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Multi-valued Semantics: Why and How

Abstract. According to Suszko's Thesis, any multi-valued semantics for a logical system can be replaced by an equivalent bivalent one. Moreover: bivalent semantics for families of logics can frequently be developed in a modular way. On the other hand bivalent semantics usually lacks the crucial property of analycity, a property which is guaranteed for the semantics of multi-valued matrices. We show that one can get both modularity and analycity by using the semantic framework of multi-valued non-deterministic matrices. We further show that for using this framework in a constructive way it is best to view "truth-values" as information carriers, or "information-values".

Keywords: Multi-valued semantics, non-deterministic matrices, Suszko's Thesis, many-valued logics, analycity, modularity.

1. Introduction

In [19] Suszko argued that there are really just two *logical* truth-values: true and false, and all other "truth-values" which are employed in (so-called) "many-valued logics" are just algebraic values, used for providing purely algebraic semantics for those logics. This claim is now known as *Suszko's Thesis*, and it got formal mathematical support by what is known as *Suszko's Reduction* (see Section 2). [1]

Suszko's Thesis has extensively been discussed in the literature (see e.g. [14, 11, 20, 8, 23]), especially from the philosophical point of view. The goal of this paper is to examine it from the *practical* point of view. The main question we try to answer is: "What can be gained by using a multi-valued semantic framework rather than only a bivalent one?". Our main answer as given in Section 3 (and the first main point this paper makes) is that the use of multi-valued matrices (i.e. algebraic structures) provides semantics which guarantees the crucial property of *analycity* [2] — a property that bivalent semantics usually lacks. On the other hand the strict algebraic framework has the disadvantage of being inflexible: finding a characteristic matrix for a logic in some family of logics usually does not help much in finding such a

[1] Bivalent interpretations of many-valued logics can also be found, e.g., in [21, 16, 11].

[2] For particular cases this property was discussed, most of the times under the name *effectivity*, in several previous papers on non-deterministic semantics. However, in this paper it is the first time it gets a general, precise definition.

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matrix for another (even closely related) member of that family.[3] The second main point of this paper (made in Section 4) is that this shortcoming of matrices-based semantics can be overcome (without losing the advantages of this framework, including analycity) by using multi-valued *non-deterministic* matrices (Nmatrices). We demonstrate by way of a representative example the great usefulness of this framework, in particular the possibility it opens to construct effective semantics for complex families of logics in a systematic and *modular* way. Now it turned out that for using Nmatrices in the way just described, one should view the "truth-values" they employ neither as logical values, nor as algebraic values (because in Nmatrices the interpretations of the connectives are not algebraic operations on the set of "truth-values"). Rather it is best to view them as information carriers, or "information-values". This third main point made by this paper is the subject of Section 5. The philosophical import of this view (if any) is left for future discussions.

2. Suszko's Thesis

We start with a short presentation of Suszko's Thesis, and its implications to the notion of a "Truth-value".

In what follows $\mathcal{L}$ is a propositional language, $\mathcal{F}(\mathcal{L})$ is its set of wffs, p, q, r denote atomic formulas, ψ, φ, θ denote arbitrary formulas (of $\mathcal{L}$), and T, S denote sets of formulas. $Fv(X)$ denotes the set of atomic formulas occurring in X. By a *substitution in* $\mathcal{L}$ we shall mean a function σ from $\mathcal{F}(\mathcal{L})$ to $\mathcal{F}(\mathcal{L})$ such that $\sigma(\diamond(\varphi_1, \ldots, \varphi_n)) = \diamond(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and formulas $\varphi_1, \ldots, \varphi_n$ of $\mathcal{F}(\mathcal{L})$.

2.1. The Fundamental Concepts

DEFINITION 1. A *(Tarskian) consequence relation* (*tcr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ and formulas of $\mathcal{L}$ that satisfies the following conditions:

Reflexivity: if $\varphi \in T$ then $T \vdash \varphi$.
Monotonicity: if $T \vdash \varphi$ and $T \subseteq T'$ then $T' \vdash \varphi$.
Transitivity: if $T \vdash \psi$ and $T, \psi \vdash \varphi$ then $T \vdash \varphi$.

[3] Thus it was shown in [7] that by deleting any rule from the standard Gentzen-type system for Classical Logic (which of course has a two-valued characteristic matrix) we get a logic which has *no* finite characteristic matrix.

DEFINITION 2. A propositional logic is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a tcr on $\mathcal{L}$ which satisfies the following: [4]

Structurality:	If $T \vdash \varphi$ then $\sigma(T) \vdash \sigma(\varphi)$, where σ is a substitution in $\mathcal{L}$.
Non-triviality:	$p \not\vdash q$, (where p and q are distinct atomic formulas).

DEFINITION 3.

- A *denotational semantics* for $\mathcal{L}$ is a pair $\mathcal{SEM} = \langle SEM, \models_{\mathcal{SEM}} \rangle$, where SEM is a nonempty set, and $\models_{\mathcal{SEM}} \subseteq SEM \times \mathcal{F}(\mathcal{L})$. If $v \models_{\mathcal{SEM}} \varphi$ (where $v \in SEM$ and $\varphi \in \mathcal{F}(\mathcal{L})$) then v is called a $\mathcal{SEM}$-*model* of φ.
- $\vdash_{\mathcal{SEM}}$, the tcr on $\mathcal{L}$ associated with the denotational semantics $\mathcal{SEM} = \langle SEM, \models_{\mathcal{SEM}} \rangle$, is defined as follows:
  $T \vdash_{\mathcal{SEM}} \varphi$ if every $\mathcal{SEM}$-model of T is also a $\mathcal{SEM}$-model of φ.

It is easy to see that for any denotational semantics $\mathcal{SEM}$, $\vdash_{\mathcal{SEM}}$ is indeed a tcr on $\mathcal{L}$. However, $\langle \mathcal{L}, \vdash_{\mathcal{SEM}} \rangle$ is not necessarily a *logic*. These observations lead to the following definition:

DEFINITION 4. A denotational semantics $\mathcal{SEM}$ for $\mathcal{L}$ is *logical* if $\langle \mathcal{L}, \vdash_{\mathcal{SEM}} \rangle$ is a logic (i.e. the conditions of structurality and non-triviality are satisfied).

Here are two well-known Examples:

Possible worlds semantics for modal logics: Here SEM is taken to be a nonempty collection of quadruples $\langle W, R, \Vdash, a \rangle$, where $W \neq \emptyset$ is a set, R (the "accessibility" relation) is a binary relation on W satisfying a certain set of conditions (which varies from one modal logic to another), $a \in W$, and $\Vdash$ (the "forcing" relation) is a relation from W to $\mathcal{F}(\mathcal{L})$ which satisfies the usual conditions on Kripke frames for modal logics (in particular: $a \Vdash \Box\varphi$ iff $b \Vdash \varphi$ for every b such that aRb). $\models_{\mathcal{SEM}}$ is defined by: $\langle W, R, \Vdash, a \rangle \models_{\mathcal{SEM}} \varphi$ iff $a \Vdash \varphi$. The resulting $\vdash_{\mathcal{SEM}}$ is the "truth" consequence relation of the corresponding modal logic. If we take instead SEM to be the set of the corresponding triples $\langle W, R, \Vdash \rangle$, and let $\langle W, R, \Vdash \rangle \models_{\mathcal{SEM}} \varphi$ iff $a \Vdash \varphi$ for every $a \in W$, then we get the "validity" consequence relation of that logic[5]. Obviously, both types of semantics are logical.

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[4]The condition of non-triviality (which was also called *consistency* in previous papers) is usually not included in the definition of a logic. However, it rules out from the class of "logics in $\mathcal{L}$" just two trivial logics (see [7]), and it is technically convenient to include it.

[5]See [1] for the differences between the "truth" and "validity" consequence relations of modal logics.

Matrices: A *matrix* for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set (of "truth values"), $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (its "designated values"), and for every n-ary ($n \geq 0$) connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $\mathcal{V}$. A (legal) $\mathcal{M}$-assignment is a function v from $\mathcal{F}(\mathcal{L})$ to $\mathcal{V}$ such that:

$$v(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

Such v *satisfies* a formula ψ in $\mathcal{M}$ ($v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$.

Given a matrix $\mathcal{M}$, the corresponding semantics is obtained by taking SEM to be the set of $\mathcal{M}$-assignments, and $\models_{\mathcal{SEM}}$ to be $\models^{\mathcal{M}}$. This can easily be generalized for defining the semantics that corresponds to a collection of matrices. It is easy to see that the semantics induced by any collection of matrices is logical.

The semantics of matrices is the paradigmatic example of a denotational semantics (in the sense of Definition 3). Thus even the possible worlds semantics for modal logics described above can be viewed as the semantics induced by a set of matrices of the form $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is the set of subsets of some nonempty set W, and $\mathcal{D}$ is either $\{W\}$ (in the case of the "validity" consequence relation), or $\{X \subseteq W \mid a \in X\}$ for some chosen element a of W (in the case of the "truth" consequence relation).[6] Theoretically, this observation is just an instance of the general principle that matrices-based semantics can replace any other type of semantics. This is due to the following theorems:

THEOREM 1. [24, 25] *Every logic is induced by some set of matrices.*

THEOREM 2. [13, 24, 25, 22] *A logic $\langle \mathcal{L}, \vdash \rangle$ has a single characteristic matrix iff it satisfies the following condition of* uniformity*: if $T \cup S \vdash \varphi$ and S is the union of $\vdash$-consistent sets that have no atomic formulas in common with one another or with T or φ, then $T \vdash \varphi$ (a set T of formulas is $\vdash$-consistent if there exists a formula ψ such that $T \not\vdash \psi$).*

Next we turn to the case where only two truth-valued are used:

DEFINITION 5. A semantics $\mathcal{SEM} = \langle SEM, \models_{\mathcal{SEM}} \rangle$ for $\mathcal{L}$ is *bivalent* if $\mathcal{SEM}$ is a set of functions from $\mathcal{F}(\mathcal{L})$ to $\{\mathbf{t}, \mathbf{f}\}$, and $v \models_{\mathcal{SEM}} \varphi$ iff $v(\varphi) = \mathbf{t}$.

Here is a nontrivial example of the use of a bivalent semantics which will serve us in this paper as a running example:

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[6] In both cases the operations in $\mathcal{O}$ are determined using an accessibility relation R on W, but this is unimportant for us here.

EXAMPLE 1. A *paraconsistent* logic is a logic which allows nontrivial inconsistent theories. There are several approaches to the problem of designing a useful paraconsistent logic. One of the oldest and best known is da Costa's approach ([10, 12]), which has led to the family of LFIs (Logics of Formal (In)consistency — see [9]). Now logical bivalent semantics has been the main semantic tool used in the study of LFIs (see e.g. [12, 9], where such semantics is called *bivaluation semantics*). We describe now as an example the bivaluation semantics of one of the most basic LFIs. This is Marcos' system **mCi**, to which the whole of section 4 of [9] is devoted. **mCi** is the logic in the language $\mathcal{L}_C = \{\wedge, \vee, \supset, \neg, \circ\}$ which is obtained from the positive fragment of classical logic (in $\mathcal{L}_C$) by adding the schemata:

(t) $\neg\varphi \vee \varphi$

(p) $\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi)$

(i) $\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)$

(cc) $\circ\neg^n\circ\varphi$ (for every $n \geq 0$

Now the bivalent semantics of **mCi** given in [9] (for which **mCi** is sound and complete) consists of the set of **mCi**-valuations, where an **mCi**-valuation is a function v from $\mathcal{F}(\mathcal{L}_C)$ to $\{\mathbf{t}, \mathbf{f}\}$ which satisfies the following conditions:

1. $v(\varphi \wedge \psi) = \mathbf{t}$ iff $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{t}$.
2. $v(\varphi \vee \psi) = \mathbf{t}$ iff $v(\varphi) = \mathbf{t}$ or $v(\psi) = \mathbf{t}$.
3. $v(\varphi \supset \psi) = \mathbf{t}$ iff $v(\varphi) = \mathbf{f}$ or $v(\psi) = \mathbf{t}$.
4. If $v(\varphi) = \mathbf{f}$ then $v(\neg\varphi) = \mathbf{t}$.
5. If $v(\varphi) = \mathbf{t}$ and $v(\neg\varphi) = \mathbf{t}$ then $v(\circ\varphi) = \mathbf{f}$.
6. If $v(\varphi) = \mathbf{f}$ or $v(\neg\varphi) = \mathbf{f}$ then $v(\neg\circ\varphi) = \mathbf{f}$.
7. $v(\circ\neg^n\circ\varphi) = \mathbf{t}$ for every $n \geq 0$.

2.2. The Two-valued Reduction

Let $\mathcal{SEM} = \langle SEM, \models_{\mathcal{SEM}}\rangle$ be a semantics for $\mathcal{L}$. For $m \in SEM$ define:

$$v^m(\varphi) = \begin{cases} \mathbf{t} & m \models_{\mathcal{SEM}} \varphi \\ \mathbf{f} & m \not\models_{\mathcal{SEM}} \varphi \end{cases}$$

Let $\mathcal{SEM}^* = \langle SEM^*, \models_{\mathcal{SEM}^*}\rangle$, where $SEM^* = \{v^m \mid m \in SEM\}$, and $v \models_{\mathcal{SEM}^*} \varphi$ iff $v(\varphi) = \mathbf{t}$. Obviously, $\mathcal{T} \vdash_{\mathcal{SEM}^*} \varphi \quad \Leftrightarrow \quad \mathcal{T} \vdash_{\mathcal{SEM}} \varphi$. This observation naturally leads to the following proposition:

PROPOSITION 1.

1. *A bivalent semantics $\langle SEM, \models_{\mathcal{SEM}}\rangle$ is logical if it satisfies the following conditions:*

   Structurality: *If $v \in SEM$ and σ is a substitution then $v \circ \sigma \in SEM$.*

   Non-triviality: *If p and q are distinct atomic formulas then there exists $v \in SEM$ such that $v(p) = \mathbf{t}$ and $v(q) = \mathbf{f}$.*

2. *For every logical semantics $\mathcal{SEM}$ there exists a logical bivalent semantics $\mathcal{SEM}^*$ such that $\vdash_{\mathcal{SEM}^*} = \vdash_{\mathcal{SEM}}$.*

It follows that in principle, the use of $\mathbf{t}$ and $\mathbf{f}$ suffices for all semantic purposes. This fact provides the basis for *Suszko's Thesis*, according to which $\mathbf{t}$ and $\mathbf{f}$ are the only real *logical truth-values*.

Suszko's Thesis might (or might not) be significant from a philosophical point of view. However, in this paper we are interested in its practical import. As stated in the introduction, the question we want to answer is: Given the second part of Proposition 1, *what can we gain by using semantics based on more than two truth-values?* Now one answer to this question is obvious already at this stage. The consequence relation which is induced by any matrix (or a family of matrices) is *guaranteed* to be logical. This is not true for arbitrary bivalent semantics. When such semantics is introduced directly (i.e. not via Suszko's reduction), one has to verify the conditions given in the first part of Proposition 1. This might not always be easy. Still, in practice in most cases it is not too difficult either . However, in the next section we discuss another crucial semantic property that the use of matrices provides for free, but its verification for a given bivalent semantics (which is not truth-functional) might be hard, or even impossible.

3. Analycity

Let us summarize the general principles that underlies a semantics which is based on the use of truth-values (of which the general bivalent semantics described in Section 2 is a special case):

- A semantics is a collection of "legal" assignments, where an assignment is a function from the set of formulas to some set of "truth-values".

- Some of the truth-values (but not all) are taken as "designated".
- A model of φ is a legal assignment v such that $v(\varphi)$ is designated.
- φ follows from a theory $\mathcal{T}$ iff every model of $\mathcal{T}$ is also a model of φ.

Now in practice we do not really want or need to assign truth values to *all* formulas in order to decide whether φ follows from $\mathcal{T}$. Partial assignments, which associate truth-values only to relevant formulas, should suffice (the method of truth-tables used in classical logic is a good example). But what are the "relevant formulas"? The *analycity principle* gives a very simple and natural answer to this question: it demands that only formulas which are subformulas of formulas in $\mathcal{T} \cup \{\varphi\}$ should be relevant to the question whether φ follows from $\mathcal{T}$. Our next task is to translate this vague idea into precise mathematical definitions.

Definition 6. A subset $\mathcal{F}$ of $\mathcal{F}(\mathcal{L})$ is *closed* if it is closed under subformulas (i.e. $\psi_1, \ldots, \psi_n \in \mathcal{F}$ whenever $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$).

Definition 7. A generalized semantics for $\mathcal{L}$ is a nonempty set $\mathcal{SEM}$ of quadruples which satisfies the following conditions:

1. If $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v \rangle \in \mathcal{SEM}$ then:
   - $\mathcal{V}$ is a nonempty set, and $\mathcal{D}$ is a proper nonempty subset of $\mathcal{V}$.
   - $\mathcal{F}$ is a closed subset of $\mathcal{F}(\mathcal{L})$.
   - v is a function from $\mathcal{F}$ to $\mathcal{V}$.
2. If $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v \rangle \in \mathcal{SEM}$, $\mathcal{A}$ is a set of atomic formulas, and a is a function from $\mathcal{A}$ to $\mathcal{V}$, then $\langle \mathcal{V}, \mathcal{D}, \mathcal{A}, a \rangle \in \mathcal{SEM}$.
3. If $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v \rangle \in \mathcal{SEM}$, $\mathcal{F}^*$ is a closed subset of $\mathcal{F}(\mathcal{L})$, and σ is an $\mathcal{L}$-substitution such that $\sigma(\varphi) \in \mathcal{F}$ for every $\varphi \in \mathcal{F}^*$, then $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}^*, v^* \rangle$ is in $\mathcal{SEM}$, where $v^*(\varphi) = v(\sigma(\varphi))$ for every $\varphi \in \mathcal{F}^*$ (i.e. $v^* = v \circ \sigma$). In particular: if $\mathcal{F}^* \subseteq \mathcal{F}$ then $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}^*, v \restriction \mathcal{F}^* \rangle \in \mathcal{SEM}$.

Definition 8. Let S be a set of matrices for $\mathcal{L}$. The *generalized semantics induced by* S is the set of all quadruples $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v \rangle$ in which $\mathcal{F}$ is closed, and there exists an element $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle \in S$ such that $v(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ whenever $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$ (where $\widetilde{\diamond}$ is the interpretation of $\diamond$ in $\mathcal{O}$).

Definition 9. A generalized semantics $\mathcal{SEM}$ is *bivalent* if $\mathcal{V} = \{\mathbf{t}, \mathbf{f}\}$ and $\mathcal{D} = \{\mathbf{t}\}$ for every $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v \rangle \in \mathcal{SEM}$.

Definition 10. Let $\mathcal{SEM}$ be a generalized semantics for $\mathcal{L}$, and let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v\rangle$ be an element of $\mathcal{SEM}$.

1. $\mathcal{M}$ is an $\mathcal{F}$-*model* in $\mathcal{SEM}$ of a formula φ (notation: $\mathcal{M} \models^{\mathcal{F}}_{\mathcal{SEM}} \varphi$) if $\varphi \in \mathcal{F}$ and $v(\varphi) \in \mathcal{D}$.

2. $\mathcal{M}$ is an $\mathcal{F}$-*model* of a theory $\mathcal{T}$ ($\mathcal{M} \models^{\mathcal{F}}_{\mathcal{SEM}} \mathcal{T}$) if it is an $\mathcal{F}$-*model* of every $\varphi \in \mathcal{T}$.

Definition 11. Let $\mathcal{SEM}$ be a generalized semantics for $\mathcal{L}$.

1. Assume that $\mathcal{F}$ is a closed subset of $\mathcal{F}(\mathcal{L})$, and that $\mathcal{T} \cup \{\varphi\} \subseteq \mathcal{F}$. Then φ $\mathcal{F}$-follows from $\mathcal{T}$ in $\mathcal{SEM}$ ($\mathcal{T} \vdash^{\mathcal{F}}_{\mathcal{SEM}} \varphi$) if $\mathcal{M} \models^{\mathcal{F}}_{\mathcal{SEM}} \varphi$ whenever $\mathcal{M} \models^{\mathcal{F}}_{\mathcal{SEM}} \mathcal{T}$.

2. φ follows from $\mathcal{T}$ in $\mathcal{SEM}$ ($\mathcal{T} \vdash_{\mathcal{SEM}} \varphi$) if $\mathcal{T} \vdash^{\mathcal{F}(\mathcal{L})}_{\mathcal{SEM}} \varphi$.

Proposition 2. *If $\mathcal{SEM}$ is a generalized semantics for $\mathcal{L}$ then $\langle \mathcal{L}, \vdash_{\mathcal{SEM}}\rangle$ is a logic.*

Definition 12. Let $\mathcal{SEM}$ be a generalized semantics for $\mathcal{L}$.

1. $\mathcal{SEM}$ is called *analytic* if for every quadruple $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v\rangle \in \mathcal{SEM}$ there exists v^* such that $v = v^* \restriction \mathcal{F}$ and $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}(\mathcal{L}), v^*\rangle \in \mathcal{SEM}$ (in other words: every partial valuation in $\mathcal{SEM}$ can be extended to a total valuation in $\mathcal{SEM}$).

2. $\mathcal{SEM}$ is called *uniform* if $\langle \bigcup_{i\in I} \mathcal{V}_i, \bigcup_{i\in I} \mathcal{D}_i, \bigcup_{i\in I} \mathcal{F}_i, \bigcup_{i\in I} v_i\rangle \in \mathcal{SEM}$ whenever $\{\langle \mathcal{V}_i, \mathcal{D}_i, \mathcal{F}_i, v_i\rangle \mid i \in I\} \subseteq \mathcal{SEM}$, and $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ for every $i, j \in I$ such that $i \neq j$ (here each v_i is taken as a set of ordered pairs).

What makes analycity crucial are the following proposition and its corollary:

Proposition 3. *Let $\mathcal{SEM}$ be an analytic generalized semantics for $\mathcal{L}$. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are closed subsets of $\mathcal{F}(\mathcal{L})$, and $\mathcal{T} \cup \{\varphi\} \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$. Then $\mathcal{T} \vdash^{\mathcal{F}_1}_{\mathcal{SEM}} \varphi$ iff $\mathcal{T} \vdash^{\mathcal{F}_2}_{\mathcal{SEM}} \varphi$.*

Proof. Suppose e.g. that $\mathcal{T} \vdash^{\mathcal{F}_2}_{\mathcal{SEM}} \varphi$. Let $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}_1, v\rangle$ be an $\mathcal{F}_1$-model of $\mathcal{T}$. By analycity there exists v^* such that $v = v^* \restriction \mathcal{F}_1$ and $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}(\mathcal{L}), v^*\rangle$ is in $\mathcal{SEM}$. Now $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}_2, v^*\restriction\mathcal{F}_2\rangle$ is in $\mathcal{SEM}$, and since $\mathcal{T} \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$, it is an $\mathcal{F}_2$-model of $\mathcal{T}$. Hence it is also an $\mathcal{F}_2$-model of φ, so $(v^*\restriction\mathcal{F}_2)(\varphi) \in \mathcal{D}$. Since $v(\varphi) = v^*(\varphi) = (v^*\restriction\mathcal{F}_2)(\varphi)$, $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}_1, v\rangle$ is an $\mathcal{F}_1$-model of φ. It follows that $\mathcal{T} \vdash^{\mathcal{F}_1}_{\mathcal{SEM}} \varphi$. The proof of the converse is identical. ■

COROLLARY 1. *Let $\mathcal{SEM}$ be an analytic generalized semantics for $\mathcal{L}$. Then $\mathcal{T} \vdash_{\mathcal{SEM}} \varphi$ iff $\mathcal{T} \vdash^{\mathcal{F}}_{\mathcal{SEM}} \varphi$ for every $\mathcal{F}$ such that $\mathcal{T} \cup \{\varphi\} \subseteq \mathcal{F}$, iff $\mathcal{T} \vdash^{\mathcal{F}}_{\mathcal{SEM}} \varphi$ for some $\mathcal{F}$ such that $\mathcal{T} \cup \{\varphi\} \subseteq \mathcal{F}$, iff $\mathcal{T} \vdash^{\mathcal{F}_0}_{\mathcal{SEM}} \varphi$, where $\mathcal{F}_0$ is the set of subformulas of $\mathcal{T} \cup \{\varphi\}$.*

Note. It follows that for an analytic generalized semantics $\mathcal{SEM}$ for $\mathcal{L}$, the question whether φ follows from $\mathcal{T}$ according to $\mathcal{SEM}$ does not really depend on the exact identity of $\mathcal{L}$. The only relevant connectives (and atomic formulas) are those which occur in $\mathcal{T} \cup \{\varphi\}$. This property of consequence relations induced by analytic semantics is so natural and essential, that it is usually taken for granted. Thus when it is claimed that φ follows from $\neg\neg\varphi$ in classical logic, but not in intuitionistic logic, nobody bothers to ask relative to what set of connectives do these facts hold!

Another important consequence of analycity, that easily follows from the last corollary, is that the logic induced by an analytic and uniform (Definition 12) generalized semantics is necessarily uniform too. Therefore by Theorem 2 such a logic has a single (usually infinite) characteristic matrix.

One more important property that frequently follows from analycity is *decidability*: if for every finite closed set $\mathcal{F}$ there is only a finite number of quadruples $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v \rangle$ in $\mathcal{SEM}$, and they can all effectively be found, then analycity implies decidability (this is of course what happens in the special case where $\mathcal{SEM}$ is induced by a finite set of finite matrices — and also when it is induced by a finite set of finite Nmatrices. See next section).

Now given a logical bivalent semantics $\mathcal{SEM}$ it is easy to derive from it an equivalent analytic bivalent generalized semantics $\mathcal{SEM}^*$ by taking the latter to be $\{\langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{F}, v \restriction \mathcal{F} \rangle \mid v \in SEM, \mathcal{F} \text{ closed}\}$. However, the analycity of this bivalent generalized semantics is useless, unless we can directly and effectively characterize what quadruples $\{\langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{F}, v \rangle$ are in it (without having to check whether or not v is the restriction to $\mathcal{F}$ of some total valuation on $\mathcal{F}(\mathcal{L})$). The existence of such a characterization is not guaranteed for the bivalent reductions described in the previous subsection, even if the original semantics $\mathcal{SEM}$ has a simple description. Showing that a plausible candidate does the job may be non-trivial (or simply false).

EXAMPLE 2. The natural generalized bivalent semantics which corresponds to the bivalent semantics of **mCi** presented in Example 1 consists of all the quadruples $\{\langle \{\mathbf{t}, \mathbf{f}\}, \{\mathbf{t}\}, \mathcal{F}, v \rangle$, where $\mathcal{F}$ is a closed subset of $\mathcal{F}(\mathcal{L}_C)$, and v is a function from $\mathcal{F}$ to $\{\mathbf{t}, \mathbf{f}\}$ which satisfies conditions 1.– 7. above, *provided all the relevant formulas are in $\mathcal{F}$*. (For example, condition 1. should be satisfied in case $\varphi \wedge \psi \in \mathcal{F}$). But is this generalized semantics analytic?

No. it isn't (we show this in a moment). Instead, some efforts are devoted in [9] to prove a *special case* of analycity (Lemma 89 of [9]), and then the *proof* of that lemma (the lemma itself is not sufficient!) is used to show that $\neg\neg p \supset p$ is not a theorem of **mCi**. This is a good example of the crucial importance of analycity: The proof of this fact in [9] basically refutes $\neg\neg p \supset p$ by taking $\mathcal{F}$ to be the closed set $\{p, \neg p, \neg\neg p, \neg\neg p \supset p\}$, and defining $v_0 : \mathcal{F} \to \{\mathbf{t}, \mathbf{f}\}$ by $v_0(p) = \mathbf{f}$, $v_0(\neg p) = v_0(\neg\neg p) = \mathbf{t}, v_0(\neg\neg p \supset p) = \mathbf{f}$. The difficult part of the proof is then to show that v_0 can be extended to an **mCi**-valuation. This would have been immediate from general analycity. However, general analycity is not available here, and so a special (relatively complicated) proof is provided for the particular case of v_0. To understand why this was absolutely needed, we "prove" by the same method that $p \vee \circ p$ is not a theorem of **mCi** (although actually it is). For this we take this time $\mathcal{F}$ to be the closed set $\{p, \circ p, p \vee \circ p\}$, and define $v_1 : \mathcal{F} \to \{\mathbf{t}, \mathbf{f}\}$ by $v_1(p) = v_1(\circ p) = v_1(p \vee \circ p) = \mathbf{f}$. The problem is that although v_1 too is a legal *partial* **mCi**-valuation, unlike v_0 it cannot be extended to a total **mCi**-valuation (because condition 4 and condition 6 in Example 1 impose conflicting constraints on the truth-value that such an extension should assign to $\neg\circ p$). [7]

In contrast to the situation described above concerning general bivalent semantics, for a matrices-based semantics we practically *get analycity for free*, because the following proposition is very easy to prove:

PROPOSITION 4. *The generalized semantics induced by a set of matrices (see Definition 8) is always analytic. The generalized semantics induced by a single matrix is always analytic and uniform.*

Proposition 4 provides the most significant advantage of matrices-based semantics over bivaluation semantics. Thus unlike the bivalent semantics for **mCi** presented above, the semantics of matrices that exists for it according to Theorem 1 is analytic.

Note. Proposition 4 will be strengthened considerably in the next section (see Proposition 5).

[7] The fact that the bivaluation semantics of **mCi** is not analytic should have been the reason why no attempt to use it for providing a decision procedure for **mCi** has been made in [9]. In fact, I do not see how this semantics can possibly be used for this goal, even though **mCi** *is* decidable (see next section).

4. Modularity

Looking again at Example 1, it should be admitted that despite the problematic aspect of analycity, in the case of **mCi** the bivalent semantic approach still seems to be superior to the multi-valued approach. Its obvious big advantage is simply that we have an effective characterization of the bivalent semantics for **mCi** that can be used (and has indeed been used in [9]) to show various facts about the system. We do not have such a characterization for the characteristic matrix for **mCi** that exists by Theorem 2 (and the existence of an adequate analytic and uniform generalized semantics for **mCi**, to be described in the next section). The proof of that theorem does not give a clue how to effectively construct such a matrix. In general, finding *effective* characteristic matrices for a given system (if one exists at all...) is a difficult task that frequently demands creativity and inspiration. In contrast, it is rather easy to understand how the bivaluation semantics of **mCi** has been discovered. The obvious main feature of this semantics is its *modularity*. The various conditions on an **mCi**-valuation correspond in a straightforward way to the axioms of the system. Thus condition 6 is clearly imposed by axiom (**i**), while condition 7 is imposed by axiom (**cc**). Now when a semantics for a system is constructed in a modular way, it is rather easy to change it when changes are made to the system. For example: let **mCic**, **mCie**, and **mCice** be obtained from **mCi** by respectively adding to it the schema **(c)** $\neg\neg\varphi \supset \varphi$, the schema **(e)** $\varphi \supset \neg\neg\varphi$, and both. Then bivalent semantics for these systems are easily obtained from that of **mCi** by letting an **mCic**-valuation be an **mCi**-valuation v such that $v(\neg\neg\varphi) = 0$ whenever $v(\varphi) = 0$, an **mCie**-valuation be an **mCi**-valuation v such that $v(\neg\neg\varphi) = 1$ whenever $v(\varphi) = 1$, and **mCice**-valuation be an **mCi**-valuation v satisfying both conditions. It is then a straightforward matter to convert the soundness and completeness proof for **mCi** given in [9] into corresponding soundness and completeness proofs for **mCic** , **mCie**, and **mCice**.

To sum up: the bivalent semantics of **mCi** and similar systems can be found in a modular and straightforward way. The proof of soundness and completeness is usually also rather straightforward (and can be done modularly and simultaneously for many systems). In fact, one might feel that everything here is *too* straightforward. And indeed: as noted in the previous section, there is a price to pay. The bivalent semantics obtained by this method is not necessarily analytic, and its proof is usually the most difficult part of the enterprise in the lucky case where it is.

Is there a semantic method which enjoys both the analycity guaranteed by the semantics of matrices and the modularity provided by the method of bivaluations? Yes. There is. It is provided by the the following generalization given in [7, 2, 3] to the concept of a matrix:

DEFINITION 13.

1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

   (a) $\mathcal{V}$ is a non-empty set of *truth values.*

   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.

   (c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.

2. A *(legal) $\mathcal{M}$-assignment* in an Nmatrix $\mathcal{M}$ is a function from $\mathcal{F}(\mathcal{L})$ to $\mathcal{V}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

3. An $\mathcal{M}$-assignment v (in an Nmatrix $\mathcal{M}$) is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$.

4. Given an Nmatrix $\mathcal{M}$, the corresponding semantics is obtained by taking SEM to be the set of $\mathcal{M}$-assignments, and $\models_{\mathcal{SEM}}$ to be $\models^{\mathcal{M}}$. This can easily be generalized to define the semantics that corresponds to a collection of Nmatrices. Like in the case of ordinary matrices, the semantics induced by a collection of Nmatrices is obviously logical. [8]

5. Let S be a set of Nmatrices for $\mathcal{L}$. The *generalized semantics induced by S* is the set of all quadruples $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v \rangle$ in which $\mathcal{F}$ is closed, and there exists an element $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle \in S$ such that $v(\diamond(\psi_1, \ldots, \psi_n))$ is in $\widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ whenever $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$ (where $\widetilde{\diamond}$ is the interpretation of $\diamond$ in $\mathcal{O}$).

Every ordinary (deterministic) matrix can be viewed as an Nmatrix in which the interpretations of the connectives always return singletons. Hence the semantics of Nmatrices is a generalization of the semantics of Matrices.

[8] In [23] a multi-valued semantics is called "structural" if its valuations are truth-functional. This terminology is unfortunate, because the semantics of legal valuations in some Nmatrix is not "structural" according to it, even though the consequence relation it induces *is* structural.

This generalization enjoys all the advantages of the more restricted semantics of ordinary matrices. Thus a logic with a finite characteristic Nmatrix is decidable, finitary ([2])[9], and there is ([6]) a uniform method for constructing cut-free calculus of n-sequents for it (where n is the number of truth-values of the Nmatrix). For our purposes here the most important property of this semantics is given in the following obvious generalization of Proposition 4:

PROPOSITION 5. *The generalized semantics induced by a set of Nmatrices is always analytic. The generalized semantics induced by a single Nmatrix is analytic and uniform.*

PROOF. The proof of the first part is almost identical to the proof in the case of ordinary matrices. So let S be a set of Nmatrices for $\mathcal{L}$, and let $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}, v \rangle$ be an element of the generalized semantics $\mathcal{SEM}$ induced by S. Then there is an element $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle \in S$ such that $v(\diamond(\psi_1, \ldots, \psi_n))$ is in $\widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ whenever $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}$ (where $\widetilde{\diamond}$ is the interpretation of $\diamond$ in $\mathcal{O}$). We extend v to a valuation v^* from $\mathcal{F}(\mathcal{L})$ to $\mathcal{V}$ such that v^* is a legal $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$-valuation (and so $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}(\mathcal{L}), v^* \rangle \in \mathcal{SEM}$). v^* is defined by induction on the complexity of formulas as follows. For p atomic we let $v^*(p) = v(p)$ if $p \in \mathcal{F}$, and $v^*(p) = a$ (where a is some fixed element of $\mathcal{V}$) otherwise. For $\varphi = \diamond(\psi_1, \ldots, \psi_n)$ we let $v^*(\varphi)$ be $v(\varphi)$ in case $\varphi \in \mathcal{F}$, and some element of the nonempty set $\widetilde{\diamond}(v^*(\psi_1), \ldots, v^*(\psi_n))$ otherwise. In the second case $v^*(\varphi)$ is in $\widetilde{\diamond}(v^*(\psi_1), \ldots, v^*(\psi_n))$ by definition. In the first case $\psi_1, \ldots, \psi_n$ are all in $\mathcal{F}$ (since $\mathcal{F}$ is closed). Hence by induction hypothesis $v^*(\psi_i) = v(\psi_i)$ $(1 \leq i \leq n)$, and so again $v^*(\varphi)$ is in $\widetilde{\diamond}(v^*(\psi_1), \ldots, v^*(\psi_n))$ by our assumption on v. Hence v^* is indeed a legal $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$-valuation which extends v.

That the generalized semantics induced by a single Nmatrix is analytic is immediate from the relevant definitions. ■

On the other hand the freedom of choices which may be allowed in the framework of Nmatrices makes it much more flexible than the framework of ordinary (deterministic) matrices. Like in the case of bivaluations, this opens the door for constructing semantics of systems in a modular way.

EXAMPLE 3. Let us return to the system **mCi** from Example 1. It follows from results in [4] that **mCi** does not have a finite characteristic matrix. However, in [5] it was shown that it has a 5-valued characteristic *Nmatrix* $\mathcal{M}_{mCi}$ defined as follows: $\mathcal{M}_{mCi} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

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[9]This means that a formula follows from a set T of formulas only if it already follows from some finite subset of T. That a logic which is characterized by a finite ordinary matrix has this property was shown in [17, 18]

- $\mathcal{V} = \{I, T, F, t, f\}$
- $\mathcal{D} = \{I, T, t\}$
- $\mathcal{O}$ is defined by:

$$a \tilde{\vee} b = \begin{cases} \{t, I\} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \{f\} & \text{if } a, b \notin \mathcal{D} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \{t, I\} & \text{if either } a \notin \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{f\} & \text{if } a \in \mathcal{D} \text{ and } b \notin \mathcal{D} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \{f\} & \text{if either } a \notin \mathcal{D} \text{ or } b \notin \mathcal{D} \\ \{t, I\} & \text{otherwise} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \{F\} & \text{if } a = T \\ \{T\} & \text{if } a = F \\ \{f\} & \text{if } a = t \\ \{t, I\} & \text{if } a = f \\ \{t, I\} & \text{if } a = I \end{cases}$$

$$\tilde{\circ} a = \begin{cases} \{F\} & \text{if } a = I \\ \{T\} & \text{if } a \neq I \end{cases}$$

Now by Proposition 5, this semantics is analytic, it immediately provides a decision procedure for **mCi** (as promised in footnote 7), and it guarantees that **mCi** is uniform. What is more: deriving from it characteristic Nmatrices for some natural extensions of **mCi** is rather easy. Thus axiom (**c**) corresponds to the condition $\tilde{\neg} f = \{t\}$ (instead of $\tilde{\neg} f = \{t, I\}$), while axiom (**e**) corresponds to the condition $\tilde{\neg} I = \{I\}$ (instead of $\tilde{\neg} I = \{t, I\}$). These two conditions are independent of each other, and so characteristic Nmatrices for the systems **mCic**, **mCie**, and **mCice** are obtained from $\mathcal{M}_{mCi}$ by respectively imposing the first condition, the second one, and both of the conditions. The completeness proofs can be done simultaneously for the four systems by a straightforward adaptation of the completeness proof given in [5] for the case of **mCi**.

A lot of other examples of the use of Nmatrices for modularly providing semantics for *thousands* of logical systems are described in [7, 2, 3, 4].

5. Truth-values as Information Carriers

There is one obvious question concerning the 5-valued Nmatrix which was used at the last section as a basis for modular semantics for **mCi** and some of its extensions: how can such an Nmatrix be discovered/constructed? It may seem plausible that (like in that example) once a characteristic Nmatrix is discovered for some basic logic **L**., it might be easy to derive from it in a modular way characteristic Nmatrices for various natural extensions of **L**. But again: how does one start looking for the basic Nmatrix which characterizes the basic system **L**? The goal of this section is to show (by way of an example) how the use of non-deterministic multi-valued semantics helps to do this in a systematic way. The key idea is to view the "truth values" used in multi-valued semantics as *information carriers*. The "truth values" which are assigned to formulas should *encode the data concerning the formulas which is relevant for the consequence relation we deal with*. Accordingly, the main problem in constructing appropriate semantics for a given logic is to determine what is the "relevant data" concerning formulas. Thus in classical logic what is important about a formula is only whether that formula is true or false (and so the two basic truth-values can be used as the relevant information carriers), while in modal logic the relevant data is what formula is true in what "world".

With this idea in mind, let us return to the example of **mCi**, and ask: what is the relevant information about formulas in the case of this system (and related ones)? Well, the semantics for LFIs described and used in [4] was based on the observation that in LFIs what is usually important about a sentence φ is whether φ is true or false, whether $\neg\varphi$ is true or false, and whether $\circ\varphi$ is true or false. Accordingly, for most of the systems considered there we used as truth-values triples in $\{0,1\}^3$ (or sometimes $\{0,1\}^2$, in case this sufficed). However, Marcos' schema (**cc**) ([15]) is concerned with formulas of a certain particular *syntactic* form, and so it is also important whether a formula has this particular form or not. Accordingly, the additional key idea in constructing multi-valued semantics for LFIs which include axioms like (**cc**) is to add more bits of information to the truth-value which is assigned to a formula, each indicating whether that formula has a certain special syntactic form. To handle (**cc**) we'll need one such bit. Thus our truth-values for the basic LFIs which include (**cc**) will be elements of $\{0,1\}^4$. The intended meaning of $v(\psi) = \langle x, y, z, u\rangle$ is then as follows:

- $x = 1$ iff ψ is "true" (i.e. $v(\psi) \in \mathcal{D}$).
- $y = 1$ iff $\neg\psi$ is "true" (i.e. $v(\neg\psi) \in \mathcal{D}$).

- $z = 1$ iff $\circ\psi$ is "true" (i.e. $v((\circ\psi) \in \mathcal{D})$.
- $u = 1$ iff ψ has the form $\neg^n \circ \varphi$.

The choice of what elements of $\{0,1\}^4$ can be used as truth-values depends on what LFI we take here as basic. In [9] the basic LFI is the system **mbC**[10] obtained from positive classical logic (in $\mathcal{L}_C$) by adding to it the axioms (**t**) and (**p**). Since the main interest in the current (more complicated) example is the treatment of (**cc**), we shall include this axiom too in the system **mC** that will be taken here as the basic system. So **mC** is the extension of **mbC** with (**cc**) (**mCi** itself is obtained from **mC** by adding axiom (**i**)). The intended meaning of the truth values, together with the content of the axioms of **mC**, naturally lead to the following type of Nmatrices as an appropriate semantic framework for **mC** and its extensions:

Definition 14. An **mC**-Nmatrix is any Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which:

- $\mathcal{V}$ is a subset of the set $\mathcal{V}_7$ of tuples $\langle x_1, x_2, x_3, x_4 \rangle \in \{1,0\}^4$ s. t.
    1. $x_1 = 0 \Rightarrow x_2 = 1$
    2. $x_1 = x_2 = 1 \Rightarrow x_3 = 0$ [11]
    3. $x_4 = 1 \Rightarrow x_3 = 1$
- $\mathcal{D} = \{\vec{x} \in \mathcal{V} \mid x_1 = 1\}$
- $\vec{x} \tilde{\vee} \vec{y} \subseteq \{\vec{z} \mid z_1 = max(x_1, y_1), z_4 = 0\}$
- $\vec{x} \tilde{\wedge} \vec{y} \subseteq \{\vec{z} \mid z_1 = min(x_1, y_1), z_4 = 0\}$
- $\vec{x} \tilde{\supset} \vec{y} \subseteq \{\vec{z} \mid z_1 = max(1 - x_1, y_1), z_4 = 0\}$
- $\tilde{\neg} \vec{x} \subseteq \{\vec{y} \in \mathcal{V} \mid y_1 = x_2, \ \ y_4 = x_4\}$
- $\tilde{\circ} \vec{x} \subseteq \{\vec{y} \in \mathcal{V} \mid y_1 = x_3, \ \ y_4 = 1\}$

Some explanations about the last definition are in order:

1. Since the first component of $v(\varphi)$ indicates whether φ is true, $v(\varphi)$ should be designated iff this component is 1. Hence the choice of $\mathcal{D}$.
2. The three constraints on $\mathcal{V}_7$ correspond in a precise way to the axioms (**t**), (**p**), and (**cc**) of **mC**:

[10] In [4] **mbC** is simply called **B**.

[11] In the presence of condition 3 this is equivalent to: $x_1 = x_2 = 1 \Rightarrow x_3 = x_4 = 0$.

- (**t**) means that if φ is false then its negation is true. Since the second component of $v(\varphi)$ indicates whether $\neg\varphi$ is true, (**t**) translates into: if the first component of $v(\varphi)$ is 0, then its second component is 1.
- (**p**) means that φ, $\neg\varphi$, and $\circ\varphi$ cannot all be true (because then *everything* is true, and this state should be ruled out). Since the third component of $v(\varphi)$ indicates whether $\circ\varphi$ is true, (**p**) translates into: the first 3 components of $v(\varphi)$ cannot all be 1.
- (**cc**) means that $\circ\psi$ is true whenever ψ is of the form $\neg^n\circ\varphi$. Given the information encoded by the forth and third components of φ, this obviously translates into the third constraint on $\mathcal{V}_7$.

It can easily be checked that together the three constraints leave only 7 legal truth-values (out of the potential 16). This is the reason for the name "$\mathcal{V}_7$".

3. The positive connectives $\vee$,$\wedge$, and $\supset$ behave completely in a classical way. Hence the truth/falsity of $\varphi * \psi$ for $* \in \{\vee, \wedge, \supset\}$ is determined by the truth/falsity of φ and ψ according to the classical truth tables for these connectives. This means that the first component of $v(\varphi * \psi)$ is determined by the first components of $v(\varphi)$ and $v(\psi)$ according to the classical truth tables. In addition, $\varphi * \psi$ cannot be of the form $\neg^n \circ\varphi$. Hence the forth component of $v(\varphi * \psi)$ should be 0.

4. The second component of $v(\varphi)$ and the first component of $v(\neg\varphi)$ should be equal, because they both indicate whether $\neg\varphi$ is true. In addition $\neg\psi$ is of the form $\neg^n \circ\varphi$ iff ψ is of this form. Hence $v(\psi)$ and $v(\neg\psi)$ should have the same forth component.

5. The third component of $v(\varphi)$ and the first component of $v(\circ\varphi)$ should be equal, because they both indicate whether $\circ\varphi$ is true. In addition $\circ\psi = \neg^0 \circ \psi$, and so it is of the form $\neg^n\circ\varphi$. Hence the forth component of $v(\circ\psi)$ should be 1.

Let $\mathcal{M}_7$ be the **mC**-Nmatrix with the highest degree of non-determinism. In other words: the set of truth-values used in $\mathcal{M}_7$ is $\mathcal{V}_7$, and the interpretations of the connectives are defined by replacing all conditions of the form $S \subseteq T$ in Definition 14 by the corresponding equalities (for example: $\vec{x} \tilde{\supset} \vec{y} = \{\vec{z} \mid z_1 = max(1 - x_1, y_1), z_4 = 0\}$, $\tilde{\neg}\vec{x} = \{\vec{y} \in \mathcal{V} \mid y_1 = x_2,\ \ y_4 = x_4\}$, etc.). Using the methods of [4] and [5], it is not difficult to show that $\mathcal{M}_7$ is a characteristic Nmatrix for **mC**. Now the additions of new axioms to

mC have the effect of reducing the degree of non-determinism in the corresponding semantics. For natural axioms the corresponding conditions can easily be computed, and modularly applied (see [4] for an extensive set of examples). Here we shall demonstrate this by showing how the characteristic 5-valued Nmatrix for **mCi** described in Example 3 is derived from $\mathcal{M}_7$. This amounts to finding the semantic effect of axiom **(i)** in the context of $\mathcal{M}_7$. To find this, it would be convenient to decompose **(i)** into two weaker axioms:

$$(\mathbf{i})_1 \quad \neg\circ\varphi \supset \varphi \qquad (\mathbf{i})_2 \quad \neg\circ\varphi \supset \neg\varphi$$

Let us examine the content of $(\mathbf{i})_1$. Obviously it means that if φ is false, then so is $\neg\circ\varphi$. Translating this into a condition concerning our truth-values, it means that if the first component of $v(\varphi)$ is 0, then the second component of $v(\circ\varphi)$ is 0 too. Hence in this case the first component of $v(\circ\varphi)$ is necessarily 1, implying that if the first component of $v(\varphi)$ is 0, then its third component should be 1. This means that $\langle 0,1,0,0\rangle$ (which is an element of $\mathcal{V}_7$) cannot be used as a truth-value. With this truth-value deleted, the condition that if the first component of $v(\varphi)$ is 0 then the second component of $v(\circ\varphi)$ is 0 too, is equivalent in **mC**-Nmatrices to the condition: $\widetilde{\circ}(\langle 0,1,1,0\rangle) = \widetilde{\circ}(\langle 0,1,1,1\rangle) = \{\langle 1,0,1,1\rangle\}$. Similarly, it is easy to see that the validity of $(\mathbf{i})_2$ implies that $\langle 1,0,0,0\rangle$ should be deleted, and without $\langle 1,0,0,0\rangle$ the validity of $(\mathbf{i})_2$ is equivalent in **mC**-Nmatrices to the condition: $\widetilde{\circ}(\langle 1,0,1,0\rangle) = \widetilde{\circ}(\langle 1,0,1,1\rangle) = \{\langle 1,0,1,1\rangle\}$. Combining all these conditions we find that the requirement of the validity of **(i)** produces from $\mathcal{M}_7$ the 5-valued Nmatrix from [5] described in the previous section, where: $I = \langle 1,1,0,0\rangle$, $T = \langle 1,0,1,1\rangle$, $F = \langle 0,1,1,1\rangle$, $t = \langle 1,0,1,0\rangle$, $f = \langle 0,1,1,0\rangle$.

6. Conclusion

On the philosophical level, Suszko's thesis seems to me correct. When it comes to *truth* of meaningful propositions there are just two possibilities: a proposition is either true or false. Therefore in my opinion there are indeed just two *truth*-values (calling them "logical" truth-values adds nothing, and I believe it is even misleading). However, for the working logicians and mathematicians this philosophical thesis has little significance. Bivalent semantics for logical systems is a too limited framework. The point in developing adequate semantics for a given logic is to *use* it for getting deeper understanding of the system and its properties. Bivalent semantics is seldom useful for this purpose. Its main (but not the only) shortcoming is that in most cases it is not analytic. The main point made in this paper is that the use of multi-valued semantics based on *information*-values (rather than "truth-values") can be much more efficient and practically useful than the use of bivalent semantics. In particular: multi-valued non-deterministic matrices provide a particularly appealing framework for constructing adequate analytic semantics for logical systems in a systematic and modular way.

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