Implicational F-Structures and Implicational Relevance Logics

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Abstract

We describe a method for obtaining classical logic from intuitionistic logic which does not depend on any proof system, and show that by applying it to the most important implicational relevance logics we get relevance logics with nice semantical and proof-theoretical properties. Semantically all these logics are sound and strongly complete relative to classes of structures in which all elements except one are designated. Proof-theoretically they correspond to cut-free hypersequential Gentzen-type calculi. Another major property of all these logic is that the classical implication can faithfully be translated into them.

The intuitionistic implicational logic is, as is well-known, the minimal logic for which the standard deduction theorem obtains. The classical implicational calculus, in turn, is a sort of a completion of the intuitionistic one, since the set of theorems of any non-trivial implicational logic which extends intuitionistic implicational logic should be a subset of the set of the classical tautologies. Now each of the various standard substructural implicational logics is also usually characterized as the minimal system which satisfies a certain deduction theorem. Since minimality does not necessarily mean optimality, it should be interesting to try to apply to implicational relevance logics the process of completion that leads from the intuitionistic implicational logic to the classical one.

But how exactly do we get classical logic from intuitionistic logic? The usual answer is that this is done by a passage from a single-conclusion sequential calculus to a multiple-conclusion one, in which the logical rules remain the same, but applications of the official structural rules are allowed on *both* sides of the sequents. However, in logics like the basic implicational relevance logic $R_{\rightarrow}$ such a passage yields no new provable sequents, since it is still impossible to deduce there sequents which are not single-conclusion. So this method of "completion" would not work. There is, however, another method for deriving classical logic from intuitionistic logic, which does not depend on any proof system: It is possible to show that $\mathcal{T} \vdash_{CL_{\rightarrow}} \varphi$ iff there exist formulae $A_1, \ldots, A_n$ ($n \geq 0$) such that $\mathcal{T} \vdash_{H_{\rightarrow}} (\varphi \rightarrow A_1) \rightarrow ((\varphi \rightarrow A_2) \rightarrow (\cdots \rightarrow ((\varphi \rightarrow A_n) \rightarrow \varphi) \cdots))$. This method *can* be applied to relevance logics!

The main goal of this paper is to examine the logics which one gets from the standard implicational relevance logics by the method which we have just described. Its main discovery is that not only do the resulting systems still have the variable-sharing property (which is characteristic for relevance logics), but they can be finitely axiomatized, cut-free Gentzen-type systems can be constructed for them, and (most important of all) they all have clear, particularly simple algebraic semantics. They correspond, in fact, to classes of structures in which *there is exactly one nondesignated element*. Such structures will be called below (implicational) F-structures, and the corresponding logics are called "F-logics".

Algebraic structures, in which the set of nondesignated elements is a singleton, have already been introduced in [Av97] and [Av9?]. They were shown there to be very useful in investigating and understanding substructural logics. More specifically: we have demonstrated that while weakening corresponds to the assumption that there is exactly one designated truth-value, contraction has strong connections with the assumption that there is exactly one *non*designated truth-value. The investigations in these two papers were done, however, for the full multiplicative language (and sometimes beyond), and the availability of a De-Morgan negation was crucial in them. In this paper we return to F-structures from a different point of view, and treats only languages and structures which are purely implicational. The conclusion is that here also F-structure are very useful for understanding implicational relevance logics.

I A Review of Basic Implicational Logics

In this section we review the most important implicational logics with contraction (We refer the reader to [Do93] for an excellent introduction to the topic and for further references). $R_{\rightarrow}$, $H_{\rightarrow}$, $RMI_{\rightarrow}$, $RM_{\rightarrow}$, and $CL_{\rightarrow}$ are, respectively, the purely implicational fragments of the relevance logic R ([AB75], [ABD92], [Du86]), Intuionistic Logic, the purely relevance system RMI ([Av90]), Dunn-McCall semi-relevant system RM ([AB75], [Du86]) and classical logic. $RM0_{\rightarrow}$ is the logic which is defined by the purely implicational axioms and rule of the standard axiomatization of RM (while $RMI_{\rightarrow}$ is the implicational fragment of the logic which is defined by the purely multiplicative axioms and rule of the same axiomatic system).

I.1 Cut-Free Gentzen-Type Representations

GR$_{\rightarrow}$:

Axioms:

$$A \Rightarrow A$$

Logical Rules:

$$\frac{?_1 \Rightarrow \Delta_1, A \quad B, ?_2 \Rightarrow \Delta_2}{?_1, ?_2, A \rightarrow B \Rightarrow \Delta_1, \Delta_2} \qquad \frac{A, ? \Rightarrow B, \Delta}{? \Rightarrow A \rightarrow B, \Delta}$$

Structural rules: Permutation and Contraction.

***GRM*0**$_{\rightarrow}$: Like $GR_{\rightarrow}$, with the addition of the expansion[1] on the *left* hand side (only!).

GH$_{\rightarrow}$: Like $GR_{\rightarrow}$, with the addition of weakening on the *left* hand side (only!).

GRMI$_{\rightarrow}$: Like $GR_{\rightarrow}$, with the addition of expansion on *both* sides. Alternatively, $GRMI_{\rightarrow}$ can be obtained from $GR_{\rightarrow}$ by adding to it *mingle* (or *relevant mix*):

$$\frac{A, ?_1 \Rightarrow \Delta_1 \qquad A, ?_2 \Rightarrow \Delta_2}{A, A, ?_1, ?_2 \Rightarrow \Delta_1, \Delta_2} \qquad \frac{?_1 \Rightarrow \Delta_1, A \qquad ?_2 \Rightarrow \Delta_2, A}{?_1, ?_2 \Rightarrow \Delta_1, \Delta_2, A, A}$$

GRM$_{\rightarrow}$: Like $GR_{\rightarrow}$, with the addition of *mix*:

$$\frac{?_1 \Rightarrow \Delta_1 \qquad ?_2 \Rightarrow \Delta_2}{?_1, ?_2 \Rightarrow \Delta_1, \Delta_2}$$

GCL$_{\rightarrow}$: Like $GR_{\rightarrow}$, with the addition of weakening on *both* sides.

I.2 Hilbert-Type Representations

(I) ***R***$_{\rightarrow}$:

Axioms:

(I)	$A \rightarrow A$	(Identity)
(B)	$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$	(Transitivity)
(C)	$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$	(Permutation)
(W)	$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$	(Contraction)

Rule of inference:

$$\frac{A \qquad A \rightarrow B}{B}$$

[1] "expansion" is the usual name for the converse of contraction.

(II) $\boldsymbol{RM0}_{\rightarrow}$: $R_{\rightarrow}$ together with

(M) $A \rightarrow (A \rightarrow A)$ (Mingle)

(III) $\boldsymbol{H}_{\rightarrow}$: $R_{\rightarrow}$ together with weakening.

The above are standard Hilbert-type counterparts of the single-conclusion Gentzen-type systems presented in the previous subsection. A Hilbert-type system for $RMI_{\rightarrow}$ with MP as the sole rule of inference can be found in [Av84], while for $RM_{\rightarrow}$ there are such formulations in [MP72] and [Av84]. $CL_{\rightarrow}$, can, as is well known, be axiomatized by adding to $H_{\rightarrow}$ Pierce's law: $((A \rightarrow B) \rightarrow A) \rightarrow A$.

I.3 The Consequence Relation

Definition. *Let L be any of the Hilbert-type systems above. The associated (Tarskian) consequence relation $\vdash_L$ is defined in the usual way: $\mathcal{T} \vdash_L A$ iff there exists a sequence $A_1, \ldots, A_n = A$ such that each A_i either belongs to $\mathcal{T}$, or is an instance of an axiom, or follows from two previous ones by MP.*

Proposition. *Let L be one of the systems above.*

1. *$\vdash_{GL} \Rightarrow A$ iff $\vdash_L A$.*

2. *$A_1, \ldots, A_n \vdash_L B$ iff $\Rightarrow B$ is derivable in GL from $\Rightarrow A_1, \ldots, \Rightarrow A_n$ (using cuts).*

3. *$\mathcal{T} \vdash_L B$ iff there exists a (possibly empty) multiset ?, all elements of which belong to $\mathcal{T}$, such that $\vdash_{GL}$? $\Rightarrow B$.*

Note. Without weakening, it is *not* the case that $A_1, \ldots, A_n \vdash_L B$ iff $\vdash_{GL} A_1, \ldots, A_n \Rightarrow B$.

Relevant Deduction theorem[2]**.** *Let L be any extension of $R_{\rightarrow}$ by axiom schemes. Then $\mathcal{T}, A \vdash_L B$ iff either $\mathcal{T} \vdash_L B$ or $\mathcal{T} \vdash_L A \rightarrow B$.*

II Implicational F-Structures

II.1 Definition. *An implicational F-structure (F.s.) is a structure $\overline{S} = \langle S, \leq, \perp, \rightarrow \rangle$ in which:*

1. *$\langle S, \leq \rangle$ is a poset with at least two elements.*

2. *$\perp$ is the least element of $\langle S, \leq \rangle$*

[2]This is an easy consequence of the relevant deduction theorem for $R_{\rightarrow}$ in [AB75] and [Du86].

3. $a \leq b$ *iff* $a \to b \neq \perp$ (*iff* $a \to b > \perp$)

4. $\to$ *is semi-commutative:* $a \to (b \to c) = b \to (a \to c)$

5. $\to$ *is left-monotonic: if* $b \leq c$ *then* $a \to b \leq a \to c$.

II.2 Lemma. *In every F.s.* $\overline{S}$ *we have:*

(0) $a \leq b \to c \Leftrightarrow b \leq a \to c$

(1) $\top = \perp \to \perp$ *is the greatest element*

(2) $\perp \to a = a \to \top = \top$ *for all* a

(3(i)) *If* $a \neq \top$ *then* $\top \to a = \perp$

(3(ii)) *If* $a \neq \perp$ *then* $a \to \perp = \perp$

(4) *If* $\perp < a \leq b$ *then* $\perp < a \to b \leq b$.

Proof: Part (0) is immediate from parts 4 and 3 of the last definition. Since $\perp \leq a \to \perp$, this implies that $a \leq \perp \to \perp$ for all a. Hence (1). Since $\perp \leq a$, $\top = \perp \to \perp \leq \perp \to a$ by left monotonicity. Hence $\top = \perp \to a$ for all a. Since $a \to \top = a \to (\perp \to \perp) = \perp \to (a \to \perp) = \top$, also $a \to \top = \top$ for all a. This proves (2). (3) is immediate from part 3 of the last definition. Now (0) and the fact that $a \to b \leq a \to b$ together implies that $a \leq (a \to b) \to b$. In case $a \neq \perp$ this means that $(a \to b) \to b > \perp$ and so $a \to b \leq b$. If in addition $a \leq b$ then also $a \to b > \perp$ by definition of an F.s. This proves part (4). $\square$

II.3 Examples.

1. Let $\mathcal{A}_\omega = \langle A_\omega, \leq, \to \rangle$ where $A_\omega = \{\top, \perp, I_0, I_1, I_2, \ldots\}$, $\perp \leq I_i \leq \top$ for all i, and $a \to b = \perp$ if $a \not\leq b$, I_i if $a = b = I_i$ and $\top$ if $a = \perp$ or $b = \top$. This is an F.s. which has been shown in [Av84] to be a characteristic matrix for $RMI_\to$. As in [Av84], we denote by $\mathcal{A}_n$ the substructure of $\mathcal{A}_\omega$ which is created by $\{\top, \perp, I_0, \ldots, I_{n-1}\}$. Each $\mathcal{A}_n$ is of course also an F.s.

2. Let $\mathcal{B}_\omega = \langle B_\omega, \leq, \to, \perp \rangle$ where $B_\omega = \{\perp, 0, 1, 2, \ldots, \top\}$, $\perp \leq 0 \leq 1 \leq 2 < \ldots \leq \top$ and $a \to b = \perp$ if $a > b$, $\top$ if $a = \perp$ or $b = \top$ and $b \perp a$ otherwise. We denote by $\mathcal{B}_n$ the substructure of $\mathcal{B}_\omega$ which is created by $\{\top, \perp, 0, 1, \ldots, n \perp 1\}$. Each of these structures is an F.s.

3. The structures $\mathcal{C}_\omega$ and $\mathcal{C}_n$ are defined like $\mathcal{B}_\omega$ and $\mathcal{B}_n$, except that in case $i, j \in N$ and $i \leq j$, $i \to j$ would be j rather than $j \perp i$.

Note. Obviously, $\mathcal{A}_0 = \mathcal{B}_0 = \mathcal{C}_0 =$ the two-valued Boolean Algebra. Also $\mathcal{A}_1 = \mathcal{B}_1 = \mathcal{C}_1 =$ Sobocinski's 3-valued logic $\mathcal{M}_3$ ([So52]) which is characteristic for $RM_{\to}$.

The example of $\mathcal{A}_\omega$ can easily be generalized. Given any set S, We can add to it two object $\top$ and $\perp$, define $a \leq b$ if either $a = \perp$ or $b = \top$ or $a = b$ and $a \to b$ to be $\top$ if $a = \perp$ or $b = \top$, a if $a \in S$ and $a = b$, and $\perp$ otherwise. By this we get an F.s. $\mathcal{A}_S$ which belongs to the following class of implicational F-structures:

II.4 Definition. *An implicational F-structure is called flat if $a \leq b$ only in case $a = \perp$ or $b = \top$ or $a = b$.*

Note. It is easy to see that if $\langle S, \leq \rangle$ is a poset with a least element $\perp$ and a greatest element $\top$ s.t. $a \leq b$ only if $a = \perp$ or $b = \top$ or $a = b$ then there is exactly one way to define an operation $\to$ on it which will make it an F.s. (This F.s. will be flat, of course).

$\mathcal{A}_n$ and $\mathcal{C}_n$ are special cases of the following class of implicational F-structures.

II.5 Definition. *An implicational F-structure is called simple if $a \to a = a$ for $a \neq \perp$.*

II.6 Proposition. *Every flat structure is simple.*

Proof: This easily follows from the note after II.4. □

II.7 An example. $\mathcal{B}_\omega$ is not simple.

II.8 A Characterization theorem. Given a poset $\langle S, \leq \rangle$ with a greatest element $\top$ and a least element $\perp$, there is exactly one way to define an operation $\to$ on S so that the resulting structure is a simple F.s.

Proof: Given $\langle S, \leq \rangle$ as above, define

$$a \to b = \begin{cases} \perp & a \not\leq b \\ \top & a = \perp \\ b & \perp < a \leq b \end{cases}$$

First we show that in every simple F.s., $\to$ is necessarily defined as above. Indeed the first two clauses are necessary by II.1 and II.2. As for the third clause, we know (from II.2 (4)) that if $\perp < a \leq b$ then $a \to b \leq b$. On the other hand, if $b = b \to b$ then $a \to b = a \to (b \to b) = b \to (a \to b)$. In case $a \leq b$ this means that $b \to (a \to b) > \perp$ and so $b \leq a \to b$. Hence $a \to b = b$ whenever $b = b \to b$ and $\perp < a \leq b$.

It remains to check that by using the above definition we indeed get a simple F.s. Most of the conditions are straightforward. The only one that needs a little more effort is semi-commutativity, but here also it is easy, once we observe that $a \to (b \to c)$ is $\top$ if $a = \bot$ or $b = \bot$, c in case $\bot < a \leq c$ and $\bot < b \leq c$ and $\bot$ otherwise. □

III The Logic of Implicational F-Structures

In this section we investigate the consequence relation which is induced by implicational F-structures.

III.0.1 Definition.

(1) *Let $\mathcal{T}$ be a set of purely implicational formulae. By an F-model of $\mathcal{T}$ we mean a pair $\langle \overline{S}, v \rangle$ where $\overline{S}$ is an F.s. and v is a $\to$-respecting valuation in S such that $v(A) \neq \bot$ for all $A \in \mathcal{T}$.*

(2) *$\mathcal{T} \models_F A$ if every F-model of $\mathcal{T}$ is also an F-model of A.*

(3) *A is F-valid if $\models_F A$.*

Examples. In the next subsection we show that all theorems of $R_\to$ are F-valid. On the other hand the mingle axiom $A \to (A \to A)$ (and so, in general, $A \to (B \to A)$) is not F-valid. To see this, take $v(p) = 1$ in $\mathcal{B}_\omega$ (see II.3). Then $v(p \to (p \to p)) = \bot$. □

III.0.2 Proposition. *$\models_F$ has the variable sharing property: $A \to B$ is F-valid only if A and B share an atomic variable.*

Proof: Suppose $A \to B$ share no atomic variable. Define in $\mathcal{A}_1$ ($=\mathcal{M}_3$) a valuation v such that $v(p) = \top$ if p occurs in A, while $v(p) = I$ if p occurs in B. Then $v(A \to B) = \bot$. □

Note. The same proof work for all the other logics we consider below (including $RM_\to!$).

III.1 The Logic of Implicational F-Structures and $R_\to$

In this subsection we investigate the strong relations between implicational F-structures and the standard implicational relevance login $R_\to$ ([Ch51] [AB75], [Du86])[3].

[3] $R_\to$ was first introduced in [Ch51]). The whole study of F-structures started, in fact, from the observation that almost every countermodel for nontheorems of $R_\to$ which has been produced by Slaney's program MaGic has been an implicational F-structure. I take here the opportunity to thank J. Slaney for his MaGic help to this research!

III.1.1 Lemma. *Every theorem of $R_\rightarrow$ is F-valid. Moreover: if $\mathcal{T} \vdash_{R_\rightarrow} \varphi$ then $\mathcal{T} \models_F \varphi$.*

Proof: We show the validity of the contraction axiom as an example. Let v be any valuation in an F.s. $\overline{S}$. If $v(A) = \perp$ then $v(A \rightarrow B) = v((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)) = \top$. If $v(A) \neq \perp$ then by II.2(4), $v(A) \rightarrow v(B) \leq v(B)$ and so $v(A) \rightarrow (v(A) \rightarrow v(B)) \leq v(A) \rightarrow v(B)$. Hence $v((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)) \neq \perp$. □

Note. The converse of III.1.1 fails (see subsection III.3.). What we do have is:

III.1.2 Theorem. *$\mathcal{T} \models_F \varphi$ iff there exist formulae $A_1, \ldots, A_n$ $(n \geq 0)$ such that:*

$$\mathcal{T} \vdash_{R_\rightarrow} (\varphi \rightarrow A_1) \rightarrow ((\varphi \rightarrow A_2) \rightarrow (\cdots \rightarrow ((\varphi \rightarrow A_n) \rightarrow \varphi) \cdots))$$

.

Proof: It is easy to see that if $v(\varphi) = \perp$ for some valuation v in $\overline{S}$ then $v((\varphi \rightarrow A_1) \rightarrow (\cdots((\varphi \rightarrow A_n) \rightarrow \varphi)\cdots)) = \perp$ regardless of the values of $v(A_1), \ldots, v(A_n)$. It follows by this and III.1.1, that if $A_1, \ldots, A_n$ like above exist then $v(\varphi) \neq \perp$ in every F-model of $\mathcal{T}$. Hence $\mathcal{T} \models_F \varphi$.

For the converse, let $\mathcal{T}$ be a theory and φ a formula such that no corresponding $A_1, \ldots, A_n$ exist. Let $\mathcal{T}' = \mathcal{T} \cup \{\varphi \rightarrow A | A \text{ is an implicational formula}\}$. By our assumption and the relevant deduction theorem (see I.3), $\mathcal{T}' \not\vdash_{R_\rightarrow} \varphi$. Extend now $\mathcal{T}'$ to a maximal theory $\mathcal{T}^* \supseteq \mathcal{T}'$ such that $\mathcal{T}^* \not\vdash_{R_\rightarrow} \varphi$. By the relevant deduction theorem again, $\mathcal{T}^* \not\vdash_{R_\rightarrow} A$ iff $\mathcal{T}^* \vdash_{R_\rightarrow} A \rightarrow \varphi$, iff (Since $\mathcal{T}^* \supseteq \mathcal{T}'$) $\mathcal{T}^* \vdash_{R_\rightarrow} A \leftrightarrow \varphi$ (by this we mean here just that both $A \rightarrow \varphi$ and $\varphi \rightarrow A$ are theorems of $\mathcal{T}^*$). This fact entails that in the standard Lindenbaum algebra of $\mathcal{T}^*$ there is exactly one element which is not designated: $[\varphi]$. Moreover; $[\varphi] \leq [A]$ for all A, since $\mathcal{T}^* \supseteq \mathcal{T}'$ (As usual, $[A] \leq [B]$ means that $\mathcal{T}^* \vdash_{R_\rightarrow} A \rightarrow B$). It is easy now to see that this Lindenbaum Algebra is an F.s. and that the canonical valuation ($v(A) = [A]$) provides an F-model of $\mathcal{T}^*$ (and so of $\mathcal{T}$) which is not an F-model of φ. □

Note. The proof that the same relation holds between $\vdash_{CL_\rightarrow}$ and $\vdash_{H_\rightarrow}$ is identical.

III.1.3 Corollary.

1. *The compactness theorem obtains for $\models_F$.*

2. *The set of F-valid formulae is recursively enumerable.*

III.2 A Translation of Classical Implication

Theorem III.1.2 implies that $(p \rightarrow q) \rightarrow p \models_F p$. $((p \rightarrow q) \rightarrow p) \rightarrow p$, on the other hand, is not F-valid, since it is easy to refute it in $\mathcal{A}_\omega$ of II.3 ($v(p) = I_1, v(q) = \perp$). It follows that the deduction theorem for $\to$ (in either its classical or relevant form) fails for $\models_F$. There exists however a definable implication connective for which the classical deduction theorem and its converse *are* valid.

III.2.1 Definition. $A \supset B =_{Df} (A \to (B \to B)) \to (A \to B)$.

Note. $A \supset B$ is equivalent to $A \to B$ in intuitionistic logic and classical logic.

III.2.2 Theorem. *Let $\mathcal{C}$ be a class of F-structures. Define $\models_{\mathcal{C}}$ in the obvious way. Then $\supset$ is an internal implication for $\models_{\mathcal{C}}$: $\mathcal{T} \models_{\mathcal{C}} A \supset B$ iff $\mathcal{T}, A \models_{\mathcal{C}} B$ (in particular, $\supset$ is an internal implication for $\models_F$).*

Proof: Suppose $\mathcal{T} \models_{\mathcal{C}} A \supset B$, and let $\langle \overline{S}_1, v\rangle$ be an F-model of $\mathcal{T} \cup \{A\}$ which belongs to $\mathcal{C}$. We want to show that $v(B) \neq \perp$. Assume otherwise. Then $v(B \to B) = \perp \to \perp = \top$, $v(A \to (B \to B)) = \top$ $v(A \to B) = \perp$ (since $v(A) \neq \perp$) and $v(A \supset B) = \top \to \perp = \perp$. This contradicts $\mathcal{T} \models_{\mathcal{C}} A \supset B$.

For the converse, suppose $\mathcal{T}, A \models_C B$. We show $\mathcal{T} \models_C A \supset B$. So let $\langle \overline{S}, v\rangle$ be an F-model of $\mathcal{T}$ which belongs to C. If $a = v(A) = \perp$ then $v(A \to B) = \top$ and so $v(A \supset B) = \top \neq \perp$. If $a \neq \perp$ then $\langle \overline{S}, v\rangle$ is a model of $\mathcal{T} \cup \{A\}$ and so $b = v(B) \neq \perp$. This entails that $b \to b \leq b$ (by II.2(4)), and so $a \to (b \to b) \leq a \to b$. It follows that $v(A \supset B) \neq \perp$. □

III.2.3 A Generalization. Let $\mathcal{C}$ be as in III.2.2. Then $\mathcal{T}, A_1, \ldots, A_n \models_{\mathcal{C}} B$ iff

$$\mathcal{T} \models_F (A_1 \to A_2 \to \cdots \to A_n \to (B \to B) \cdots) \to (A_1 \to A_2 \to \cdots \to (A_n \to B) \cdots)$$

Proof: Similar to that of III.2.2. □

III.2.4 Theorem. *$\supset$ is a faithful interpretation in $\models_F$ of classical implication.*

Proof: Suppose $A_1, \ldots, A_n, B$ are formulae in the language of $\{\supset\}$. We want to show that $A_1, \ldots, A_n \models_F B$ iff B classically follows from $A_1, \ldots, A_n$. One direction is trivial: the classical two-valued algebra is an F.s. so if $A_1, \ldots, A_n \models_F B$ then B classically follows from $A_1, \ldots, A_n$. For the converse, assume $A_1, \ldots, A_n \not\models_F B$. Let $\langle \overline{S}, v\rangle$ be a model of $A_1, \ldots, A_n$ for which $v(B) = \perp$. Define a classical valuation v^* by

$$v^*(A) = \begin{cases} \top & v(A) \neq \perp \\ \perp & v(A) = \perp \end{cases}$$

obviously $v^*(A_i) = \top$ $(i = 1, \ldots, n)$ while $v^*(B) = \perp$. It remains to show that v^* is indeed a valuation, i.e.: that $v^*(C \supset D) = v^*(C) \supset v^*(D)$ for all C, D. Well, if $v(C) = \perp$ then both sides are $\top$. If $v(C) \neq \perp$ but $v(D) = \perp$ then both sides are $\perp$. Finally, if $d = v(D) \neq \perp$

then $v^*(C) \supset v^*(D) = v^*(C) \supset \top = \top$. On the other hand in this case $d \to d \leq d$ and so $c \to (d \to d) \leq c \to d$ for all c. Hence $v(C \supset D) \neq \perp$ and $v^*(C \supset D) = \top$ as well. □

Note. To get a translation of the full classical propositional logic, we should switch to the language of $\{\to, \perp\}$, and translate $\neg A$ as $A \supset \perp$. The additional propositional constant $\perp$ has an obvious interpretation in implicational F.s., so this switch can be done in a very natural way. Details will be given elsewhere.

From III.2.2 it follows that MP for $\supset$ is valid in $\models_F$. The next proposition examines the status of this rule in $RMI_\to$ and $RM_\to$.

III.2.5 Proposition. *M.P. for $\supset$ is is admissible in $RMI_\to$ and $RM_\to$, but it is not derivable in them.*

Proof: It is easy to see that if A and $A \supset B$ are valid in $\mathcal{A}_1$ then so is B. Hence MP for $\supset$ is admissible in $RM_\to$. A similar argument applies for $RMI_\to$, using $\mathcal{A}_\omega$ instead of $\mathcal{A}_1$.

To show that the rule is not derivable, assume for contradiction that $p, p \supset q \vdash_{RM_\to} q$ (p, q-atomic). Then the relevant deduction theorem easily entails that $\vdash_{RM_\to} (p \supset q) \to (p \to q)$. Hence $(p \supset q) \to (p \to q)$ should be valid in $\mathcal{A}_1$ (see note after II.3). This is false though, as the valuation $v(p) = \top$, $v(q) = I$ demonstrates. □

Note. This shows that the completeness of $RM_\to$ relative to $\mathcal{A}_1$ is only *weak* completeness (completeness only w.r.t. theoremhood). Similarly with $RMI_\to$ and $\mathcal{A}_\omega$.

Open Problem 1. Is MP for $\supset$ admissible in $R_\to$?

Note. MP for $\supset$ is not admissible in R or RM. Otherwise it would have been admissible in their $\{\to, \wedge\}$ fragment, and so *derivable* there, by [MS92]. This implies that $p, p \supset q \vdash_R q$. But letting $v(p) = 2$, $v(q) = \perp 1$ we get a model of $\{p, p \supset q\}$ in Sugihara matrix (see [AB75] or [Du86]) which is not a model of q. Hence q does not follow from $\{p, p \supset q\}$ even in RM.[4]

III.3 Corresponding Formal Systems

III.3.1 Hilbert-type systems

From III.1.2 it immediately follows That a Hilbert-type axiomatization of $\models_F$ is given by $R_\to + ?$, where ? consists of the following inference rules, $n \geq 1$:

$$\gamma_n : (B \to A_1) \to ((B \to A_2) \to (\ldots \to ((B \to A_n) \to B)\ldots))/B$$

[4]Sugihara matrix is weakly characteristic for RM by a famous result of Meyer (see [AB75]).

It is possible, however, to construct a complete *finite* system if we take MP for $\supset$ as the rule of inference. It would be more elegant and useful to do this in a language in which both $\rightarrow$ and $\supset$ are primitives:

The system $HF_{\rightarrow}$:

Axioms:

1. $A \supset (B \supset A)$
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
3. $A \rightarrow A$
4. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
5. $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
6. $(A \rightarrow B) \supset (A \supset B)$
7. $((A \rightarrow (B \rightarrow B)) \rightarrow (A \rightarrow B)) \rightarrow (A \supset B)$
8. $(A \supset B) \rightarrow ((A \rightarrow (B \rightarrow B)) \rightarrow (A \rightarrow B))$
9. $((A \rightarrow B) \supset C) \supset ((A \supset C) \supset C)$

Rule of inference: MP for $\supset$.

III.3.2 Soundness and strong completeness theorem:

$$\mathcal{T} \vdash_{HF_{\rightarrow}} \varphi \qquad \text{iff} \qquad \mathcal{T} \models_F \varphi$$

Proof: Soundness is easy: axioms 1-2 means that the deduction theorem holds for $\supset$, and this was proved in III.2.2. 3-6 are all theorems of $R_{\rightarrow}$, and so valid by III.1.1. 7-8 just repeat the definition of $\supset$ in terms of $\rightarrow$. Finally, (9) is valid, by III.2.2, iff $((A \rightarrow B) \supset C)$, $A \supset C \models_F C$. So suppose we are given a model of $\{(A \rightarrow B) \supset C, A \supset C\}$. If A is true then so is C since $A, A \supset C \models_F C$. If A is not true (i.e. $v(A) = \perp$) then $A \rightarrow B$ is true. But $(A \rightarrow B) \supset C$, $A \rightarrow B \models_F C$. Hence again C is true.

For completeness, assume $\mathcal{T} \not\vdash_{HF_{\rightarrow}} \varphi$. We construct an implicational F.s. and a valuation v such that $v(A) \neq \perp$ for every $A \in \mathcal{T}$, but $v(\varphi) = \perp$. For this extend $\mathcal{T}$ to a maximal theory $\mathcal{T}^*$ such that $\mathcal{T}^* \not\vdash_{HF_{\rightarrow}} \varphi$. The maximality of $\mathcal{T}^*$ and the deduction theorem for $\supset$ entail that

$A \notin \mathcal{T}^*$ iff $A \supset \varphi \in \mathcal{T}^*$. Define now the Lindenbaum Algebra of $\mathcal{T}$ using $\to$ in the standard way: First, let $A \sim B$ iff both $A \to B$ and $B \to A$ are theorems of $\mathcal{T}^*$. By axioms 3-6 this is a congruence relation w.r.t. $\to$, and so also w.r.t. $\supset$, by axioms 7-8. Let [A] denote the equivalence class of A, and let M be the set of equivalence classes. The operations $\to$ and $\supset$ are defined on M in the obvious way. Define also $[A] \leq [B]$ iff $A \to B \in \mathcal{T}^*$. $\leq$ is well defined and is a partial order by axiom 3-6. Now, by axiom 9 and the main property of $\mathcal{T}^*$, if $A \notin \mathcal{T}^*$ then $A \to B \in \mathcal{T}^*$ for all B (otherwise we'll get $\varphi \in \mathcal{T}^*$). It follows that all the non-elements of $\mathcal{T}^*$ form one equivalence class ($[\varphi]$), which is the least element w.r.t. $\leq$. Denote this class by $\perp$. $\langle M, \leq, \perp, \to \rangle$ is an implicational F.s. (by axioms 3-6 again). It is obvious that by defining $v(A) = [A]$ we get a valuation in it as desired. Finally, axioms 7 and 8 entail that $\supset$ is defined in M as it should (i.e.: $a \supset b = ((a \to b) \to b) \to (a \to b)$). □

Let us return now to the system $R_\to + ?$ which was mentioned above. Since $\vdash_{R_\to + \Gamma} = \models_F$, each axiom of $HF_\to$ is provable in $R_\to + ?$ using a finite number of its rules. MP for $\supset$ is also easily seen to be derivable already in the presence of γ_1. It follows that $R_\to + ?$ is equivalent to $R_\to + \{\gamma_1, \ldots, \gamma_n\}$ for some n. Moreover, since $((B \to A_n) \to B) \to ((B \to B) \to ((B \to A_n) \to B))$ is a theorem of $R_\to$, γ_{n+1} implies γ_n (within $\vdash_{R_\to}$) for all n, and so γ_n implies γ_k within $\vdash_{R_\to}$ whenever $n \geq k$. Hence $R_\to + ?$ is equivalent to $\models_F = \vdash_{R_\to + \{\gamma_n\}}$ for some n. [5]

Open Problem 2: What can the smallest value for n be? In particular, can it be equal to 1? (In other words: is $R_\to + \{\gamma_1\}$ a complete axiomatization of $\models_F$?)

Note. $\models_F$ is at least as strong as $R_\to$, but the relevant deduction theorem fails for it (as we noted at the beginning of III.2). It follows that there does not exist a sound and strongly complete axiomatization of $\models_F$ having only axiom-schemes and MP for $\to$ as the sole rule of inference. Hence a rule like γ_n (or MP for $\supset$) is necessary to get a strongly complete axiomatization of $\models_F$.

Open Problem 3. Is there a finite axiomatization of $\models_F$, having only axiom-schemes and MP for $\to$ as the sole rule of inference, which is sound and *weakly* complete (i.e. has exactly the F-valid formulae as theorems)? Note that the argument above does not exclude such a possibility! From the next theorem, which is about the status of γ_1 in relevance logics, it immediately follows that $R_\to$ itself does not have this property.

III.3.3 Theorem. *The rule* $\frac{(A \to B) \to A}{A}$ *is:*

1. *Admissible but not derivable in* $RMI_\to$ *and its various extensions (including* $RM_\to$*), except classical logic.*

[5] We like to thank an anonymous referee for these observations.

2. *Not admissible in any system between* $R_\rightarrow$ *and* $H_\rightarrow$ *(like* $R_\rightarrow$, $RM0_\rightarrow$, $H_\rightarrow$*) or any system between* $R_\rightarrow \cup \{A \wedge B \rightarrow A, A \wedge B \rightarrow B\}$ *and* RM.[6]

Proof:

1. The rule is valid in $\mathcal{A}_n$ $(0 \leq n \leq \omega)$. Since any extension of $RMI_\rightarrow$ (including $RMI_\rightarrow$ itself) has one of the $\mathcal{A}_n$'s as a characteristic matrix ([Av84]), the rule is admissible in each of them. On the other hand, the rule is not sound in Sugihara matrix (take $v(A) = \perp 1, v(B) = \perp 2$), while $RM_\rightarrow$ is sound in that matrix. Hence the rule is not derivable in $RM_\rightarrow$. Since by [Av84] $RM_\rightarrow$ is the strongest extension of $RMI_\rightarrow$ (except $CL_\rightarrow$), the rule is not derivable in any of these extensions.

2. Let $\psi = ((A \rightarrow B) \rightarrow A) \supset A$. Then $\vdash_{R_\rightarrow} (\psi \rightarrow B) \rightarrow \psi$ but $\not\vdash_{H_\rightarrow} \psi$. This entails the first part. For the second part take $A = ((C \rightarrow E) \wedge ((C \rightarrow D) \rightarrow E)) \rightarrow E$, $B = C \rightarrow D$. Then $\not\vdash_{RM} A$ (take in Sugihara matrix $v(C) = v(E) = \perp 1$, $v(D) = \perp 2$), but $(A \rightarrow B) \rightarrow A$ is provable in $R_\rightarrow \cup \{\varphi \wedge \psi \rightarrow \varphi, \varphi \wedge \psi \rightarrow \psi\}$. To see this, note that by using the scheme $\varphi \wedge \psi \rightarrow \varphi$ we easily get $C \rightarrow A$ in this system. But $\vdash_{R_\rightarrow} (C \rightarrow A) \rightarrow ((A \rightarrow (C \rightarrow D)) \rightarrow (((C \rightarrow D) \rightarrow E) \rightarrow E))$. Using $(C \rightarrow E) \wedge ((C \rightarrow D) \rightarrow E) \rightarrow ((C \rightarrow D) \rightarrow E)$ and the provability of $C \rightarrow A$ we easily get from this $(A \rightarrow (C \rightarrow D)) \rightarrow A$. □

Note. III.3.3(1) means, among other things, that $RM_\rightarrow$ is not "structurally closed". The implication-conjunction fragment of RM, on the other hand *is* structurally closed (see [MS92]). This shows that $L_{\{\rightarrow,\wedge\}}$ can be closed without $L_\rightarrow$ being closed.

Open Problem 4. Is γ_1 admissible in $LL_\rightarrow$? $BCK_\rightarrow$?

III.3.4 A Gentzen-type system.

In order to get a cut-free Gentzen-type system we need to use a calculus of *hypersequents*. Such a calculus resembles ordinary sequential calculi in its logical rules, but is richer in structural rules[7]. In general, a hypersequent is a syntactic structure of the form $?_1 \Rightarrow \Delta_1 \mid ?_2 \Rightarrow \Delta_2 | \cdots | ?_n \Rightarrow \Delta_n$ (where $?_i \Rightarrow \Delta_i$ is an ordinary sequent). For the logic of F-structures we shall employ only hypersequents with single-conclusion components (i.e.: hypersequents of the form $?_1 \Rightarrow A_1 \mid ?_2 \Rightarrow A_2 \mid \cdots | ?_n \Rightarrow A_n$, where A_i is a sentence). We use G, H as metavariables for hypersequents, S for sequents.

[6] The claim is true, in fact also for almost all the extensions of RM. The only exceptions are classical logic and the 3-valued extension of RM. This follows from Dunn's characterizations of all these extensions in [Du70] and the proof below.

[7] See [Av95] for an introduction to this method and many examples.

The system $GF_{\rightarrow}$:

Axioms:

$$A \Rightarrow A$$

External Structural rules:

$$\frac{G}{G|H}(EW) \qquad \frac{G|S|S|H}{G|S|H}(EC) \qquad \frac{G|S_1|S_2|H}{G|S_2|S_1|H}(EP)$$

(External Weakening, Contraction and Permutation, respectively).

Internal Structural rules:

$$\frac{G|?_1, A, B, ?_2 \Rightarrow C|H}{G|?_1, B, A, ?_2 \Rightarrow C|H} \quad (IP) \qquad \frac{G|A, A, ? \Rightarrow B|H}{G|A, ? \Rightarrow B|H} \quad (IC)$$

$$\frac{G|?_1, ?_2 \Rightarrow A|H}{G|?_1 \Rightarrow A|?_2, \Delta \Rightarrow B|H} \quad (ww) \qquad \frac{G_1|?_1 \Rightarrow A|H_1 \qquad G_2|A, ?_2 \Rightarrow B|H_2}{G_1|G_2|?_1, ?_2 \Rightarrow B|H_1|H_2} \quad (Cut)$$

(Internal Permutation, Internal Contraction, weak weakening and Cut)

Logical rules:

$$\frac{G_1|?_1 \Rightarrow A|H_1 \qquad G_2|B, ?_2 \Rightarrow C|H_2}{G_1|G_2|?_1, A \rightarrow B, ?_2 \Rightarrow C|H_1|H_2} \quad (\rightarrow\Rightarrow) \qquad \frac{G|?, A \Rightarrow B|\Delta}{G|? \Rightarrow A \rightarrow B|\Delta} \quad (\Rightarrow\rightarrow)$$

Note. IC is derivable, in fact, in the presence of the other rules, since from $G|A, A, ? \Rightarrow B|H$ one can infer, using ww, $G|A, ? \Rightarrow B|A, ? \Rightarrow B|H$, and from this $G|A, ? \Rightarrow B|H$ follows by EC.

An example:

$$(ww)\ \underline{A \Rightarrow A}$$
$$\underline{\Rightarrow A \mid A \Rightarrow B}$$
$$\underline{\Rightarrow A \mid \Rightarrow A \rightarrow B \qquad A \Rightarrow A}$$
$$\underline{\Rightarrow A \mid (A \rightarrow B) \rightarrow A \Rightarrow A \qquad A \Rightarrow A}$$
$$\underline{(A \rightarrow B) \rightarrow A \Rightarrow (A \rightarrow B) \rightarrow A \qquad \Rightarrow A \mid (A \rightarrow B) \rightarrow A, A \rightarrow A \Rightarrow A}$$
$$(IC)\ \underline{\Rightarrow A \mid ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow A), (A \rightarrow B) \rightarrow A, (A \rightarrow B) \rightarrow A \Rightarrow A}$$
$$\underline{\Rightarrow A \mid ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow A), (A \rightarrow B) \rightarrow A \Rightarrow A \qquad A \Rightarrow A}$$
$$\underline{(A \rightarrow B) \rightarrow A \Rightarrow A \rightarrow (B \rightarrow A) \qquad A \rightarrow A \Rightarrow A \mid ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow A), (A \rightarrow B) \rightarrow A \Rightarrow A}$$
$$\underline{((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow A), (A \rightarrow B) \rightarrow A \Rightarrow A \mid ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow A), (A \rightarrow B) \rightarrow A \Rightarrow A}$$
$$\underline{((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow A), (A \rightarrow B) \rightarrow A) \Rightarrow A}$$
$$\underline{((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow A) \Rightarrow ((A \rightarrow B) \rightarrow A) \rightarrow A}$$
$$\Rightarrow ((A \rightarrow B) \rightarrow A) \supset A$$

III.3.5 Cut elimination theorem. *The cut rule is admissible in the presence of the other rules of $GF_{\rightarrow}$.*

Proof: Details are similar to those in the proof of cut-elimination for GRM in [Av87]. □

III.3.6 Definition.

1. *The translation of a sequent $A_1, \ldots, A_n \Rightarrow B$ is the sentence $A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_n \rightarrow B)\cdots)$.*

2. *A hypersequent $?_1 \Rightarrow A_1|\cdots|?_n \Rightarrow A_n$ is true in a model $(\overline{S}, v)$ (where $\overline{S}$ is an implicational F.s. and v – a valuation in S) if for some $1 \leq i \leq n$, the translation of $?_i \Rightarrow A_i$ is true in $(\overline{S}, v)$.*

3. *Let $\mathcal{T}$ be a theory (i.e. a set of sentences) and let $G = ?_1 \Rightarrow A_1|\cdots|?_n \Rightarrow A_n$. We say that G follows from $\mathcal{T}$ in $GF_{\rightarrow}$ ($\mathcal{T} \vdash_{GF_{\rightarrow}} G$) iff there exist $\Delta_1, \ldots, \Delta_k \subseteq \mathcal{T}$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$ (not necessarily distinct) such that:*

$$\vdash_{GF_{\rightarrow}} \Delta_1, ?_{i_1} \Rightarrow A_{i_1}|\cdots|\Delta_k, ?_{i_k} \Rightarrow A_{i_k} \ .$$

4. *A sentence A follows in $GF_{\rightarrow}$ from a theory $\mathcal{T}$ if there exists $\Delta_1, \ldots, \Delta_k \subseteq \mathcal{T}$ such that $\vdash_{GF_{\rightarrow}} \Delta_1 \Rightarrow A|\cdots|\Delta_k \Rightarrow A$ (i.e. if $\mathcal{T} \vdash_{GF_{\rightarrow}} \Rightarrow A$).*

III.3.7 Soundness and completeness theorem. *$\mathcal{T} \vdash_{GF_{\rightarrow}} G$ iff G is true in every model of $\mathcal{T}$.*

Proof: As usual, the soundness part is relatively easy. We show the validity of *ww* as an example. So suppose $G|?_1, ?_2 \Rightarrow A|H$ is true in $(\overline{S}, v)$. If one of the components of G or of H is true in $(\overline{S}, v)$ we are done. Otherwise $?_1, ?_2 \Rightarrow A$ is true. If all the sentence in $?_2$ are true this entails that $?_1 \Rightarrow A$ is true. If not then $v(C) = \bot$ for some C in $?_2$, and so $v(?_2, \Delta \Rightarrow B) = \top$ for all Δ, B, by II.2(2).

For the converse, let $G = ?_1 \Rightarrow A_1|\cdots|?_n \Rightarrow A_n$ and suppose $\mathcal{T} \not\vdash_{GF_{\rightarrow}} G$. We construct a model $(\overline{S}, v)$ of $\mathcal{T}$ in which G is not true. For this extend $\mathcal{T}$ to a maximal theory $\mathcal{T}^*$ such that $\mathcal{T}^* \not\vdash_{GF_{\rightarrow}} G$. Obviously, $A \notin \mathcal{T}^*$ iff there exist $\Delta_1, \ldots, \Delta_k \subseteq \mathcal{T}^*$ and $1 \leq i_1, \ldots, i_k \leq n$ such that

$$\vdash_{GF_{\rightarrow}} A, \Delta_1, ?_{i_1} \Rightarrow A_{i_1}|\cdots|A, \Delta_j, \ldots, ?_{i_j} \Rightarrow A_{i_j}|\Delta_{j+1}, ?_{i_{j+1}} \Rightarrow A_{i_{j+1}}|\cdots|\Delta_k, ?_{i_k} \Rightarrow A_{i_k} \ .$$

This easily entails, using cuts, that

(i) If $\mathcal{T}^* \vdash_{GF_\rightarrow} \Delta_1 \Rightarrow B_1|\cdots|\Delta_k \Rightarrow B_k$ and $\Delta_1,\ldots,\Delta_k \subseteq \mathcal{T}^*$ then $B_i \in \mathcal{T}^*$ for some $1 \leq i \leq k$.

Since $GR_\rightarrow$, the standard Gentzen-type formulation of $R_\rightarrow$ is contained in $GF_\rightarrow$, III.1.1 and (i) entail that:

(ii) If $? \vdash_{R_\rightarrow} C$ and $? \subseteq \mathcal{T}^*$ then $C \in \mathcal{T}^*$.

Since $\vdash_{GF_\rightarrow} \Rightarrow A| \Rightarrow A \rightarrow B$ (because from $A \Rightarrow A$ one can infer $\Rightarrow A|A \Rightarrow B$ by (ww)), another corollary of (i) is:

(iii) For every A, B, either $A \in \mathcal{T}^*$ or $A \rightarrow B \in \mathcal{T}^*$.

Define now the Lindenbaum algebra $\overline{S}$ of $\mathcal{T}^*$ and the canonical valuation v in it as in the proof of III.3.2. Like in that proof, facts (ii) and (iii) easily imply that $\overline{S}$ is an F.s., and v is a valuation in it such that $v(A) \neq \perp$ iff $\mathcal{A} \in \mathcal{T}^*$. Hence $(\overline{S}, v)$ is a model of $\mathcal{T}^*$. G, on the other hand, is false in $(\overline{S}, v)$. Indeed, suppose one of the components of G, say $?_1 \Rightarrow A_1$, is true in $(\overline{S}, v)$. This means that $v(?_1 \rightarrow A_1) \neq \perp$ (where $?_1 \rightarrow A_1$ is the translation of $?_1 \Rightarrow A_1$). Hence $?_1 \rightarrow A_1 \in \mathcal{T}^*$. But $\vdash_{GR_\rightarrow} ?_1 \rightarrow A_1, ?_1 \Rightarrow A_1$, and so $\vdash_{GF_\rightarrow} ?_1 \rightarrow A_1, ?_1 \Rightarrow A_1$. It follows that $\mathcal{T}^* \vdash_{GF_\rightarrow} ?_1 \Rightarrow A_1$, and so $\mathcal{T}^* \vdash_{GF_\rightarrow} G$. A contradiction. □

III.3.8 Corollaries.

- *$\mathcal{T} \models_F A$ iff there exist $?_1, \ldots, ?_n \subseteq \mathcal{T}$ such that $\vdash_{GF_\rightarrow} ?_1 \Rightarrow A|?_2 \Rightarrow A|\cdots|?_n \Rightarrow A$*
- *If $?$ is finite then $? \models_F A$ iff $\vdash_{GF_\rightarrow} ? \Rightarrow A| \Rightarrow A$.*
- *A is valid in every implicational F.s. iff $\vdash_{GF_\rightarrow} \Rightarrow A$.*

Proof: By definition, $\Rightarrow A$ is true in every model of $\mathcal{T}$ iff $\mathcal{T} \models_F A$. Hence (1) is a special case of the last theorem. (3), in turn, is a special case of (1), when we take $\mathcal{T} = \emptyset$ (and use external contractions). (2) also easily follows from (1), using n applications of ww, followed by external contractions. For example, if $?_1, ?_2 \subseteq ?$ then from $?_1 \Rightarrow A|?_2 \Rightarrow A$ one can infer $? \Rightarrow A| \Rightarrow A|?_2 \Rightarrow A$ by ww, then $? \Rightarrow A| \Rightarrow A|? \Rightarrow A| \Rightarrow A$ by another application of ww. Then $? \Rightarrow A| \Rightarrow A$ follows, using external contractions and permutations. □

IV The Logics of Simple and of Flat F-Structures

Simple (implicational) F-structures have been defined and characterized in II.5 and II.8. We now show that the logic which corresponds to them has the same relations with $RM0_\rightarrow$ as

$\models_F$ has with $R_\to$ (note that the characteristic mingle axiom of $RM0_\to$ is obviously valid in simple F-structures).

IV.1 Definition. *$\models_{SF}$ is defined using simple F-structures exactly as $\models_F$ is defined using general F-structures (III.0.1).*

IV.2 Theorem. *$\mathcal{T} \models_{SF} \varphi$ iff there exist $A_1, \ldots, A_n$ $(n \geq 0)$ such that $\mathcal{T} \vdash_{RMO_\to} (\varphi \to A_1) \to ((\varphi \to A_2) \to (\cdots \to (\varphi \to A_n) \to \varphi)\cdots)$.*

Proof: Like that of III.1.2. It only remains to show that the F.s. which is constructed there is simple if we use $RM0_\to$ instead of $R_\to$. This is obvious because of the validity in it of the mingle axiom $A \to (A \to A)$. □

IV.3 Corollary. *$\models_{SF}$ is compact (or "finitary").*

IV.4 Theorem. *$\models_{SF}$ is decidable.*

Proof: From the proof of II.8 it follows that in any simple F.s., $a \to b \in \{\top, \bot, b\}$. It follows that if $p_1, \ldots, p_n$ are the atomic formulae in φ and v is a valuation in a simple F.s. $\overline{S} = \langle S, \leq, \to, \top, \bot \rangle$ then $v(\varphi) \in S' = \{\top, \bot, v(p_1), \ldots v(p_k)\}$. Moreover: S' together with the induced partial order and implication operator is a substructure of $\overline{S}$ (and so a simple F.s. itself). It follows that if $\not\models_{SF} \varphi$ then φ has a countermodel with at most $n+2$ elements. This implies decidability. □

Note. Since the deduction theorem for $\supset$ obtains for $\models_{SF}$, the last theorem means that the question whether $? \models_{SF} \varphi$ when ? is finite is also decidable.

Open Problem 5 Is $\models_F$ is decidable?

We turn now to the corresponding formal systems.

IV.5 Theorem. *By adding the mingle axiom to $HF_\to$ we get a sound and strongly complete axiomatization, $HSF_\to$, for $\models_{SF}$.*

Proof: Almost identical to that of III.3.2. The extra axiom is needed for making the structures built there simple. □

IV.6 Theorem.

1. *A sound and complete Gentzen-type system, $GSF_\to$, for $\models_{SF}$ is obtained if we add to $GF_\to$ the following hypersequential version of the mingle rule of $GRM0_\to$:*

$$\frac{G_1 | ?_1 \Rightarrow A | H_1 \qquad G_2 | ?_2 \Rightarrow A | H_2}{G_1 | G_2 | ?_1, ?_2 \Rightarrow A | H_1 | H_2}$$

2. *The cut elimination theorem obtains for $GSF_{\rightarrow}$.*

The proof of IV.6 is similar to the proofs in III.3., and we omit it. □

Unlike the case of $R_{\rightarrow}$, to $RM0_{\rightarrow}$ we can apply *both* methods of classical "completion". As we have just seen, the method of this paper gives $\models_{SF}$. On the other hand the method of passing to a multiple-conclusion Gentzen-type calculus leads to $RMI_{\rightarrow}$. What is the relation between these two logics? Well, since $\mathcal{A}_\omega$ (the characteristic matrix of $RMI_{\rightarrow}$) is simple, every valid formula of $\models_{SF}$ is a theorem of $RMI_{\rightarrow}$. The converse fails: $((B \rightarrow B) \rightarrow (A \rightarrow A)) \rightarrow (((A \rightarrow B) \rightarrow A) \rightarrow A)$ is an example of a formula which is valid in $\mathcal{A}_\omega$, but not in the simple F-structure $\mathcal{C}_\omega$ (or even $\mathcal{C}_2$). From the point of view of *theoremhood* $\models_{SF}$ is therefore weaker than $RMI_{\rightarrow}$. However, if we consider also the *consequence relation* we find that the two logics are not comparable: $(A \rightarrow B) \rightarrow A \models_{SF} A$ but this is not true for $\vdash_{RMI_{\rightarrow}}$.

What happens if we try to "complete" $RMI_{\rightarrow}$ itself by the method of this paper? It is easily seen that we get by this the logic of *flat* F-structures (see II.4), or (equivalently) the logic of the flat F-structure $\mathcal{A}_\omega$. Indeed, the logic of flat F-structures is based on $RMI_{\rightarrow}$ exactly as that of simple F-structures is based on $RM0_{\rightarrow}$, and that of general F-structures is based on $R_{\rightarrow}$. The connections in this case are, however, stronger:

- Unlike in the previous two cases, this time we do have at least *weak* completeness: a formula A is valid in every flat F-structure (equivalently: A is valid in $\mathcal{A}_\omega$) iff $\vdash_{RMI_{\rightarrow}} A$ (iff $\Rightarrow A$ is provable in $GRMI_{\rightarrow}$). This was shown already in [Av84]. We still do not have, however, *strong* completeness, since again $(A \rightarrow B) \rightarrow A \models_{\mathcal{A}_\omega} A$ but $A \rightarrow (B \rightarrow A) \not\vdash_{RMI_{\rightarrow}} A$.

- We have the following strengthening of theorems III.1.2 and IV.2:

IV.7 Theorem. *$\mathcal{T} \models_{\mathcal{A}_\omega} A$ iff there exists B such that $\mathcal{T} \vdash_{RMI_{\rightarrow}} (A \rightarrow B) \rightarrow A$.*

Proof: In [Av97] it is proved that in the full multiplicative language, $\mathcal{T} \models_{\mathcal{A}_\omega} A$ iff there exists B such that $\mathcal{T} \vdash_{RMI_m} A \otimes B$. Here B can be assumed to be a purely implicational formula (since if $p_1, \ldots, p_k$ are the atomic variables in B then $B \rightarrow ((p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_k) \rightarrow (p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_k))$ is valid in $\mathcal{A}_\omega$ and so provable in RMI_m). But $\vdash_{R_m} A \otimes B \rightarrow ((A \rightarrow (B \rightarrow A)) \rightarrow A)$. Hence if $\mathcal{T} \models_{\mathcal{A}_\omega} A$ then there is a purely implicational formula $B' = B \rightarrow A$ such that $\mathcal{T} \vdash_{RMI_m} (A \rightarrow B') \rightarrow A$. Since RMI_m is a conservative extension of $RMI_{\rightarrow}$ ([Av84]), the theorem follows (the other direction is trivial, since $(A \rightarrow B) \rightarrow A \models_F A$). □

IV.8 Corollary. *By adding to $RMI_{\rightarrow}$ either the rule $\frac{(A \rightarrow B) \rightarrow A}{A}$ or the rule $\frac{A \quad A \supset B}{B}$ we get a strongly complete axiomatization of the logic of implicational flat F-structures.*

Proof: This is immediate from IV.7 and the fact that $((A \rightarrow B) \rightarrow A) \supset A$ is valid in $\mathcal{A}_\omega$, and so provable in $RMI_{\rightarrow}$. □

Note. Everything we have said concerning $RMI_{\rightarrow}$ and $\mathcal{A}_\omega$ holds also for $RM_{\rightarrow}$ and $\mathcal{A}_1$. In particular, $\models_{\mathcal{A}_1}$ is the "completion" of $RM_{\rightarrow}$, and a corresponding cut-free Gentzen-type system is obtained by adding to GA_ω the hypersequential version of the mix rule (see [Av97], [Av87]).

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