ON STRICT STRONG CONSTRUCTIBILITY WITH A COMPASS ALONE

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We show that every point in the plane which can be constructed by a compass and a ruler, given a set S of points, can be constructed using a compass alone so that the following restriction is met. Let O and K be two arbitrarily chosen distinct points of S. Then every point is obtained as a proper intersection of two circles that are either completely symmetrical with respect to the line OK or have both their centers on this line.

In [1] we have shown that every point in the plane which can be constructed by a compass and a ruler, given a set S of points, can be constructed using a compass alone so that the centers of all the circles used are on a particular line OK, where O and K are two arbitrarily chosen distinct points of S. This was a strengthening of a famous theorem of Mascheroni and Mohr. There was, however, a serious drawback in our construction: points on the line OK itself were (necessarily) obtained only as the *tangent* points of two circles and not as proper intersection points. The original proofs of Mascheroni and Mohr, in contrast, took special care to avoid using tangent points.(1) In this paper we remedy this shortcoming. For this we shall somewhat relax, of course, the restriction above. Nevertheless, the needed relaxation turned out to be minimal: points outside OK are still obtained as the (proper!) intersection points of two circles with centers on OK, but the points of OK itself are obtained as the intersection points of two circles which are *completely symmetrical* with respect to OK.

In the definition below S is a set of points in the plane, O and K are two distinct points of S.

(1) This point was called to our attention by the referee of [1]. We take the opportunity here to thank him.

DEFINITION. 1) A construction with a compass alone of a point B from S is a sequence $A_1, \ldots, A_n = B$ of points such that for each $1 \leq i \leq n$ either $A_i \in S$ or there exist in the sequence points $A_{i_1},\ A_{i_2},\ A_{i_3},\ A_{i_4},\ A_{i_5},\ A_{i_6}$ such that $i_j < i \quad (1 \leq j \leq 6)$, $A_{i_2} \neq A_{i_3},\ A_{i_5} \neq A_{i_6}$ and A_i is an intersection point of $A_{i_1}(A_{i_2}A_{i_3})$ and $A_{i_4}(A_{i_5}A_{i_6})$. (2)

2) Two circles are *completely symmetrical* with respect to a line ℓ iff their centers are symmetrical with respect to ℓ and their radii are equal.

3) We call a construction from S with a compass alone *permissible* (relative to O and K) if any point it uses which is not in S (including the final one) is obtained as a proper intersection of two circles which are either completely symmetrical with respect to OK or have both their centers on OK.

4) We shall call a point C - *constructible* (from S relative to O and K) if it can be obtained from S by a permissible construction (relative to O and K).

THEOREM. Every point of the plane that can be constructed from S using a ruler and a compass is C - constructible from S relative to O and K where O and K are arbitrarily chosen two distinct points of S.

Proof. The proof closely follows that given in [1], though some of the constructions there need to be changed. We leave to the reader the task of checking that every construction we use below is permissible. Again we employ a Cartesian coordinate system in which $O = (0,0),\ K = (1,0)$.

Fact 1. Suppose $A,\ B,\ C$ are on the X-axis, and $AB = BC$. Then if A and B are C - constructible then so is C.

Proof. Let C_1 and C_2 be the intersection points of $A(AB)$ and $B(AB)$. Then $C_1C_2 = \sqrt{3}AB$, and C is one of the two intersection points of $C_1(C_1C_2)$ and $C_2(C_1C_2)$.

Fact 2. (Corollary): if $(x,0)$ is C-constructible, then so is $(nx,0)$ for every integer n.

Fact 3. If (x,y) is C-constructible then so is $(x,-y)$.

Proof. Immediate from the definition of C-constructibility.

Fact 4. If $(x,0)$ is C-constructible so is $(0,x)$.

Proof. Exactly like in Lemma 5 of [1]: By Fact 2 $(-x,0)$ is C-constructible. Let $A = (x,0),\ B = (-x,0)$. Then $AB = 2x$. Now $A(2x)$ and $B(2x)$ intersect at $(0,\sqrt{3}x)$, $A(\sqrt{3}x),\ B(\sqrt{3}x)$ intersect at $(0,\sqrt{2}x)$ and $A(\sqrt{2}x),\ B(\sqrt{2}x)$ intersect at $(0,x)$.

(2) $A(BC)$ is the circle with center at A and radius BC.

Fact 5. If $(0, x)$ is C-constructible so is $(x, 0)$.

Proof. If we examine the proof of Fact 4, we find that except $(x, 0)$ and $(-x, 0)$ all the points used are obtained as the intersection points of two circles that are completely symmetrical with respect to the Y-axis. Now if $(0, x)$ is C-constructible, so is $(0, -x)$ by Fact 3. Hence we can change the roles of the X-axis and the Y-axis in the proof of Fact 3 to get a permissible construction of $(x, 0)$.

Fact 6. If $(x, 0)$ and $(y, 0)$ are C-constructible so are (x, y) and $(x, -y)$.

Proof. By Fact 5 $D = (0, y)$ is constructible. Let $A = (x, 0)$. $A(y)$ and $O(AD)$ intersect at (x, y) and $(x, -y)$.

Fact 7. If $(x, 0)$ and $(y, 0)$ are C-constructible so are $(x + y, 0)$ and $(x - y, 0)$.

Proof. By Facts 6 and 4, $A = (x, y)$, $B = (x, -y)$ and $(0, y)$ are C-constructible. Now the distance between $(y, 0)$ and $(0, y)$ is $\sqrt{2}y$ and $A(\sqrt{2}y)$, $B(\sqrt{2}y)$ intersect at $(x + y, 0)$ and $(x - y, 0)$.

Fact 8. If (x, y) is C-constructible, so is $(2x, 0)$.

Proof. Let $A = (x, y)$. The case $y = 0$ follows from Fact 2. If $y \neq 0$ then $B = (x, -y)$ is C-constructible (Fact 3) and $A(OA)$, $B(OA)$ intersect at $(2x, 0)$.

Fact 9. If (x, y) is C-constructible so is $(x, 0)$.

Proof. By Fact 8, if (x, y) is C-constructible, so are $A = (2x, 0)$ and $B = (4x, 0)$. Now $O(OA)$ and $B(OB)$ intersect at $(x/2, \sqrt{15}x/2)$. It follows therefore from Fact 8 (again) that $(x, 0)$ is C-constructible.

Fact 10. If (x, y) is C-constructible then so is $(y, 0)$

Proof. Let $A = (x, y)$. By Fact 9 $B = (x, 0)$ is C-constructible. Now $O(AB)$ and $B(OA)$ intersect at $(0, y)$. Fact 5 entails therefore that $(y, 0)$ is C-constructible.

Fact 11. (x, y) is C-constructible iff both $(x, 0)$ and $(y, 0)$ are.

Proof. From Facts 6, 9, and 10.

Fact 12. If $(x, 0)$ is C-constructible so is $(x/2, 0)$.

Proof. This follows from the proof of Fact 9 and the Fact itself.

Proof of the theorem. From this point on we can just follow the proof in [1]. We repeat it here, though, to make this paper self-contained: It is well known that a point (x, y) is constructible from S using a ruler and a compass iff both x and y belong to the smallest set which contains the coordinates of each $p \in S$ and is closed under $+, -, \times, :$ and $\sqrt{}$. By Fact 11 and Fact 7, it remains therefore to show that the set of x such that $(x, 0)$ is C-constructible is closed under $\cdot, :$ and $\sqrt{}$. Call such an x *achievable*. Since $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$ it suffices by Facts 12 and 7 to show that if $a > 0$ and a is achievable, then so are a^2, $1/a$ and $\sqrt{a}$. Now if $A = (a, 0)$ is C-constructible and b is achievable where $0 < b < 2a$ then $O(b)$ and $A(a)$ intersect at $(\frac{b^2}{2a}, \cdots)$. It follows by Fact 8 that $\frac{b^2}{a}$ is achievable in this case. Suppose now that a and b are arbitrary achievable positive numbers. By Archimedes' axiom there is an integer n such that $b < na$. By what we have just shown and Fact 2, $\frac{b^2}{na}$ is achievable. Hence, by Fact 2 again, b^2/a is achievable. In particular a^2 and $1/a$ are achievable whenever a is.
Suppose, finally, that $x > 0$ is achievable. Then so are $\frac{1+x}{2}$ and $\frac{|x-1|}{2}$. Let $A = (|x-1|, 0)$. $O(\frac{1+x}{2})$ and $A(\frac{1+x}{2})$ intersect at $(\frac{|x-1|}{2}, \sqrt{x})$. It follows from Fact 10 that $\sqrt{x}$ is also achievable

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