THE METHOD OF HYPERSEQUENTS IN THE PROOF THEORY OF PROPOSITIONAL NON-CLASSICAL LOGICS

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I. An Introduction

Until not too many years ago, all logics except classical logic (and, perhaps, intuitionistic logic too) were considered to be things esoteric. Today this state of affairs seems to have completely been changed. There is a growing interest in many types of nonclassical logics: modal and temporal logics, substructural logics, paraconsistent logics, non-monotonic logics – the list is long. The diversity of systems that have been proposed and studied is so great that a need is felt by many researchers to try to put some order in the present logical jungle. Thus [Cl91], [Ep90] and [Wo88] are three recent books in which an attempt is made to develop a general theoretical framework for the study of logics. On the more pragmatic side, several systems have been developed with the goal of providing a *computerized* logical framework in which many different logical systems can be implemented in a uniform way. An example is the Edinburgh LF([HHP91]).

It is clear that there is no limit to the number of logics that logicians (and non-logicians) can produce. Logical frameworks should only be expected, therefore, to be able to handle those that are "good" or "interesting". But what *is* a "good" logic? One simple answer might be: a logic which has applications. This answer is not satisfactory, though. First, systems of logics are frequently introduced *before* they find actual applications. Moreover: there is a tendency to choose for application exactly those that are "good" in some sense. Second: Logic is an autonomous mathematical discipline, and as such should have its own independent criteria. One such criterion is the existence of simple, illuminating semantics. This indeed is always a very good sign. A more important criterion (in my opinion, and since logics deals above all with *proofs*) is the existence of a good proof system. Such a system should make it easier to find proofs *in* the system, to prove results *about* it, and, above all, should have the subformula property.

An important line of research in the general study of logics should, according to the discussion above, be a search for a general proof-theoretical framework, within which "good" proof-systems could be constructed. Such a framework should have the following properties:

(1) It should be able to handle a great diversity of logics of different types. We expect that most logics which logicians have found interesting for other reasons will get good proof systems within the framework (something that will indicate, in turn, that these logics are really worth further investigations and use). On the other hand, the construction of the framework might suggest new logics that should be important.

(2) Because of the proof-theoretical nature and the expected generality, the framework should be independent of any particular semantics. One should not be able to guess, just from the form of the structures which are used, the intended semantics of a given proof system (recent frameworks for many-valued logics and for modal logics violate this principle – see below).

(3) The structures used in the framework should be built from the formulae of the logic and should not be too complicated (for human understanding and for computer implementation). Most important – the subformula property they allow should be a *real* one.[1]

Beyond these three basic demands, the following properties are also desirable:

(4) The rules of inference should have a small, fixed number of premises, and their application should have a local nature. In other words: the applicability of a rule should depend only on the structure of the premises and not on the way they have been

[1] A use of "structural connectives" that can arbitrarily be nested usually violates this principle. It seems to me that this is the weak point of Belnap's framework of Display Logic [Be82], [AB92], which otherwise has all the other properties.

obtained.

(5) Since there should be something common to all the various connectives, we call "conjunction", "disjunction", "implication" and "negation", the corresponding rules should be as standard as possible. The difference between logics should be due to some other rules, which are independent of any particular connective. Such rules are usually called "structural rules".[2]

(6) The proof systems constructed within the framework should give us better understanding of the corresponding logics and the difference between them.

The best example of a framework which has properties (1)-(6) above is, of course, that of cut-free proofs in Gentzen-style systems. Indeed, with respect to (1), it is amazing how many and extremely different logics can be captured within this framework: classical logic, intuitionistic logic, the most important modal logics (except, perhaps, $S5$), relevance logics, many-valued logics and more. On the other hand, the framework has indeed led to new interesting logics (for example – Linear Logic [Gi87]).

Powerful as it is, the framework of ordinary sequents is not capable of handling all interesting logics. There are logics with nice, simple semantics and obvious interest for which no decent, cut-free formulation seems to exist (we shall see many examples below). It would be an exaggeration to reject them as worthless just because of this fact. Larger, but still satisfactory frameworks should therefore be sought.[3] This paper is devoted to a description, with many examples, of one particular framework of this sort: that of hypersequents. We shall show that this framework is indeed stronger than that of ordinary sequents, but still has properties (1)-(6) above. In addition, the following two points concerning it should be noted:

(a) Hypersequents are finite sets (or sequences) of usual sequents. As such, they

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[2] This principle was put forward in [Do89]. Došen's paper contains a general characterization of the basic standard connectives. A similar characterization, together with a discussion of the general role of these connectives, which explains their universal importance (and so why they usually are picked) can be found in [Av91d].

[3] Two frameworks which were proposed and deserve mentioning here are that of higher-order sequents ([Do85], [Du73], [AB75]) and Display Logic ([Be82], [AB92]). Both are somewhat weak, I believe, with respect to point (3) above.

form a simple and natural generalization of the sequential framework. Obviously, every standard, sequential calculus is also a hyperseqential calculus. There are, however, logics (like Dummet's LC and Lukasiewicz 3-valued logic) the treatment of which requires proper hypersequents. It might be added here that although a hypersequent is certainly a more complex data structure than an ordinary sequent, it is not *much* more complicated, and goes in fact just one step further.[4]

(b) As we shall see, doing proofs within the hypersequential framework allows a high degree of parallelism. Certain hypersequential calculi (and this admittedly is still no more than a speculation) might provide therefore good models of parallel computations (exactly as the sequential calculus for intuitionistic logic is recognized as a good model of sequential computation).

The structure of the rest of this paper is as follows. In section II we give a general description of hypersequential calculi and the way they are meant to be used. The other sections are devoted to examples. We have tried to bring examples from different types of logics, and also that each example will represent some new aspect of the use of hypersequents or of logic in general. In more detail:

Section III is devoted to two of the most important intermediate logics: Dummet's LC (which corresponds to an infinite-valued matrix of Gödel) and a weaker system we call LCW. In both, the aspect of parallelism is well demonstrated. While LC is the most famous intermediate logic, we were led to LCW through a natural calculus of hypersequents for which we have discovered a very simple semantics. (As it happens this semantics and a corresponding Hilbert-type system had in fact been known long before.)

Section IV is devoted to two substructural Logics (which belong, in fact, to the relevance family). For the first, RM, the use of hypersequents makes it possible to give a constructive proof of a crucial theorem for which only semantical, non-constructive proofs were known before. The second example, RMI_m, demonstrates the difference between weak and strong completeness and between theoremhood and consequence. An ordinary Gentzen-type system corresponds here to the first, but only a hypersequential one captures

[4] Hypersequents were first introduced in [Po83], and independently in [Av87].

the second.

Section V deals with modal logic, with $S5$ as the principal example. Here the importance of modalized versions of structural rules is revealed (previously this idea was used only in Linear Logic).

All the examples in sections III-V were of logics which do not have a finite characteristic matrix (although they do have the finite model property). In section VI we show that this is not a necessary feature by giving two examples of 3-valued logics: Lukasiewicz L_3 and RM_3 – which is a maximal paraconsistent logic.

One final note: this is for the most part a review paper, so except for section V (on modal logics), in which most of the material is new, we omit proofs and only give references to the original papers. Exceptions are proofs of claims which were never proved before or proofs which are particularly short and illuminating.

II. The Framework of Hypersequents – A General Description

Definition 1. A *hypersequent* is a structure of the form:

$$?_1 \Rightarrow \Delta_1 | ?_2 \Rightarrow \Delta_2 | \cdots | ?_n \Rightarrow \Delta_n$$

where $?_i$ and Δ_i ($i = 1, \ldots, n$) are finite sequences of formulae (i.e.: $?_i \Rightarrow \Delta_i$ is an ordinary sequent). $?_i \Rightarrow \Delta_i$ ($i = 1, \ldots, n$) are called the *components* of the hypersequent. If for all i Δ_i consists of a single formula the hypersequent is called *single-conclusioned*.

We shall use G and H as variables for (possibly empty) hypersequents.

The standard interpretation of the "|" symbol is usually disjunctive. Intuitively a hypersequent is true in a certain state iff one of its components is true in that state (relative to some semantics which makes the last statement meaningful).

Axioms and rules. In most hypersequential calculi, the only axioms are of the form $A \Rightarrow A$ (or even $p \Rightarrow p$, p atomic) – exactly as in standard sequential calculi. And like in sequential calculi one needs to add axioms to deal with propositional constants (for example: $\perp, ? \Rightarrow \Delta$) in case the language contains any.

Again like in ordinary sequential calculi, the rules of inference are usually divided into *logical* rules and *structural* rules. The guiding idea is that the logical rules should essentially be identical to those used in ordinary calculi, and that the difference between the various logics should mainly be due to differences in their structural rules. This in general is a very important principle in modern research on logics – especially substructural logics. Thus the difference between classical logic, intuitionistic logic, linear logic, various Relevance logics, BCK logic and so on are all due to differences in the structural rules (concerning *ordinary* sequents) which are allowed in each. As we shall see, the richer structure of hypersequents makes it possible to introduce *new* types of structural rules, and so to extend the applicability of the above principle as well as to allow greater versatility in developing interesting logical systems.

According to the above principles the only difference between the rules we shall employ for the standard connectives and the usual rules, will be that in the hypersequential version extra *side components* will be allowed. For example, in all the systems below the rules of conjunction will have either the multiplicative form:

$$\frac{G|A,B,? \Rightarrow \Delta|H}{G|A \wedge B,? \Rightarrow \Delta|H} \qquad \frac{G_1|?_1 \Rightarrow \Delta_1, A|H_1 \qquad G_2|?_2 \Rightarrow \Delta_2,\ B|H_2}{G_1|G_2|?_1,?_2 \Rightarrow \Delta_1,\Delta_2, A \wedge B|H_1|H_2}$$

or else the additive form:[5]

$$\frac{G|A,? \Rightarrow \Delta|H}{G|A \wedge B,? \Rightarrow \Delta|H} \qquad \frac{G|B,? \Rightarrow \Delta|H}{G|A \wedge B,? \Rightarrow \Delta|H} \qquad \frac{G|? \Rightarrow \Delta, A|H \qquad G|? \Rightarrow \Delta, B|H}{G|? \Rightarrow \Delta, A \wedge B|H}$$

As usual, the two sets of rules are equivalent in the presence of the standard structural rules (see below), but not if one of them is omitted.

The standard structural rules in ordinary Gentzen-type systems are permutation, contraction, weakening and cut. In calculi of hypersequents we have two versions of the first three: an *external* version and an *internal* version. The external one treats *components* within a hypersequent. The internal one treats *formulae* within some component. For example, the external contraction rule is:

$$\frac{G|? \Rightarrow \Delta|? \Rightarrow \Delta|H}{G|? \Rightarrow \Delta|H}$$

while the internal contraction rules are:

$$\frac{G|A,A,? \Rightarrow \Delta|H}{G|A,? \Rightarrow \Delta|H} \qquad \frac{G|? \Rightarrow \Delta, A, A|H}{G|? \Rightarrow \Delta, A|H}$$

[5] The terminology is due to [Gi87].

In all the examples below, all the three external structural rules and internal permutation are valid. Internal contraction and internal weakening are sometimes omitted, though (usually only one of them).

As for the cut rule – it has only internal versions. The more natural and useful one is the multiplicative version:

$$\frac{G_1|?_1 \Rightarrow \Delta_1, A|H_1 \qquad\qquad G_2|A, ?_2 \Rightarrow \Delta_2|H_2}{G_1|G_2|?_1, ?_2 \Rightarrow \Delta_1, \Delta_2|H_1|H_2}$$

For all the logics dealt with below we give a version in which this cut rule is admissible (i.e.: the cut-elimination theorem obtains).

We have noted above that the use of hypersequents opens the door to new types of structural rules. One example which is characteristic to hypersequents and has turned to be particularly useful is that of *splitting* rules. We are using below four rules of this type:

$$(S_c) \quad \frac{G|?_1, ?_2 \Rightarrow \Delta_1, \Delta_2|H}{G|?_1 \Rightarrow \Delta_1|?_2 \Rightarrow \Delta_2|H} \qquad\qquad (S_I) \quad \frac{G|?, \Delta \Rightarrow A|H}{G|? \Rightarrow A|\Delta \Rightarrow A|H}$$

$$(MS) \quad \frac{G|\Box ?, ?' \Rightarrow \Delta', \Box\Delta|H}{G|\Box ? \Rightarrow \Box\Delta|?' \Rightarrow \Delta'|H} \qquad\qquad (ES) \quad \frac{G|?_1, ?_2 \Rightarrow \Delta_1, \Delta_2|H}{G|?_1 \Rightarrow \Delta_1|?', ?_2 \Rightarrow \Delta_2, \Delta'|H}$$

(S_c) is the basic form of splitting. (S_I) is its single-conclusioned (or "intuitionistic") version. We shall see that LC, for example, is obtained from intuitionistic logic by adding this rule to it. (MS) is the *modalized* version of the rule. Here only modalized formulae (those which begin with either $\Box$ or $\Diamond$ [6]) are allowed to be split from a given component. This limitation is close in spirit to the limitation of the standard structural rules to modalized (or "exponential") formulae in Linear Logic. We shall see that $S5$ is obtained from S4 (or even just T) by adding to it this version of splitting. Finally, (ES) is an extended form of splitting which is useful in logics which do not permit the usual internal weakening, but do allow it partially in a combination with splitting.

Other types of rules which are useful might be called "shuffling" rules. Here two components of different premises are shuffled and then split in a certain way to produce

[6] For simplicity, $\Box$ is mentioned in the formulation above, but this is not a necessary limitation.

two or more components of the conclusion. Two examples which are used below are:

(communication) $$\frac{G_1|?_1 \Rightarrow A_1|H_1 \qquad G_2|?_2 \Rightarrow A_2|H_2}{G_1|G_2|?_1 \Rightarrow A_2|?_2 \Rightarrow A_1|H_1|H_2}$$

(mixing) $$\frac{G_1|?_1, ?_2, ?_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3|H_1 \qquad G_2|?'_1, ?'_2, ?'_3 \Rightarrow \Delta'_1, \Delta'_2, \Delta'_3|H_2}{G_1|G_2|?_1, ?'_1 \Rightarrow \Delta_1, \Delta'_2|?_2, ?'_2 \Rightarrow \Delta_1, \Delta'_2|?_3, ?'_3 \Rightarrow \Delta_3, \Delta'_3|H_1|H_2}$$

A final remark: as was noted already in the introduction, almost all the rules above treat exactly one component in each premise. Hence most of the time activity in one component is completely independent of the activity in another, and rules can frequently be applied concurrently. The only rule (among those which are described above or used below) which brings moments of synchronization into proofs is external contraction. This rule should therefore be very important in a use of a hypersequential calculus as a computational model. This rule is also the one that causes most trouble in the proofs of cut-elimination, and its presence is the explanation why in hypersequential calculi cut-elimination usually does not imply the Craig interpolation theorem.

III. Intermediate Logics

III.1 LC: Parallelism in Action.

III.1.1 General Background.

In [Go33] Gödel introduced a sequence $\{G_n\}$ of n-valued logics, as well as an infinite-valued matrix G_ω in which all the $G_n s$ can be embedded. He used these matrices to show some important properties of intuitionistic logic. The logic of G_ω was later axiomatized by Dummet in [Du59] and is known since then as Dummet's LC. It probably is the most important intermediate logic, one that turns up in several places, like the provability logic of Heyting's Arithmetics ([Vi82]) and relevance logic ([DM71]).

Semantically LC corresponds to linearly ordered Kripke structures. It also corresponds of course to G_ω, which is the matrix $\langle N \cup \{\omega\}, \leq, \rightarrow, \neg, \vee, \wedge \rangle$ where $\leq$ is the usual order on N extended by a maximal element ω, $a \rightarrow b$ is ω if $a \leq b$ and b otherwise, $\neg a$ is simply $a \rightarrow 0$, and $\wedge$ and $\vee$ are, respectively, the min and max operations on $\langle N \cup \{\omega\}, \leq \rangle$. The consequence relation $\vdash_{LC}$ is defined as follows: $A_1, \ldots, A_n \vdash_{LC} B$ iff

$\min\{v(A_1), \ldots, v(A_n)\} \le v(B)$ for every v in G_ω.[7] This is equivalent ([Av91a, p.236]) to taking ω as the only designated element and defining: $A_1, \ldots, A_n \vdash_{LC} B$ iff for every v in G_ω either $v(B) = \omega$ or $v(A_i) \neq \omega$ for some $1 \le i \le n$.

A Hilbert-type axiomatization for LC can be obtained from intuitionistic logic by adding to it the axiom $(A \to B) \vee (B \to A)$ ([Du59]). A cut-free formulation was first given by Sonobo in [So75]. His system has, however, the serious drawback of using a rule with arbitrary number of premises, all of which contain formulae which are essential for the inference (strictly speaking, this is an infinite set of rules).

A cut-free formulation of LC which uses hypersequents and does not have the above drawback has been given in [Av91a]. Like intuitionistic logic, it uses single-conclusioned hypersequents.

III.1.2 The System *GLC*.

Axioms. As in intuitionistic logic.

Rules. (1) The standard external rules and the standard single-conclusioned internal rules.

(2) The hypersequential version of the intuitionistic logical rules.

(3) The intuitionistic splitting rule and the communication rule of the previous section:

$$(S_I) \quad \frac{G|?,\Delta \Rightarrow A|H}{G|? \Rightarrow A|\Delta \Rightarrow A|H} \qquad (Com) \quad \frac{G_1|?_1 \Rightarrow A_1|H_1 \qquad G_2|?_2 \Rightarrow A_2|H_2}{G_1|G_2|?_1 \Rightarrow A_2|?_2 \Rightarrow A_1|H_1|H_2}$$

An example of a proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B|B \Rightarrow A}\,(\mathrm{Com})}{\Rightarrow A \to B|B \Rightarrow A}}{\Rightarrow (A \to B) \vee (B \to A)|B \Rightarrow A}}{\Rightarrow (A \to B) \vee (B \to A)| \Rightarrow B \to A}}{\dfrac{\Rightarrow (A \to B) \vee (B \to A)| \Rightarrow (A \to B) \vee (B \to A)}{\Rightarrow (A \to B) \vee (B \to A)}}$$

[7] As usual, in case $n = 0$ the "minimal element" is taken to be ω.

III.1.3 Notes. (1) In the previous section we have used a slightly different version of (Com), in which $G_1 = G_2$ and $H_1 = H_2$. Obviously, the two versions are equivalent because of the external structural rules.

(2) In the above example of a proof the work on the two components (before they are merged into one by external contraction at the last line) can be done completely in parallel. The order chosen above (first working on the left component, and after finishing with it working with the other) is completely arbitrary. This situation is typical.

(3) The name "communication rule" is due to the intuition that if we take a hypersequent as representing a *multiprocess* then the rule corresponds to a communication, or an exchange of information between two such multiprocesses.

(4) The two extra rules of LC (S_I and Com) can be replaced by a *single* rule, that may take either of the following forms:

$$\frac{G|?_1, ?_2 \Rightarrow A|H \qquad G|?_1, ?_2 \Rightarrow B|H}{G|?_1 \Rightarrow A|?_2 \Rightarrow B|H} \qquad \frac{G|?_1, \Delta_1 \Rightarrow A|H \qquad G|?_2, \Delta_2 \Rightarrow B|H}{G|?_1, ?_2 \Rightarrow A|\Delta_1, \Delta_2 \Rightarrow B|H}$$

(The second rule is what was originally used in [Av91a] and called there the communication rule, the first is a simplification suggested by Mints). The two forms are obviously equivalent, since the first can be derived from the second using internal contractions, while the second can be derived from the first using internal weakenings. Obviously, splitting can be derived from the first version by taking $A = B$ and using external contraction, while the communication rule is derivable from it with the help of internal weakening. Finally, the first version is derivable in GLC as follows: first, split $G|?_1, ?_2 \Rightarrow A|H$ and $G|?_1, ?_2 \Rightarrow B|H$ to $G|?_1 \Rightarrow A|?_2 \Rightarrow A|H$ and $G|?_1 \Rightarrow B|?_2 \Rightarrow B|H$ respectively. Apply now communication to these two hypersequents to get $G|?_1 \Rightarrow A|G|?_2 \Rightarrow B|?_1 \Rightarrow A|H|?_2 \Rightarrow B|H$. Finish by external permutations and contractions.

III.1.4. Semantics and Main Results. Let the interpretation of a *sequent* $A_1, A_2, \ldots, A_n \Rightarrow B$ be $A_1 \to (A_2 \to \cdots \to (A_n \to B)\cdots)$. The interpretation of a hypersequent $?_1 \Rightarrow \Delta_1|\cdots|?_n \Rightarrow \Delta_n$ is $\varphi_{\Gamma_1 \Rightarrow \Delta_1} \vee \cdots \vee \varphi_{\Gamma_n \Rightarrow \Delta_n}$, where for each i $\varphi_{\Gamma_i \Rightarrow \Delta_i}$ is the interpretation of $?_i \Rightarrow \Delta_i$.

The two main theorems concerning GLC are:

Theorem 1. *The cut elimination theorem obtains for GLC.*

Theorem 2. *A hypersequent is provable in GLC iff its interpretation is valid in G_ω (iff it is provable in Dummet's system LC).*

The proofs of both theorems can be found in [Av91a]. We note that the second easily follows from the first if we use Dummet's completeness theorem, but it is possible to give a direct proof. The proof of cut-elimination is not that easy. As in all hypersequential calculi, the main obstacle is the presence of external contraction. Still, in the case of GLC the proof is not as complicated as in the case of RM which is described in the next section.

III.2. LCW: The Destructive Power of Conjunction.

III.2.1. Background. The hypersequential system for LC has *two* special rules. A natural question that the use of the hypersequential framework leads to now is: what system do we obtain if we use only the first of them (the splitting rule, which seems to be the most typical hypersequential rule)? The answer is somewhat surprising:

Theorem 1. *The system which is obtained from GLC by omitting the communication rule is equivalent to GLC, but the cut-elimination theorem fails for it.*

Proof: We can derive the communication rules in the resulting system with the help of cuts as follows:

$$(\mathrm{Cuts})\frac{\dfrac{A_2 \Rightarrow A_2}{A_1 \wedge A_2 \Rightarrow A_2} \qquad (S_I)\dfrac{\dfrac{G|?_1 \Rightarrow A_1|H \qquad G|?_2 \Rightarrow A_2|H}{G|?_1, ?_2 \Rightarrow A_1 \wedge A_2|H}}{G|?_1 \Rightarrow A_1 \wedge A_2|?_2 \Rightarrow A_1 \wedge A_2|H} \qquad \dfrac{A_1 \Rightarrow A_1}{A_1 \wedge A_2 \Rightarrow A_1}}{G|?_1 \Rightarrow A_2|?_2 \Rightarrow A_1|H}$$

(in the description of this proof we did the two cuts on the same line, since they can be done in parallel. From now on we shall use such descriptions frequently).

To see that without communication we lose cut-elimination we note that in the system without it, any hypersequent which consists only of atomic formula is provable without cuts only if one of its components is of the from $? \Rightarrow p$ where $p \in ?$ (easy). Hence $p \Rightarrow q \mid q \Rightarrow p$, which is valid in GLC, cannot have a cut-free proof. ∎

So is the communication rule needed only to get cut elimination? Not really. In its derivation above conjunction and its rules have a crucial role. It turns out that without conjunction the situation is quite different. First, the communication rule is not derivable any more. Second, we do have cut elimination.

III.2.2. The System *GLCW*.

Axioms. As in the intuitionistic logic.

Rules. (i) The standard external and single-conclusioned internal structural rules.

(ii) The hypersequential version of the intuitionistic rules for $\to, \vee$ and $\neg$ (The propositional constants $\top, \bot$ are optional).

(iii) Intuitionistic splitting (S_I)

An Example of a Proof.

$$\dfrac{\dfrac{\dfrac{(S_I)\,\dfrac{\dfrac{A \Rightarrow A \qquad\qquad B \Rightarrow B}{A, A \to B \Rightarrow B}}{A \Rightarrow B | A \to B \Rightarrow B}}{\Rightarrow A \to B | \Rightarrow (A \to B) \to B}}{\Rightarrow (A \to B) \vee ((A \to B) \to B) | \Rightarrow (A \to B) \vee ((A \to B) \to B)}}{\Rightarrow (A \to B) \vee ((A \to B) \to B)}$$

Theorem 2.

1) The cut-elimination theorem obtains for GLCW.

2) GLCW is strictly weaker than the corresponding fragment of GLC.

The proof of cut-elimination can be found in [Av91a]. The second part follows from the first exactly as in the proof of theorem 1.

Why is conjunction the problematic connective? Well, here it is because unlike the introduction rules on the r.h.s of the other connectives (which have all exactly one premise), $(\Rightarrow \wedge)$ has *two* premises. This blocks the proof of cut-elimination for $GLCW$ (see [Av91a] for details). It should be emphasized, though, that this is not the only known case in which the addition of conjunction is unconservative. In relevance logic, for example, the addition of a connective $\wedge$ for which $A \wedge B \to A$, $A \wedge B \to B$ and $A \to (B \to A \wedge B)$ are all valid has even more catastrophic consequences. Thus in $R_{\to}$ (the implicational fragment of R) this addition gives the $\{\to, \wedge\}$ fragment of intuitionistic logic. In $R_{\underset{\to}{\sim}}$ (the purely multiplicative fragment of R) it simply gives full classical logic. The moral is that conjunction is far from being a simple, "innocent" connective. The more general conclusion is that one should be careful with the addition of connectives – even the most standard ones.

III.2.3 Semantics. Returning to $GLCW$, one may ask whether it too corresponds to some nice semantics. The answer is positive. It corresponds to upper semi-lattices with a greatest element (ω) and a smallest element (0). The definitions of $\rightarrow$ and $\neg$ are exactly as in G_ω, while $\vee$ corresponds here to the operator of sup (rather then to max, which does not always exist). If we limit ourselves to the purely implicational fragment then we can use the larger class of posets with a greatest element: For negation we also need the existence of 0.

Theorem 3. *(1) $\vdash_{GLCW} G$ iff for every valuation v in a structure as above G has a component $? \Rightarrow B$ such that $v(B) = \omega$ or $v(A) \leq v(B)$ for some $A \in ?$.*
(2) $\vdash_{GLCW} G$ iff for every v as above G has a component $? \Rightarrow B$ such that $v(B) = \omega$ or $v(A) \neq \omega$ for some $A \in ?$.

For proofs see again [Av91a]. It should be noted that the two characterizations hold for the full language of $GLCW$ as well as for its various fragments relative to their broader semantics.

For the full language (with $\vee$) we also have the following characterization:

Theorem 4. *$\vdash_{GLCW} G$ iff φ_G is valid, where φ_G is the translation of φ as defined in the case of LC.*

Proof: It is easy to see that if G is provable then so is $\Rightarrow \varphi_G$, since $\Rightarrow \varphi_G$ can be derived from G. Hence if $\vdash_{GLCW} G$ then φ_G is valid, by theorem 3. For the converse, assume φ_G is valid. By theorem 3 $\Rightarrow \varphi_G$ is provable. By definition of φ_G and the easy fact that $H| \Rightarrow A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_2 \rightarrow B)\ldots)$ is provable iff $H|A_1, \ldots, A_n \Rightarrow B$ is provable, all we need to show is that if $\vdash \Rightarrow A_1 \vee A_2 \vee \cdots \vee A_k$ then $\vdash \Rightarrow A_1 | \Rightarrow A_2 | \cdots | \Rightarrow A_k$. So assume that $\Rightarrow A_1 | \Rightarrow A_2 | \cdots | \Rightarrow A_k$ is not provable. Then there exists an upper semi-lattice M (with 0 and ω) and a valuation v in M s.t. $v(A_i) \neq \omega$ for all $1 \leq i \leq k$. Construct M' from M by adding a new element ω' so that $\omega' < \omega$ but $a \leq \omega'$ for all $a \neq \omega$. Define v' in M' by $v'(p) = v(p)$. By induction on the complexity of A prove that $v'(A) = v(A)$ if $v(A) \neq \omega$,

$v'(A) \in \{\omega, \omega'\}$ if $v(A) = \omega$. Since $v(A_i) \neq \omega$ for all i, $v'(A_i) = v(A_i) \leq \omega'$ for all i, and so $v(A_1 \vee A_2 \vee \cdots \vee A_n) \leq \omega'$, and $A_1 \vee \cdots \vee A_n$ is not valid. Hence $\Rightarrow A_1 \vee \cdots \vee A_n$ is not provable in this case. ■

III.2.4 Hilbert-Type Formulation. A corresponding Hilbert type system LCW can be obtained by adding to intuitionistic logic the axiom $(A \to B) \vee ((A \to B) \to B))$ which we have proved above, or $(W) : ((A \to B) \to C) \to ((((A \to B) \to B) \to C) \to C)$ in the fragments without disjunction. The exact connection is the following:

Theorem 5. *(1) $A_1, \ldots, A_n \vdash_{LCW} B$ iff the sequent $A_1, \ldots, A_n \Rightarrow B$ is provable in $GLCW$ (this is true also for the fragments without $\vee$)*
(2) $\vdash_{GLCW} G$ iff $\vdash_{LCW} \varphi_G$.

The theorem follows easily from theorems 3,4 and the soundness and completeness theorem for LCW which is proved in [Av91a].

We have here a good opportunity to correct a mistake from [Av91a]. It is stated there (p.139) that $G_1, \ldots, G_n \vdash_{GLCW} G$ (using cuts) iff $\varphi_{G_1}, \ldots, \varphi_{G_n} \vdash_{LCW} \varphi_G$. This is true if $n = 0$ (theorem 5), and a corresponding theorem does obtain for LC. Here, however, if we take $G_1 = \Rightarrow A \vee B$, $G = \Rightarrow A| \Rightarrow B$ then $\varphi_{G_1} = \varphi_G$ but it is easy to construct a model of $\Rightarrow A \vee B$ which is not a model of $\Rightarrow A| \Rightarrow B$.[8]

I.2.5 Historical Notes. The fact that by adding to the implicational intuitionistic logic the axiom (W) above we get a system which corresponds to posets with a greatest element (with $\to$ defined as above) was first shown in [Bu64]. The fact that by adding conjunction to this system we get the full implication-conjunction fragment of LC (and so this addition is not conservative) was first shown in [Pr64].[9] These two facts were rediscovered in [Av91a]. I believe that the failure of the proof of cut-elimination in $GLCW$ when the rules for conjunction are added provides the deep explanation why the addition of conjunction to LCW is not conservative (something which is difficult to understand otherwise).

[8] Note, however, that $\Rightarrow A| \Rightarrow B$ is provable without assumptions whenever $\Rightarrow A \vee B$ is, by theorem 4 and its proof.

[9] I am indebted to Kosta Došen for bringing these two papers to my attention.

IV. Substructural Logics

The name "substructural logics" is used nowadays for logics which are obtained from classical logic by deleting some of the standard structural rules. The main examples are linear logic, relevance logics and BCK logic. The term is not precisely defined, in fact. Thus we shall see in section VI that Lukasiewicz 3-valued logic should also be called a substructural logic, since the contraction rule is not valid there. In this section we shall treat, however, examples which are more "standard", since they belong (or are strongly related to) the "relevance" family.

IV.1. *RM*: A Constructive Proof of the Disjunctive Syllogism.

IV.1.1. General Background. The most characteristic feature of relevance logics (see [AB75], [Du84], [AB92]) is the rejection of the weakening rule. Thus, the purely multiplicative (i.e. the $\{\neg, \rightarrow\}$ fragment) of R is obtained from the corresponding fragment of classical logic in this way.[10] R itself is obtained by adding also $\wedge$ and $\vee$ in their additive version, and a corresponding distribution axiom. The last is the source of many troubles for R. With it no cut-free Gentzen-type formulation is known, and the system is undecidable ([Ur84]). The same problems exist in most other major systems created by Anderson & Belnap's relevantist school.[11] There is one exception though: the Dunn-McCall system RM. This system is by far the best understood among the systems of this school: it has been proved by Meyer to be decidable and to have a nice, simple semantics (which we describe below). Still, no corresponding cut-free formulation was known for a long time, although the existence of one should have been expected because of these nice properties. Such a formulation was finally given in [Av87] using hypersequents.

RM is obtained from R by adding to it the "mingle" axiom $A \rightarrow (A \rightarrow A)$. This axiom is equivalent to $(A \rightarrow B) \rightarrow (A \rightarrow A \rightarrow B)$, and its validity is equivalent to that of the converse of contraction in Gentzen-type systems. The idea (at least on the multiplicative level) is that assumptions in deductions come in *sets* rather than in sequences or multisets (as the multiplicative fragments of linear logic and of R suggest).

[10] This was first discovered by Kripke [(Kr59b]).

[11] They all are usually presented in a Hilbert-type form. See [AB75], [Du84].

Semantically, RM was shown by Meyer to correspond to Sugihara matrix S_z. S_z consists of the integers, equipped with the following operations:

$$\neg a = \perp a$$

$$a \vee b = \max(a, b)$$

$$a \wedge b = \min(a, b)$$

$$a \to b = \begin{cases} \max(|a|, |b|) & a \leq b \\ \perp \max(|a|, |b|) & a > b \end{cases}$$

Meyer's main result is that $\vdash_{RM} A$ iff $v(A) \geq 0$ for every valuation v in S_z. Another interesting result that deserves mentioning here is due to Dunn ([Du70]): RM does not have a finite characteristic matrix, but every proper extension of it in its language does (this is known as "the Scrogg's property").

IV.1.2. The System *GRM*.

Axioms. $A \Rightarrow A$ (and optional axioms for propositional constants).

Rules. (1) The standard external structural rules.

(2) The standard internal structural rules, *except weakening* (the converse of contraction is derivable, though).

(3) The hypersequential versions of the multiplicative rules for $\neg$ and $\to$ and the additive rules for $\vee$ and $\wedge$.

(4) The following *Mingle* rule (M) and splitting rule (S_c):

$$(M) \quad \frac{G|?_1 \Rightarrow \Delta_1|H \qquad G|?_2 \Rightarrow \Delta_2|H}{G|?_1, ?_2 \Rightarrow \Delta_1, \Delta_2|H} \qquad\qquad (S_c) \quad \frac{G|?_1, ?_2 \Rightarrow \Delta_1, \Delta_2|H}{G|?_1 \Rightarrow \Delta_1|?_2 \Rightarrow \Delta_2|H} \;.$$

Note. The mingle rule (which is called "mix" in [Gi87]) is essentially a sequential structural rule. Indeed, if we take the sequential versions of the rules above (except (S_c), of course) we get a cut-free system for RM without the distribution axiom.

An Example of a Proof.

$$
\dfrac{
\dfrac{A \Rightarrow A}{A \wedge (B \vee C) \Rightarrow A}
\quad
\dfrac{
\dfrac{
C \Rightarrow C \quad
\dfrac{
\dfrac{B \Rightarrow B \qquad C \Rightarrow C}{B, C \Rightarrow B, C}\,(\mathrm{M})
}{B \Rightarrow C | C \Rightarrow B}\,(\mathrm{S_c})
\quad B \Rightarrow B
}{B \vee C \Rightarrow C | B \vee C \Rightarrow B}
}{A \wedge (B \vee C) \Rightarrow C | A \wedge (B \vee C) \Rightarrow B}
\quad
\dfrac{A \Rightarrow A}{A \wedge (B \vee C) \Rightarrow A}
}{
\dfrac{
\dfrac{A \wedge (B \vee C) \Rightarrow A \wedge C | A \wedge (B \vee C) \Rightarrow A \wedge B}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C) | A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)}
}{
\dfrac{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)}{\Rightarrow A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)}
}
}
$$

The above is a cut-free proof of the problematic distribution axiom. Note that in the third line $B \Rightarrow C|C \Rightarrow B$ is derived. The same hypersequent was derived also in the section on LC, but using different rules (from this point on it is possible to proceed to show $(B \rightarrow C) \vee (C \rightarrow B)$ exactly as in the LC-example).

IV.1.3. Main Results.

Theorem 1. *The cut-elimination theorem is valid for GRM.*

Theorem 2. *Let the interpretation of a sequent $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_k$ be $\neg A_1 + \neg A_2 + \cdots + \neg A_m + B_1 + \cdots + B_k$ (where $A + B =_{Df} \neg A \rightarrow B$. This connective is commutative and associative in linear logic and its extensions). Let also the interpretation φ_G of a sequent $G = ?_1 \Rightarrow \Delta_1 | \cdots | ?_n \Rightarrow \Delta_n$ be $\varphi_{\Gamma_1 \Rightarrow \Delta_1} \vee \varphi_{\Gamma_2 \Rightarrow \Delta_2} \vee \cdots \vee \varphi_{\Gamma_n \Rightarrow \Delta_n}$, where $\varphi_{\Gamma_i \Rightarrow \Delta_i}$ $(i = 1, \ldots, n)$ is the interpretation of $?_i \Rightarrow \Delta_i$. Then $\vdash_{GRM} G$ iff φ_G is a theorem of RM. In particular $\vdash_{RM} A$ iff $\vdash_{GRM} \Rightarrow A$.*

The proof of both theorems can be found in [Av87]. The proof there of theorem 1 is constructive, and quite complex. It uses a special method for constructive proofs of cut-elimination in hypersequential calculi which is called the "history" method. Another place in which an even more complex application of this method is described in detail is [Av91b]. There a cut-free hypersequential formulation of the purely relevant logic RMI of [Av90] is presented (for an easier description of this logic and its importance see [Av92]).

IV.1.4. An Application. The admissibility of the disjunctive syllogism (if $\vdash \neg A$ and $\vdash A \vee B$ then $\vdash B$) has been considered as one of the major problems in relevance logic. It has first been proved for many of the relevant systems in [MD69] and many other proofs have been found since then (see [Du84] for more details). None of these proofs was constructive, though. This fact was strongly emphasized in [AB75]: "The Meyer-Dunn argument...guarantees the existence of a proof of B, but there is no guarantee that the proof of B is related in any plausible way to that of $\neg A$ and $A \vee B$". Using the above hypersequential calculus, such a guarantee was provided for the first (and so far, the only) time in one of the major systems, namely: RM. There the following constructive proof of the disjunctive syllogism is given.

Theorem 3. *If* $\vdash_{RM} \neg A$ *and* $\vdash_{RM} A \vee B$ *then* $\vdash_{RM} B$.

Proof: The assumption implies by theorem 2, that $A \Rightarrow$ and $\Rightarrow A | \Rightarrow B$ are theorems of GRM. Hence so is also $\Rightarrow | \Rightarrow B$, by theorem 1. It is easy however to show constructively (using theorem 1), that if $\vdash_{GRM} \Rightarrow |G$ then $\vdash_{GRM} G$. It follows that $\vdash_{GRM} \Rightarrow B$ and so $\vdash_{RM} B$.

IV.2. *SRMI*: The Difference Between Theoremhood and Consequence.

IV.2.1. General Background.

The multiplicative fragment LL_m of Linear Logic ([Gi87]) consists of the standard axioms and rules for the multiplicative connectives ($\neg$ and $\rightarrow$ suffice [12]). In addition, there is only one structural rule: permutation. This means that both sides of a sequent can be treated as multisets. In R_m, the multiplicative fragment of R, contraction is added, but the two sides of a sequent are still only multisets. In order to be able to treat them as *sets* also the converse of contraction should be added. By doing so we get the system RMI_m.[13]

[12] We don't include here the "multiplicative constants" of [Gi87] in LL_m, but is is possible to include the "additive" ones.

[13] Recall that this precisely has been the idea that led from R to RM. Indeed, in its standard Hilbert-type formulation RMI_m consists of the purely multiplicative axioms and rule of RM. It was shown, however, in [Pa72] (see also [AB75, p. 148]) that RM is not a conservative extension of RMI_m. We note also that in [Av84] RMI_m was called $RMI_{\underset{\rightarrow}{\sim}}$.

RMI_m was studied in [Av84] and shown there to admit cut-elimination and to have a simple, nice semantic. It turned out that $\vdash_{RMI_m} \varphi$ if $v(\varphi) \neq \perp$ for every valuation in the following structure A_ω:

Truth Values. $\{\top, \perp, I_1, I_2, I_3, \ldots, \}$. All elements except $\perp$ are designated.

Operations. (i) $\neg\top = \perp$, $\neg\perp = \top$, $\neg I_j = I_j$ $(1 \leq j < \infty)$.

$$\text{(ii)}a \to b : \begin{cases} \top & a = \perp \text{ or } b = \top \\ I_j & a = b = I_j \\ \perp & \text{otherwise .} \end{cases}$$

Other important properties of RMI_m are the variable-sharing property which is characteristic to relevance logics ($\vdash A \to B$ only if A and B share an atomic variable. Note that RM lacks this property) and Scrogg's property: it has no finite characteristic matrix, but every proper extension in its language does (in fact, a submatrix of A_ω which consists of $\{\top, \perp, I_1, \ldots, I_n\}$ for some n).

It seems, accordingly, that we have just described a simple structure with a corresponding simple, cut-free *standard* Gentzen-type system (called $GRMI_m$ below). So what else can be said and why is another, more complicated hypersequential system needed (as the fact that this example is included here suggests)? The answer is that the above correspondence between A_ω and RMI_m is only a *weak* one. It is true that $\vdash_{RMI_m} A$ (or $\vdash_{GRMI_m} \Rightarrow A$) iff $\models_{A_\omega} A$. It is *not* true that $B_1, \ldots, B_n \vdash_{RMI_m} A$ (or $\Rightarrow B_1, \ldots, \Rightarrow B_n \vdash_{GRMI_m} \Rightarrow A$) iff $B_1, \ldots, B_n \models_{A_\omega} A$. For example $\neg(p \to q) \models_{A_\omega} p$, but $\neg(p \to q) \not\vdash_{RMI_m} p$. In [Av94] it is shown that in order to get a system which is *strongly* complete relative to A_ω one has to add to RMI_m the rules $\neg(A \to B)/A$ and $\neg(A \to B)/\neg B$ (just one of them suffices, in fact. [14]) The resulting system is called there $SRMI$. It has the same logical *theorems* as RMI_m, but a stronger *consequence relation*.

It is in order to have a cut-free Gentzen-type system which corresponds to $\vdash_{SRMI}$ ($= \models_{A_\omega}$) that we need to use a hypersequential calculus.

IV.2.2. The System *GSRMI* and its Main Properties

(I) Axioms: $A \Rightarrow A$.

[14] In [Av94] the somewhat more intuitive form $A \otimes B/A$ is used, where $A \otimes B = \neg(A \to \neg B)$.

(II) The standard external structural rules.

(III) The hypersequential versions of internal permutation, contraction and its converse and cut (but not weakening!).

(IV) The standard hypersequential versions of the multiplicative rules for $\neg$ and $\rightarrow$.

(V) The extended splitting rule (see section II):

$$(ES) \qquad \frac{G|?_1, ?_2 \Rightarrow \Delta_1, \Delta_2|H}{G|?_1 \Rightarrow \Delta_1|?', ?_2 \Rightarrow \Delta_2, \Delta'|H}$$

An Example of a Proof.

$$\frac{\dfrac{\text{(ES)}\ \dfrac{A \Rightarrow A}{A \Rightarrow B, A| \Rightarrow A}}{\Rightarrow (A \rightarrow B), A| \Rightarrow A}}{\neg(A \rightarrow B) \Rightarrow A| \Rightarrow A}$$

Note. Using a cut, this example shows that $\Rightarrow \neg(A \rightarrow B) \vdash_{GSRMI} \Rightarrow A$.

Theorem 1. *The cut-elimination theorem obtains for $GSRMI$.*

The proof of this theorem and of all the results mentioned below can be found in [Av94].

Theorem 2. *$? \Rightarrow \Delta$ is provable in classical logic iff $? \Rightarrow \Delta| \Rightarrow$ is provable in $GSRMI$.*

Theorem 3. *$? \Rightarrow \Delta$ is provable in $GRMI_m$ iff it is provable in $GSRMI$. In other words: if on ordinary sequent is provable in the hypersequential calculus $GSRMI$ then it can be proved in its purely sequential fragment, without a need for a detour through proper hypersequents.*

The proof of the last result in [Av94] is not constructive, and uses semantic considerations. No method is given of transferring a given hypersequential proof of a sequent into a purely sequential one. It is difficult, in fact, to imagine what such a method can look like. The reason is that an example is given in [Av94] of a sequent $? \Rightarrow A$ which has a hypersequential proof which belongs to the "intuitionistic" fragment (i.e. all the hypersequents used are single-conclusioned), but every possible purely sequential proof should contain a sequent with more than one formula on the r.h.s.

IV.2.3. The Semantics and Consequence Relation.

The characterizations of the consequence relation $\models_{A_\omega}$ which $GSRMI$ provides are summarized in the next theorem. It is instructive to compare its content with the example above and the note that follows it.

Theorem 4. *(1)* $? \models_{A_\omega} B$ *iff there exits* $?_1, \ldots, ?_k \subseteq ?$ *such that* $?_1 \Rightarrow B | ?_2 \Rightarrow B | \cdots | ?_n \Rightarrow B$ *is provable (without cuts) in* $GSRMI$*.*

(2) $? \models_{A_\omega} B$ *iff* $? \Rightarrow B | \Rightarrow B$ *is provable (without cuts) in* $GSRMI$*.*

(3) $? \models_{A_\omega} B$ *iff* $\Rightarrow B$ *can be derived (with cuts!) in* $GSRMI$ *from* $(\Rightarrow A_1), \ldots, (\Rightarrow A_n)$*, where* $? = A_1, \ldots, A_n$*.*

The system $GSRMI$ can in fact be used to show more concerning A_ω:

Theorem 5. $?_1 \Rightarrow \Delta_1 | \cdots | ?_n \Rightarrow \Delta_n$ *is provable in* $GSRMI$ *iff for every valuation* v *in* A_ω *there exists* $1 \leq i \leq n$ *such that* $v(\varphi_{\Gamma_i \Rightarrow \Delta_i}) \neq \perp$ *(where* $\varphi_{\Gamma_i \Rightarrow \Delta_i}$ *is the translation of* $?_i \Rightarrow \Delta_i$ *into the language, as given in the previous subsection).*

Corollary. *$GSRMI$ is decidable.*

Note. We have not given above a translation of a full hypersequent into the language. Such a translation exists, though. It is possible to define in the language a connective $\vee$ so that $v(A_1 \vee \cdots \vee A_n) \neq \perp$ iff $v(A_i) \neq \perp$ for some i. In fact, $SRMI_m$ and $(GSRMI)$ has the following interesting stronger property.

Theorem 6. *There is a strong translation of positive classical logic into* $SRMI_m$*, where by "strong" we mean that the interpretation preserves the consequence relation (not just the theorems) of classical logic, and the classical connectives are interpreted by definable connectives of the language. If we add to the language a constant* $\perp$ *(corresponding to Girard's* 0*) with the axiom* $\perp \rightarrow A$ $(\perp, ? \Rightarrow \Delta$ *in* $GSRMI)$ *then all of the above theorems 1-5 remain valid, but now full classical logic can be strongly translated into the resulting system.*

Note. No such translations are known for any other substructural logic (including linear logic and the standard relevance logics).

One more property of $\models_{A_\omega}$ (and the corresponding formal systems) which deserves mentioning here is the fact that it is *paraconsistent*: $\neg p, p \not\models_{A_\omega} q$ when p and q are atomic.

V. Modal Logic

V.1. S5: The Use of Modalized Structural Rules.

V.1.1. General Background.

Most of the important systems in propositional modal logic (like $K, K4, T, S4$ and the provability logics GL and Grz) have ordinary, cut-free Gentzen-type formulations. The sequential system for $S4$, for example, is obtained from that of classical logic by adding to it the following two rules for $\Box$ (for simplicity we deal only with $\Box$, taking $\Diamond$ as a definable connective):

$$(\Box \Rightarrow)\quad \frac{?\,, A \Rightarrow \Delta}{?\,, \Box A \Rightarrow \Delta} \qquad (\Rightarrow \Box)\quad \frac{\Box ? \Rightarrow A}{\Box ? \Rightarrow \Box A}$$

($\Box ?$ is a sequence of formulae which begin with $\Box$. If $? = B_1, \ldots, B_k$ then $\Box ? = \Box B_1, \ldots, \Box B_k$).

There is one very important modal system for which no such cut-free system is known: $S5$. In its usual formulation the $(\Rightarrow \Box)$ rule of S4 is strengthened to: $\frac{\Box\Gamma \Rightarrow A, \Box\Delta}{\Box\Gamma \Rightarrow \Box A, \Box\Delta}$. It is easy to see, however, that $p \Rightarrow \Box\neg\Box\neg p$ is derivable in this system using a cut on $\Box\neg p$, but it has no cut-free proof.[15]

The problem of providing a cut-free formulation for $S5$ can be a solved with the help of hypersequents. In a sense, the idea goes back at least to [Kr59a], where a semantic tableaux for $S5$ is presented. This tableaux system can easily be presented in the form of a Gentzen-type calculus (see, e.g. [Mi74] and [Mi92]), which in turn can be viewed as a hypersequential calculus. Following Kripke, Mints' calculus uses formulae labeled with worlds, and one of these worlds is designated. The hypersequential form is therefore only implicit here. It is made explicit in [Po83], where hypersequents and nothing beyond (like

[15] What *can* be shown is that only analytic cuts are needed here. This means: only cuts on subformulae of the proved sequent.

a "designated" component) are used for the first time. Nevertheless, it is not difficult to see that the systems of Mints and Pottinger are essentially equivalent.

We present now our version of a hypersequential calculus for $S5$, which is both simpler (at least in our opinion) and in line with the other systems which are described in this paper (this is important if we seek unity and a useful but general framework).

V.1.2. The System $GS5$.

(I) Axioms: $A \Rightarrow A$.

(II) The standard external and internal structural rules (including cut).

(III) The hypersequential version of the rules of S4 (or just T).

(IV) The modalized splitting rule

$$(MS) \quad \frac{G|\Box ?_1, ?_2 \Rightarrow \Box\Delta_1, \Delta_2|H}{G|\Box ?_1 \Rightarrow \Box\Delta_1|?_2 \Rightarrow \Delta_2|H}$$

As noted in section II, (MS) is a modalized version of the splitting structural rule, which follows the spirit of the exponential versions of the structural rules in Linear Logic.

An Example of a Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{(\mathrm{MS})\ \dfrac{\dfrac{\dfrac{p \Rightarrow p}{p, \neg p \Rightarrow}}{p, \Box\neg p \Rightarrow}}{p \Rightarrow \ |\Box\neg p \Rightarrow}}{p \Rightarrow \ | \Rightarrow \neg\Box\neg p}}{p \Rightarrow \ | \Rightarrow \Box\neg\Box\neg p}}{p \Rightarrow \Box\neg\Box\neg p | p \Rightarrow \Box\neg\Box\neg p}}{p \Rightarrow \Box\neg\Box\neg p}$$

Notes. (1) In the presence of (MS) the $(\Rightarrow \Box)$ can be strengthened to the usual one of $S5$: $\frac{G|\Box\Gamma \Rightarrow \Box\Delta, A|H}{G|\Box\Gamma \Rightarrow \Box\Delta, \Box A|H}$. Here is the derivation:

$$\dfrac{\dfrac{\dfrac{(\mathrm{MS})\ \dfrac{G|\Box ? \Rightarrow \Box\Delta, A|H}{G|\Box ? \Rightarrow A| \Rightarrow \Box\Delta|H}}{G|\Box ? \Rightarrow \Box A| \Rightarrow \Box\Delta|H}}{G|\Box ? \Rightarrow \Box\Delta, \Box A|\Box ? \Rightarrow \Box\Delta, \Box A|H}}{G|\Box ? \Rightarrow \Box\Delta, \Box A}$$

By the same token, a much weakened version, $\frac{G|\Rightarrow A|H}{G|\Rightarrow \Box A|H}$, suffices for getting the full power of $(\Rightarrow \Box)$:

$$\begin{array}{c} G|\Box ? \Rightarrow A|H \\ \hline G|\Box ? \Rightarrow | \Rightarrow A|H \\ \hline G|\Box ? \Rightarrow | \Rightarrow \Box A|H \\ \hline G|\Box ? \Rightarrow \Box A|\Box ? \Rightarrow \Box A|H \\ \hline G|\Box ? \Rightarrow \Box A|H \end{array}$$

(2) Pottinger's main rule in [Po83] was basically a combination of $(\Box \Rightarrow)$ and the following rule: [16]

$$\frac{G|\Box A, ?_1 \Rightarrow \Delta_1|?_2 \Rightarrow \Delta_2|H}{G|?_1 \Rightarrow \Delta_1|\Box A, ?_2 \Rightarrow \Delta_2|H} \; .$$

This rule can be derived in $GS5$ as follows:

$$\begin{array}{c} \text{(MS)} \quad G|\Box A, ?_1 \Rightarrow \Delta_1|?_2 \Rightarrow \Delta_2|H \\ \hline C|?_1 \Rightarrow \Delta_1|\Box A \Rightarrow |?_2 \Rightarrow \Delta_2|H \\ \hline C|?_1 \Rightarrow \Delta_1|\Box A, ?_2 \Rightarrow \Delta_2|\Box A, ?_2 \Rightarrow \Delta_2|H \\ \hline G|?_1 \Rightarrow \Delta_1|\Box A, ?_2 \Rightarrow \Delta_2|H \end{array}$$

The equivalence of $S5$ and $GS5$ is established by the next definition and the theorem that follows it.

Definition. The translation of the hypersequent $?_1 \Rightarrow \Delta_1|?_2 \Rightarrow \Delta_2|\cdots|?_n \Rightarrow \Delta_n$ is the sentence $\Box\varphi_{\Gamma_1\Rightarrow\Delta_1} \vee \Box\varphi_{\Gamma_2\Rightarrow\Delta_2} \vee \cdots \vee \Box\varphi_{\Gamma_n\Rightarrow\Delta_n}$, where $\varphi_{\Gamma\Rightarrow\Delta}$ is the standard classical translation of $? \Rightarrow \Delta$ into a sentence.

Theorem 1. $\vdash_{GS5} G$ *iff* $\vdash_{S5} \varphi_G$.

Proof: Let $G = ?_1 \Rightarrow \Delta_1|\cdots|?_n \Rightarrow \Delta_n$. Suppose $\vdash_{S5} \varphi_G$. Then $\Rightarrow \Box\varphi_{\Gamma_1\Rightarrow\Delta_1}, \Box\varphi_{\Gamma_2\Rightarrow\Delta_2}$, $\Box\varphi_{\Gamma_n\Rightarrow\Delta_n}$ is derivable in the standard *ordinary* sequential formulation of $S5$ (described at the beginning of the subsection), and so also in $GS5$ (which extends that calculus). Applying (ES) we get that $\vdash_{GS5} \Rightarrow \Box\varphi_{\Gamma_1\Rightarrow\Delta_1}| \Rightarrow \Box\varphi_{\Gamma_2\Rightarrow\Delta_2}|\cdots| \Rightarrow \Box\varphi_{\Gamma_n\Rightarrow\Delta_n}$. But

[16] In Pottinger's system the standard external and internal structural rules (except cut) are built into the axioms and rules. As a result, his form of the rule is a little bit more complicated.

$\vdash_{GS5} \Box\varphi \Rightarrow \varphi$. Hence (using cuts) $\vdash_{GS5} \Rightarrow \varphi_{\Gamma_1 \Rightarrow \Delta_1} | \cdots | \Rightarrow \varphi_{\Gamma_n \Rightarrow \Delta_n}$. Since $\varphi_{\Gamma \Rightarrow \Delta}, ? \Rightarrow \Delta$ is a tautology, we get $\vdash_{GS5} G$ (using cuts again).

For the converse, we prove by induction on length of proofs, that if $\vdash_{GS5} G$ then $\vdash_{S5} \varphi_G$. The case of the axioms is trivial, while for most induction steps all we need is classical logic and the fact that if $\vdash_{S5} A_1 \Rightarrow (A_2 \to \cdots \to (A_n \to B)\cdots)$ then $\Box A_1 \to (\Box A_2 \to \cdots \to (\Box A_n \to \Box B)\cdots)$. For example, since $\varphi_{A,\Gamma \Rightarrow \Delta} \to \varphi_{\Gamma \Rightarrow \Delta, \neg A}$ is a classical tautology, $\vdash_{S_5} \Box\varphi_{A,\Gamma \Rightarrow \Delta} \to \Box\varphi_{\Gamma \Rightarrow \Delta, \neg A}$. From this the induction step in the case of the $(\Rightarrow \neg)$ easily follows (For the $(\Box \Rightarrow)$ we also need the axiom $\Box A \to A$, of course). The more difficult cases are $(\Rightarrow \Box)$ and (MS). In the case of $(\Rightarrow \Box)$ we may limit ourselves to the special case described in note (1) above, and this special case is handled by the axiom $\Box A \to \Box\Box A$. Finally, the (MS) case follows from the following theorems of S5 .

$$(i) \quad \Box\,(\Box A \vee B) \to \Box\,\Box A \vee \Box B \qquad (ii) \quad \Box(\neg\Box A \vee B) \to \Box\neg\Box A \vee \Box B \qquad \blacksquare$$

The above proof relies on the fact that cut is one of the rules of $GS5$. We now give a constructive proof that cut is not necessary.

Theorem 2. *$GS5$ admits cut-elimination.*

Proof: We prove by induction on length of proofs that if $G_1 | ?_1 \Rightarrow \Delta_1, A, \ldots, A | \cdots | ?_n \Rightarrow \Delta_n, A, \ldots, A | H_1$ and $G_2 | A, \ldots, A, ?'_1 \Rightarrow \Delta'_1 | \cdots | A, \ldots, A, ?'_k \Rightarrow \Delta'_k | H_2$ are provable in $GS5$ then:

(1) $\vdash_{GS5} G_1 | G_2 | ?_1, \ldots, ?_n, ?'_1, \ldots, ?'_k \Rightarrow \Delta_1, \ldots, \Delta_n, \Delta'_1, \ldots, \Delta'_k | H_1 | H_2$

(2) If $A = \Box B$ then $\vdash_{GS5} G_1 | G_2 | ?_1 \Rightarrow \Delta_1 | \cdots | ?_n \Rightarrow \Delta_n | ?'_1 \Rightarrow \Delta'_1 | \cdots | ?'_k \Rightarrow \Delta'_k | H_1 | H_2$ (note that the hypersequent in (1) follows from that in (2) using internal weakenings and external contractions).

Some explanations are in order here. The generalization to the case in which the cut-formula appears in more than one component is needed for handling the external contraction rule. The need for a special treatment of the case in which the cut formula is of the form $\Box B$ arises in the following situation (for brevity we omit side components etc.):

$$\dfrac{\dfrac{\Box ?, ?_1 \Rightarrow \Delta_1,\ \Box\Delta,\ \Box A}{?_1 \Rightarrow \Delta_1 \mid \Box ? \Rightarrow \Box\Delta, \Box A} \qquad \Box A, ?_2 \Rightarrow \Delta_2}{?_1 \Rightarrow \Delta_1 \mid \Box ?, ?_2 \Rightarrow \Box\Delta, \Delta_2}$$

If we apply the induction hypothesis (1) to the two premises we get only $\Box ?, ?_1, ?_2 \Rightarrow \Box\Delta, \Delta_1, \Delta_2$ and from this one cannot, in general, deduce the desired result. By applying the induction hypothesis (2) we can infer, in contrast, from the two premises $\Box ?, ?_1 \Rightarrow \Delta_1, \Box\Delta | ?_2 \Rightarrow \Delta_2$. From this follows, by (MS), $?_1 \Rightarrow \Delta_1 | \Box ? \Rightarrow \Box\Delta | ?_2 \Rightarrow \Delta_2$, and from this the desired conclusion easily follows. Similarly in the following case:

$$\dfrac{?' \Rightarrow \Delta', \Box A \qquad \dfrac{\Box A, \Box ? \Rightarrow B}{\Box A, \Box ? \Rightarrow \Box B}}{\Box ?, ?' \Rightarrow \Delta', \Box B}$$

An application of the induction hypothesis (1) to the premises yields $?', \Box ? \Rightarrow \Delta', B$. One cannot apply (MS) or $(\Rightarrow \Box)$ to this component, though. An application of (2), on the other hand, gives $?' \Rightarrow \Delta' | \Box ? \Rightarrow B$. From this follows $?' \Rightarrow \Delta' | \Box ? \Rightarrow \Box B$ and then $\Box ?, ?' \Rightarrow \Delta', \Box B$.

It is straightforward (though tedious) to check that the proof goes through in all cases. It should be emphasized that because of (2) one should be careful even in the basic case, when one of the premises is an axiom $\Box A \Rightarrow \Box A$. An application of (2) to this and to (say) $\Box A, ?_1, \Rightarrow \Delta_1 | \Box A, ?_2 \Rightarrow \Delta_2 | G$ yields $\Box A \Rightarrow | ?_1 \Rightarrow \Delta_1 | ?_2 \Rightarrow \Delta_2 | G$. This can, of course, be derived directly from the second premise using several application of (MS) and external contraction. ■

V.1.3. Relating $GS5$ to the Semantics.

In the previous subsection the equivalence of $GS5$ with $S5$ was proved constructively, using only proof-theoretical notions and methods. In this subsection we attack the same issues from the semantic point of view, giving alternative (though less constructive) proofs. The reason is that we believe that this sheds further light on the system and the prospect of generalizing the method to other modal systems (also some may find the proofs here easier).

Definition [Po83]. A hypersequent G is *valid* if given any Kripke model (M, v) there exists a component of G which is valid in this model (i.e.: true in any $a \in M$, where $? \Rightarrow \Delta$ is true in a iff $v(\varphi_{\Gamma \Rightarrow \Delta}, a) = T$).

Notes. (1) Again the meaning of | is disjunctive, but there is a difference here between this meaning and that of the comma on the r.h.s of a component. Given $a \in M$, the meaning of $\Rightarrow A_1, \ldots, A_n$ is that one of the A_s' is *true* in a (relative to v), while the meaning of $G_1|\cdots|G_k$ is that one of the G_i's is *valid* in (M, v) (i.e. true for all $a \in M$).

(2) The connection between the last definition and the translation given in the last subsection is obvious.

Soundness Theorem. If $\vdash_{GS5} H$ then H is valid. Moreover: if G can be derived in $GS5$. ¿From $H_1, \ldots, H_n$ then G is valid in any Kripke model in which all the H_i's are valid.

Proof: Straightforward. We only note that the soundness of (MS) is due to the fact that in an $S5$-model (M, v) a formula of the form $\Box A$ is either true in all elements of M, or false in all them. ∎

The completeness of $GS5$ and the cut-elimination theorem are proved simultaneously in the next theorem. Its proof was essentially given in [Kr59a] (see also [Mi92]). Since it was not stated as such there, we give below a simplified version.

Theorem 3. *If a hypersequent G is valid then it has a cut-free proof.*

Proof: Call a hypersequent H *saturated* if the following conditions are satisfied.

(1) If $? \Rightarrow \Delta$ is a component of G and $A \vee B \in \Delta$ then both A and B are in Δ, while if $A \vee B \in ?$ then either $A \in ?$ or $B \in ?$. Similar conditions should obtain for the other classical connectives.

(2) If $? \Rightarrow \Delta$ is a component of G and $\Box A \in ?$ then $A \in ?'$ for every component $?' \Rightarrow \Delta'$ of G.

(3) If $? \Rightarrow \Delta$ is a component of G and $\Box A \in \Delta$ then there exists a component $?' \Rightarrow \Delta'$ of G such that $A \in \Delta'$.

(4) $? \cap \Delta = \emptyset$ for every component of G.

Lemma 1. *A saturated hypersequent G is not valid.*

Proof of Lemma 1. Take M to consist of the components of G and let $v(p, ? \Rightarrow \Delta) = T$ if $p \in ?$, $v(p, ? \Rightarrow \Delta) = F$ if $p \in \Delta$ (if $p \notin ? \cup \Delta$ define $v(p)$ arbitrarily). It is easy to see that $v(A, ? \Rightarrow \Delta) = T$ if $A \in ?$, $v(A, ? \Rightarrow \Delta) = F$ if $A \in \Delta$. Hence every component is false (as a sequent) in itself (taken as a world).

Lemma 2. *If H has no cut-free proof then there is a saturated hypersequent G that can be derived from it using external and internal weakenings.*

Proof of Lemma 2. By a double induction on the complexity of a maximal formula for which one of conditions (1)-(3) (in the definition of a saturated formula) is violated, and the number of such maximal formulae. If there are no such formulae then H itself is saturated since the fact that it has no cut-free proof entails condition (4).

Assume there is such a maximal formula of the form $A \vee B$ which appears in the component $? \Rightarrow \Delta$. If $A \vee B \in \Delta$ obtains H^* from H by adding A and B to Δ. Obviously H^* does not have a cut-free proof. If $A \vee B \in ?$ then by adding either A or B to $?$ we obtain a hypersequent H^* which also does not have a cut-free proof. In either case we can apply the induction hypothesis to H^*.

The cases of the other classical connectives are treated similarly.

Suppose next that there is a maximal formula of the form $\Box A$. If $\Box A \in \Delta$ for some component $? \Rightarrow \Delta$, obtain H^* from H by adding to it a component $?^* \Rightarrow A$, where $B \in ?^*$ iff $\Box B$ belongs to the l.h.s of some component of H. If $\Box A \in ?$ add A to the left hand side of all components of H. In both cases H can be derived from H^* (Pottinger's rule of note (2) of V.1.2. is very useful in showing this) and so does not have a cut-free proof. Apply now the induction hypothesis to H^*.

Theorem 3 follows now easily from the two lemmas and the soundness theorem. ■

Note. We needed to add $?^*$ in the last case to ensure that no new formula of the previous higher complexity (or even higher) is created.

V.2. Other Modal Systems.

In [Po83] Pottinger applied the hypersequential framework also to $S4$ and T. His system for $S4$ was (essentially) like ours for $S5$ – but without the modal splitting rule. Without it, however, the only way to create new components is through external weakening and this does not really add anything to the power of the system. It is easy, in fact, to see that in Pottinger's system for T and $S4$ a hypersequent G is provable iff one of its components is provable in the usual sequential calculi and with a proof which is contained in the proof of G. The passage to hypersequents is therefore artificial and unnecessary there.

In Mints' approach in [Mi92] (following [Kr59]), more structure is added to that of a hypersequent. A hypersequent is not just a set of ordinary sequents, but this set comes equipped also with a relation R. The way rules of the system are applied depends on this R. For example, external contraction of S_1 and S_2 might be allowed only if $S_1 R S_2$ or $S_2 R S_1$.[17] Semantically, a hypersequent $S_1|S_2|\cdots|S_k$ is valid iff for every model (M, R^*, v) of the appropriate type and for every choice of elements $I_1, \ldots, I_k$ of M which satisfies $I_i R^* I_j$ whenever $S_i R S_j$, S_i is true in I_i for at least one i. This approach seems more promising for generalization. Still, in the case of $S4$ and T the resulting systems can easily be seen to be equivalent to those of Pottinger. In any case, the addition of the relation R means building the semantics into the formal systems (something that goes against the desiderata at set forth in section II), and it takes us out of the simple hypersequential framework dealt with in this paper.

VI. 3-Valued Logics

There is something which is common to all examples described so far: None of them have a finite characteristic matrix. We end therefore with two examples that show that the application of the hypersequential method is not limited to such logics.

There exists a well-known method of systemizing (finite) many-valued logics using structures that superficially resembles hypersequents. (See e.g. [Ca91] and [Za93]. The latter contains extensive references to the literature on Gentzen-type calculi for many-

[17] This is not the way the systems are described, but they can be reformulated like this.

valued logics.) There are crucial differences, though. In any hypersequential calculus there exist hypersequents with an arbitrary number of components. In the framework we have just mentioned, the number of components is always fixed, and depends on the logic in question: 3 for 3-valued logics, 4 for four-valued logic etc. Also a component is not an ordinary sequent, but a sequence of formulae. The structures used are therefore straightforward generalizations of ordinary sequents, but ordinary sequents are not a special case of the structure used for, say, 3-valued logics.

This approach is very successful in handling many-valued logics. From the point of view which was described in the framework it has however, serious drawbacks:

(1) The semantics is strongly built into the proof theory.

(2) As a result, this is only a framework for finite-valued logics, not a general framework.

(3) The method applied, practically, to every possible finite-valued logic. This strong advantage is also a source of weakness: one cannot use the framework for distinguishing between valuable logics and those that probably are not.

(4) The rules for the standard connectives are frequently complex and unnatural, and it might be difficult to tell what makes a certain connective an "implication" or a "disjunction".

Because of these drawbacks we believe that cut-free hypersequential versions, that follow the lines described in the introduction, are preferable whenever they exist. As usual, we expect them to exist only in special, important cases. Now there are well-known, important 3-valued logics that have nice, cut-free *standard* sequential formalisms (Kleene 3-valued logics and Sobociński 3-valued logic [So52] are two examples.) We now give two examples of (relatively) well-known 3-valued logics which apparently do not have such formalisms but do have cut-free hypersequential calculi.

VI.1. General Background.

We assume that the 3 truth values are $\{\top, I, \bot\}$, ordered as follows: $\bot < I < \top$. In both examples $a \vee b$ is $\max(a, b)$, $a \wedge b$ is $\min(a, b)$, $\neg \bot = \top$, $\neg \top = \bot$, $\neg I = I$. The differences are with respect to the definition of $\rightarrow$ and what values are taken as designated. In L_3 (Lukasiewicz 3-valued logic) only $\top$ is designated and $a \rightarrow b$ is $\top$ if $a \leq b$, $\bot$ if $a = \top$ and

$b = \perp$, I otherwise. In RM_3 (the 3-valued extension of RM, and the strongest logic in the relevance family) both $\top$ and I are designated, while $a \to b$ is $\perp$ if $a > b$, I if $a = b = I$, $\top$ otherwise.

It might seem that we have given a full description of L_3 and RM_3. This is false, since we have not defined yet the intended consequence relations. Suppose we simply define that ? $\models A$ if every valuation which gives designated values to all the elements in ? does the same to A. In such a case we can do with *ordinary* sequents – provided we change our choice of *primitive* connectives (without changing the expressive power of the language). All we have to do is to define $A \supset B =_{Df} B \vee (A \to (A \to B))$ (this is equivalent to $A \to (A \to B)$ in the case of L_3 and to $B \vee (A \to B)$ in RM_3). The language of $\{\neg, \vee, \wedge, \supset\}$ is equivalent, in both cases, to that of $\{\neg, \vee, \wedge, \to\}$. Indeed, in both cases $A \to B$ is equivalent to $(A \supset B) \wedge (\neg B \supset \neg A)$. Now it is not difficult (see [Av91c]) to construct cut-free *sequential* calculi for both systems in the new language. The calculi are almost identical. The only difference is with respect to the axioms: $A \Rightarrow A$ and $A, \neg A \Rightarrow$ in "L_3", $A \Rightarrow A$ and $\Rightarrow A, \neg A$ in "RM_3". Obviously, if we are interested only in theoremhood of formulae this would be completely sufficient.

There is a certain amount of cheating in this approach. The reason is that the consequence relation that is used in it does not correspond to the official implication connectives of L_3 and RM_3 – while these implications are the characteristic connectives of these logics. The real consequence relation which reflects the spirit of L_3 and RM_3 should be defined as follows: $A_1, \ldots, A_n \models B$ iff $A_1 \to (A_2 \to \cdots \to (A_n \to B) \cdots)$ is valid. Moreover, an appropriate Gentzen-type formulation should reflect *this* consequence relation. In other words: $A_1, \ldots, A_n \Rightarrow B$ should be provable iff $A_1 \to (A_2 \to \cdots \to (A_n \to B) \cdots)$ is valid. In order to achieve this we need to employ hypersequential calculi (first presented in [Av91c]) in which not all standard internal structural rules are allowed (in this sense they are substructural logics too, like those presented in section IV).

VI.2. The System GRM_3.

This system can be described as the union of GRM and $GSRMI$ of section IV:

Axioms. $A \Rightarrow A$.

Rules.

(1) The standard external structural rules.

(2) The standard internal structural rules, except weakening.

(3) The hypersequential versions of the multiplicative rules for $\neg$ and $\rightarrow$ and the additive rules for $\vee$ and $\wedge$.

(4) The mingle rule (of GRM) and the extended splitting rule (of $GSRMI$):

$$(M) \quad \frac{G|?_1 \Rightarrow \Delta_1|H \qquad G|?_2 \Rightarrow \Delta_2|H}{G|?_1, ?_2 \Rightarrow \Delta_1, \Delta_2|H} \qquad\qquad (ES) \quad \frac{G|?_1, ?_2 \Rightarrow \Delta_1, \Delta_2|H}{G|?_1 \Rightarrow \Delta_1|?', ?_2 \Rightarrow \Delta_2, \Delta'|H}$$

An Example of a Proof.

$$\begin{array}{l}
(\mathrm{ES})\underline{A \Rightarrow A} \\
\underline{\Rightarrow A|A \Rightarrow B} \\
\underline{\Rightarrow A \vee (A \rightarrow B)| \Rightarrow A \rightarrow B} \\
\underline{\Rightarrow A \vee (A \rightarrow B)| \Rightarrow A \vee (A \rightarrow B)} \\
\Rightarrow A \vee (A \rightarrow B)
\end{array}$$

Main Properties:

(1) The cut-elimination theorem obtains.

(2) $\vdash_{GRM_3} A_1, \ldots, A_n \Rightarrow B$ if $A_1 \rightarrow (\cdots \rightarrow (A_n \rightarrow B)\cdots)$ is valid in the matrix for RM_3. On the other hand, B follows from $?$ in the sense that if $v(A) \neq \perp$ for all $A \in ?$ then $v(B) \neq \perp$ iff $\vdash_{GRM_3} ?_1 \Rightarrow B|?_2 \Rightarrow B|\cdots|?_n \Rightarrow B$ for some $?_1, \ldots, ?_n \subseteq ?$.

(3) Let $\varphi_{\Gamma\Rightarrow\Delta}$ be a standard translation (in classical logic) of $? \Rightarrow \Delta$ in terms of $\neg$ and $\rightarrow$. Then $\vdash_{GRM_3} ?_1 \Rightarrow \Delta_1|\cdots|?_k \rightarrow \Delta_k$ iff $\varphi_{\Gamma_1\Rightarrow\Delta_1} \vee \varphi_{\Gamma_2\Rightarrow\Delta_2} \vee \cdots \vee \varphi_{\Gamma_n\Rightarrow\Delta_n}$ is valid in RM_3.

The proofs of these results are similar to those in the case of $GSRMI$. See [Av94] for more details.

VI.3. The System GL_3.

Axioms. $A \Rightarrow A$.

Rules.

(I) The standard external structural rules.

(II) The standard internal structural rules, *except contraction*.

(III) The same rules for the connectives as in GRM_3.

(IV) The *mixing* (MX) rule from section II:

$$\frac{G|?_1, ?_2, ?_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3|H \qquad\qquad G|?'_1, ?'_2, ?'_3 \Rightarrow \Delta'_1, \Delta'_2, \Delta'_3|H}{G|?_1, ?'_1 \Rightarrow \Delta_1, \Delta'_1|?_2, ?'_2 \Rightarrow \Delta_2, \Delta'_2|?_3, ?'_3 \Rightarrow \Delta_3, \Delta'_3|H}$$

An Example of a Proof.

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, B}}{\Rightarrow A, A \to B} \qquad \dfrac{\text{(MX)}\ \dfrac{A \Rightarrow A \qquad\quad B \Rightarrow B}{B \Rightarrow\ |\Rightarrow A\ |\ A \Rightarrow B}}{B \Rightarrow\ |\Rightarrow A\ |\Rightarrow A \to B} \qquad \dfrac{B \Rightarrow B \qquad A \Rightarrow A}{B, B \to A \Rightarrow A}}{\dfrac{\dfrac{(A \to B) \to B \Rightarrow A\ \ |\Rightarrow A\ \ |\ \ (A \to B) \to B, B \to A \Rightarrow A}{(A \to B)\to B, B\to A \Rightarrow A|(A\to B)\to B, B\to A \Rightarrow A|(A\to B)\to B, B\to A \Rightarrow A}}{\dfrac{(A \to B) \to B, B \to A \Rightarrow A}{(A\to B)\to B \Rightarrow (B\to A)\to A}}}$$

Main Properties ([Av91c]):

(1) The cut-elimination theorem obtains.

(2) $\vdash_{GL_3} A, \ldots, A_n \Rightarrow B$ iff $A_n \to (\cdots \to (A_n \to B)\cdots)$ is valid in L_3

(3) $\vdash_{GL_3} ?_1 \Rightarrow \Delta_1|\cdots|?_n \to \Delta_n$ iff $\varphi_{\Gamma_1 \Rightarrow \Delta_1} \vee \cdots \vee \varphi_{\Gamma_n \to \Delta_n}$ is valid in L_3, where $\varphi_{\Gamma \Rightarrow \Delta}$ is, again, a standard translation of $? \Rightarrow \Delta$ in terms of $\to$ and $\neg$.

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