A Simple Proof of Completeness and Cut-elimination for Propositional Gödel Logic

Arnon Avron
School of Computer Science, Tel-Aviv University
http://www.math.tau.ac.il/ aa/

August 24, 2008

Abstract

We provide a constructive, direct, and simple proof of the completeness of the cut-free part of the hypersequential calculus for Gödel logic (thereby proving both completeness of the calculus for its standard semantics, and the admissibility of the cut rule in the full calculus). We then extend the results and proofs to derivations from assumptions, showing that such derivations can be confined to those in which cuts are made only on formulas which occur in the assumptions.

The paper is self-contained, and no previous knowledge concerning **HG** (or even Gödel logic) is needed for understanding it.

1 introduction

In [8] Gödel introduced a sequence $\{G_n\}$ $(n \geq 2)$ of n-valued Matrices in the language of propositional intuitionistic logic. He used these matrices to show some important properties of intuitionistic logic. An infinite-valued matrix G_ω in which all the $G_n s$ can be embedded was later introduced by Dummett in [7]. G_ω, in turn, can naturally be embedded in a matrix $G_{[0,1]}$, the truth values of which are the real numbers between 0 and 1 (inclusive). It has not been difficult to show that the logics of G_ω and $G_{[0,1]}$ are identical, and both are known today as "Gödel Logic" [1]. It is probably the most important intermediate logic, which turns up in several places[2]. Recently it has again

[1]It is also called Gödel-Dummett Logic, because it was first introduced and axiomatized in [7]. The name Dummett himself has used (and is still in use for the propositional fragment of the logic) is *LC*. Still another common name is "intuitionistic fuzzy logic".

[2]Viewed as an intermediate logic, i.e. a logic between intuitionistic logic and classical logic, it is best characterized as the logic of *linear* intuitionistic Kripke frames.

attracted a lot of attention because of its recognition as one of the three most basic fuzzy logics [9].

A cut-free Gentzen-type proof system **HG** for propositional Gödel Logic, having exactly the same *logical* rules as the usual Gentzen-type system for propositional intuitionistic logic, was first introduced in [1] using *hypersequents*. **HG** was later extended by Baaz, Ciabattoni, and Fermüller, to provide appropriate proof systems for extensions of propositional Gödel Logic with quantifiers of various types and modalities (see [3] for a survey). Now in all the works about **HG** and its extensions the proofs of completeness (either for the Gödel's many-valued semantics or for the Kripke semantics) and the proofs of cut-elimination have completely been separated. Completeness has been shown for the full calculus (including cut), while cut-elimination has been proved syntactically by some type of induction on complexity of proofs [3] [4]. In this paper we provide for the first time a constructive, direct, and simple proof of the completeness of the cut-free part of **HG** for its intended semantics (thereby proving both completeness of the calculus and the admissibility of the cut rule in it). We then extend the results to derivations from assumptions, showing that such derivations can be confined to those in which cuts are made only on formulas which occur in the assumptions.

The paper is self-contained, and no previous knowledge concerning **HG** (or even Gödel logic) is needed for understanding it.

2 Review of the System and its Semantics

2.1 The Language

Definition 1 The language $\mathcal{L}_{LC}$ of propositional Gödel logic is the propositional language based on the binary connectives $\rightarrow$, $\vee$, and $\wedge$, and the propositional constant $\perp$.

Definition 2

1. A *sequent* is a construct of the form $\Gamma \Rightarrow \varphi$ where Γ is a finite set of formulas, and φ is a formula.[5]

[3]The syntactic methods are notoriously prone to errors. Thus there has been a gap in the proof given in [1] in its handling of the case of disjunction. The first proof (in [5]) of cut-elimination for the first-order **HG** was also erroneous.

[4]Recently an algebraic proof of cut-elimination for **HG** has been given in [6]. However, the semantics used there is not any of the usual ones, and the proof is not constructive: unlike the proof given below, it does not find countermodels for invalid formulas.

[5]In this paper the term "sequent" refers only to single-conclusion sequents. Note that

2. A *hypersequent* is a finite set of sequents. The elements of a hypersequent G are also called *components* of G. [6]

Notation: We shall denote the hypersequent $\{\Gamma_1 \Rightarrow \varphi_1, \ldots, \Gamma_k \Rightarrow \varphi_k\}$ by $\Gamma_1 \Rightarrow \varphi_1 \mid \ldots \mid \Gamma_k \Rightarrow \varphi_k$. G and H are used as metavariables for (possibly empty) hypersequents.

2.2 The Hypersequential Calculus for Gödel Logic

The hypersequential system **HG** for the language $\mathcal{L}_{LC}$ is defined as follows:

Axioms:

$$\varphi \Rightarrow \varphi \qquad \bot \Rightarrow \varphi$$

Structural Rules:

$$(IW)\ \frac{\Gamma \Rightarrow \varphi \mid G}{\psi, \Gamma \Rightarrow \varphi \mid G} \qquad (EW)\ \frac{G}{\Gamma \Rightarrow \varphi \mid G}$$

$$(com)\ \frac{\Gamma, \Delta \Rightarrow \varphi \mid G \quad \Gamma, \Delta \Rightarrow \psi \mid G}{\Gamma \Rightarrow \varphi \mid \Delta \Rightarrow \psi \mid G}$$

$$(cut)\ \frac{\Gamma \Rightarrow \varphi \mid G \quad \varphi, \Gamma \Rightarrow \psi \mid G}{\Gamma \Rightarrow \psi \mid G}$$

Logical Rules:

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \varphi \mid G \quad \psi, \Gamma \Rightarrow \theta \mid G}{\varphi \rightarrow \psi, \Gamma \Rightarrow \theta \mid G} \qquad (\Rightarrow\rightarrow)\ \frac{\Gamma, \varphi \Rightarrow \psi \mid G}{\Gamma \Rightarrow \varphi \rightarrow \psi \mid G}$$

$$(\vee \Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \theta \mid G \quad \psi, \Gamma \Rightarrow \theta \mid G}{\varphi \vee \psi, \Gamma \Rightarrow \theta \mid G} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \varphi \mid \Gamma \Rightarrow \psi \mid G}{\Gamma \Rightarrow \varphi \vee \psi \mid G}$$

$$(\wedge \Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \theta \mid G}{\varphi \wedge \psi, \Gamma \Rightarrow \theta \mid G} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \varphi \mid G \quad \Gamma \Rightarrow \psi \mid G}{\Gamma \Rightarrow \varphi \wedge \psi \mid G}$$

Note. An equivalent version is obtained if $(\Rightarrow \vee)$ is split into the two obvious hypersequential versions of the usual intuitionistic rules for $(\Rightarrow \vee)$.

---

in other calculi Γ in $\Gamma \Rightarrow \varphi$ may be a multiset of formulas or even a sequence of formulas. Here it is most convenient to view it as a set. In addition, we prefer not to deal with sequents of the form $\Gamma \Rightarrow$, identifying them with $\Gamma \Rightarrow \bot$.

[6] Again in other calculi a hypersequent may be a finite multiset of sequents, or a sequence of sequents. Our choices mean that both internal contraction and external contraction (the main sources of problems in syntactic proofs of cut-elimination) are here built into the data structures we use. What is more: they cause no problems for our proofs.

Definition 3

1. A hypersequent G follows in **HG** from a set S of hypersequents (notation: $S \vdash_{\mathbf{HG}} G$) if there is a proof in **HG** of G from S.

2. A formula φ follows in **HG** from a finite set Γ of formulas (notation: $\Gamma \vdash_{\mathbf{HG}} \varphi$) if the (hyper)sequent $\Gamma \Rightarrow \varphi$ is provable in **HG** (from the empty set of hypersequents).

2.3 The Many-valued Semantics

Definition 4 A *Gödel valuation* (for Gödel logic) is a triple $\mathcal{V} = \langle V, \leq, v \rangle$ where $\langle V, \leq \rangle$ is a nonempty linearly ordered set with a maximal element 1 and a minimal element 0, and v is a function from $\mathcal{L}_{LC}$ to V satisfying:

- $v(\perp) = 0$
- $v(\varphi \vee \psi) = max\{v(\varphi), v(\psi)\}$
- $v(\varphi \wedge \psi) = min\{v(\varphi), v(\psi)\}$
- $v(\varphi \rightarrow \psi) = \begin{cases} 1 & v(\varphi) \leq v(\psi) \\ v(\psi) & v(\varphi) > v(\psi) \end{cases}$

Definition 5

1. A Gödel valuation $\mathcal{V} = \langle V, \leq, v \rangle$ is a *Gödel model* of a sequent $\Gamma \Rightarrow \varphi$ (notation: $\mathcal{V} \models_{Go} \Gamma \Rightarrow \varphi$) if either Γ is empty and $v(\varphi) = 1$, or there exists $\psi \in \Gamma$ such that $v(\psi) \leq v(\varphi)$.

2. A Gödel valuation $\mathcal{V}$ is a *Gödel model* of a hypersequent G (notation: $\mathcal{V} \models_{Go} G$) if it is a model of at least one of its components. It is a Gödel model of a set S of hypersequents (notation: $\mathcal{V} \models_{Go} S$) if it is a Gödel model of each element of S.

Definition 6

1. A hypersequent G *Gödel-follows* from a set S of hypersequents (notation: $S \vdash^{LC}_{\mathbf{Go}} G$) if every Gödel model of S is a Gödel model of G.

2. A formula φ Gödel-follows from a finite set Γ of formulas (notation: $\Gamma \vdash^{LC}_{\mathbf{Go}} \varphi$) if the (hyper)sequent $\Gamma \Rightarrow \varphi$ Gödel-follows from the empty set of hypersequents.

Note. The following two well-known facts (which are *not* needed in what follows) are not difficult to prove:

1. $\Gamma \vdash^{LC}_{\mathbf{Go}} \varphi$ if every Gödel model of each formula in Γ is also a Gödel model of φ (where $\langle V, \leq, v\rangle$ is a Gödel model of a formula ψ if it is a Gödel model of $\Rightarrow \psi$, i.e.: if $v(\psi) = 1$).
2. The two consequence relations (between hypersequents and between formulas) we denote by $\vdash^{LC}_{\mathbf{Go}}$ remain the same if in the previous definition we consider only Gödel valuations of the form $\langle [0,1], \leq, v\rangle$ (where $[0,1]$ is the unit interval, and $\leq$ is the usual order of the real numbers).

Proposition 1 *Let S be a finite set of hypersequents, G a hypersequent, Γ a finite set of formulas, and φ a formula.*

1. *If $S \vdash_{\mathbf{HG}} G$ then $S \vdash^{LC}_{\mathbf{Go}} G$.*
2. *If $\Gamma \vdash_{\mathbf{HG}} \varphi$ then $\Gamma \vdash^{LC}_{\mathbf{Go}} \varphi$.*

Proof: The second part is immediate from the first. The proof of the first is by a routine induction. We do here only the case of (*com*). Obviously, it suffices to show that if $\langle V, \leq, v\rangle \models_{Go} \Gamma, \Delta \Rightarrow \varphi$ and $\langle V, \leq, v\rangle \models_{Go} \Gamma, \Delta \Rightarrow \psi$, then $\langle V, \leq, v\rangle \models_{Go} \Gamma \Rightarrow \varphi \mid \Delta \Rightarrow \psi$. By linearity, either $v(\varphi) \leq v(\psi)$ or $v(\psi) \leq v(\varphi)$. Assume e.g. the former. Then if $v(\varphi) = 1$ or $v(\theta) \leq v(\varphi)$ for some $\theta \in \Gamma$ then obviously $\langle V, \leq, v\rangle \models_{Go} \Gamma \Rightarrow \varphi \mid \Delta \Rightarrow \psi$. If not, then $v(\theta) \leq v(\varphi)$ for some $\theta \in \Delta$ (because $\langle V, \leq, v\rangle \models_{Go} \Gamma, \Delta \Rightarrow \varphi$). Hence $v(\theta) \leq v(\psi)$ for some $\theta \in \Delta$, and so again $\langle V, \leq, v\rangle \models_{Go} \Gamma \Rightarrow \varphi \mid \Delta \Rightarrow \psi$. $\square$

2.4 The Kripke-style Semantics

Definition 7 A *Kripke frame* (for Gödel logic) is a triple $\mathcal{W} = \langle W, \leq, v\rangle$ where W is a nonempty set linearly ordered by $\leq$, and v is a function from $W \times \mathcal{L}_{LC}$ to $\{t, f\}$ which satisfies the following conditions for every $w, u \in W$:

- If p is atomic, $v(w, p) = t$, and $w \leq u$ then $v(u, p) = t$.
- $v(w, \bot) = f$
- $v(w, \varphi \vee \psi) = t$ iff either $v(w, \varphi) = t$ or $v(w, \psi) = t$.
- $v(w, \varphi \wedge \psi) = t$ iff $v(w, \varphi) = t$ and $v(w, \psi) = t$.
- $v(w, \varphi \to \psi) = t$ iff for every $u \geq w$, either $v(u, \varphi) = f$ or $v(u, \psi) = t$.

A well-known fact: If $\langle W, \leq, v\rangle$ is a Kripke frame then for every $w, u \in W$ and every formula φ: if $v(w, \varphi) = t$, and $w \leq u$ then $v(u, \varphi) = t$.

Definition 8

1. A Kripke frame $\mathcal{W} = \langle W, \leq, v\rangle$ is a *Kripke model* of a sequent $\Gamma \Rightarrow \varphi$ (notation: $\mathcal{W} \models_{Kr} \Gamma \Rightarrow \varphi$) if for every $w \in W$, either $v(w, \psi) = f$ for some $\psi \in \Gamma$, or $v(w, \varphi) = t$.

2. A Kripke frame $\mathcal{W}$ is a *Kripke model* of a hypersequent G (notation: $\mathcal{W} \models_{Kr} G$) if it is a model of at least one of its components. It is a Kripke model of a set S of hypersequents (notation: $\mathcal{W} \models_{Kr} S$) if it is a Kripke model of each element of S.

Definition 9

1. A hypersequent G *Kripke-follows* from a set S of hypersequents (notation: $S \vdash^{LC}_{\mathbf{Kr}} G$) if every Kripke model of S is a Kripke model of G as well.

2. A formula φ Kripke-follows from a finite set Γ of formulas (notation: $\Gamma \vdash^{LC}_{\mathbf{Kr}} \varphi$) if the (hyper)sequent $\Gamma \Rightarrow \varphi$ Kripke-follows from the empty set of hypersequents (i.e.: if every Kripke frame is a Kripke model of $\Gamma \Rightarrow \varphi$).

Proposition 2 *Let S be a finite set of hypersequents, G a hypersequent, Γ a finite set of formulas, and φ a formula.*

1. *If $S \vdash^{LC}_{\mathbf{Go}} G$ then $S \vdash^{LC}_{\mathbf{Kr}} G$.*

2. *If $\Gamma \vdash^{LC}_{\mathbf{Go}} \varphi$ then $\Gamma \vdash^{LC}_{\mathbf{Kr}} \varphi$.*

Proof: The second part is immediate from the first. For the first it suffices to show that for every Kripke frame $\mathcal{W} = \langle W, \leq, v\rangle$ there exists a Gödel valuation $\mathcal{V} = \langle V, \leq_V, v^*\rangle$ such that for every hypersequent G, $\mathcal{V} \models_{Go} G$ iff $\mathcal{W} \models_{Kr} G$. Construct $\mathcal{V}$ as follows: Let V be the set of all upwards closed subsets of W (a subset X of W is upwards closed if $u \in X$ whenever $w \in X$ and $w \leq u$). Take $\leq_V$ to be $\subseteq$, and define $v^*(\varphi) = \{w \in W \mid v(w, \varphi) = t\}$ for every formula φ. It is a routine matter to show that $\langle V, \leq_V, v^*\rangle$ is a Gödel valuation, and that it has the required property. $\square$

Note. The main point to note about the last proof (which is the reason why we have included it here) is its strong constructive character: Given a (finite) Kripke countermodel of $S \vdash^{LC}_{\mathbf{Kr}} G$, it constructs from it a (finite) Gödel countermodel of $S \vdash^{LC}_{\mathbf{Kr}} G$.

3 Completeness and Cut-elimination

Proposition 3 *A hypersequent is valid in all Kripke frames for Gödel logic (i.e. each of them is a Kripke model of it) iff it has a cut-free proof in* **HG**.

Proof: The "if" part is immediate from Propositions 1 and 2. For completeness, assume that G_0 has no cut-free proof in **HG**. We construct a Kripke frame in Which G_0 is not valid.

Let $\mathcal{F}$ be the finite set of subformulas of formulas of G_0, and let G be a maximal extension of G_0 such that G contains only formulas from $\mathcal{F}$, and G has no cut-free proof in **HG** (G exists and is finite because $\mathcal{F}$ is finite, and so there are only finitely many sequents of formulas from $\mathcal{F}$). Obviously, G has the following property:

> (*) If s is a sequent consisting only of formulas from $\mathcal{F}$, then either $s \in G$ or $s \mid G$ has a cut-free proof.

Let $\mathcal{F}' = \{\varphi \in \mathcal{F} \mid \exists \Gamma \subseteq \mathcal{F}.(\Gamma \Rightarrow \varphi) \in G\}$. For $\varphi \in \mathcal{F}'$ let Γ_φ be the union of all the sets Δ such that $(\Delta \Rightarrow \varphi) \in G$.

Lemma 1 *Let $\varphi \in \mathcal{F}'$. Then $(\Delta \Rightarrow \varphi) \in G$ iff $\Delta \subseteq \Gamma_\varphi$.*

Proof of Lemma 1: One direction is trivial. For the converse, suppose that $\Gamma_\varphi = \bigcup_{i=1}^{k} \Delta_i$, where $(\Delta_i \Rightarrow \varphi) \in G$ for $i = 1, \ldots, k$. Assume for contradiction that $(\Gamma_\varphi \Rightarrow \varphi) \notin G$. Then by (*) $\Gamma_\varphi \Rightarrow \varphi \mid G$ has a cut-free proof. By repeatedly applying the splitting rule (the special case of (com) in which the two premises are identical) to this hypersequent we get a cut-free proof of $\Delta_1 \Rightarrow \varphi \mid \ldots \mid \Delta_k \Rightarrow \varphi \mid G$, and so a cut-free proof of G (because $(\Delta_i \Rightarrow \varphi) \in G$ for every $1 \leq i \leq k$). It follows that $(\Gamma_\varphi \Rightarrow \varphi) \in G$. Now let $\Delta \subseteq \Gamma_\varphi$. Assume that $(\Delta \Rightarrow \varphi) \notin G$. Then $\Delta \Rightarrow \varphi \mid G$ has a cut-free proof. Since $\Gamma_\varphi \Rightarrow \varphi$ is derivable from $\Delta \Rightarrow \varphi$ using (IW), we get that $\Gamma_\varphi \Rightarrow \varphi \mid G$ has a cut-free proof, implying that G has a cut-free proof.

Now we define a Frame $\mathcal{W} = \langle W, \leq, v\rangle$ as follows:

- $W = \{\Gamma_\varphi \mid \varphi \in \mathcal{F}'\}$
- $\leq = \subseteq$
- For every atomic formula p and $w \in W$, let $v(w, p) = t$ iff $p \in w$. For other formulas v is defined according to the conditions stated in Definition 7.

Obviously, $\mathcal{W}$ is a legal intuitionistic Kripke frame.

Lemma 2 *$\mathcal{W}$ is linear.*

Proof of Lemma 2: Let $\Gamma_\varphi, \Gamma_\psi \in W$, and assume that $\Gamma_\varphi \not\subseteq \Gamma_\psi$ and $\Gamma_\psi \not\subseteq \Gamma_\varphi$. Then $\Gamma_\varphi \cup \Gamma_\psi$ is a proper extension of both Γ_φ and Γ_ψ. Hence (by Lemma 1) both $\Gamma_\varphi \cup \Gamma_\psi \Rightarrow \varphi \mid G$ and $\Gamma_\varphi \cup \Gamma_\psi \Rightarrow \psi \mid G$ have cut-free proofs. By applying (com) to these two hypersequents we get a cut-free proof of $\Gamma_\varphi \Rightarrow \varphi \mid \Gamma_\psi \Rightarrow \psi \mid G$. This contradicts Lemma 1 and the fact that G does not have a cut-free proof.

Lemma 3 *For every $\psi \in \mathcal{F}$ and $\Gamma_\varphi \in W$:*

(i) *If $\psi \in \Gamma_\varphi$ then $v(\Gamma_\varphi, \psi) = t$.*

(ii) *If $(\Gamma_\varphi \Rightarrow \psi) \in G$ then $v(\Gamma_\varphi, \psi) = f$.*

Proof of Lemma 3: By induction on the complexity of ψ.

- Suppose ψ is the atomic formula p. Then (i) holds by definition of v. To prove (ii), assume that $(\Gamma_\varphi \Rightarrow p) \in G$. Had $p \in \Gamma_\varphi$ then G would have a cut-free proof from the axiom $p \Rightarrow p$ (using (IW) and (EW)). Hence $p \notin \Gamma_\varphi$, and so $v(\Gamma_\varphi, p) = f$ by definition of v.

- Suppose $\psi = \bot$. Then (ii) is trivially true. (i), on the other hand is vacuously true, since had $\bot \in \Gamma_\varphi$ then G could have been derived from the axiom $\bot \Rightarrow \varphi$ using just (IW) and (EW).

- Suppose ψ is $\psi_1 \rightarrow \psi_2$.

  (i) Suppose $\psi \in \Gamma_\varphi$ and $\Gamma_\varphi \leq w'$. We want to show that either $v(w', \psi_1) = f$ or $v(w', \psi_2) = t$. Let $w' = \Gamma_{\varphi'}$. Since $\Gamma_\varphi \subseteq \Gamma_{\varphi'}$, $\psi_1 \rightarrow \psi_2 \in \Gamma_{\varphi'}$. Therefore by induction hypothesis it suffices to show that in this case either $(\Gamma_{\varphi'} \Rightarrow \psi_1) \in G$ or $\psi_2 \in \Gamma_{\varphi'}$. Assume otherwise. Then both $(\Gamma_{\varphi'} \Rightarrow \psi_1) \mid G$ and (by Lemma 1) $\psi_2, \Gamma_{\varphi'} \Rightarrow \varphi' \mid G$ have cut-free proofs. By applying $(\rightarrow\Rightarrow)$ to these hypersequents we get a cut-free proof of $\psi_1 \rightarrow \psi_2, \Gamma_{\varphi'} \Rightarrow \varphi' \mid G$. Since $\psi_1 \rightarrow \psi_2 \in \Gamma_{\varphi'}$, and $(\Gamma_{\varphi'} \Rightarrow \varphi') \in G$ (by Lemma 1 again), this means that G has a cut-free proof. A contradiction.

  (ii) Suppose $(\Gamma_\varphi \Rightarrow \psi) \in G$. Then $(\Gamma_\varphi, \psi_1 \Rightarrow \psi_2) \in G$, since otherwise $\Gamma_\varphi, \psi_1 \Rightarrow \psi_2 \mid G$ would have a cut-free proof, and so (using $(\Rightarrow\rightarrow)$) $\Gamma_\varphi \Rightarrow \psi \mid G$ would have a cut-free proof, contradicting $(\Gamma_\varphi \Rightarrow \psi) \in G$. It follows that $\psi_2 \in \mathcal{F}'$, and

$\Gamma_\varphi \cup \{\psi_1\} \subseteq \Gamma_{\psi_2}$. Hence $\Gamma_\varphi \leq \Gamma_{\psi_2}$, and by induction hypothesis $v(\Gamma_{\psi_2}, \psi_1) = t$. The induction hypothesis and Lemma 1 also entail that $v(\Gamma_{\psi_2}, \psi_2) = f$. From the last three facts it follows that $v(\Gamma_\varphi, \psi) = f$.

- Suppose $\psi = \psi_1 \vee \psi_2$.

  (i) Assume that $\psi \in \Gamma_\varphi$. Then either $\psi_1 \in \Gamma_\varphi$ or $\psi_2 \in \Gamma_\varphi$ (Otherwise Lemma 1 would have implied that both of $\psi_1, \Gamma_\varphi \Rightarrow \varphi \mid G$ and $\psi_2, \Gamma_\varphi \Rightarrow \varphi \mid G$ have cut-free proofs. Using $(\vee \Rightarrow)$ this implies that $\psi_1 \vee \psi_2, \Gamma_\varphi \Rightarrow \varphi \mid G$ has such a proof, contradicting Lemma 1 and $\psi \in \Gamma_\varphi$). By induction hypothesis it follows that either $v(\Gamma_\varphi, \psi_1) = t$ or $v(\Gamma_\varphi, \psi_2) = t$. In both cases $v(\Gamma_\varphi, \psi) = t$.

  (ii) Assume that $(\Gamma_\varphi \Rightarrow \psi) \in G$. Then $\Gamma_\varphi \Rightarrow \psi_1$ and $\Gamma_\varphi \Rightarrow \psi_2$ are both in G (Suppose for example that $(\Gamma_\varphi \Rightarrow \psi_1) \notin G$. Then $\Gamma_\varphi \Rightarrow \psi_1 \mid G$ would have a cut-free proof. Using (EW) and $(\Rightarrow \vee)$ we would get a cut-free proof of G). It follows by the induction hypothesis that $v(\Gamma_\varphi, \psi_1) = f$ and $v(\Gamma_\varphi, \psi_2) = f$. Hence $v(\Gamma_\varphi, \psi) = f$.

- Suppose $\psi = \psi_1 \wedge \psi_2$.

  (i) Assume that $\psi \in \Gamma_\varphi$. Then $\{\psi_1, \psi_2\} \subseteq \Gamma_\varphi$ (Otherwise Lemma 1 would have implied that $\psi_1, \psi_2, \Gamma_\varphi \Rightarrow \varphi \mid G$ has a cut-free proof. Using $(\wedge \Rightarrow)$ this implies that $\psi_1 \wedge \psi_2, \Gamma_\varphi \Rightarrow \varphi \mid G$ has such a proof, contradicting Lemma 1 and $\psi \in \Gamma_\varphi$). By induction hypothesis it follows that $v(\Gamma_\varphi, \psi_1) = t$ and $v(\Gamma_\varphi, \psi_2) = t$. Hence $v(\Gamma_\varphi, \psi) = t$.

  (ii) Assume that $(\Gamma_\varphi \Rightarrow \psi) \in G$. Then either $(\Gamma_\varphi \Rightarrow \psi_1) \in G$ or $(\Gamma_\varphi \Rightarrow \psi_2) \in G$ (Suppose otherwise. Then both $\Gamma_\varphi \Rightarrow \psi_1 \mid G$ and $\Gamma_\varphi \Rightarrow \psi_2 \mid G$ would have cut-free proofs. Using $(\Rightarrow \wedge)$ we would get a cut-free proof of G). It follows by the induction hypothesis that either $v(\Gamma_\varphi, \psi_1) = f$ or $v(\Gamma_\varphi, \psi_2) = f$. Hence $v(\Gamma_\varphi, \psi) = f$.

To end the proof, we show that G_0 is not valid in $\mathcal{W}$ (and so, by Lemma 2, it is not valid in all linear Kripke frames). So let $(\Delta \Rightarrow \varphi) \in G_0$. Then $(\Delta \Rightarrow \varphi) \in G$. Hence $\varphi \in \mathcal{F}'$, $\Gamma_\varphi \in W$, and $\Delta \subseteq \Gamma_\varphi$. Therefore Lemma 3 implies that $v(\Gamma_\varphi, \psi) = t$ for every $\psi \in \Delta$. By the same Lemma (and Lemma 1), $v(\Gamma_\varphi, \varphi) = f$. It follows that $\mathcal{W}$ is not a Kripke model of $\Delta \Rightarrow \varphi$. Since this is true for every $(\Delta \Rightarrow \varphi) \in G_0$, $\mathcal{W}$ is not a Kripke model of G_0. □

The following two theorems are easy corollaries of Propositions 1, 2, and 3:

Theorem 1 *For every hypersequent G, $\vdash_{\mathbf{HG}} G$ iff $\vdash^{LC}_{\mathbf{Go}} G$ iff $\vdash^{LC}_{\mathbf{Kr}} G$.*

Theorem 2 *The cut-elimination theorem is valid for* **HG**.

Next we show that Proposition 3 and the last two theorems can be generalized to consequences from assumptions:

Proposition 4 *Let S be a finite set of hypersequents and G a sequent. Then $S \vdash^{LC}_{\mathbf{Kr}} G$ iff there is a proof of G from S in which all cuts are made only on* formulas *(not even proper subformulas of formulas) that occur in S.*

Proof: The proof closely follows that of Proposition 3, provided we define this time $\mathcal{F}$ to be the set of all subformulas of formulas which occur in G_0 *or in S*, and everywhere replace "cut-free proof" by "a proof in which all cuts are made on formulas that occur in S" (henceforth such cuts, as well as proofs from S that use only such cuts, will be called "permissible"). The three lemmas from the proof of Proposition 3 are proved exactly as before, and so the resulting frame $\mathcal{W}$ is again a countermodel of G_0. What remains to be done is to show that $\mathcal{W}$ is a model of every element of S. So let H be such an element. Had all components of H been also components of G then G could have been derived from H using only (EW). So there exists a component $\Delta \Rightarrow \psi$ of H which is not a component of G (implying that $\Delta \Rightarrow \psi \mid G$ has a permissible proof). We show that $\mathcal{W}$ is a Kripke model of $\Delta \Rightarrow \psi$. So let Γ_φ be an element of W. We should show that either $v(\Gamma_\varphi, \psi) = t$, or $v(\Gamma_\varphi, \theta) = f$ for some $\theta \in \Delta$. By Lemma 3 the first option obtains in case $\psi \in \Gamma_\varphi$. So assume that $\psi \notin \Gamma_\varphi$, implying that $\psi, \Gamma_\varphi \Rightarrow \varphi \mid G$ has a permissible proof. This in turn implies that $\Gamma_\varphi \Rightarrow \psi$ is in G (because otherwise $\Gamma_\varphi \Rightarrow \psi \mid G$ would have had a permissible proof, and then a permissible cut on ψ of this hypersequent and $\psi, \Gamma_\varphi \Rightarrow \varphi \mid G$ would have given $\Gamma_\varphi \Rightarrow \varphi \mid G$, and so, by Lemma 1, G would have had a permissible proof). It follows that $\Delta \not\subseteq \Gamma_\varphi$ (since otherwise $\Gamma_\varphi \Rightarrow \psi \mid G$ could have been derived from the permissibly provable $\Delta \Rightarrow \psi \mid G$ using (IW)). Accordingly, let θ be an element of Δ such that $\theta \notin \Gamma_\varphi$. Like in the case of ψ, $\Gamma_\varphi \Rightarrow \theta$ is necessarily in G. Hence $v(\Gamma_\varphi, \theta) = f$ by Lemma 3. $\square$

Again the following theorems follow from Propositions 1, 2, and 4:

Theorem 3 $\vdash_{\mathbf{HG}} = \vdash^{LC}_{\mathbf{Go}} = \vdash^{LC}_{\mathbf{Kr}}$.

Theorem 4 **The strong cut-elimination theorem:** *If G is derivable in* **HG** *from a set S of hypersequents, then G has a proof in* **HG** *from S in which all cuts are on formulas which occur in S.*

Note. The strong cut-elimination theorem for classical logic was stated and proved in [2]. Note that the usual cut-elimination theorem is the special case of the strong one in which S is taken to be empty.

An open Problem: To find a purely semantic (direct) proof of the (strong) cut-elimination theorem for the first-order extension of **HG** (which at the same time will demonstrate the completeness of this calculus for its usual semantics. See [4]).

References

[1] Avron A., *Using Hypersequents in Proof Systems for Non-classical Logics*, Annals of Mathematics and Artificial Intelligence 4 (1991), 225–248.

[2] Avron A., *Gentzen-Type Systems, Resolution and Tableaux*, Journal of Automated Reasoning 10 (1993), 265–281.

[3] Baaz M., Ciabattoni A., and Fermüller C. G., *Hypersequent Calculi for Gödel Logics - a Survey*, Journal of Logic and Computation 13 (2003), 835–861.

[4] Baaz M., Preining N., and Zach R., *First-order Gödel Logics*, Annals of Pure and Applied Logic 147 (2007), 23–47.

[5] Baaz M., and Zach R., *Hypersequent and the Proof Theory of Intuitionistic Fuzzy Logic*, in **Proceedings of CSL 2000**, vol. 1862 of LNCS, 187-201

[6] Ciabattoni A., Galatos N., and Terui K., *From Axioms to Analytic Rules in Nonclassical Logics*, in **Proceedings of LICS 2008**, 229-240

[7] Dummett M., *A Propositional Calculus with a Denumerable matrix*, Journal of Symbolic Logic 24 (1959), 96–107.

[8] Gödel K., *On the Intuitionistic Propositional Calculus, 1933*. In **Collected Work, Vol. 1**, edited by S. Feferman et al, Oxford University Press, 1986.

[9] Hájek P., **Metamathematics of Fuzzy Logic**, Kluwer Academic Publishers, 1998.