Logical Non-determinism as a Tool for Logical Modularity: An Introduction

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1 Introduction

It is well known that every propositional logic which satisfies certain very natural conditions can be characterized semantically using a multi-valued matrix ([Łos and Suszko, 1958; Wójcicki, 1988; Urquhart, 2001]). However, there are many important decidable logics whose characteristic matrices necessarily consist of an infinite number of truth values. In such a case it might be quite difficult to find any of these matrices, or to use one when it is found. Even in case a logic does have a finite characteristic matrix it might be difficult to discover this fact, or to find such a matrix. The deep reason for these difficulties is that in an ordinary multi-valued semantics the rules and axioms of a system should be considered as a whole, and there is no method for separately determining the semantic effects of each rule alone.

In this introductory paper (which is based on several previous papers of the author) it is shown that by allowing the use of non-deterministic operations, one can provide in a lot of cases simple modular semantics of rules of inference, so that the semantics of a system is obtained by joining the semantics of its rules in the most straightforward way. Our main tool for this task is the use of finite Nmatrices ([Avron and Lev, 2001; Avron and Lev, 2004; Avron and Lev, 2005]. Nmatrices are multi-valued structures in which the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. The use of finite structures of this sort has the benefit of preserving all the advantages of logics with ordinary finite-valued semantics (in particular: decidability and compactness), while it is applicable to a much larger family of logics. The central idea in using Nmatrices for providing semantics for rules is that the main effect of a "normal" rule is to reduce the degree of non-determinism of operations, by forbidding some options (in non-deterministic computations of truth values) which we could have had otherwise. Many examples of applying this idea are provided below.

2 Basic Concepts

2.1 Consequence Relations, Logics, and Pure Rules

DEFINITION 1.

1. A *Scott consequence relation* (*scr* for short) for a language $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ that satisfies the following three conditions:

   strong reflexivity: if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$.
   monotonicity: if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$.
   Transitivity (*cut*): if $\Gamma \vdash \psi, \Delta$ and $\Gamma', \psi \vdash \Delta'$ then $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

2. An scr $\vdash$ for $\mathcal{L}$ is *structural* if for every uniform $\mathcal{L}$-substitution σ and every Γ and Δ, if $\Gamma \vdash \Delta$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$. $\vdash$ is *finitary* if the following condition holds for all $\Gamma, \Delta \subseteq \mathcal{W}$: if $\Gamma \vdash \Delta$ then there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash \Delta'$. $\vdash$ is *consistent* (or *non-trivial*) if there exist non-empty Γ and Δ s.t. $\Gamma \not\vdash \Delta$. [1]

3. A propositional *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language and $\vdash$ is an scr for $\mathcal{L}$ which is structural and consistent. The logic $\langle \mathcal{L}, \vdash \rangle$ is finitary if $\vdash$ is finitary.

DEFINITION 2.

1. A *pure rule* in a propositional language $\mathcal{L}$ is any ordered pair $\langle \Gamma, \Delta \rangle$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$ (We shall usually denote such a rule by $\Gamma \Rightarrow \Delta$ rather than by $\langle \Gamma, \Delta \rangle$).

2. Let $\langle \mathcal{L}, \vdash_1 \rangle$ be a propositional logic, and let S be a set of rules in a propositional language $\mathcal{L}'$. By the *extension* of $\langle \mathcal{L}, \vdash_1 \rangle$ by S we mean the logic $\langle \mathcal{L}^*, \vdash^* \rangle$, where $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}'$, and $\vdash^*$ is the least *structural* scr $\vdash$ such that $\Gamma \vdash \Delta$ whenever $\Gamma \vdash_1 \Delta$ or $\langle \Gamma, \Delta \rangle \in S$.

REMARK 3. Obviously, the extension of $\langle \mathcal{L}, \vdash_1 \rangle$ by S is well-defined (i.e. a logic) only if $\vdash^*$ is consistent. In all the cases we consider below this will easily be guaranteed by the semantics we provide (and so we shall not even mention it).

REMARK 4. It is easy to see that $\vdash^*$ is the closure under cuts and weakenings of the set of all pairs $\langle \sigma(\Gamma), \sigma(\Delta) \rangle$, where σ is a uniform substitution in

[1] See [Avron and Lev, 2001; Avron and Lev, 2005] for the importance of the consistency property.

$\mathcal{L}^*$, and either $\Gamma \vdash_1 \Delta$ or $\langle \Gamma, \Delta \rangle \in S$. This in turn implies that an extension of a finitary logic by a set of pure rules is again finitary.

CONVENTION: To emphasize the fact that the presence of a rule in a system means the presence of all its instances, we shall usually describe a rule using the metavariables φ, ψ, θ rather than the atomic formulas $p_1, p_2, \ldots$. Thus although formally the rule ($\supset\Rightarrow$) is the rule $p_1, p_1 \supset p_2 \Rightarrow p_2$, we shall write it as $\varphi, \varphi \supset \psi \Rightarrow \psi$.

REMARK 5. Suppose that the formula θ occurs in a pure rule of a logic $\mathcal{L}$, and we decide to select θ as the "principal formula" of that rule. Assume e.g. that the rule is of the form $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_k, \theta$ (the consideration in the other case is similar). Suppose further that $\Gamma_i \vdash \Delta_i, \varphi_i$ for $i = 1, \ldots, n$ and $\psi_j, \Gamma_j \vdash \Delta_j$ for $j = 1, \ldots, k$. Then $\Gamma_1, \ldots, \Gamma_n \vdash \Delta_1, \ldots, \Delta_k, \theta$ (by n+k cuts). It follows that $\mathcal{L}$ is closed in this case under the Gentzen-type rule:

$$\frac{\Gamma_i \Rightarrow \Delta_i, \varphi_i \;\; (i = 1, \ldots, n) \qquad \psi_j, \Gamma_j \Rightarrow \Delta_j \;\; (j = 1, \ldots, k)}{\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_k, \theta}$$

Conversely, if $\mathcal{L}$ is closed under this Gentzen-type rule then by applying it to the reflexivity axioms $\varphi_i \vdash \varphi_i$ $(i = 1, \ldots, n)$ and $\psi_j \vdash \psi_j$ $(j = 1, \ldots, k)$ we get $\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_k, \theta$. It follows that every pure rule in the sense of Definition 2 is equivalent to some *multiplicative* (in the terminology of [Girard, 1987]) or *pure* (in the terminology of [Avron, 1991]) Gentzen-type rule. Moreover: it is easy to see that most standard rules used in Gentzen-type systems are equivalent to finite sets of pure rules in the sense of Definition 2. For example: the usual ($\supset\Rightarrow$) rule of classical logic is equivalent by what we have just shown to the pure rule $\varphi, \varphi \supset \psi \Rightarrow \psi$. The classical ($\Rightarrow\supset$), in turn, can be split into the following two rules:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi \supset \psi} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

Hence ($\Rightarrow\supset$) is equivalent to the set $\{\psi \Rightarrow \varphi \supset \psi, \quad \Rightarrow \varphi, \varphi \supset \psi\}$. [2]

2.2 Non-deterministic Matrices

Our main semantic tool in what follows will be the following generalization from [Avron and Lev, 2001; Avron and Lev, 2004; Avron and Lev, 2005] of the concept of a matrix: [3]

[2] Recall that formally we should have written here $\{p_2 \Rightarrow p_1 \supset p_2, \quad \Rightarrow p_1, p_1 \supset p_2\}$.

[3] A special two-valued case of this definition was essentially introduced in [Batens *et al.*, 1999]. Another particular case of the same idea, using a similar name, was used in [Crawford and Etherington, 1998]. It should also be noted that Carnielli's "possible-translations semantics" (see [Carnielli and Marcos, 2002]) was originally called "non-deterministic semantics", but later the name was changed to the present one.

DEFINITION 6.

1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

   (a) $\mathcal{V}$ is a non-empty set of *truth values*.

   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.

   (c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.

   We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{V}$.

2. Let $\mathcal{W}$ be the set of formulas of $\mathcal{L}$. A *(legal) valuation* in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{V}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{W}$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

3. A valuation v in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model* in $\mathcal{M}$ of a set Γ of formulas (notation: $v \models^{\mathcal{M}} \Gamma$) if it satisfies every formula in Γ.

4. $\vdash_{\mathcal{M}}$, the consequence relation induced by the Nmatrix $\mathcal{M}$, is defined by: $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every v such that $v \models^{\mathcal{M}} \Gamma$, there is $\varphi \in \Delta$ such that $v \models^{\mathcal{M}} \varphi$.

5. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is *sound* for an Nmatrix $\mathcal{M}$ (where $\mathcal{L}$ is the language of $\mathcal{M}$) if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{M}}$. $\mathbf{L}$ is *complete* for $\mathcal{M}$ if $\vdash_{\mathbf{L}} \supseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is *characteristic* for $\mathbf{L}$ if $\mathbf{L}$ is both sound and complete for it (i.e.: if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{M}}$). $\mathcal{M}$ is *weakly-characteristic* for $\mathbf{L}$ if for every formula φ of $\mathcal{L}$, $\vdash_{\mathbf{L}} \varphi$ iff $\vdash_{\mathcal{M}} \varphi$.

REMARK 7. We shall identify an ordinary (deterministic) matrix with an Nmatrix whose functions in $\mathcal{O}$ always return singletons.

THEOREM 8 ([Avron and Lev, 2005; Avron and Lev, 2004]). *A logic which has a finite characteristic Nmatrix is finitary and decidable.*

The following Definition and Theorem are from [Avron, 2005a]:

DEFINITION 9. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$.

1. A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F : \mathcal{V}_1 \to \mathcal{V}_2$ such that:

   (a) For every $x \in \mathcal{V}_1$, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$.

(b) $F(y) \in \widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $x_1, \ldots, x_n, y \in \mathcal{V}_1$ such that $y \in \widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n)$.

2. $\mathcal{M}_1$ is a *refinement* of $\mathcal{M}_2$ if there exists a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$.

THEOREM 10. *If* $\mathcal{M}_1$ *is a refinement of* $\mathcal{M}_2$ *then* $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Proof. Assume that F is a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. We first show that if v is a legal valuation in $\mathcal{M}_1$ then $v' = F \circ v$ (the composition of F and v) is a legal valuation in $\mathcal{M}_2$. Indeed, let $\diamond$ be an n-ary connective of $\mathcal{L}$, and let $\varphi_1, \ldots, \varphi_n$ be n formulas of $\mathcal{L}$. We show that $v'(\diamond(\varphi_1, \ldots, \varphi_n)) \in \widetilde{\diamond}_{\mathcal{M}_2}(v'(\varphi_1), \ldots, v'(\varphi_n))$. Let $y = v(\diamond(\varphi_1, \ldots, \varphi_n))$, and $x_i = v(\varphi_i)$ $(i = 1, \ldots, n)$. Then $y \in \widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n)$, and so $F(y) \in \widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n))$. Since $v'(\diamond(\varphi_1, \ldots, \varphi_n) = F(y)$ and $v'(\varphi_i) = F(x_i)$ $(i = 1, \ldots, n)$, our claim follows.

Now assume that $\Gamma \vdash_{\mathcal{M}_2} \Delta$. We show that $\Gamma \vdash_{\mathcal{M}_1} \Delta$ as well. So let v be a model of Γ in $\mathcal{M}_1$. Then $v(\varphi) \in \mathcal{D}_1$ for every $\varphi \in \Gamma$. Hence $F(v(\varphi)) \in \mathcal{D}_2$ for every $\varphi \in \Gamma$. Since $F \circ v$ is a legal valuation in $\mathcal{M}_2$, this means that $F \circ v$ is a model of Γ in $\mathcal{M}_2$, and so $F(v(\psi)) = (F \circ v)(\psi) \in \mathcal{D}_2$ for some $\psi \in \Delta$. Since F is a reduction function, this implies that $v(\psi) \in \mathcal{D}_1$ for some $\psi \in \Delta$, as required. ■

REMARK 11. An important case in which $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ is a refinement of $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ is when $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, and $\widetilde{\diamond}_{\mathcal{M}_1}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(\vec{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{V}_1^n$. It is easy to see that the identity function on $\mathcal{V}_1$ is in this case a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. A refinement of this sort will be called *simple.* [4]

3 Canonical Rules and 2-Valued Nmatrices

In this section we establish a strong connection between the simplest type of pure rules, and the simplest type of Nmatrices.

DEFINITION 12. A *2-Nmatrix* is an Nmatrix in which $\mathcal{V} = \{t, f\}$ and $\mathcal{D} = \{t\}$ (Obviously, any two-valued Nmatrix is isomorphic to some 2-Nmatrix).

DEFINITION 13. A *canonical* (propositional) rule is a pure rule of the form $\diamond(p_1, \ldots, p_n), \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta, \diamond(p_1, \ldots, p_n)$, where $\diamond$ is an n-ary connective, $p_1, \ldots, p_n$ are n distinct atomic formulas, $\Gamma \subseteq \{p_1, \ldots, p_n\}$, $\Delta \subseteq \{p_1, \ldots, p_n\}$, and $\Gamma \cap \Delta = \emptyset$.

[4] What we call here "a simple refinement" is what was called "a refinement" in [Avron, 2004]. The present definition of "a refinement" is a refinement of the definition given to that concept there.

EXAMPLE 14. The three pure rules for implication described in Remark 5 ($\varphi, \varphi \supset \psi \Rightarrow \psi$, $\psi \Rightarrow \varphi \supset \psi$, and $\Rightarrow \varphi, \varphi \supset \psi$) are all canonical.

DEFINITION 15. A set S of canonical rules is *coherent* if whenever both $\diamond(p_1, \ldots, p_n), \Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2, \diamond(p_1, \ldots, p_n)$ are rules of S, we have that $(\Gamma_1 \cup \Gamma_2) \cap (\Delta_1 \cup \Delta_2) \neq \emptyset$.

EXAMPLE 16. The three pure rules for implication from Example 14 form a coherent set of rules. In contrast, the following two pure rules for the famous connective "tonk" of Prior ([Prior, 1960]) form an incoherent set: $\{p \Rightarrow pTq, \quad pTq \Rightarrow q\}$.

REMARK 17. It is easy to see that if we extend a logic by an incoherent set of canonical rules we get an inconsistent scr. Hence coherence is a minimal constraint on sets of canonical rules.

DEFINITION 18. A propositional logic $\langle \mathcal{L}, \vdash \rangle$ is called *canonical* if it is an extension of the trivial logic $\langle \mathcal{L}, \vdash_{triv} \rangle$ (where $\Gamma \vdash_{triv} \Delta$ iff $\Gamma \cap \Delta \neq \emptyset$) by a coherent set of canonical rules.

THEOREM 19. *A propositional logic is canonical iff it has a characteristic 2-Nmatrix.*

Proof. Suppose first that $\langle \mathcal{L}, \vdash \rangle$ is an extension of $\langle \mathcal{L}, \vdash_{triv} \rangle$ by a coherent set S of canonical rules. We define a corresponding 2-Nmatrix $\mathcal{M}_S$ as follows. Given an n-ary connective $\diamond$ of $\mathcal{L}$ and $x_1, \ldots, x_n \in \{t, f\}$, we let:

- $\widetilde{\diamond}(x_1, \ldots, x_n) = \{f\}$ if there is a rule $\diamond(p_1, \ldots, p_n), \Gamma \Rightarrow \Delta$ in S such that $x_i = t$ if $p_i \in \Gamma$, and $x_i = f$ if $p_i \in \Delta$.

- $\widetilde{\diamond}(x_1, \ldots, x_n) = \{t\}$ if there is a rule $\Gamma \Rightarrow \Delta, \diamond(p_1, \ldots, p_n)$ in S such that $x_i = t$ if $p_i \in \Gamma$, and $x_i = f$ if $p_i \in \Delta$.

- $\widetilde{\diamond}(x_1, \ldots, x_n) = \{t, f\}$ otherwise.

Since S is coherent, $\widetilde{\diamond}$ is well-defined. It is also easy to see that $\Gamma \vdash_{\mathcal{M}_S} \Delta$ whenever $\Gamma \Rightarrow \Delta$ is an instance of a rule in S. Hence $\langle \mathcal{L}, \vdash \rangle$ is sound with respect to $\mathcal{M}_S$. To prove completeness, suppose $\Gamma_0 \not\vdash \Delta_0$. We construct a model of Γ_0 in $\mathcal{M}_S$ which is not a model of any formula in Δ_0. For this extend Γ_0 to a maximal set T of formulas such that $T \not\vdash \Delta_0$. Obviously, $\varphi \notin T$ iff $T, \varphi \vdash \Delta_0$. This entails that T is $\vdash$-prime, in the sense that for every finite set Δ, if $T \vdash \Delta$ then $T \vdash \varphi$ for some $\varphi \in \Delta$ (otherwise we could have derived $T \vdash \Delta_0$ using n cuts, where n is the number of formulas in Δ). Define now a valuation v_T by $v_T(\varphi) = t$ iff $\varphi \in T$. The facts that T is $\vdash$-prime, and that $\Gamma \vdash \Delta$ whenever $\Gamma \Rightarrow \Delta$ is (an instance of) a rule of S, together imply that v_T is a legal valuation in $\mathcal{M}_S$. Since obviously $T \cap \Delta_0 = \emptyset$, v_T is the required model of Γ_0 which is not a model of Δ_0.

For the converse, let $\mathcal{M}$ be a 2-Nmatrix. We show that $\vdash_{\mathcal{M}}$ is canonical. Let $S(\mathcal{M})$ be the set of all canonical rules having one of the following forms:

- $\diamond(p_1, \ldots, p_n), \Gamma \Rightarrow \Delta$, where $\Gamma \cap \Delta = \emptyset$, $\Gamma \cup \Delta = \{p_1, \ldots, p_n\}$, and $\diamond$ is an n-ary connective of the language of $\mathcal{M}$ such that $\widetilde{\diamond}(x_1, \ldots, x_n) = \{f\}$, where $\langle x_1, \ldots, x_n \rangle$ is defined by: $x_i = t$ if $p_i \in \Gamma$, $x_i = f$ if $p_i \in \Delta$.

- $\Gamma \Rightarrow \Delta, \diamond(p_1, \ldots, p_n)$, where $\Gamma \cap \Delta = \emptyset$, $\Gamma \cup \Delta = \{p_1, \ldots, p_n\}$, and $\diamond$ is an n-ary connective of the language of $\mathcal{M}$ such that $\widetilde{\diamond}(x_1, \ldots, x_n) = \{t\}$, where $\langle x_1, \ldots, x_n \rangle$ is defined by: $x_i = t$ if $p_i \in \Gamma$, $x_i = f$ if $p_i \in \Delta$.

It is easy to see that $\mathcal{M}_{S(\mathcal{M})} = \mathcal{M}$. Therefore it follows from the first part of this proof that $\vdash_{\mathcal{M}}$ is the extension of $\vdash_{triv}$ by the set $S(\mathcal{M})$. ■

REMARK 20. Using the procedure described in Remark 5 (where the principal formula is taken to be $\diamond(p_1, \ldots, p_n)$), we can see that every canonical rule in the sense used here is equivalent to what was called in [Avron and Lev, 2001] a *separated canonical* Gentzen-type rule. A much more extensive class of canonical Gentzen-type rules and systems was investigated in [Avron and Lev, 2001; Avron and Lev, 2005]. Accordingly, a more complicated coherence criterion was used there (which applies to all type of sets of canonical rules, and is easily seen to be equivalent to the one used here in the case of a set of separated canonical rules). Since it was proved in [Avron and Lev, 2001] that every canonical Gentzen-type rule is equivalent to a finite set of separated canonical Gentzen-type rules, Theorem 19 is actually equivalent to the characterization given in [Avron and Lev, 2001; Avron and Lev, 2005] to the whole class of canonical Gentzen-type systems.

REMARK 21. In [Avron and Lev, 2005] it is shown that any logic which has a characteristic 2-Nmatrix $\mathcal{M}$ can have a finite characteristic (ordinary) matrix if and only if $\mathcal{M}$ itself is deterministic (in which case it is unique). It follows that a canonical propositional logic is either a fragment of classical logic, or else it has no finite characteristic deterministic matrix (although it has of course a characteristic 2-valued non-deterministic matrix).

4 Negation Rules and 4-Valued Nmatrices

The most common type of non-canonical rules used in proof systems for the classical connectives, are rules which handle combinations of negation with other connectives. In this section we show how to use Nmatrices with four values (or less) in order to provide modular semantics for many rules of this sort. For simplicity of presentation, we investigate only rules involving the unary connective $\neg$, and the binary connective $\supset$ (where $\neg$ has some features of classical negation, while $\supset$ has some features of classical implication). Extending the methods for similar rules involving other connectives (like disjunction or conjunction) is a straightforward matter. [5]

DEFINITION 22.

1. Let NIR be the the following set of rules:

$(\neg\Rightarrow)$	$\neg\varphi, \varphi \Rightarrow$
$(\Rightarrow\neg)$	$\Rightarrow \neg\varphi, \varphi$
$(\supset\Rightarrow)$	$\varphi, \varphi \supset \psi \Rightarrow \psi$
$(\Rightarrow\supset)_1$	$\psi \Rightarrow \varphi \supset \psi$
$(\Rightarrow\supset)_2$	$\Rightarrow \varphi \supset \psi, \varphi$
$(\Rightarrow\neg\neg)$	$\varphi \Rightarrow \neg\neg\varphi$
$(\neg\neg\Rightarrow)$	$\neg\neg\varphi \Rightarrow \varphi$
$(\neg\supset\Rightarrow)_1$	$\neg(\varphi \supset \psi) \Rightarrow \varphi$
$(\neg\supset\Rightarrow)_2$	$\neg(\varphi \supset \psi) \Rightarrow \neg\psi$
$(\Rightarrow\neg\supset)$	$\varphi, \neg\psi \Rightarrow \neg(\varphi \supset \psi)$

2. For $S \subseteq NIR$ let $\mathbf{L}[S]$ be the extension by S of the trivial logic in the language $\{\neg, \supset\}$ (i.e. $\mathbf{L}[S]$ is the minimal logic for which all the rules in S are valid).

The basic idea in providing semantics for the logics $\mathbf{L}[S]$ ($S \subseteq NIR$) is to let the value assigned to a sentence φ provide information not only about the truth/falsity of φ, but also about the truth/falsity of its negation. This leads to the use of elements from $\{0,1\}^2$ as our truth-values, where the intended intuitive meaning of $v(\varphi) = \langle x, y\rangle$ is the following:

- $x = 1$ iff φ is "true" (i.e. $v(\varphi) \in \mathcal{D}$).
- $y = 1$ iff $\neg\varphi$ is "true" (i.e. $v(\neg\varphi) \in \mathcal{D}$).

This interpretation of the components of the truth-values dictates the following constraint on any valuation v (where $P_1(\langle x_1, \ldots, x_k\rangle) = x_1$ and $P_2(\langle x_1, \ldots, x_k\rangle) = x_2$:

$$P_1(v(\neg\varphi)) = P_2(v(\varphi))$$

[5] The material in this section extends and systematizes results and methods from [Avron, 2004; Avron, 2005a].

In terms of Nmatrices this constraint translates into the condition:

$$\text{(NEG)} \quad \tilde{\neg} a \subseteq \{y \mid P_1(y) = P_2(a)\}$$

We start our semantic investigation of NIR with the weakest Nmatrix which satisfies Condition (NEG):

DEFINITION 23. The Nmatrix $\mathcal{M}_4^B = \langle \mathcal{V}_4, \mathcal{D}_4, \mathcal{O}_4^B \rangle$ is defined as follows:

- $\mathcal{V}_4 = \{t, \top, \bot, f\}$ where:

$$\begin{array}{lcl} t & = & \langle 1, 0 \rangle \\ \top & = & \langle 1, 1 \rangle \\ \bot & = & \langle 0, 0 \rangle \\ f & = & \langle 0, 1 \rangle \end{array}$$

- $\mathcal{D}_4 = \{a \in \mathcal{V}_4 \mid P_1(a) = 1\} = \{t, \top\}$
- Let $\mathcal{V} = \mathcal{V}_4$, $\mathcal{D} = \mathcal{D}_4$, $\mathcal{F} = \mathcal{V}_4 - \mathcal{D}$. The operations in $\mathcal{O}_4^B$ are:

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } P_2(a) = 1 \quad \text{(i.e. } a \in \{f, \top\}) \\ \mathcal{F} & \text{if } P_2(a) = 0 \quad \text{(i.e. } a \in \{t, \bot\}) \end{cases}$$

$$a \tilde{\supset} b = \mathcal{V}$$

Below we shall see that every logic which is defined by some subset of NIR is characterized by some simple refinement of $\mathcal{M}_4^B$. In fact, it is quite easy to compute for each rule in NIR the semantic condition on such refinement which corresponds to that rule:

DEFINITION 24.

1. The general refining conditions induced by the conditions in NIR are:

   C($\neg \Rightarrow$) : If $P_1(a) = 1$ then $P_2(a) = 0$

   C($\Rightarrow \neg$) : If $P_1(a) = 0$ then $P_2(a) = 1$

   C($\supset\Rightarrow$) : If $P_1(a) = 1$ and $P_1(b) = 0$ then $a \tilde{\supset} b \subseteq \{x \mid P_1(x) = 0\}$
   (Equivalently: if $a \in \mathcal{D}$ and $b \in \mathcal{F}$ then $a \tilde{\supset} b \subseteq \mathcal{F}$)

   C($\Rightarrow\supset)_1$: If $P_1(b) = 1$ then $a \tilde{\supset} b \subseteq \{x \mid P_1(x) = 1\}$
   (Equivalently: if $b \in \mathcal{D}$ then $a \tilde{\supset} b \subseteq \mathcal{D}$)

   C($\Rightarrow\supset)_2$: If $P_1(a) = 0$ then $a \tilde{\supset} b \subseteq \{x \mid P_1(x) = 1\}$
   (Equivalently: if $a \in \mathcal{F}$ then $a \tilde{\supset} b \subseteq \mathcal{D}$)

   C($\Rightarrow \neg\neg$) : If $P_1(a) = 1$ then $\tilde{\neg} a \subseteq \{x \mid P_2(x) = 1\}$

C($\neg\neg\Rightarrow$) : If $P_1(a) = 0$ then $\tilde{\neg} a \subseteq \{x \mid P_2(x) = 0\}$

C($\neg\supset\Rightarrow$)$_1$: If $P_1(a) = 0$ then $a \tilde{\supset} b \subseteq \{x \mid P_2(x) = 0\}$

C($\neg\supset\Rightarrow$)$_2$: If $P_2(b) = 0$ then $a \tilde{\supset} b \subseteq \{x \mid P_2(x) = 0\}$

C($\Rightarrow\neg\supset$) : If $P_1(a) = 1$ and $P_2(b) = 1$ then $a \tilde{\supset} b \subseteq \{x \mid P_2(x) = 1\}$

2. For $S \subseteq NIR$, let $C(S) = \{\mathrm{C}r \mid r \in S\}$

The conditions in $C(NIR)$ were formulated in a way which can be applied whenever we employ finite sequences of 0's and 1's as the truth-values, the designated elements are those for which the first component is 1, and condition (NEG) is satisfied (We present another, more complicated example of this type of semantics in the next section). In the case of simple refinements of $\mathcal{M}_4^B$ these conditions can easily be translated into the following more specific ones:

C($\neg\Rightarrow$) : Use only t, f and $\bot$

C($\Rightarrow\neg$) : Use only t, f and $\top$

C($\supset\Rightarrow$) : If $a \in \{t, \top\}$ and $b \in \{f, \bot\}$ then $a \tilde{\supset} b \subseteq \{f, \bot\}$

C($\Rightarrow\supset$)$_1$: If $b \in \{t, \top\}$ then $a \tilde{\supset} b \subseteq \{t, \top\}$

C($\Rightarrow\supset$)$_2$: If $a \in \{f, \bot\}$ then $a \tilde{\supset} b \subseteq \{t, \top\}$

C($\Rightarrow\neg\neg$) : $\tilde{\neg} t = \{f\}$, $\tilde{\neg}\top = \{\top\}$

C($\neg\neg\Rightarrow$) : $\tilde{\neg} f = \{t\}$, $\tilde{\neg}\bot = \{\bot\}$

C($\neg\supset\Rightarrow$)$_1$: If $a \in \{f, \bot\}$ then $a \tilde{\supset} b \subseteq \{t, \bot\}$

C($\neg\supset\Rightarrow$)$_2$: If $b \in \{t, \bot\}$ then $a \tilde{\supset} b \subseteq \{t, \bot\}$

C($\Rightarrow\neg\supset$) : If $a \in \{t, \top\}$ and $b \in \{\top, f\}$ then $a \tilde{\supset} b \subseteq \{\top, f\}$

Here are some examples of how the conditions have been derived:

C($\neg\Rightarrow$) : The rule means that if $v(\varphi)$ is designated (i.e. $P_1(v(\varphi)) = 1$) then $v(\neg\varphi)$ is not (i.e, $P_1(v(\neg\varphi)) = P_2(v(\varphi)) = 0$). This is the case iff there is no $a \in \mathcal{V}$ such that $P_1(a) = P_2(a) = 1$, i.e.: if $\top$ is not available.

C($\Rightarrow \neg\neg$) : The rule is valid iff $\neg\neg a \in \mathcal{D}$ whenever $a \in \mathcal{D}$ (where $\neg x$ denotes some element in $\tilde{\neg}x$). This means that if $P_1(a) = 1$ then $P_1(\neg\neg a) = P_2(\neg a) = 1$. It follows, e.g., that $\tilde{\neg}t \subseteq \{f, \top\}$. On the other hand, condition (NEG) above implies that $\tilde{\neg}t \subseteq \{f, \bot\}$. Hence $\tilde{\neg}t = \{f\}$. The considerations in the case of $\tilde{\neg}\top$ are similar.

C($\neg \supset\Rightarrow$)$_2$: The rule is valid if $\neg(a \supset b)$ is in $\mathcal{F}$ whenever $\neg b$ is in $\mathcal{F}$ (where $x \supset y$ denotes some element in $x\tilde{\supset}y$). This is equivalent to: if $P_2(b) = 0$ then $P_2(a \supset b) = 0$. This is exactly the condition C($\neg \supset\Rightarrow$)$_2$.

DEFINITION 25. For $S \subseteq NIR$, let $\mathcal{M}_S$ be the weakest simple refinement (Remark 11) of $\mathcal{M}_4^B$ in which the conditions in $C(S)$ are all satisfied. In other words: $\mathcal{M}_S = \langle \mathcal{V}_S, \mathcal{D}_S, \mathcal{O}_S \rangle$, where $\mathcal{V}_S$ is the set of values from $\mathcal{V}_4$ which are not rejected by any condition in S, $\mathcal{D}_S = \mathcal{D}_4 \cap \mathcal{V}_S$, and for any connective $\diamond \in \mathcal{O}$ and any $\vec{x} \in \mathcal{V}_S^n$ (where n is the arity of $\diamond$), the interpretation in $\mathcal{O}_S$ of $\diamond$ assigns to $\vec{x}$ the set of all the values in $\tilde{\diamond}_B(\vec{x})$ which are not forbidden by any condition in $C(S)$ (it is straightforward to check that for $S \subseteq NIR$ this set is never empty. The same is true for $\mathcal{D}_S$).

EXAMPLES 26.

1. Let $C_{min} = \{(\supset\Rightarrow), (\Rightarrow\supset)_1, (\Rightarrow\supset)_2, (\Rightarrow \neg), (\neg\neg \Rightarrow)\}$. Then $\mathcal{M}_{C_{min}}$ is the three-valued Nmatrix[6] $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:
   - $\mathcal{V} = \{t, \top, f\}$
   - $\mathcal{D} = \{t, \top\}$
   - The operations in $\mathcal{O}$ are:

$$\tilde{\neg}a = \begin{cases} \mathcal{D} & \text{if } a = \top \\ \{f\} & \text{if } a = t \\ \{t\} & \text{if } a = f \end{cases}$$

$$a\tilde{\supset}b = \begin{cases} \mathcal{D} & \text{if } a = f \text{ or } b \in \mathcal{D} \\ \{f\} & \text{otherwise} \end{cases}$$

2. Let $\mathcal{FOUR} = NIR - \{(\neg \Rightarrow), (\Rightarrow \neg)\}$. Then $\mathcal{M}_{\mathcal{FOUR}}$ is the 4-valued (deterministic) *matrix* $\langle \mathcal{V}_4, \mathcal{D}_4, \mathcal{O}_4 \rangle$, where the operations in $\mathcal{O}_4$ are:[7]

$$\tilde{\neg}t = \{f\} \quad \tilde{\neg}f = \{t\} \quad \tilde{\neg}\top = \{\top\} \quad \tilde{\neg}\bot = \{\bot\}$$

[6] What is denoted here by $L(C_{min})$ is the negation-implication fragment of the paraconsistent logic C_{min} studied in [Carnielli and Marcos, 1999]. The 3-valued Nmatrix for this logic described here was first introduced in [Avron and Lev, 2004; Avron and Lev, 2005].

[7] The connective $\tilde{\supset}$ of $\mathcal{O}_4$ was introduced in [Arieli and Avron, 1996]. The soundness and completeness of the logic $\mathcal{FOUR}$ (augmented with disjunction and conjunction) for $\mathcal{M}_{\mathcal{FOUR}}$ was also first stated and proved there.

$$a\tilde{\supset}b = \begin{cases} t & \text{if } a \notin \mathcal{D} \\ b & \text{otherwise} \end{cases}$$

THEOREM 27. *For $S \subseteq NIR$, $\mathcal{M}_S$ is a characteristic Nmatrix for* $\mathbf{L}[S]$.

Proof. It is easy to verify that for any $r \in NIR$, the satisfaction of C(r) in some simple refinement of $\mathcal{M}_4^B$ guarantees the validity of r in that refinement. This entails the soundness of $\mathbf{L}[S]$ with respect to $\mathcal{M}_S$.

For the converse, assume $\Gamma \not\vdash_{\mathbf{L}[S]} \Delta$. We construct a model of Γ in $\mathcal{M}_S$ which is not a model of any formula in Δ. For this extend Γ to a maximal set Γ^* of formulas such that $\Gamma^* \not\vdash_{\mathbf{L}[S]} \Delta$. Then Γ^* is closed under the rules in S, and $\varphi \notin \Gamma^*$ iff $\Gamma^*, \varphi \vdash_{\mathbf{L}[S]} \Delta$. Define now a valuation v by $v(\varphi) = \langle x(\varphi), y(\varphi)\rangle$, where:

$$x(\varphi) = \begin{cases} 1 & \varphi \in \Gamma^* \\ 0 & \varphi \notin \Gamma^* \end{cases} \qquad y(\varphi) = \begin{cases} 1 & \neg\varphi \in \Gamma^* \\ 0 & \neg\varphi \notin \Gamma^* \end{cases}$$

It is trivial that v is a legal valuation in $\mathcal{M}_4^B$. To show that it is also a legal valuation in $\mathcal{M}_S$, we need to check that it respects the conditions in $C(S)$. We do half of the cases, leaving the other half for the reader:

C($\neg \Rightarrow$) : Assume $(\neg \Rightarrow) \in S$. Then there can be no sentence φ such that $\{\varphi, \neg\varphi\} \subseteq \Gamma^*$. Hence $v(\varphi) \neq \top$ for all φ.

C($\Rightarrow \neg$) : Assume $(\Rightarrow \neg) \in S$. Suppose that there is φ such that $v(\varphi) = \perp$. Then $\varphi \notin \Gamma^*$ and $\neg\varphi \notin \Gamma^*$. It follows that $\Gamma^*, \varphi \vdash_{\mathbf{L}[S]} \Delta$ and $\Gamma^*, \neg\varphi \vdash_{\mathbf{L}[S]} \Delta$. Since also $\Gamma^* \vdash_{\mathbf{L}[S]} \varphi, \neg\varphi$ in this case, we get that $\Gamma^* \vdash_{\mathbf{L}[S]} \Delta$. A contradiction. Hence $v(\varphi) \neq \perp$ for all φ.

C($\Rightarrow \neg\neg$) : Assume $(\Rightarrow \neg\neg) \in S$.

- Suppose $v(\varphi) = t$. Then $\varphi \in \Gamma^*$ and $\neg\varphi \notin \Gamma^*$. By $(\Rightarrow \neg\neg)$, also $\neg\neg\varphi \in \Gamma^*$. Hence $v(\neg\varphi) = f$ by definition of v.
- Suppose $v(\varphi) = \top$. Then $\varphi \in \Gamma^*$ and $\neg\varphi \in \Gamma^*$. By $(\Rightarrow \neg\neg)$, also $\neg\neg\varphi \in \Gamma^*$. Hence $v(\neg\varphi) = \top$ by definition of v.

C($\supset\Rightarrow$) Assume $(\supset\Rightarrow) \in S$. Suppose that $v(\varphi) \in \mathcal{D}$ and $v(\psi) \notin \mathcal{D}$. Then $\varphi \in \Gamma^*$ and $\psi \notin \Gamma^*$. It follows that $\varphi \supset \psi \notin \Gamma^*$ (since Γ^* is closed under $(\supset\Rightarrow)$ in this case), and so $v(\varphi \supset \psi) \notin \mathcal{D}$.

C($\neg \supset\Rightarrow$)$_1$: Assume $(\neg \supset\Rightarrow)_1 \in S$. Suppose that $v(\varphi) \notin \mathcal{D}$. Then $\varphi \notin \Gamma^*$, and so also $\neg(\varphi \supset \psi) \notin \Gamma^*$. It follows that $v(\varphi \supset \psi) \in \{t, \perp\}$.

Obviously, v is a model of Γ in $\mathcal{M}_S$ which is not a model of Δ. ∎

COROLLARY 28. $\mathbf{L}[S]$ *is decidable for every $S \subseteq NIR$.*

5 Logics of Formal (In)consistency

In this section we present an example of a family of logics for which the modular semantic analysis provided by Nmatrices has proved to be particularly useful: the paraconsistent logics of da Costa's school (see [da Costa., 1974; Carnielli and Marcos, 2002; Carnielli *et al.*, 2006]). The main ideas of this school is to limit the applicability of the $(\neg \Rightarrow)$ rule (which amounts to "a single contradiction entails everything") to the case where φ is "consistent", and to express the assumption of this consistency of φ within the language. The easiest way to implement these ideas is to include in the language a special connective $\circ$, with the intended meaning of $\circ\varphi$ being "φ is consistent". Then one can explicitly add the assumption of the consistency of φ to the problematic (from a paraconsistent point of view) classical (and intuitionistic) rule $(\neg \Rightarrow)$, getting the rule called (**b**) below. Various other rules concerning $\neg$, $\circ$ and other connectives can then be added, leading to a large family of logics known as "Logics of Formal Inconsistency" (LFIs - see [Carnielli and Marcos, 2002; Carnielli *et al.*, 2006]). For simplicity of presentation, we again consider only the binary connective $\supset$ in addition to the unary connectives $\neg$ and $\circ$, and investigate some of the most common rules connected with these 3 connectives which have been investigated in the literature concerning LFIs. Extending the methods for similar rules involving other connectives (like disjunction or conjunction) is again a straightforward matter. [8]

DEFINITION 29.

1. Let FCR be the the following set of rules:

   (b) $\circ\varphi, \neg\varphi, \varphi \Rightarrow$

   (d1) $\Rightarrow \circ\varphi, \varphi$

   (d2) $\Rightarrow \circ\varphi, \neg\varphi$

   (i1) $\neg\circ\varphi \Rightarrow \varphi$

   (i2) $\neg\circ\varphi \Rightarrow \neg\varphi$

   $(\mathbf{a}_\neg)$ $\circ\varphi \Rightarrow \circ\neg\varphi$

   $(\mathbf{a}_\supset)$ $\circ\varphi, \circ\psi \Rightarrow \circ(\varphi \supset \psi)$

   $(\mathbf{o}^1_\supset)$ $\circ\varphi \Rightarrow \circ(\varphi \supset \psi)$

   $(\mathbf{o}^2_\supset)$ $\circ\psi \Rightarrow \circ(\varphi \supset \psi)$

   (4) $\circ\varphi \Rightarrow \circ\circ\varphi$

[8] The material in this section extends and systematizes results and methods from [Avron, 2005a].

2. Let $LFIR = NIR \cup FCR$.

3. For $S \subseteq LFIR$ let $\mathbf{L}[S]$ be the extension by S of the trivial logic in the language $\{\neg, \circ, \supset\}$ (i.e. $\mathbf{L}[S]$ is the minimal logic for which all the rules in S are valid).

This time the basic idea in providing semantics for the logics of the form $\mathbf{L}[S]$ ($S \subseteq LFIR$) is to let the value assigned to a sentence φ provide information not only about the truth/falsity of φ and $\neg\varphi$, but also about the truth/falsity of $\circ\varphi$. This leads to the use of elements from $\{0,1\}^3$ as our truth-values, where now the intended intuitive meaning of $v(\varphi) = \langle x, y, z \rangle$ is the following:

- $x = 1$ iff φ is "true" (i.e. $v(\varphi) \in \mathcal{D}$).
- $y = 1$ iff $\neg\varphi$ is "true" (i.e. $v(\neg\varphi) \in \mathcal{D}$).
- $z = 1$ iff $\circ\varphi$ is "true" (i.e. $v(\circ\varphi) \in \mathcal{D}$).

In addition to (NEG), which remains unchanged, this interpretation dictates this time also the following condition:

$$\text{(CON)} \qquad \widetilde{\circ} a \subseteq \{y \mid P_1(y) = P_3(a)\}$$

Accordingly, this time we start our semantic investigation of $LFIR$ with the weakest Nmatrix which satisfies both (NEG) and (CON). Then we show that every logic which is defined by some subset of $LFIR$ is characterized by some (easily computable) simple refinement of that Nmatrix.

DEFINITION 30.

1. The Nmatrix $\mathcal{M}_8^B = \langle \mathcal{V}_8, \mathcal{D}_8, \mathcal{O}_8^B \rangle$ is defined as follows:

   - $\mathcal{V}_8 = \{0,1\}^3$
   - $\mathcal{D}_8 = \{a \in \mathcal{V}_8 \mid P_1(a) = 1\}$
   - Let $\mathcal{V} = \mathcal{V}_8$, $\mathcal{D} = \mathcal{D}_8$, $\mathcal{F} = \mathcal{V}_8 - \mathcal{D}$. The operations in $\mathcal{O}_8^B$ are:

$$\widetilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } P_2(a) = 1 \\ \mathcal{F} & \text{if } P_2(a) = 0 \end{cases}$$

$$\widetilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } P_3(a) = 1 \\ \mathcal{F} & \text{if } P_3(a) = 0 \end{cases}$$

$$a \widetilde{\supset} b = \mathcal{V}$$

2. The general refining conditions induced by the conditions in NIR are identical to those given in Definition 24.

3. The general refining conditions induced by the conditions in FCR are:

   C(**b**): If $P_1(a) = 1$ and $P_2(a) = 1$ then $P_3(a) = 0$
   C(**d1**): If $P_1(a) = 0$ then $P_3(a) = 1$
   C(**d2**): If $P_2(a) = 0$ then $P_3(a) = 1$
   C(**i1**): If $P_1(a) = 0$ then $\widetilde{\circ}a \subseteq \{x \mid P_2(x) = 0\}$
   C(**i2**): If $P_2(a) = 0$ then $\widetilde{\circ}a \subseteq \{x \mid P_2(x) = 0\}$
   C($\mathbf{a}_\neg$): If $P_3(a) = 1$ then $\widetilde{\neg}a \subseteq \{x \mid P_3(x) = 1\}$
   C($\mathbf{a}_\supset$): If $P_3(a) = 1$ and $P_3(b) = 1$ then $a\widetilde{\supset}b \subseteq \{x \mid P_3(x) = 1\}$
   C($\mathbf{o}^1_\supset$): If $P_3(a) = 1$ then $a\widetilde{\supset}b \subseteq \{x \mid P_3(x) = 1\}$
   C($\mathbf{o}^2_\supset$): If $P_3(b) = 1$ then $a\widetilde{\supset}b \subseteq \{x \mid P_3(x) = 1\}$
   C(**4**): If $P_3(a) = 1$ then $\widetilde{\circ}a \subseteq \{\langle 1,0,1\rangle, \langle 1,1,1\rangle\}$

4. For $S \subseteq LFIR$, let $C(S) = \{\mathrm{C}r \mid r \in S\}$, and let $\mathcal{M}_S$ be the weakest simple refinement of $\mathcal{M}^B_8$ in which the conditions in $C(S)$ are all satisfied (again it is not difficult to check that this is well-defined for every $S \subseteq LFIR$).

THEOREM 31. *For $S \subseteq LFIR$, $\mathcal{M}_S$ is a characteristic Nmatrix for* $\mathbf{L}[S]$.

Proof. The proof is similar to that of Theorem 27, using the intended interpretation of a triple $\langle x, y, z\rangle$. ■

COROLLARY 32. $\mathbf{L}[S]$ *is decidable for every $S \subseteq LFIR$.*

EXAMPLES 33.

1. Let $\mathbf{B} = \{(\supset\Rightarrow), (\Rightarrow\supset)_1, (\Rightarrow\supset)_2, (\Rightarrow\neg), (b)\}$. Then $\mathbf{L}[\mathbf{B}]$ is the basic logic of formal inconsistency *mbC* from [Carnielli *et al.*, 2006] [9]. By Theorem 31, the following Nmatrix $\mathcal{M}^B_5 = \langle \mathcal{V}_5, \mathcal{D}_5, \mathcal{O}^B_5\rangle$ is a characteristic Nmatrix for this logic:

   - $\mathcal{V}_5 = \{t, t_I, I, f_I, f\}$ where:

$$\begin{array}{rcl} t & = & \langle 1,0,1\rangle \\ t_I & = & \langle 1,0,0\rangle \\ I & = & \langle 1,1,0\rangle \\ f & = & \langle 0,1,1\rangle \\ f_I & = & \langle 0,1,0\rangle \end{array}$$

[9] More precisely: it is the $\{\supset, \neg, \circ\}$-fragment of *mbC*. *mbC* itself has in addition conjunction and disjunction, together with their usual classical rules.

- $\mathcal{D}_5 = \{t, I, t_i\}$ $(= \{\langle x, y, z\rangle \in \mathcal{V}_5 \mid x = 1\})$.
- Let $\mathcal{D} = \mathcal{D}_5$, $\mathcal{F} = \mathcal{V}_5 - \mathcal{D}$. The operations in $\mathcal{O}_5^B$ are defined by:

$$a\tilde{\supset}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$\tilde{\neg}a = \begin{cases} \mathcal{D} & \text{if } a \in \{I, f, f_I\} \\ \mathcal{F} & \text{if } a \in \{t, t_I\} \end{cases}$$

$$\tilde{\circ}a = \begin{cases} \mathcal{D} & \text{if } a \in \{t, f\} \\ \mathcal{F} & \text{if } a \in \{I, t_I, f_I\} \end{cases}$$

2. Let $S = \mathbf{B} \cup \{(\Rightarrow \neg \supset), (\mathbf{i1}), (\mathbf{a}_\neg)\}$. Then $\mathcal{M}_S = \langle \mathcal{V}_S, \mathcal{D}_S, \mathcal{O}_S \rangle$, where:

   - $\mathcal{V}_S = \{t, t_I, I, f\}$
   - $\mathcal{D}_S = \{t, I, t_i\}$
   - $a\tilde{\supset}b = \begin{cases} \mathcal{D}_S & \text{if either } a = f \text{ or } b \in \{t, t_I\} \\ \{I\} & \text{if } a \in \mathcal{D}_S \text{ and } b = I \\ \{f\} & \text{if } a \in \mathcal{D}_S \text{ and } b = f \end{cases}$
   - $\tilde{\neg}t = \tilde{\neg}t_I = \{f\}$ $\quad \tilde{\neg}I = \mathcal{D}_S$ $\quad \tilde{\neg}f = \{t\}$
   - $\tilde{\circ}t = \mathcal{D}_S$ $\quad \tilde{\circ}t_I = \tilde{\circ}I = \{f\}$ $\quad \tilde{\circ}f = \{t, t_I\}$

3. Let $\mathbf{Cia} = \{(\supset\Rightarrow), (\Rightarrow\supset)_1, (\Rightarrow\supset)_2, (\Rightarrow \neg), (\neg\neg \Rightarrow), (\mathbf{b}), (\mathbf{i1}), (\mathbf{i2}), ((\mathbf{a}_\supset)\}$.[10] Then $\mathcal{M}_{Cia} = \langle \mathcal{V}_{Cia}, \mathcal{D}_{Cia}, \mathcal{O}_{Cia} \rangle$, where:

   - $\mathcal{V}_{Cia} = \{t, I, f\}$
   - $\mathcal{D}_{Cia} = \{t, I\}$
   - $a\tilde{\supset}b = \begin{cases} \{f\} & \text{if } a \in \{t, I\} \text{ and } b = f \\ \{t\} & \text{if either } a = f,\ b \in \{f, t\} \text{ or } a = t,\ b = t \\ \{t, I\} & \text{otherwise} \end{cases}$
   - $\tilde{\neg}t = \{f\}$ $\quad \tilde{\neg}I = \{I\}$ $\quad \tilde{\neg}f = \{t\}$
   - $\tilde{\circ}t = \tilde{\circ}f = \{t\}$ $\quad \tilde{\circ}I = \{f\}$

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[10] **L**[**Cia**] is the $\{\supset, \neg, \circ\}$-fragment of the logic *Cia* from [Carnielli and Marcos, 2002; Carnielli *et al.*, 2006].

6 Further Research

We have demonstrated the potential of using non-deterministic matrices in providing modular semantics for logics. This leaves us with two main research problems. The first is to develop a general theory determining the type of systems for which the methods exemplified here work, and under what conditions (such a theory has been developed so far only for the special case of systems defined by a set of canonical rules - see Section 3). The second is to extend the methods for other types of logical systems. Here are some specific directions requiring further research:

1. Every set S of rules dealt with in Sections 4 and 5 has the property that the corresponding Nmatrix $\mathcal{M}_S$ actually exists. This should not always be the case. One obvious case in which this may fail is when the conditions in $C(S)$ leave no designated element or no non-designated one. In this case the resulting system should be trivial. A simple example in which this happens is when we add to $LFIR$ the rule $\circ\varphi \Rightarrow \varphi$ (corresponding to the modal axiom usually denoted by (T)). The corresponding condition is: if $P_1(a) = 0$ then $P_3(a) = 0$ (thus rejecting $\langle 0,1,1\rangle$ and $\langle 0,0,1\rangle$). Together with C(**d1**), (T) leaves no non-designated element, and indeed it is easy to verify that a system with both (T) and (**d1**) is trivial. A generalization of the criterion of coherence for canonical systems is therefore needed here. A more subtle possible problem is that the conditions in $C(S)$ may leave in some cases only an empty set of options for applications of operations. Here is an example: suppose we add to NIR (see Section 4) the rule (MT): $\neg\psi, \varphi \supset \psi \Rightarrow \neg\varphi$. The corresponding condition, $C(MT)$, is: if $P_2(a) = 0$ and $P_2(b) = 1$ then $a \tilde{\supset} b \subseteq \{x \mid P_1(x) = 0\}$. Thus $\langle 0,1\rangle \tilde{\supset} \langle 1,1\rangle$ is not defined in a system which contains both $(\Rightarrow\supset)_1$ and (MT), although neither $\langle 0,1\rangle$ nor $\langle 1,1\rangle$ is rejected by any of the two rules. However, in this case the resulting system is not trivial, and can in fact be characterized by a finite Nmatrix (obtained as the product of simpler Nmatrices of the type used in Section 4). Hence an even more sophisticated coherence theory is needed to deal with this phenomenon, and a more general theory for constructing characteristic Nmatrices is needed to overcome the difficulty it causes.

2. Not every logic defined by a finite system of rules has a finite characteristic Nmatrix. Let, for example, (**l**) be the following rule from [Carnielli and Marcos, 2002; Carnielli *et al.*, 2006]:

   (**l**) $\neg(\varphi \wedge \neg\varphi) \vdash \circ\varphi$

In [Avron, 2005b] it was shown that $\mathbf{L}[\{(\Rightarrow\ \neg), (\neg\neg\ \Rightarrow), (\mathbf{b}), (\mathbf{l})\}]$ (which is called there Cl) has no finite characteristic Nmatrix. However, it does have an infinite characteristic Nmatrix $\mathcal{M}$ with is "semifinite" in the sense that determining the validity in this logic of any formula requires only a finite sub-Nmatrix of $\mathcal{M}$ (which can be determined in advance by the formula). Accordingly, a general theory is needed when a system has a finite characteristic Nmatrix, and when it has a "semifinite" characteristic Nmatrix.

3. This paper deals only with Scott consequence relations defined by a set of *pure* rules like in Definition 2. It is not clear how in general to extend it to deal with rules that are not treated as pure (like the necessitation rule in modal logics). An important case in which this can be done (using both possible worlds semantics and Nmatrices) is described in [Avron, 2004], where extensions of positive intuitionistic logic by subsets of NIR are investigated.

4. The idea of modular non-deterministic semantics has been applied so far mainly to propositional logics. It is important to extend it also to logics with quantifiers (some steps in this direction have been made in [Avron and Zamansky, 2005; Avron and Zamansky, 2004]).

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