The Structure of Interlaced Bilattices

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Abstract

Bilattices were introduced and applied by Ginsberg and Fitting for a diversity of applications, such as truth maintenance systems, default inferences and logic programming. In this paper we investigate the structure and properties of a particularly important class of bilattices called interlaced bilattices (introduced by Fitting). The main results are that every interlaced bilattice is isomorphic to the Ginsberg-Fitting product of two bounded lattices and that the variety of interlaced bilattices is equivalent to the variety of bounded lattices with two distinguishable distributive elements which are complements of each other . This implies that interlaced bilattices can be characterized using a finite set of equations. Our results generalize to interlaced bilattices results of Ginsberg, Fitting and Jónsson for distributive bilattices.

Introduction

The notion of a bilattice was introduced by Ginsberg in [Gi88] as a general framework for a diversity of applications, such as truth maintenance systems, default inferences and others. The notion was further investigated and applied for logic programming by Fitting ([Fi89], [Fi90], [Fi91], [Fi94]). The main idea behind bilattices is to use structures in which there are two partial order relations, having different interpretations. There should, of course, be a connection between the two relations. Ginsberg uses for this an operation of negation which is order preserving w.r.t. one, an involution w.r.t. to the other. Another connection he considered is distributive laws (12 altogether). This was generalized by Fitting who introduced the notion of an interlaced bilattice, in which all the basic bilattice operations are order preserving w.r.t. both partial orders.

The structure of distributive bilattices is well understood since Ginsberg and Fitting proved a characterization theorem for them, to the effect that each such bilattice is isomorphic to a certain product of two distributive bounded lattices. Fitting has further observed that if we apply that construction to any two bounded lattices (not necessarily distributive) we get an interlaced bilattice. The converse, however, was not established, and the structure of interlaced bilattices has not, so far, been well understood.

In this paper we thoroughly investigate the structure of interlaced bilatttices (with and without negation), and their important properties. Our main result is a generalization to interlaced bilattices of the Ginsberg-Fitting characterization of distributive bilattices (i.e. the converse of Fitting's observation). Other central results are characterizations using equational bases and the equivalence of the variety of interlaced bilattices with the variety of bounded lattices with two complementary, distributive elements. (This generalizes to interlaced bilattice results of Jónsson in [Jo94] for distributive bilattices.)

Note

In order to make the presentation complete and self-contained, we repeat, together with the appropriate references, some short proofs which have already appeared elsewhere.

1 General Background

1.1 Definition [Fi90]

An *interlaced bilattice* (IBL) is a structure $\mathcal{B} = \langle B, \leq_t, \leq_k, \wedge, \vee, \otimes, \oplus, t, f, \top, \bot \rangle$ such that

1. $\langle B, \leq_t, \wedge, \vee, t, f \rangle$ is a bounded lattice (with $\leq_t$ the order relation, $\wedge$ and $\vee$ the meet and join operations, and t, f the maximal and minimal elements, respectively).

2. $\langle B, \leq_k, \otimes, \oplus, \top, \bot \rangle$ is also a bounded lattice.

3. Each of the four operations $\vee, \wedge, \oplus, \otimes$ is order preserving with respect to both $\leq_t$ and $\leq_k$ (e.g. if $a \leq_k b$ then $a \wedge c \leq_k b \wedge c$).

Note

In the original definitions of Ginsberg and Fitting $\langle B, \leq_t \rangle$ and $\langle b, \leq_k \rangle$ are required to be *complete* lattices. We shall call below IBLs with this property *complete*.

1.2 Definition [Gi88]

An IBL is called *distributive* if all the twelve possible distributive laws concerning $\wedge, \vee, \otimes$ and $\oplus$ hold.

Note

Fitting has observed (see, e.g. [Fi90]) that the distributive laws imply condition (3) of 1.1. For example if $a \leq_t b$ then $b = a \vee b$, and so $c \otimes b = c \otimes (a \vee b) = (c \otimes a) \vee (c \otimes b)$ by the distributive laws. Hence $c \otimes a \leq_t c \otimes b$ in this case.

1.3 Definition

A unary operation $\sim$ on an IBL $\mathcal{B}$ is called *negation* if it is order-preserving with respect to $\leq_k$ and an involution w.r.t. $\leq_t$. In other words, $\sim$ is negation if

$$\begin{array}{ll} \text{(i)} & \sim\sim a = a \\ \text{(ii)} & a \leq_t b \Rightarrow \sim b \leq_t \sim a \\ \text{(iii)} & a \leq_k b \Rightarrow \sim a \leq_t \sim b\ . \end{array}$$

Notes

1. Ginsberg's original definition of a bilattice in [Gi88] is: a structure which satisfies the first two conditions in definition 1 (+ completeness) and has a negation.

2. Obviously, if $\sim$ is a negation then $\sim(a \wedge b) = \sim a \vee \sim b$, $\sim(a \vee b) = \sim a \wedge \sim b$, $\sim(a \oplus b) = \sim a \oplus \sim b$, $\sim(a \otimes b) = \sim a \otimes \sim b$, $\sim t = f$, $\sim f = t$, $\sim\top = \top$, $\sim\bot = \bot$.

The main method of constructing an IBL is described in the following definition. It was essentially introduced by Ginsberg in [Gi88], and further generalized by Fitting in [Fi90].

1.4 Definition

Let $\mathcal{L} = \langle L, \leq_L, \sqcup_L, \sqcap_L, 1_L, 0_L \rangle$ and $\mathcal{R} = \langle R, \leq_R, \sqcup_R, \sqcap_R, 1_R, 0_R \rangle$ be two bounded lattices. Their product, $\mathcal{L} \odot \mathcal{R}$, is the structure $\langle L \times R, \leq_t, \leq_k, \wedge, \vee, \otimes, \oplus, t, f, \top, \bot \rangle$ defined as follows:

$$\begin{array}{ll} \text{(i)} & (a_1, b_1) \leq_k (a_2, b_2) \Leftrightarrow a_1 \leq_L a_2 \quad \text{and} \quad b_1 \leq_R b_2 \\ \text{(ii)} & (a_1, b_1) \leq_t (a_2, b_2) \Leftrightarrow a_1 \leq_L a_2 \quad \text{and} \quad b_1 \geq_R b_2 \\ \text{(iii)} & (a_1, b_1) \vee (a_2, b_2) = (a_1 \sqcup_L a_2, b_1 \sqcap_R b_2) \\ \text{(iv)} & (a_1, b_1) \wedge (a_2, b_2) = (a_1 \sqcap_L a_2, b_1 \sqcup_R b_2) \\ \text{(v)} & (a_1, b_1) \oplus (a_2, b_2) = (a_1 \sqcup_L a_2, b_1 \sqcup_R b_2) \\ \text{(vi)} & (a_1, b_1) \otimes (a_2, b_2) = (a_1 \sqcap_L a_2, b_1 \sqcap_R b_2) \\ \text{(vii)} & t = \langle 1_L, 0_R \rangle \qquad f = \langle 0_L, 1_R \rangle \qquad \top = \langle 1_L, 1_R \rangle \qquad \bot = \langle 0_L, 0_R \rangle \end{array}$$

1.5 Theorem

1. [Fi90,91], [Gi88] *$\mathcal{L} \odot \mathcal{R}$ is an IBL. Moreover, if both $\mathcal{L}$ and $\mathcal{R}$ are distributive lattices then so is $\mathcal{L} \odot \mathcal{R}$, and if both $\mathcal{L}$ and $\mathcal{R}$ are complete lattices then so is $\mathcal{L} \odot \mathcal{R}$.*

2. [Gi88] *If $\mathcal{L}$ is a bounded lattice then the operation $\sim$, defined by $\sim(x, y) = (y, x)$, is a negation on $\mathcal{L} \odot \mathcal{L}$.*

The proof of this theorem is straightforward.

For distributive bilattices, a converse to Theorem 1.5 was proved by Ginsberg and Fitting (in [Gi88], [Fi90]): if $\mathcal{B}$ is a distributive bilattice then there exist distributive bounded lattices $\mathcal{L}$ and $\mathcal{R}$ such that $\mathcal{B}$ is isomorphic to $\mathcal{L} \odot \mathcal{R}$. These $\mathcal{L}$ and $\mathcal{R}$ are unique (up to isomorphism). Ginsberg-Fitting's result will easily follow from a generalization we prove below for every IBL.

2 Basic Properties of $IBLs$

In this section $\mathcal{B}$ is a fixed interlaced bilattice.

2.1 Definition

$$(1) \qquad [a, b]_t = \{x \in B \mid a \leq_t x \leq_t b\}$$
$$(2) \qquad [a, b]_k = \{x \in B \mid a \leq_k x \leq_k b\}$$

2.2 Proposition

1. *If $a \leq_t b$ then $[a, b]_t = [a \otimes b, a \oplus b]_k$.*

2. *If $a \leq_k b$ then $[a, b]_k = [a \wedge b, a \vee b]_t$.*

Proof

We show the first formula as an example. Assume first that $y \in [a, b]_t$, so $a \leq_t y \leq_t b$. Since $\mathcal{B}$ is interlaced, $a \otimes (a \otimes b) \leq_t y \otimes (a \otimes b) \leq_t b \otimes (a \otimes b)$. Hence $a \otimes b \leq_t y \otimes (a \otimes b) \leq_t a \otimes b$ and so $y \otimes (a \otimes b) = a \otimes b$. It follows that $a \otimes b \leq_k y$. Similarly $y \leq_k a \oplus b$. Hence $y \in [a \otimes b, a \oplus b]_k$.

For the converse, assume that $a \leq_t b$ and $a \otimes b \leq_k y \leq_k a \oplus b$. Then $a \wedge (a \otimes b) \leq_k a \wedge y \leq_k a \wedge (a \oplus b)$. But, if $a \leq_t b$ then $a = a \otimes a \leq_t a \otimes b$ and so $a \wedge (a \otimes b) = a$. Similarly, $a \wedge (a \oplus b) = a$. It follows that $a \leq_k a \wedge y \leq_k a$ and so $a \wedge y = a$ and $a \leq_t y$. Similarly, $y \leq_t b$ and so $y \in [a, b]_t$. □

2.3 Corollary

$[a,b]_t$ and $[a,b]_k$ are closed under $\wedge, \vee, \oplus$ and $\otimes$. Moreover, in case $a \leq_t b$ then $[a,b]_t$ is an interlaced bilattice, with the same order relations and bilattice operations as $\mathcal{B}$ (but with $a, b, a \otimes b, a \oplus b$ taking the roles of $f, t, \bot, \top$ respectively). Similarly, if $a \leq_k b$ then $[a,b]_k$ is an IBL with the same order relations and bilattice operations as $\mathcal{B}$.

2.4 Corollary

If $a \leq_t b$ then $b \otimes f \leq_k a \leq_k b \oplus f$ and $a \otimes t \leq_k b \leq_k a \oplus t$. Similarly, if $a \leq_k b$ then $b \wedge \bot \leq_t a \leq_t b \vee \bot$ and $a \wedge \top \leq_t b \leq_t a \vee \top$.

Proof

The first part follows from 2.2 and the fact that if $a \leq_t b$ then $a \in [f,b]_t$ and $b \in [a,t]_t$. Similar considerations apply for the second part.

2.5 Corollary [Fi90]

$$\text{(i)} \qquad t \otimes f = \bot \; , \qquad t \oplus f = \top$$
$$\text{(ii)} \qquad \bot \wedge \top = f \; , \qquad \bot \vee \top = t$$

Proof

Since $B = [f,t]_t$, we have by 2.2 that $B = [f \otimes t, f \oplus t]_k$. But also $B = [\bot, \top]_k$. Hence, the two equations in (i). The proof of (ii) is similar.

2.6 Proposition

$$\text{(i)} \qquad x \wedge \bot = x \otimes f \; , \qquad x \vee \bot = x \otimes t \; ,$$
$$\text{(ii)} \qquad x \wedge \top = x \oplus f \; , \qquad x \vee \top = x \oplus t \; .$$

Proof

We show the first equality. The proof of the rest is similar.

Since $x \wedge \bot \leq_t x$ we have by 2.4 that $x \otimes f \leq_k x \wedge \bot$. To show the converse it is enough to show that $x \wedge \bot \leq_k x$ and $x \wedge \bot \leq_k f$ (since $\otimes$ is the meet operation of $\leq_k$). The first inequality follows from $\bot \leq_k x$, since $\bot \leq_k x \Rightarrow x \wedge \bot \leq_k x \wedge x = x$. The second inequality follows from $\bot \leq_k f$, since $\bot \leq_k f \Rightarrow x \wedge \bot \leq_k x \wedge f = f$. $\square$

Notes

1. For distributive bilattices 2.6 was shown in [Jo94].

2. If we substitute, e.g. $\top$ for x in the first equality we get another proof of $\top \wedge \bot = f$. Hence 2.5 is a corollary of 2.6.

2.7 Proposition

$$
\begin{array}{ll}
(1) & \textit{If } x \geq_k b \textit{ then } x = (x \wedge b) \oplus (x \vee b) \\
(2) & \textit{If } x \leq_k b \textit{ then } x = (x \wedge b) \otimes (x \vee b) \\
(3) & \textit{If } x \geq_t b \textit{ then } x = (x \otimes b) \vee (x \oplus b) \\
(4) & \textit{If } x \leq_t b \textit{ then } x = (x \otimes b) \wedge (x \oplus b)
\end{array}
$$

Proof

Again we prove only (1).

Assume $x \geq_k b$. Then $x \wedge x \geq_k x \wedge b$ and so $x \geq_k x \wedge b$. Similarly, $x \geq_k x \vee b$. By combining these two inequalities we get $x \geq_k (x \wedge b) \oplus (x \vee b)$. On the other hand $x \in [x \wedge b, x \vee b]_t$, and so , by 2.2, $x \in [(x \wedge b) \otimes (x \vee b), (x \wedge b) \oplus (x \vee b)]_k$. Hence, $x \leq_k (x \wedge b) \oplus (x \vee b)$, and the quality follows. □

2.8 Corollary

$$
\begin{array}{ll}
(1) & x = (x \wedge \bot) \oplus (x \vee \bot) = (x \otimes f) \oplus (x \otimes t)\ , \\
(2) & x = (x \wedge \top) \otimes (x \vee \top) = (x \oplus f) \otimes (x \oplus t)\ , \\
(3) & x = (x \otimes f) \vee (x \oplus f) = (x \wedge \bot) \vee (x \wedge \top)\ , \\
(4) & x = (x \otimes t) \wedge (x \oplus t) = (x \vee \bot) \wedge (x \vee \top)\ .
\end{array}
$$

Proof

Immediate from 2.7 and 2.6.

Note

For distributive bilattices 2.8 is an immediate corollary of 2.5 and 2.6 (an so – of 2.6)

We end this section with a result which was shown for distributive bilattices in [Jo94].

2.9 Proposition

$$(1) \qquad \leq_k = \leq_t \ \textit{iff} \ f = \bot \ \textit{iff} \ t = \top$$
$$(2) \qquad \leq_k = \geq_t \ \textit{iff} \ f = \top \ \textit{iff} \ t = \bot$$

Proof

1. Obviously, if $\leq_k = \leq_t$ then $f = \bot$ and $t = \top$. For the converse, assume, e.g. that $f = \bot$. Then, for all a, b:

$$b \leq_t a \Leftrightarrow b \in [f, a]_t \Leftrightarrow b \in [\bot, a]_t \overset{2.2}{\Leftrightarrow} b \in [\bot \otimes a, \bot \oplus a]_k \Leftrightarrow b \in [\bot, a]_k \Leftrightarrow b \leq_k a \ .$$

2. The proof is similar. □

3 The Characterization Theorems

In this section we show that the Ginsberg-Fitting characterization of distributive billatices apply to IBLs in general, with the same construction.

We assume, again, that $\mathcal{B}$ is a fixed IBL.

3.1 Notation

$$L_{\mathcal{B}} = \{x \mid x \geq_t \bot\} \qquad R_{\mathcal{B}} = \{x \mid x \leq_t \bot\}$$

3.2 Proposition

1. $L_{\mathcal{B}} = \{x \mid x \leq_k t\}$, $\qquad R_{\mathcal{B}} = \{x \mid x \leq_k f\}$

2. *The relations $\leq_t$ and $\leq_k$ on $L_{\mathcal{B}}$ are identical while on $R_{\mathcal{B}}$ they are inverse to each other.*

Proof

1. $R_{\mathcal{B}} = [f, \bot]_t \overset{2.2}{=} [f \otimes \bot, f \oplus \bot]_k = [\bot, f]_k = \{x \mid x \leq_k f\}$. The proof for $L_{\mathcal{B}}$ is similar.

2. Immediate from Part 1, 2.9 and 2.3.

3.3 Theorem

If $\mathcal{B}$ is an IBL then there are bounded lattices $\mathcal{L}, \mathcal{R}$ such that $\mathcal{B}$ is isomorphic to $\mathcal{L} \odot \mathcal{R}$. These $\mathcal{L}$ and $\mathcal{R}$ are unique up to isomorphism.

Proof

We first prove uniqueness. So assume $\mathcal{B}$ is isomorphic to $\mathcal{L} \odot \mathcal{R}$, where $\mathcal{L}$ and $\mathcal{R}$ are bounded lattices. Now, $\mathcal{L}$ is obviously isomorphic to the sublattice $\langle \{(x, 0_R) \mid x \in \mathcal{L}\}, \leq_t \rangle$ of $\mathcal{L} \odot \mathcal{R}$ (where $\leq_t$ here is that of $\mathcal{L} \odot \mathcal{R}$). But $\{(x, 0_R) \mid x \in \mathcal{L}\}$ is exactly $L_{\mathcal{L}\odot\mathcal{R}}$, which is isomorphic to L_B (since $\mathcal{L} \odot \mathcal{R}$ is isomorphic to $\mathcal{B}$). Hence, $\mathcal{L}$ is isomorphic to $\langle L_B, \leq_t \rangle$ (where here $\leq_t$ is that of $\mathcal{B}$) and so is unique up to isomophism. Similarly, $\mathcal{R}$ should be equivalent to $\langle R_{\mathcal{B}}, \geq_t \rangle$, and so is also unique.

To prove existence, we use the two condidates that are naturally suggested by the proof of uniqueness (and are the ones used also in [Gi88] and [Fi90] for distributive bilattices):

$$\begin{aligned} \text{Let} \quad \mathcal{L}_{\mathcal{B}} &= \langle L_{\mathcal{B}}, \leq_t \rangle \quad (= \langle L_{\mathcal{B}}, \leq_k \rangle \text{ , by 3.2)} \\ \mathcal{R}_{\mathcal{B}} &= \langle R_{\mathcal{B}}, \geq_t \rangle \quad (= \langle R_{\mathcal{B}}, \leq_k \rangle \text{ , by 3.2)} . \end{aligned}$$

Define $g : B \to L_B \times R_B$ by $g(x) = (x \vee \bot, x \wedge \bot)$. We show that g is an isomorphism of $\mathcal{B}$ on $\mathcal{L}_{\mathcal{B}} \odot \mathcal{R}_{\mathcal{B}}$.[1]

That g is one-one is immediate from 2.8(1).

To show that g is onto, let $(a, b) \in L_B \times R_B$. We show that $g(a \oplus b) = (a, b)$. In other words, we show that if $a \geq_t \bot \geq_t b$ then, (i) $(a \oplus b) \vee \bot = a$; (ii) $(a \oplus b) \wedge \bot = b$. For (i) note that since $a \leq_k a \oplus b$, $a \leq_t (a \oplus b) \vee \bot$ by 2.4. On the other hand, since $a \geq_t b$, $a = a \oplus a \geq_t a \oplus b$. Since here also $a \geq_t \bot$ it follows that $a \geq_t (a \oplus b) \vee \bot$. Hence $a = (a \oplus b) \vee \bot$. The proof of (ii) is similar.

Next we need to show that $a \leq_t b \Leftrightarrow g(a) \leq_t g(b)$ and $a \leq_k b \Leftrightarrow g(a) \leq_k g(b)$. We show the second quivalence (the proof of the first is similar). So assume $a \leq_k b$. Then $a \vee \bot \leq_k b \vee \bot$ and $a \wedge \bot \leq_k b \wedge \bot$. By 3.2 this means that $a \vee \bot \leq_{\mathcal{L}_{\mathcal{B}}} b \vee \bot$ and $a \wedge \bot \leq_{\mathcal{R}_{\mathcal{B}}} b \wedge \bot$. Hence $g(a) = (a \vee \bot, a \wedge \bot) \leq_k (b \vee \bot, b \wedge \bot) = g(b)$. Conversely, assume $g(a) \leq_k g(b)$ and so $a \vee \bot \leq_k b \vee \bot$, $a \wedge \bot \leq_k b \wedge \bot$. This immediately entails that $a \stackrel{2.8}{=} (a \vee \bot) \oplus (a \wedge \bot) \leq_k (b \vee \bot) \oplus (b \wedge \bot) \stackrel{2.8}{=} b$. □

Other characterizations are easy consequences of 3.3.

3.4 Proposition

A structure $\mathcal{B} = \langle B, \leq_t, \leq_k \rangle$ is a complete IBL iff there exist two complete lattices $\mathcal{L}, \mathcal{R}$ such that $\mathcal{B}$ is isomorphic to $\mathcal{L} \odot \mathcal{R}$.

[1] Most of the details of the proof are exactly as in [Fi90]. The proof that g is surjective is the only true innovation. Of course, what makes the reproducing of that proof possible is the demonstration in the previous section that the relevant facts obtain also for interlaced bilattices.

Proof

The "if" part is straightforward. For the "only if" we note that if $\mathcal{B}$ is complete then by an obvious generalization of 2.3, so are $\mathcal{L}_\mathcal{B}$ and $\mathcal{R}_B$ of the proof of 3.3. $\square$

3.5 Proposition ([Gi88],[Fi90])

A structure $\mathcal{B} = \langle B, \leq_t, \leq_k \rangle$ is a (complete) distributive bilattice iff there exist (complete) distributive bounded lattices $\mathcal{L}, \mathcal{R}$ such that $\mathcal{B}$ is isomorphic to $\mathcal{L} \odot \mathcal{R}$.

Proof

Again the "if" part is easy, while the "only if" part follows from the proof of 3.3 and the fact that if $\mathcal{B}$ is a distributive bilattice Then $\mathcal{L}_B, \mathcal{R}_B$ are necessarily distributive lattices as well. $\square$

3.6 Proposition

An IBL $\mathcal{B}$ is distributive iff $\wedge$ and $\vee$ (or $\oplus$ and $\otimes$) are distributive over each other.

Proof

Since $\mathcal{L}_\mathcal{B}$ and $\mathcal{R}_B$ are defined in terms of $\leq_t$ alone (or $\leq_k$ alone), if $\langle \mathcal{B}, \leq_t \rangle$ is a distributive lattice so are $\mathcal{L}_\mathcal{B}$ and $\mathcal{R}_\mathcal{B}$. Hence so is also $\mathcal{L}_\mathcal{B} \odot \mathcal{R}_\mathcal{B}$, which is isomorphic to $\mathcal{B}$. $\square$

Note

This means that for IBLs, the 12 conditions in the definition of a distributive bilattice can be replaced by just two.

We now establish a converse to Theorem 1.5, part (2).

3.7 Proposition

Suppose $\mathcal{B}$ is a (complete, distributive) IBL with negation. Then there exist a (complete, distributive) bounded lattice $\mathcal{L}$, so that $\langle \mathcal{B}, \sim \rangle$ is equivalent to $\mathcal{L} \odot \mathcal{L}$, equipped with Ginsberg's negation (i.e. $\sim(x, y) = (y, x)$).

Proof

The proof is practically identical to that in [Fi94] for the distributive case: The function g defined in the proof of 3.3 ($g(x) = (x \vee \bot, x \wedge \bot)$) is an isomorphism of $\mathcal{B}$ on $\mathcal{L}_\mathcal{B} \odot \mathcal{R}_\mathcal{B}$. On the other and the function $\lambda x.{\sim}x$ is easily seen to be an isomorphism between the lattices $\mathcal{L}_\mathcal{B}$ and $\mathcal{R}_\mathcal{B}$. Hence the function h defined by $h(x, y) = (x, {\sim}y)$ is an isomorphism between $\mathcal{L}_\mathcal{B} \odot \mathcal{R}_\mathcal{B}$ and $\mathcal{L}_\mathcal{B} \odot \mathcal{L}_\mathcal{B}$. It follows that $f = h \circ g$ is an isomorphism between $\mathcal{B}$ and $\mathcal{L}_\mathcal{B} \odot \mathcal{L}_\mathcal{B}$. It remains only to show that f preserves also the negation operator. Now from the definitions, $f(x) = (x \vee \bot, {\sim}(x \wedge \bot)) = (x \vee \bot, {\sim}x \vee \bot)$ (see note (2) after Definition 1.3). Hence $f({\sim}x) = ({\sim}x \vee \bot, {\sim}{\sim}x \vee \bot) = ({\sim}x \vee \bot, x \vee \bot) = {\sim}f(x)$ (according to Ginsberg's definition of $\sim$ in $\mathcal{L}_\mathcal{B} \otimes \mathcal{L}_\mathcal{B}$). □

4 Applications

In this section we give some examples of the power of the characterization theorem 3.3, which allows us to reduce claims about bilattices to simple calculations.

4.1 Definition

1. An element a of an IBL is called *distributive* if each equation of the form

$$x *_1 (y *_2 z) = (x *_1 y) *_2 (x *_1 z)$$

   (were $*_1, *_2 \in \{\vee, \wedge, \oplus, \otimes\}$) obtains in case $x = a$ or $y = a$ or $z = a$.

2. Distributive elements of a *lattice* are defined similarly.

4.2 Theorem

$\top, \bot, t$ and f are all distributive (in any IBL $\mathcal{B}$).

Proof

It is enough, by 3.3, to check it for IBLs of the form $\mathcal{L} \odot \mathcal{R}$. Obviously, the equation obtains there iff the corresponding distribution equations obtain for each component separately. Since $\top, \bot, t, f$ are all defined in terms of the extreme elements of $\mathcal{L}$ and $\mathcal{B}$, and since such elements of a *lattice* are always distributive (trivial), the claim follows. □

An example

Suppose $*_1 = \vee$, $*_2 = \otimes$, $x = (x_1, x_2)$, $y = t = (1_L, 0_R)$, $z = (z_1, z_2)$,

$$\begin{aligned} (x_1,x_2) \vee ((1_L,0_R) \otimes (z_1,z_2)) &= (x_1,x_1) \vee (1_L \sqcap_L z_1, 0_R \sqcap_R z_2) = (x_1,x_2) \vee (z_1, 0_R) \\ &= (x_1 \sqcup_L z_1, x_2 \sqcap_R 0_R) = (x_1 \sqcup_L z_1, 0_R) \\ ((x_1,x_2) \vee (1_L,0_R)) \otimes ((x_1,x_2) \vee (z_1,z_2)) &= (1_L,0_R) \otimes (x_1 \sqcup_L z_1, x_2 \sqcap_R z_2) = (x_1 \sqcup_L z_1, 0_R) \ . \end{aligned}$$

Note

In [Fi90], Fitting pointed out that in his proof of the analogue of 3.3 for distributive bilattices, only instances involving $\bot$ of the distributive laws are used. Hence the theorem is valid for every IBL in which $\bot$ is distributive (and similarly for $t, f, \top$). It follows that an alternative proof of 3.3 can be achieved if we prove 4.2 directly. This is possible, but the proof is longer.

One of the remarkable properties of all known finite IBLs is that they can be represented by a two-dimensional graph, in which one axis represents the $\leq_t$ relation, while the other the $\leq_k$ relation. Moreover, the same edge on the graph between points A and B means that A and B are immediate successors according to *both* relations (although it is possible that A is an immediate $\leq_t$-successor of B while B is an immediate $\leq_k$-successor of A). In [Av95] we have shown that all finite IBLs have in fact such a graphic representation. The key for this was the next result. With the help of 3.3, we can obtain a new easy proof of it.

4.3 Proposition

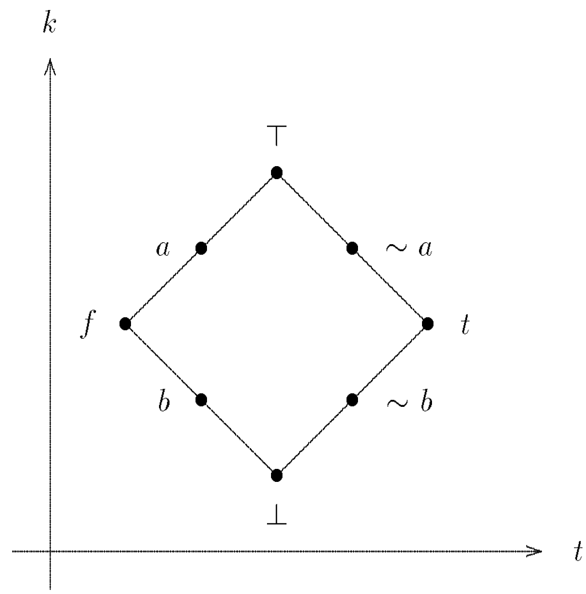
Let $a <_t^1 b$ ($a <_k^1 b$) mean that b is an immediate $\leq_t$ ($\leq_k$) successor of a. Then if $a <_t^1 b$, then $a <_k^1 b$ or $b <_k^1 a$. Similarly, if $a <_k^1 b$ then $a <_t^1 b$ or $b <_t^1 a$.

Proof

Again it is enough to show this for bilattices of the form $\mathcal{L} \odot \mathcal{R}$. So suppose that $(a_1, b_1) <_k^1 (a_2, b_2)$. It is easy to see that this can happen only if either $a_1 = a_2$ and $b_1 <_{\mathcal{R}}^1 b_2$ or if $b_1 = b_2$ and $a_1 <_{\mathcal{L}}^1 a_2$. In the first case $(a_2, b_2) <_t^1 (a_1, b_1)$, while in the second $(a_1, b_1) <_t^1 (a_2, b_2)$. □

Note

In [Av95] it has been conjectured that every finite bilattice which has the property described in 4.3 (such bilattices were called there "precise") is necessarily interlaced. This conjecture is wrong, even if the bilattice has a negation. A counterexample is given by the following bilattice $EIGHT$:



That $EIGHT$ is not interlaced follows immediately from the following proposition.

4.4 Proposition

A finite bilattice $\mathcal{B}$ is interlaced only if $|B| = |L_{\mathcal{B}}| \cdot |R_{\mathcal{B}}|$ (where $L_B = \{x \in B \mid x \geq_t \bot\}$, $R_{\mathcal{B}} = \{x \in B \mid x \leq_t \bot\}$).

Proof

Immediate from the proof of 3.3, which shows that if $\mathcal{B}$ is interlaced then $\mathcal{B}$ is isomorphic to $\mathcal{L}_{\mathcal{B}} \odot \mathcal{R}_{\mathcal{B}}$. □

4.5 Corollary

If $\mathcal{B}$ is a finite IBL which has a prime number of elements then either $\leq_t = \leq_k$ or $\leq_t = \geq_k$.

Proof

Immediate from 4.4 and 2.9.

4.5.1 An Example

The bilattice $DEFAULT$ of Ginsberg ([Gi88]) has seven elements and is not trivial. Hence it is not interlaced.

A similar negative criterion for IBLs with negation is given in the next corollary.

4.6 Corollary

If $\mathcal{B}$ is a finite IBL with negation then the number of elements in $\mathcal{B}$ is a perfect square.

Proof

Immediate from 3.7.

An Example

In [Fi91], Fitting presented an IBL called SIX as an example of an IBL with no negation. By 4.6, the fact that SIX has no negation follows simply from the fact that it has six elements.

5 An Equational Basis

In [Jo94], Jónsson shows that the variety of distributive bilattices is equivalent to the variety of all algebras $\langle A, \oplus, \otimes, \top, \bot, f, t\rangle$ in which $\langle A, \oplus, \otimes, \top, \bot\rangle$ is a bounded distributive lattice (with $\top$ and $\bot$ as the upper and lower bounds) and f, t are two complementaty elements of A (i.e. $t \oplus f = \top$, $t \otimes f = \bot$). We proceed now to show an analogous result for the variety of IBLs.

5.1 Definition

A structure $\mathcal{L} = \langle B, \leq_k, \otimes, \oplus, \bot, \top, t, f\rangle$ is a potential IBL *if:*

1. *$\langle B, \leq_k, \otimes, \oplus, \bot, \top\rangle$ is a bounded lattice;*
2. *t, f are distributive elements of $\mathcal{L}$ which are complements of each other. In other words, the following equations obtain:*

$$\text{(I)} \qquad t \oplus f = \top \qquad\qquad t \otimes f = \bot$$

$$\text{(II)} \qquad a \otimes (t \oplus b) = (a \otimes t) \oplus (a \otimes b) \qquad\qquad a \oplus (t \otimes b) = (a \oplus t) \otimes (a \oplus b)$$

$$a \otimes (f \oplus b) = (a \otimes f) \oplus (a \otimes b) \qquad\qquad a \oplus (f \otimes b) = (a \oplus f) \otimes (a \oplus b)$$

$$\text{(III)} \qquad t \otimes (a \oplus b) = (t \otimes a) \oplus (t \otimes b) \qquad\qquad t \oplus (a \otimes b) = (t \oplus a) \otimes (t \oplus b)$$

$$f \otimes (a \oplus b) = (f \otimes a) \oplus (f \otimes b) \qquad\qquad f \oplus (a \otimes b) = (f \oplus a) \otimes (f \oplus b)$$

Note

Since the variety of bounded lattices has an equational basis (i.e. can be defined by a set of equations – see [Bi48]), so does the variety of potential IBLs.

Our next proposition shows that the set of equations in the equational basis for potential IBLs can be reduced somewhat.

5.2 Proposition

The set of 6 equations in (I) *and* (II) *of 5.1 can be replaced by the following two equations.*

$$\begin{array}{lrcl} \text{(i)} & x & = & (x \otimes t) \oplus (x \otimes f) \\ \text{(ii)} & x & = & (x \oplus t) \otimes (x \oplus f) \end{array}$$

Proof

For one direction, assume (I) and (II). Then

$$x = x \otimes \top \overset{(I)}{=} x \otimes (t \oplus f) \overset{(II)}{=} (x \otimes t) \oplus (x \otimes f)\ , \qquad x = x \oplus \bot \overset{(I)}{=} x \oplus (t \otimes f) \overset{(II)}{=} (x \oplus t) \otimes (x \oplus f)\ .$$

For the converse, assume (i),(ii) and the equations in III of 5.1. Substituting $\top$ for x in (i) and $\bot$ for x in (ii), we get the equations in (I). Next we prove the first equation in II as an example, leaving the rest to the reader:

$$\begin{array}{rcl} a \otimes (t \oplus b) & \overset{(i)}{=} & [(a \otimes (t \oplus b)) \otimes t] \oplus [(a \otimes (t \oplus b)) \otimes f] \\ & = & [a \otimes ((t \oplus b) \otimes t)] \oplus [a \otimes ((t \oplus b) \otimes f)] \\ & \overset{(III)}{=} & (a \otimes t) \oplus [a \otimes ((t \otimes f) \oplus (b \otimes f))] \\ & = & (a \otimes t) \oplus [a \otimes (\bot \oplus (b \otimes f))] \text{ by (I), which has already been proved,} \\ & = & (a \otimes t) \oplus (a \otimes b \otimes f) \\ (a \otimes t) + (a \otimes b) & \overset{(i)}{=} & (a \otimes t) \oplus [((a \otimes b) \otimes t) \oplus (a \otimes b) \otimes f] \\ & = & [(a \otimes t) \oplus ((a \otimes t) \otimes b)] \oplus ((a \otimes b) \otimes f) \\ & = & (a \otimes t) \oplus ((a \otimes b) \otimes f) \quad \text{since } \oplus, \otimes \text{ are lattice operations.} \end{array}$$

Hence $a \otimes (t \oplus b) = (a \otimes t) \oplus (a \otimes b)$. □

The main theorem of this section is the following.

5.3 Theorem

The varieties of IBLs and of potential IBLs are equivalent. Specifically:

1. *If $\mathcal{B}$ is an IBL then the reduct $\langle B, \leq_k, \oplus, \otimes, \top, \bot, t, f \rangle$ is a potential IBL.*

2. *In any potential IBL $\mathcal{B}$, it is possible to define, in a unique way, a partial order $\leq_t$ so that the resulting structure is an IBL with t and f as the upper and lower bounds of $\leq_t$.*

Proof

1. Immediate from 2.5 and 4.2 (Note: using 5.2 and 2.8 one need only show the equations in III of 5.2. This can in fact be done directly, without appealing to the characterization theorem. For example, since $t \geq_t b$, $t \otimes b \geq_t b \otimes b = b$. Similarly, $t \otimes c \geq_t c$. Hence, $(t \otimes b) \oplus (t \otimes c) \geq_t b \oplus c$. By 2.4 this entails: $(t \otimes b) \oplus (t \otimes c) \geq_k t \otimes (b \oplus c)$. The converse is true since $\langle B, \leq_k, \otimes, \oplus \rangle$ is a lattice.)

2. Uniqueness is obvious from the following two equations, the validity of which in every IBL can easily be checked using 3.3:[2]

$$
\begin{aligned}
(*) \quad x \vee y &= (x \otimes t) \oplus (y \otimes t) \oplus (f \otimes x \otimes y) \\
(**) \quad x \wedge y &= (x \otimes f) \oplus (y \otimes f) \oplus (t \otimes x \otimes y)
\end{aligned}
$$

   It remains to show that by using equations $(*)$ and $(**)$ to define $\wedge$ and $\vee$ in a given potential IBL $\mathcal{B}$, we really get an IBL.

To make reading easier we shall write (until the end of this proof) $+$ instead of $\oplus$, xy instead of $x \otimes y$, and we shall omit parentheses in the usual way. For example, $(*)$ and $(**)$ above will be rewritten as follows:

$$
(*) \quad x \vee y = tx + ty + fxy \qquad (**) \quad x \wedge y = fx + fy + txy
$$

Lemma 1

$\wedge$ and $\vee$ as defined in $(*)$ and $(**)$ are associative, commutative and idempotent.

Proof of Lemma 1

We do the case of $\vee$. Commutativity is trivial, while idempotency easily follows from 5.2(i). It remains to to show associativity. Well, by definition:

$$
(x \vee y) \vee z = t(tx + ty + fxy) + tz + f(tx + ty + fxy)z
$$

[2] Jónsson uses in [Jo94] other formulas, which are also valid in every IBL, but those presented here are better for our needs.

using the distributivity of t and f this reduces to

$$ttx + tty + tfxy + tz + (ftx + fty + ffxy)z$$

since $ft = \bot$, $\bot a = \bot$, $\bot + a = a$ and $aa = a$, this reduces to $tx + ty + tz + fxyz$. A similar computation shows that $x \vee (y \vee z)$ reduces to the same expression. Hence the equality.

Lemma 2

$$x \vee y = y \quad \textit{iff} \quad x \wedge y = x \ .$$

Proof of Lemma 2

Assume, for example, that $x \vee y = y$, then:

$$\begin{aligned}
x \wedge y &= fx + fy + txy = fx + f(x \vee y) + tx(x \vee y) \\
&= fx + f(tx + ty + fxy) + xt(tx + ty + fxy) \\
&= fx + ftx + fty + ffxy + x(ttx + tty + tfxy) \ , \ \text{since } t, f \text{ are distributive,} \\
&= fx + fxy + x(tx + ty) \ , \ \text{since } tf = \bot \\
&= ((fx) + (fx)y) + xt(x + y) \ , \ \text{since } t \text{ is distributive} \\
&= fx + t(x(x + y)) \quad \text{by the absorption laws in a lattice} \\
&= fx + tx \quad \text{by the absorption laws in a lattice} \\
&= x \quad \text{by 5.2(i)} \ .
\end{aligned}$$

Lemmas 1 and 2 together imply (see [Bi48]) that by defining

$$x \leq_t y \Leftrightarrow_{Df} x \vee y = y \qquad (\Leftrightarrow x \wedge y = x)$$

we get a lattice $\langle B, \leq_t \rangle$ in which $\wedge$ and $\vee$ are the lattice operations. We show next that this lattice is bound by t and f. Indeed, for every $x \in B$

$$\begin{aligned}
t \vee x &= tt + tx + ftx = (t + tx) + \bot x = t + \bot = t \\
f \vee x &= tf + tx + ffx = \bot + (tx + fx) = x \qquad \text{by 5.2(i)} \ .
\end{aligned}$$

It remains to show that $\langle B, \leq_k, \leq_t \rangle$ is interlaced. Now the fact that if $x \leq_k y$ then $x \vee z \leq_k y \vee z$ and $x \wedge z \leq_k y \wedge z$ easily follows from the definitions of $\vee$ and $\wedge$ and the fact that the lattice operations $\oplus$ and $\otimes$ are order preserving w.r.t. $\leq_k$. The proof of the theorem will be concluded, therefore, by the following two lemmas.

Lemma 3

If $x \leq_t y$ then $x + z \leq_t y + z$.

Lemma 4

If $x \leq_t y$ then $xz \leq_t yz$.

Proof of Lemma 3

We are assuming that $y = x \vee y = tx + ty + fxy$. We want to prove that $(x+z) \vee (y+z) = y+z$

$$\begin{aligned}
(x+z) \vee (y+z) &= t(x+z) + t(y+z) + f(x+z)(y+z) \\
&= tx + tz + ty + tz + (x+z)(f \cdot (tx + ty + fxy + z)) \text{ , since } y = x \vee y \\
&= tx + ty + tz + (x+z)(ftx + fty + ffxy + fz) \\
&= tx + ty + tz + (x+z)(fxy + fz) \\
&= tx + ty + tz + f(x+z)(xy+z)
\end{aligned}$$

but $x + z \geq_k xy + z$. Hence $(x+z)(xy+z) = xy + z$. It follows that:

$$\begin{aligned}
(x+z) \vee (y+z) &= tx + ty + tz + f(xy+z) \\
&= (tx + ty + fxy) + (tz + z) \\
&= y + z \qquad (\text{since } tx + ty + fxy = y)
\end{aligned}$$

Proof of Lemma 4

Again we assume that $y = x \vee y$. We show this time that $xz \wedge yz = xz$.

$$\begin{aligned}
xz \wedge yz &= fxz + fyz + txzyz \\
&= fxz + f(tx + ty + fxy)z + t(tx + ty + fxy)xz \\
&= fxz + (ftx + fty + fxy)z + (tx + ty + tfxy)xz \\
&= fxz + fxyz + (tx + ty)xz \\
&= ((fxz) + (fxz) \cdot y) + t((x+y)x)z \quad \text{since } t \text{ is distributive,} \\
&= fxz + txz \quad (\text{by the absorption laws}) \\
&= xz \quad (\text{by } 5.2)
\end{aligned}$$

□

5.4 Corollary [Jo94]

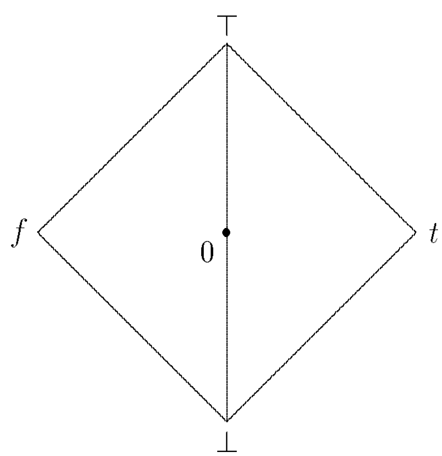
Suppose $\langle B, \leq_k, \otimes, \oplus, \perp, \top, f, t \rangle$ is a structure in which $\langle B, \leq_k, \otimes, \oplus, \perp, \top \rangle$ is a bounded distributive lattice and t, f are two complementary elements of B. Then there exists a unique partial order $\leq_t$ on B such that $\mathcal{B} = \langle B, \leq_k, \otimes, \oplus, \perp, \top, \leq_t \wedge, \vee, f, t \rangle$ is an IBL. This IBL is distributive.

Proof

t and f are trivially distributive here, so the existence of a unique $\leq_t$ which provides $\mathcal{B}$ follows from 5.3. It remains to show that this $\mathcal{B}$ is distributive. This follows immediately from 3.6. □

Note

Unlike in the distributive case, the conditions $t \oplus f = \top$, $t \otimes f = \perp$ are *not* sufficient in the general interlaced case, and we do need the extra condition that t, f should be distributive elements. Thus in the following bounded lattice:



f, t are complementary, but there is no way to define an appropriate $\leq_t$. This is an immediate consequence of 4.5.

5.5 Corollary

A finite lattice which has a prime number of elements and two complementary elements, different from the l.u.b and g.l.b of the lattice, cannot be distributive.

Proof

Again, this follows from 5.4 and 4.5. □

We end with the following observation:

5.6 Theorem

The variety of interlaced bilattices can be defined by a set of equations.

Proof

The results of this section provide several alternative equational bases for the variety of IBLs (with or without negation): In the form of potential IBLs we have already provided it above with two such bases. A basis for the full signature (with $\vee$ and $\wedge$ but, of course, without $\leq_t, \leq_k$) can be obtained from them just by adding equations $(*)$ and $(**)$ from the proof of 5.3. □

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