\}$ iff it is closed.

Proof: Obviously, every three-valued operation which is representable in the language $\{\neg, \wedge, \supset, f\}$ is closed. For the converse, replace in the definition of Ψ^g from the proof of 2.9 the constant $\top$ with the formula:

$$\top_n = (p_1 \vee \neg p_1) \wedge (p_2 \vee \neg p_2) \wedge \ldots \wedge (p_n \vee \neg p_n)$$

It is easy to verify that $\top_n$ has the following property:

$$\nu(\top_n) = \begin{cases} \top & \exists 1 \leq i \leq n \;\; \nu(p_i) = \top \\ t & \text{otherwise} \end{cases}$$

This implies that if g is closed and $g(\vec{x}) = \top$ then $F^n_{\top_n}(\vec{x}) = \top$. This easily entails that the new Ψ^g represents g.

Note: The language $\{\neg, \wedge, \supset, f\}$ is equivalent to the language used in the paraconsistent system J_3 (see [DO85, Ep95]).

Theorem 2.11 A 3-valued operation g is representable in the language $\{\neg, \wedge, \supset, \top\}$ iff it is free.

Proof: Again, the "if" part is obvious. For the converse, assume that g is free, and let ϕ^g_f, $\phi^g_\top$ be the formulas in the language of $\{\neg, \wedge, \supset\}$ which characterize (respectively) $\{\vec{\top}\} \cup g^{-1}(\{f\})$ and $g^{-1}(\{\top\})$ (such formulas exist by theorem 2.2). Define: $\Phi^g = (\phi^g_f \supset f_n) \wedge (\phi^g_\top \supset \top)$, where f_n is the formula which was introduced in the proof of 2.2. We show that Φ^g represents g. Let $\vec{x} \in THREE^n$ and assume that $\nu(p_i) = x_i$ for $i = 1, \ldots, n$.

Case 1: $g(\vec{x}) = t$. Since g is free, $\vec{x} \neq \vec{\top}$. This and the fact that $g(\vec{x}) \neq f$ imply that $\vec{x} \notin \{\vec{\top}\} \cup g^{-1}(\{f\})$. Therefore $\nu(\phi^g_f) = f$, and so $\nu(\phi^g_f \supset f_n) = t$. The fact that $\nu(\phi^g_\top \supset \top) = t$ follows similarly. Hence $\nu(\Phi^g) = t = g(\vec{x})$.

Case 2: $g(\vec{x}) = f$. Again, since g is free, $\vec{x} \neq \vec{\top}$, and so $\nu(f_n) = f$. In addition, $\nu(\phi^g_f) \in \{t, \top\}$ in this case, and so $\nu(\phi^g_f \supset f_n) = f$. It follows that $\nu(\Phi^g) = f = g(\vec{x})$.

Case 3a: $g(\vec{x}) = \top$ and $\vec{x} \neq \vec{\top}$. Then $\nu(\phi^g_\top) \in \{t, \top\}$ and so $\nu(\phi^g_\top \supset \top) = \top$. On the other hand, by the same arguments as in case 1, $\nu(\phi^g_f \supset f_n) = t$. Hence $\nu(\Phi^g) = \top = g(\vec{x})$.

Case 3b: $\vec{x} = \vec{\top}$. Since Φ^g is in the language of $\{\neg, \wedge, \supset\}$, $\nu(\Phi^g) = \top = g(\vec{x})$.

Theorem 2.12 A 3-valued operation g is representable in the language $\{\neg, \wedge, \supset\}$ iff it is closed and free.

Proof: The "if" is trivial. The proof of the converse is a combination of the previous two proofs (i.e.: we use f_n as a substitute for f, $\top_n$ as a substitute for $\top$, ϕ^g_f instead of ψ^g_f, and $\phi^g_\top$ instead of $\psi^g_\top$). Details are left for the reader.

Theorem 2.13 A 3-valued operation g is representable in the language $\{\neg, \wedge, \top, f\}$ iff it is monotonic.

Proof: The "if" is trivial. For the converse, let g be a monotonic function of arity n. Define:

$$g_t = \{\vec{x} \in THREE^n \mid g(\vec{x}) \geq_k t\}$$

$$g_f = \{\vec{x} \in THREE^n \mid g(\vec{x}) \geq_k f\}$$

Since g is monotonic, both g_t and g_f are cones. Moreover: if $g(\vec{x}) = t$ then $\vec{x}$ is a stable element of g_t, while if $g(\vec{x}) = f$ then $\vec{x}$ is a stable element of g_f. Let ψ_t and ψ_f be, respectively, the formulas which characterize these cones according to theorem 2.7. Define:

$$\Psi^g = (\psi_t \wedge \top) \vee (\neg\psi_f \wedge \top) \vee (\psi_t \wedge \neg\psi_f) \quad (= \psi_t \oplus \neg\psi_f)$$

Now if $g(\vec{x}) = t$ then $F^n_{\psi_t}(\vec{x}) = t$ (since $\vec{x}$ is stable) while $F^n_{\psi_f}(\vec{x}) = f$ (since $\vec{x} \notin g_f$). It follows that $F^n_{\Psi^g}(\vec{x}) = t$ in this case. Similarly, if $g(\vec{x}) = f$ then $F^n_{\Psi^g}(\vec{x}) = f$. Finally, if $g(\vec{x}) = \top$ then $\vec{x} \in g_t$ and $\vec{x} \in g_f$, and so $F^n_{\psi_t}(\vec{x}) \in \{t, \top\}$ and also $F^n_{\psi_f}(\vec{x}) \in \{t, \top\}$. This implies that $F^n_{\Psi^g}(\vec{x}) = \top$ in this case. Hence $F^n_{\Psi^g}(\vec{x}) = g(\vec{x})$ in all cases, and so Ψ^g represents g.

Theorem 2.14 A 3-valued operation g is representable in the language $\{\neg, \wedge, \top\}$ iff it is monotonic and free.

Proof: The proof is almost identical to that of 2.13, only we use theorem 2.8 instead of 2.7. This is possible because in this case both g_t and g_f are nonempty: since g is free, $\vec{\top}$ belong to both. Moreover: if $g(\vec{x}) = t$ then $\vec{x}$ is a stable element of g_t which is different from $\vec{\top}$, and similarly for g_f. Hence the exceptional case in 2.8 is not relevant here.

Theorem 2.15 A 3-valued operation g is representable in the language $\{\neg, \wedge, f\}$ iff it is monotonic and closed.

Proof: The proof is again almost identical to that of 2.13, only in the definition of Ψ^g we substitute $\top_n$ (from the proof of 2.10) for $\top$. In case $\vec{x} \notin \{t, f\}^n$ this makes no difference, since $\top_n$ is equivalent to $\top$ for such $\vec{x}$. On $\{t, f\}^n$, on the other hand, g is a two-valued function (because it is closed) and it is easy to see that our Ψ^g indeed represents g in the two-valued case.

Theorem 2.16 A 3-valued operation g is representable in the language $\{\neg, \wedge\}$ iff it is monotonic, free, and closed.

Proof: The proof is almost identical to that of of the previous one, only again we rely on theorem 2.8 rather than on theorem 2.7.

The eight theorems that were proved in this subsection provide a full characterization of the representation power of all the subsets of $\{\neg, \wedge, \supset, f, \top\}$ which include $\neg$ and $\wedge$. A precise correspondence has been found between these sets of connectives (which correspond to the 8 subsets of $\{\supset, f, \top\}$) and the 8 possible combinations of monotonicity, freedom, and closure. These eight theorems can therefore be summarized as follows:

Theorem 2.17 Let $L = \{\neg, \wedge\}$ and suppose that Ξ is a subset of $\{\supset, f, \top\}$. A function $g : THREE^n \rightarrow THREE$ is representable in $L \cup \Xi$ iff it satisfies those conditions from the list: "monotonicity", "freedom", and "closure" that all the connectives in Ξ satisfy.

Note: Most of the theorems which were proved in this subsection are equivalent to theorems that have been published before, sometimes with different set of connectives[4]. Thus except for theorems 2.11, 2.12 and 2.17, equivalents of all the other 6 theorems are proved in [Th92]. [5] In addition, equivalents of 2.9 can be found also in [He83, Bl86, La88], of 2.10 in [He83, vB88], Of 2.13 in [Fi75, Bl86], of 2.15 in [vB88], and of 2.14 in [vB88, La88]. Theorems 2.11 and 2.12, on the other

[4]Especially the expressive power of $\supset$, though crucial in J_3, does not seem to have been investigated before.

[5]Instead of our $\supset$ Thijsse has used another connective: $\sim$, which (given $\neg$ and $\wedge$) is equivalent in its expressive power to the combination of $\supset$ and f: $\sim p = p \supset f$.

hand, are new (to the best of my knowledge) and provide a solution to the two cases which remain open in [Th92]. The ease in which we got them here (and the similarity of their proofs to those of the two theorems that precede them) demonstrate the power of the notion of characterizability. Moreover: nothing like theorem 2.17 can be found in the literature (as far as I know). In fact a failure to find a set of connectives which satisfies a theorem of this sort is explicitly noted in [Th92] (the splitting of $\sim$ into $\supset$ and f is what makes theorem 2.17 possible!).

The next theorem provide simple generalizations of some theorems of this subsection.

Theorem 2.18

1. By adding to $\{\neg, \wedge, \supset\}$ any connective which is not free we get a language in which every closed connective is representable.
2. By adding to $\{\neg, \wedge, \supset\}$ any connective which is not closed we get a language in which every free connective is representable.
3. Let L be a language in which $\{\neg, \wedge, \supset\}$ are all representable, as well as at least one connective which is not closed and one (not necessarily distinct) connective which is not free. Then L is functionally complete (for three-valued operations).

Proof:

1. Obviously, if some connective which is not free is definable, then there is such a unary connective $\mathcal{C}$. By 2.10, it suffices now to check that f is representable by $\neg(p \supset p) \wedge \neg(\mathcal{C}(p) \supset \mathcal{C}(p))$.
2. Suppose $g(a_1, \ldots, a_n) = \top$, where $a_i \in \{t, f\}$ for all i. let $h(x) = g(h_1(x), \ldots, h_n(x))$, where $h_i(x) = \neg x \vee x$ if $a_i = t$, $\neg x \wedge x$ if $a_i = f$. Then $h(t) = h(f) = \top$. Obviously, if g is representable in some extension of $\{\neg, \wedge\}$, then so is h. Assume that $\psi(p)$ represents h. Then $(\neg p \vee p) \wedge (\neg\psi(p) \vee \psi(p))$ is equivalent to $\top$. Hence the claim follows from 2.11.

We leave the proof of the third part to the reader.

Corollary 2.19 Denote by CF the set of the closed, free 3-valued connectives, by C the set of the closed 3-valued connectives, and by F the set of the free 3-valued connectives. The following relations obtain in the set of all the sets of 3-valued connectives which are closed under composition:

1. The set of all 3-valued connectives is the only proper extension of C. [6]

2. The set of all 3-valued connectives is the only proper extension of F.

3. The set of all 3-valued connectives, C, and F are the only proper extensions of CF.

Note: In contrast, in [Bl86] it is proved there is exactly one intermediate class of connectives between the set of monotonic connectives and the set of all 3-valued connectives. Hence the set of monotonic connectives does not have in $THREE$ the maximality property that C and F have.

We end this section with a theorem concerning the independence of our five basic connectives.

Theorem 2.20 With the exception of $\wedge$, each of the connectives in $\{\neg, \wedge, \supset, f, \top\}$ is not definable in terms of the rest. $\wedge$, in contrast *is* definable in terms of $\{\neg, \supset, f, \top\}$, and so this set is a minimal functionally complete set of 3-valued connectives (by "minimal" we mean that no proper subset of it is functionally complete).

Proof: By what has been proved above, each element of $\{\supset, f, \top\}$ lacks a property (monotonicity, freedom or closure) which is shared by the other four connectives (and everything which is definable from them). Hence they are all independent of the rest. We describe now a similar non-property of $\neg$: Define a unary function D as follows. $D(f) = f$, $D(t) = D(\top) = t$. Call a 3-valued operation g of arity n *extensional* if $D(g(\vec{x})) = D(g(\vec{D}(\vec{x})))$ for all $\vec{x}$, where $\vec{D}(x_1, \ldots, x_n) = (D(x_1), \ldots, D(x_n))$. It is not difficult to see that every connective which is definable from $\{\wedge, \supset, f, \top\}$ is extensional. $\neg$, in contrast, is not (take $x = \top$).

The proof that $\{\neg, \supset, f, \top\}$ is functionally complete is similar to that of theorem 2.9. We only have to use theorem 2.4 instead of theorem 2.3, and the connective $\bar{\wedge}$ from the proof of that theorem instead of $\wedge$ in the definition of Ψ^g.

3 Expressive Power in Four-Valued Logic

We have many more languages which are defined by subsets of $\mathcal{L}_{all}$ in the four-valued case than the eight we have in the 3-valued case. In this work we concentrate on those that contain either

[6] This particular result has first been proved in [He83].

the standard classical connectives $\{\neg, \wedge, \vee, \supset\}$ or the basic bilattice operations $\{\neg, \wedge, \vee, \oplus, \otimes\}$.

For our investigations we shall need from time to time appropriate substitutes for the basic constants. We list now some possible candidates, together with their main properties.

- $f_n = p_1 \wedge \neg p_1 \wedge p_2 \wedge \neg p_2 \wedge \ldots \wedge p_n \wedge \neg p_n$

$$\nu(f_n) = \begin{cases} \top & \forall 1 \leq i \leq n \;\; \nu(p_i) = \top \\ \bot & \forall 1 \leq i \leq n \;\; \nu(p_i) = \bot \\ f & \text{otherwise} \end{cases}$$

- $f_n^* = \neg(p_1 \supset p_1) \wedge \neg(p_2 \supset p_2) \ldots \wedge \neg(p_n \supset p_n)$

$$\nu(f_n^*) = \begin{cases} \top & \forall 1 \leq i \leq n \;\; \nu(p_i) = \top \\ f & \text{otherwise} \end{cases}$$

- $\top_n = (p_1 \supset p_1) \wedge (p_2 \supset p_2) \wedge \ldots \wedge (p_n \supset p_n)$

$$\nu(\top_n) = \begin{cases} \top & \exists 1 \leq i \leq n \;\; \nu(p_i) = \top \\ t & \text{otherwise} \end{cases}$$

- $\top_n^* = (p_1 \supset p_1) \wedge (p_2 \supset p_2) \wedge \ldots \wedge (p_n \supset p_n) \wedge (\bot p_1 \supset \bot p_1) \wedge \ldots \wedge (\bot p_n \supset \bot p_n)$

$$\nu(\top_n^*) = \begin{cases} \top & \exists 1 \leq i \leq n \;\; \nu(p_i) \notin \{t, f\} \\ t & \text{otherwise} \end{cases}$$

- $\top_n^{**} = \neg p_1 \oplus p_1 \oplus \neg p_2 \oplus p_2 \oplus \cdots \oplus \neg p_n \oplus p_n$

$$\nu(\top_n^{**}) = \begin{cases} \bot & \forall 1 \leq i \leq n \;\; \nu(p_i) = \bot \\ \top & \text{otherwise} \end{cases}$$

- $\bot_n^* = \bigvee_{i=1}^{n} (p_i \wedge ((p_i \vee \neg p_i) \supset f_n^*))$

$$\nu(\bot_n^*) = \begin{cases} \top & \forall 1 \leq i \leq n \;\; \nu(p_i) = \top \\ \bot & \exists 1 \leq i \leq n \;\; \nu(p_i) = \bot \\ f & \text{otherwise} \end{cases}$$

- $\bot_n^{**} = \neg p_1 \otimes p_1 \otimes \neg p_2 \otimes p_2 \otimes \cdots \otimes \neg p_n \otimes p_n$

$$\nu(\bot_n^{**}) = \begin{cases} \top & \forall 1 \leq i \leq n \;\; \nu(p_i) = \top \\ \bot & \text{otherwise} \end{cases}$$

3.1 Characterization of subsets of $FOUR^n$

We start with the following analogues of theorems 2.2–2.4:

Theorem 3.1 A subset S of $FOUR^n$ is characterizable by some formula in the language of $\{\neg, \supset\}$ (or $\{\neg, \wedge, \otimes, \oplus, \supset, \top\}$) iff $\vec{\top} \in S$.

Theorem 3.2 Every subset of $FOUR^n$ is characterizable in $\{\neg, \supset, f\}$ (and so also in $\{\neg, \supset, \perp\}$ and in $\{\neg, \supset, \perp\}$, by identities (vi) and (viii) from the introduction).

The proofs of these theorems are almost identical to their 3-valued counterparts. Only the definition of $\psi_i^{\vec{a}}$ should somewhat be changed. In the case of theorem 3.1 it should be:

$$\psi_i^{\vec{a}} = \begin{cases} p_i \mathbin{\bar{\wedge}} \neg p_i & \text{if } a_i = \top \\ p_i \mathbin{\bar{\wedge}} (\neg p_i \supset f_n^*) & \text{if } a_i = t \\ \neg p_i \mathbin{\bar{\wedge}} (p_i \supset f_n^*) & \text{if } a_i = f \\ (\neg p_i \supset f_n^*) \mathbin{\bar{\wedge}} (p_i \supset f_n^*) & \text{if } a_i = \perp \end{cases}$$

In the case of theorem 3.2 one should use f instead f_n^* in the above definition. [7]

Definition 3.3 $\mathcal{L}_4 = \{\neg, \wedge, \otimes, \oplus, f, \top, \perp\}$.

Note: By the identities in the introduction, $\mathcal{L}_4$ is equivalent to the language $\{\neg, \wedge, \top, \perp\}$.

Theorem 3.4 Any subset of $FOUR^n$ which can be characterized by some formula in $\mathcal{L}_4$ is a cone. Conversely, every cone $\mathcal{C}$ in $FOUR^n$ can be characterized by a formula ψ_C in $\{\neg, \wedge, f\}$.

Proof: It is easy to see that all the connectives of $\mathcal{L}_4$ correspond to $\leq_k$-monotonic functions. Since $\mathcal{D}$ itself is a cone in $FOUR$, every subset of $FOUR^n$ which is characterized by some formula of $\mathcal{L}_4$ is necessarily a cone.

For the converse, assume that $\mathcal{S}$ is a cone in $FOUR^n$. If $\mathcal{S}$ is empty then the formula f characterizes it. If not, then since $\mathcal{S}$ is a cone, it is the union of all the subsets of $FOUR^n$ of the form $\{\vec{x} \in FOUR^n \mid \vec{x} \geq_k \vec{a}\}$, where $\vec{a} \in \mathcal{S}$. By Lemma 2.1 it suffices therefore to show that every set of this form is characterizable in $\mathcal{L}_4$. It is easy however to see that $\{\vec{x} \in FOUR^n \mid \vec{x} \geq_k \vec{a}\}$ is characterized by $\psi_{a_1} \wedge \psi_{a_2} \wedge \cdots \wedge \psi_{a_n}$, where:

$$\psi_{a_i} = \begin{cases} p_i \wedge \neg p_i & a_i = \top \\ p_i & a_i = t \\ \neg p_i & a_i = f \\ t & a_i = \perp \end{cases}$$

[7] Theorems 3.1 and 3.2 have first been proved, using the same argument, in [AA98].

Note: In case $\top$ is available we can of course use it (rather than t) in the last case of the definition of ψ_{a_i} (or any other formula φ s.t. $F^n_\varphi(\vec{x}) \in \mathcal{D}$ for all $\vec{x} \in FOUR^n$). In fact, except for the case $\vec{a} = \vec{\bot}$, we could have done without the conjuncts of the form t, and just delete them from the formula above. Similarly, if $\bot$ is available we can use it (instead of f) for characterizing the empty cone.

Theorem 3.5 Any subset of $FOUR^n$ which can be characterized by some formula in $\{\neg, \wedge, \otimes, \oplus\}$ is a cone which is nonempty and nontrivial. Conversely, every cone of this sort is characterized by some formula in $\{\neg, \wedge\}$.

Proof: The condition is obviously necessary, since $\vec{\top} \in S^n_\varphi$ and $\vec{\bot} \notin S^n_\varphi$ for every φ in the language of $\{\neg, \wedge, \otimes, \oplus\}$. The proof of the converse is very similar to that of Theorem 3.4, only here we do not need to consider the case where $\mathcal{S} = \emptyset$, while in the other case we should replace t by $t_n = \neg f_n$. t_n has the property that $F^n_{t_n}(\vec{x}) \in \mathcal{D}$ unless $\vec{x} = \vec{\bot}$. Since $\vec{\bot} \notin \mathcal{S}$ (because $\mathcal{S}$ is not trivial), this exceptional case is harmless here.

Theorem 3.6

1. A subset $\mathcal{S}$ of $FOUR^n$ is characterizable in the language of $\{\neg, \wedge, \top\}$ (or even $\{\neg, \wedge, \otimes, \oplus, \top\}$) iff it is a nonempty cone.
2. A subset $\mathcal{S}$ of $FOUR^n$ is characterizable in the language of $\{\neg, \wedge, \bot\}$ (or even $\{\neg, \wedge, \otimes, \oplus, \bot\}$) iff it is a nontrivial cone.

The proof is left to the reader.

Theorem 3.7 Let $\mathcal{L}$ be obtained from $\mathcal{L}_4$ by adding to it a connective such that the corresponding function $\mathcal{N}$ [8] is not monotonic. Then every subset of $FOUR^n$ is characterizable in $\mathcal{L}$.

Proof: By Lemma 2.1 it is enough to show that every singleton $\{(a_1, \ldots, a_n)\}$ is characterizable. For this, in turn, it suffices to show that $\{a\}$ is characterizable for all $a \in FOUR$. Indeed, if $\varphi_a(p)$ characterizes $\{a\}$ then obviously $\varphi_{a_1}(p_1) \wedge \varphi_{a_2}(p_2) \wedge \cdots \wedge \varphi_{a_n}(p_n)$ characterizes $\{(a_1, \ldots, a_n)\}$.

[8] For convenience we use the same symbol for an $n\bot$ary connective and the corresponding function on $FOUR^n$.

Now $\{\top\}$ is characterized by $p_1 \wedge \neg p_1$. For the other three singletons we note first that if any of them is characterizable then so are the other two. Indeed, if $\{t\}$ is characterized by the formula $\varphi_t(p_1)$ then $\{f\}$ is characterized by $\varphi_t(\neg p_1)$ while $\{\bot\}$ is characterized by $\varphi_t(p_1 \oplus t) \wedge \varphi_t(\neg p_1 \oplus t)$. The case where $\{f\}$ is characterizable is similar. Finally, if $\{\bot\}$ is characterized by the formula φ_- then $\{t\}$ is characterized by $\varphi_-(p_1 \wedge \bot) \wedge p_1$.

Another obvious observation is that one may assume that $\mathcal{N}$ is unary. This follows from the fact that if $\mathcal{N}'$ is an n-ary connective s.t. $\mathcal{N}'$ is not monotonic, then there exist $a_1, \ldots, a_{i-1}, a_{i+1}, \cdots, a_n,\ b$ and c such $b \leq_k c$ but $\mathcal{N}'(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \not\leq_k \mathcal{N}'(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n)$. It follows that $\mathcal{N}(a_1, \ldots, a_{i-1}, p_1, a_{i-1}, \ldots, a_n)$ defines a unary connective which is not monotonic.

So assume that $\mathcal{N}$ is unary and $\mathcal{N}$ is not monotonic. Since $\mathcal{N}(x) \leq_k \mathcal{N}(x)$ for all $x \in FOUR$, we have five cases to consider:

Case 1: $\mathcal{N}(t) \not\leq_k \mathcal{N}(\top)$. Then $\mathcal{N}(t) \neq \bot$.

- **subcase 1.1:** $\mathcal{N}(t) = t$. Then $\mathcal{N}(\top) \in \{\bot, f\}$ and so $p_1 \wedge \mathcal{N}(p_1)$ characterizes $\{t\}$.
- **subcase 1.2:** $\mathcal{N}(t) = f$. Then $\mathcal{N}(\top) \in \{\bot, t\}$ and $p_1 \wedge \neg\mathcal{N}(p_1)$ characterizes $\{t\}$.
- **subcase 1.3:** $\mathcal{N}(t) = \top$. Then $\mathcal{N}(\top) \neq \top$ and $p_1 \wedge \mathcal{N}(p_1) \wedge \neg\mathcal{N}(p_1)$ characterizes $\{t\}$.

Case 2: $\mathcal{N}(f) \not\leq_k \mathcal{N}(\top)$. Similar.

Case 3: $\mathcal{N}(\bot) \not\leq_k \mathcal{N}(\top)$. Then $\mathcal{N}(\bot) \neq \bot$.

- **subcase 3.1:** $\mathcal{N}(\bot) = t$. Then $\mathcal{N}(\top) \in \{\bot, f\}$, and so $\mathcal{N}(p_1 \oplus \neg p_1)$ characterizes $\{\bot\}$.
- **subcase 3.2:** $\mathcal{N}(\bot) = f$. Then $\mathcal{N}(\top) \in \{\bot, t\}$ and $\neg\mathcal{N}(p_1 \oplus \neg p_1)$ characterized $\{\bot\}$.
- **subcase 3.3:** $\mathcal{N}(\bot) = \top$. Then $\mathcal{N}(\top) \neq \top$ and $\mathcal{N}(p_1 \oplus \neg p_1) \wedge \neg\mathcal{N}(p_1 \oplus \neg p_1)$ characterizes $\{\bot\}$.

Case 4: $\mathcal{N}(\bot) \not\leq_k \mathcal{N}(t)$. Then $\mathcal{N}(\bot) \neq \bot$.

- **subcase 4.1:** $\mathcal{N}(\bot) = t$. Then $\mathcal{N}(t) \in \{\bot, f\}$, and so $\mathcal{N}(p_1 \vee \bot) \wedge \mathcal{N}(\neg p_1 \vee \bot)$ characterizes $\{\bot\}$.
- **subcase 4.2:** $\mathcal{N}(\bot) = f$. Then $\mathcal{N}(t) \in \{\bot, t\}$ and $\neg\mathcal{N}(p_1 \vee \bot) \wedge \neg\mathcal{N}(\neg p_1 \vee \bot)$ characterizes $\{\bot\}$.

subcase 4.3: $\mathcal{N}(\perp) = \top$. Then $\mathcal{N}(t) \neq \top$, and so $\mathcal{N}(p_1 \vee \perp) \wedge \neg\mathcal{N}(p_1 \vee \perp) \wedge \mathcal{N}(\neg p_1 \vee \perp) \wedge \neg\mathcal{N}(\neg p_1 \vee \perp)$ characterizes $\{\perp\}$.

Case 5: $\mathcal{N}(\perp) \not\leq_k \mathcal{N}(f)$. Similar.

3.2 Representation of operations on $FOUR^n$

In what follows we write $\top$-closed instead $\{t, f, \top\}$-closed, $\perp$-closed instead of $\{t, f, \perp\}$-closed, and classically closed instead of $\{t, f\}$-closed.

We begin with languages which contain $\{\neg, \wedge, \supset\}$.

Theorem 3.8 The language $L^* = \{\neg, \wedge, \supset, \perp, \top\}$ is functionally complete for $FOUR$.

Proof: Let $g : FOUR^n \to FOUR$. Since $f = \neg(\perp \supset \perp)$, by Theorem 3.2 every subset of $FOUR^n$ is characterizable in L^*. Let, accordingly, ψ^g_f, $\psi^g_\top$, and $\psi^g_\perp$ characterize $g^{-1}(\{f\})$, $g^{-1}(\{\top\})$, and $g^{-1}(\{\perp\})$, respectively. Define: $\Psi^g = (\psi^g_f \supset f) \wedge (\psi^g_\top \supset \top) \wedge (\psi^g_\perp \supset \perp)$. It is easy to verify that Ψ^g represents g.

The identities in the introduction imply that relative to $\{\neg, \wedge, \supset\}$ the connectives $\top$ and $\oplus$ are interdefinable, while $\perp$ is equivalent in expressive strength to the combination of $\otimes$ and f. It follows that the set $\{\neg, \wedge, \supset, \otimes, \oplus, f\}$ is also functionally complete. The next theorem show that there is a nice correspondence between subsets of this set which contain $\{\neg, \wedge, \supset\}$ and combinations of basic properties.

Theorem 3.9 Let $L = \{\neg, \wedge, \supset\}$ and suppose that Ξ is a subset of $\{\otimes, \oplus, f\}$. A function $g : FOUR^n \to FOUR$ is representable in $L \cup \Xi$ iff it satisfies those conditions from the list: "$\top$-freedom", "$\top$-closure", and "$\perp$-closure" that all the connectives in Ξ satisfy.

Proof: The proofs closely follows that of Theorem 3.8, and are very similar to the proofs of 2.9-2.12. The following changes should be made:

1. If f is not available we use f^*_n as a substitute. In addition, instead of ψ^g_f, $\psi^g_\top$, and $\psi^g_\perp$ (which are not available in this case) we use the formulas in the language of $\{\neg, \wedge, \supset\}$ which characterize $\{\vec{\top}\} \cup g^{-1}(\{f\})$, $\{\vec{\top}\} \cup g^{-1}(\{\top\})$, and $\{\vec{\top}\} \cup g^{-1}(\{\perp\})$ (such formulas exist by Theorem 3.1).

2. If $\top$ is not available (i.e., $\oplus \notin \Xi$) then we use $\top_n$ as a substitute.

3. If $\bot$ is not available (i.e., $\{\otimes, f\} \not\subseteq \Xi$) and $\otimes \in \Xi$ we use $\bot_n^{**}$ as a substitute.

4. If $\bot$ is not available and $\otimes \notin \Xi$ we use $\bot_n^{*}$ as a substitute.

Following these guidelines, it is not difficult to prove the theorem.

Note: The last two theorems were first proved in [AA98]. We have repeated them here for the sake of completeness. Theorems which are equivalent to 3.8 have been proved in [He83, La88, Gi90, Th92]. Of the seven other claims included in 3.9, equivalent theorems have been proved in [La88] for the set of $\top$-closed connectives and in [La88, Th92] for the set of connectives which are both $\top$-closed and $\bot$-closed (such connectives are called *generally closed* in [Th92]). Instead of our $\supset$ these works use the connectives $\sim$, which is definable in our languages by: $\sim p = p \supset f$.

We turn to our first two maximality results in the context of $FOUR$.

Theorem 3.10 Any proper extension of the set of $\top$-closed connectives is functionally complete. The same applies to the set of $\bot$-closed connectives.

Proof: By duality, it suffices to prove the second part. Suppose then that g is not $\bot$-closed. Then there are $a_1, \ldots, a_n \in \{t, f, \bot\}$ such that $g(a_1, \ldots, a_n) = \top$. It follows that $\top$ is definable from $\{g, t, f, \bot\}$. Hence, by theorem 3.8, the set $\{g, t, f, \bot, \neg, \wedge, \supset\}$ is functionally complete. In this set all elements except g are $\bot$-closed. This entails the theorem.

Theorem 3.11 Any proper extension of the set of $\top$-free connectives is functionally complete. The same applies to the set of $\bot$-free connectives.

Proof: By duality, it suffices to prove the first part. Suppose then that g is not $\top$-free. Without loss in generality we may assume that g is unary and $g(\top) \neq \top$. Hence $\neg(a \supset a) \wedge \neg(g(a) \supset g(a)) = f$ for all $a \in FOUR$. It follows that f is definable from $\{g, \neg, \wedge, \supset\}$. Hence, by theorem 3.9, the set $\{g, \neg, \wedge, \supset, \oplus, \otimes\}$ is functionally complete. In this set all elements except g are $\top$-free. This entails the theorem.

Theorem 3.12 An operation g is representable in $\{\neg, \wedge, \supset, \perp\}$ iff it is classically closed.

Proof: All the connectives in $\{\neg, \wedge, \supset, \perp\}$ are classically closed. Hence the "only if" part. For the converse, note first that f is definable in the language (by identity (viii) from the introduction). Assume now that g is classically closed, and let Ψ^g be defined as in the proof of theorem 3.8, but with $\top_n^*$ and $\perp\top_n^*$ instead of $\top$ and $\perp$ (respectively). Assume, e.g., that $g(\vec{x}) = \perp$. Since g is classically closed, there exists i such that $x_i \notin \{t, f\}$. hence $F_{\neg\top_n^*}(\vec{x}) = \perp$, and so also $F_{\Psi^g}(\vec{x}) = \perp = g(\vec{x})$. The case where $g(\vec{x}) = \top$ is similar, while the cases where $g(\vec{x}) = t$ and $g(\vec{x}) = f$ are exactly as in the proof of theorem 3.8.

Note: Theorems which are equivalent to theorem 3.12 have been proved in [He83, Th92]. The next theorem has also first been proved in [He83].

Theorem 3.13 Any proper extension of the set of classically closed connectives is functionally complete.

Proof: Suppose g is not classically closed. Then there are $a_1, \ldots, a_n \in \{t, f\}$ such that $g(a_1, \ldots, a_n)$ is in $\{\top, \perp\}$. It follows that $\top$ and $\perp$ are both definable from $\{g, t, f, \perp\}$. Hence, by theorem 3.8, the set $\{g, t, f, \perp, \neg, \wedge, \supset\}$ is functionally complete. In this set all elements except g are classically closed. This entails the theorem.

Corollary 3.14 Any proper extension of $\{\neg, \wedge, \supset, \perp\}$ is functionally complete.

The last corollary entails that theorems 3.9 and 3.12 provide full characterizations of the definability power of all subsets of $\mathcal{L}_{all}$ which contain $\{\neg, \wedge, \supset\}$. We turn now to subsets of $\mathcal{L}_{all}$ that contain $\{\neg, \wedge, \vee, \oplus, \otimes\}$ but not $\supset$.

Definition 3.15 Let $g : FOUR^n \to FOUR$. Define:

$$g_t = \{\vec{x} \in FOUR^n \mid g(\vec{x}) \geq_k t\}$$

$$g_f = \{\vec{x} \in FOUR^n \mid g(\vec{x}) \geq_k f\}$$

The following lemma is immediate from the definitions:

Lemma 3.16 If g is monotonic then g_t and g_f are cones. If g is monotonic and $\top$-free they are nonempty cones, while if it is monotonic and $\bot$-free they are non-trivial cones.

The key for obtaining our next results is the following simple lemma:

Lemma 3.17 Let $\mathcal{L}$ be a propositional language which contains $\neg, \wedge, \otimes$ and $\top$, and let $g : FOUR^n \to FOUR$. Assume that g_t and g_f are characterized by formulas in $\mathcal{L}$. Then g itself is representable in $\mathcal{L}$.

Proof: Suppose g_t and g_f are characterized by ψ_t and ψ_f, respectively. We claim that the formula $\psi = (\psi_t \wedge \top) \otimes (\neg\psi_f \vee \top)$ represents g. Indeed:

- If $g(\vec{x}) = \top$ then $F^n_{\psi_t}(\vec{x}) \in \{t, \top\}, \neg F^n_{\psi_f}(\vec{x}) \in \{f, \top\}$ and so $F^n_{\psi}(\vec{x}) = \top \otimes \top = \top$.
- If $g(\vec{x}) = t$ then $F^n_{\psi_t}(\vec{x}) \in \{t, \top\}, \neg F^n_{\psi_f}(\vec{x}) \in \{t, \bot\}$ and so $F^n_{\psi}(\vec{x}) = \top \otimes t = t$.
- If $g(x) = f$ then $F^n_{\psi_t}(\vec{x}) \in \{\bot, f\}$, $\neg F^n_{\psi_f}(\vec{x}) \in \{f, \top\}$ and so $F^n_{\psi}(\vec{x}) = f \otimes \top = f$.
- If $g(x) = \bot$ then $F^n_{\psi_t}(\vec{x}) \in \{\bot, f\}$, $\neg F^n_{\psi_f}(\vec{x}) \in \{\bot, t\}$ and so $F^n_{\psi}(\vec{x}) = f \otimes t = \bot$.

We turn to some corollaries of the last lemma and its proof. We begin with a result which is attributed in [Th92] to R. Muskens ([Mu89]).

Theorem 3.18 An operation g is representable in $\mathcal{L}_4$ (Definition 3.3) iff it is monotonic.

Proof: The condition is necessary since the operations which correspond to the connectives of $\mathcal{L}_4$ are all $\leq_k$-monotonic. Its sufficiency follows from Lemma 3.16, theorem 3.4 and lemma 3.17.

Theorem 3.19 An operation $g : FOUR^n \to FOUR$ is representable in the language of $\{\neg, \wedge, \otimes, \top\}$ (=$\{\neg, \wedge, \otimes, \oplus, \top\}$) iff it is monotonic and $g(\vec{\top}) = \vec{\top}$.

Proof: The conditions are obviously necessary. Their sufficiency easily follows from Lemma 3.16, theorem 3.6 and lemma 3.17.

Theorem 3.20 1. An operation $g : FOUR^n \to FOUR$ is representable in the language of $\{\neg, \wedge, \otimes, \oplus\}$ iff it is monotonic, $\top$-free, and $\bot$-free.

2. An operation $g : FOUR^n \to FOUR$ is representable in the language of $\{\neg, \wedge, \oplus, \bot\}$ iff it is monotonic and $\bot$-free.

Proof:

1. The three conditions are obviously necessary. On the other hand if g satisfies all of them then g_t and g_f are characterizable in $\{\neg, \wedge, \otimes, \oplus\}$ by Theorem 3.5. Now Lemma 3.17 is not directly applicable here, since $\top$ is not available. We can however mimic its proof in this case, using $\top_n^{**}$ instead of $\top$. The only case that might be problematic in the reproduction of the proof of Lemma 3.17 is when $\vec{x} = \vec{\bot}$. The $\bot$-freedom of g ensures, however, that in this case too we have the desired result.

2. We leave the proof to the reader.

The next theorem is one of the main results of this paper. It shows a striking difference with respect to monotonic functions between the three-valued case and the four-valued one.

Theorem 3.21 Any proper extension of $\mathcal{L}_4$ is functionally complete.

Proof: Immediate from Theorem 3.7 and Lemma 3.17.

Theorem 3.22 The set $\mathcal{L}^* = \{\neg, \wedge, \otimes, \bot\}$ is functionally complete for $FOUR$. Moreover, it is a minimal such set in the sense that none of its proper subsets has this property.

Proof: By identities (viii)-(x) from the introduction, every connective of $\mathcal{L}_4$ is definable in $\mathcal{L}^*$. Since $\bot$ is not monotonic this implies, by Theorem 3.21 , that $\mathcal{L}^*$ is functionally complete. The fact that $\bot$ is not monotonic entails also that it is undefinable in terms of $\neg, \wedge$ and $\otimes$. By duality, $\neg$ is not definable in terms of $\bot, \wedge, \otimes$. That $\otimes$ is not definable in terms of $\neg, \wedge$ and $\bot$ follows from the fact that every $g : FOUR^n \to FOUR$ which is representable in $\{\neg, \wedge, \bot\}$ is classically closed, while $\otimes$ is not. That $\wedge$ is not definable in $\{\neg, \otimes, \bot\}$ follows by duality.

The last theorem and the identities from the introduction imply that we have characterized all the subsets of $\mathcal{L}_{all}$ which include $\{\neg, \wedge, \otimes, \oplus\}$ but not $\supset$.

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