On the Proof Theory of Natural Many-valued Logics

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Abstract. We claim that Proof Systems for natural many-valued logics, whether finite-valued or infinite-valued should be similar in their structure to proof systems of any other natural logic: one should not be able to tell from the structures which are used in a proof system the intended semantics. It is also preferable that standard connectives will be used, with corresponding standard rules. We demonstrate this thesis with some examples in which cut-free Gentzen-type systems, which employ either ordinary sequents or hypersequents, are used both for 3-valued logics and for infinite-valued logics.

1 The Methodological Approach

In recent years there is a growing Interest in many types of nonclassical logics: modal and temporal logics, substructural logics, constructive logics, many-valued logics, paraconsistent logics, non-monotonic logics – the list is long. Obviously, there is no limit to the number of logics that logicians (and non-logicians) can produce. Some creteria are needed, therefore, to distinguish those that are "natural" or "interesting" in some sense (and so deserve studying). It seems to me that the following are widely accepted virtues of a "natural" logic:

- Natural primitives. In other words: the primitive connectives and quantifiers of the language of the logic should intuitively correspond to concepts which are informally used outside the realm of formal logic, like: implication, negation, conjunction, necessity etc. The language might have several "conjunctions" (say), each corresponding to a different interpretation of the informal concept, but it should not include as primitives artificial constructs, tailored for a specific semantics (for examples: unary connectives which correspond to certain nonclassical truth values, as in [13]).
- The existence of a simple, illuminating semantics. On the propositional level such a semantics should provide (so I believe) a decision procedure for the *consequence relation* of the logic (and so, of course, also to its set of valid formulas).
- The existence of a nice proof system. Such a system should make it easier to find proofs *in* the system, to prove results *about* it, and, should have the subformula property. Here again the proof system should determine not only the set of valid formulas of the logic, but also its *consequence relation*.

To my opinion, having both a simple semantics and a nice proof system is the strongest indication that we really have a "natural" (or important) logic. This is so, however, only if the semantics and the proof system are independent, coming from completely different sources, so that the correspondence between them is a kind of a "surprise" (or, at least, not an obvious matter).[1] The way to achieve such an independence is to use for the proof system a general proof-theoretical framework.[2] Such a proof-theoretical framework should have the following properties:

- It should be able to handle a great diversity of logics of different types, including most logics which logicians have found interesting in the past.
- Because of the proof-theoretical nature and the expected generality, the framework should be independent of any particular semantics. One should not be able to guess, just from the form of the structures which are used, the intended semantics of a given proof system.
- Since there should be something common to all the various connectives that we call "conjunction", "disjunction" etc., the corresponding rules should be as standard as possible (otherwise the first "virtue" above of a natural logic is meaningless).

Another very important methodological principle which will guide us in what follows (and has already been hinted twice above) is the modern view of a logic as a language together with a *consequence relation* (For simplicity, and since we shall deal with many-valued semantics, we shall restrict ourselves to Tarskian Consequence relations). It should be emphasized that the set of valid formulas does not always determine a logic. There are, for example, important 3-valued logics (like Kleene's) which does not have any! Accordingly, when we speak about soundness and completeness of a given semantics or a given proof system for a given logic we mean that the semantics or the proof system characterizes the consequence relation. When it characterizes only the set of theorems we shall call it *weak* soundness and completeness (or just weak completeness, for short).

2 What Is a Many-valued Logic?

Our answer to the title of this section is that it is in fact somewhat misleading to talk about "many-valued logics". One should talk instead about logics with many-valued semantics. To see our point, consider the following 3 questions:

1. Is Lukasiewicz infinite-valued logic L_ω a many-valued logic?
2. Is Dummet intermediate logic LC ([6]) a many-valued logic?

[1] The tableaux and sequents systems for finitely-many valued logics in [5] or [14] (see there for further references) are examples of calculi which violate this principle, since the semantics is built there into the proof theory. In fact, the methods there apply to any finite-valued logic, so it cannot distinguish the natural from the unnatural.

[2] Using such a framework is also very important for implementing logics on a computer, using a uniform logical framework like the Edinburgh LF ([10]).

3. Is the modal logic $S5$ a many-valued logic?

The answers that most people will give are "yes" to the first question (this answer is, in fact, a part of the formulation of the question...) and "no" to the third. As for the second— it might depend on the replier's background and knowledge. In 1959 it probably would have been "yes". Today many, I guess, will answer "no". Still, there is no objective difference between these three logics. Each of them has an infinite-valued semantics according to the definition below (which is the most restrictive I know). The differences are that L_ω has been *defined* by this semantics, and no better semantics is known (as far as I know). LC was has also been defined by its many-valued semantics, but today the use of a possible-worlds semantics for it is more popular. Finally, $S5$ was originally defined by a proof system, and its possible-worlds semantics is much better known than its infinite-valued semantics (although the latter was discovered first (See [11])). I do not think, however, that historical motivations should be important for mathematical classifications. The existence of an alternative semantics should not be a factor either, since it depends on our present knowledge, which might be accidental. So from an objective point of view, all the three logics above have the same right to be called "many-valued" (or rather, to have the "many-valuedness" property). I proceed next to define this notion in precise terms. For simplicity, I shall refer only to propositional logics.

Definition 1. 1. A *matrix* $\mathcal{M}$ for a propositional language L is a triple $< M, D, O >$ such that:
   (a) M is a nonempty set (of "truth-values").
   (b) D is a proper, nonempty subset of M (the "designated values").
   (c) O is a set of operations on M, so that for each connective of L there is a corresponding operation on M and vice versa.
2. Let $\mathcal{M}$ be a matrix for L. $\vdash_{\mathcal{M}}$, The consequence relation induced by $\mathcal{M}$, is defined by: $T \vdash_{\mathcal{M}} \phi$ iff $v(\phi) \in D$ for every valuation v in M which respects the operations and such that $v(B) \in D$ for every $B \in T$.
3. A logic L is called (weakly) n-valued (where $1 \leq n \leq \aleph_0$) if there exists a matrix $\mathcal{M}$ for L such that:
   (a) M has exactly n elements.
   (b) $\vdash_{\mathcal{M}} = \vdash_L$ ($\vdash_{\mathcal{M}} \phi$ iff $\vdash_L \phi$ for every sentence ϕ).
   (c) For every finite Γ and every ϕ (for every ϕ) there is a finite submatrix $\mathcal{M}^*$ of $\mathcal{M}$ such that $\Gamma \vdash_L \phi$ ($\vdash_L \phi$) iff $\Gamma \vdash_{\mathcal{M}^*} \phi$ ($\vdash_{\mathcal{M}^*} \phi$).

Notes.

1. The main factor in our definition of a "many-valued" logic is the existence of a *single* characteristic matrix. The second demand, on the other hand, guarantees that every propositional many-valued logic is decidable, and it makes even infinite-valued logics semi-finite in a certain sense. It is possible, of course, to consider a definition where this demand is dropped, but I believe that it does reflect the spirit of the generalization from finite-valued logics to infinite-valued ones.

2. Obviously, The same logic may be n-valued for several different values of n (It is obvious, for example, that classical logic is 2^m-valued for every m). In such a case we might take the minimal such n as the principal one. Note also that a logic might be weakly n-valued for a certain n, but k-valued only for some k greater than n. We shall see examples of this below.[3]

3 The Proof-theoretical Framework

Among the various proof-theoretical frameworks, Gentzen calculi of sequents seems to me the most successful, general and intuitive. I strongly believe that the existence of a cut-free Gentzen-type proof system having the subformula property is the main proof-theoretical test for the naturality of a logic. This framework has indeed all the properties which we have listed above. It can successfully handle a diversity of important logics, it is independent of any semantics and each of the standard connectives has in it a small stock of rules that are characteristic for it. The rules for conjunction, for example, may sometimes have a "multiplicative" (or "intensional") form and sometimes an "additive" form (in the terminology of [9]). There might be cases in which a mixture of the two forms is used and still others in which there are also rules for the combination of conjunction with negation. Still, we can always identify a connective as a conjunction according to its rules alone, regardless of any corresponding semantics. If we cannot— then it is not a conjunction!

Some people might argue that the fact that Gentzen-type systems usually treats structures with *two* sides *is* connected with the two-valued semantics of classical logic. This impression is wrong, though. This is demonstrated by the fact that many other logics (including n-valued logics with $n = 3, 4, \aleph_0$ and logics which are not many-valued at all, like intuitionistic logic) also have cut-free Gentzen-type formulations with the subformula property. What really stands behind Gentzen's sequents is again the fact that it is consequence relations which Logic is all about. Dealing with single-conclusion sequents is therefore the most natural thing to do, since a calculus G of such sequents naturally defines a corresponding consequence relation $\vdash_G$, where : $T \vdash_G \phi$ iff there exists a finite list Γ of elements of T such that $\Gamma \Rightarrow \phi$ is a theorem of G. A generalization to calculi of multiple-conclusion sequents is then another natural step, which allows us to take advantage of the symmetries of logic. Note that the definition of the Tarskian consequence relation $\vdash_G$ induced by G remains unchanged when we make this step, and that the original (single-conclusion) sequents are now particular cases of the extended notion of a sequent.

Exactly as the class of single-conclusion sequents can successfully be enlarged to the class of multiple-conclusion ones, one might consider further extensions. The main properties which characterizes Gentzen-type systems should be preserved, though. For example: the stock of rules for the standard connectives should practically remain the same, and the use of the usual sequents should be

[3] It is possible, in principle, also that a logic might be weakly many-valued without being many-valued, but I know no example of this kind.

a part of the extended framework. An extension of this sort which proved to be fruitful (especially for many-valued logic) is that of *hypersequents*:[4]

Definition 2. Let L be a language. A hypersequent is a creature of the form:

$$\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2 | \dots | \Gamma_n \Rightarrow \Delta_n$$

where Γ_i, Δ_i are finite sequences of formulae of L. The $\Gamma_i \Rightarrow \Delta_i$'s will be called the *components* of the hypersequent.

We shall use G, H as metavariables for (possibly empty, i.e., without components) hypersequents.

Like in ordinary sequential calculi, the rules of inference for hypersequents are usually divided into *logical* rules and *structural* rules. The guiding idea is that the logical rules should essentially be identical to those used in the ordinary calculi, and that the difference between the various logics should mainly be due to differences in their structural rules. For example, the rules for implication are usually the following:

$$\frac{G|A, \Gamma \Rightarrow B, \Delta|H}{G|\Gamma \Rightarrow A \rightarrow B, \Delta|H} \qquad \frac{G_1|\Gamma_1 \Rightarrow \Delta_1, A|H_1 \qquad G_2|B, \Gamma_2 \Rightarrow \Delta_2|H_2}{G_1|G_2|A \rightarrow B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H_1|H_2}$$

The other usual rules, both structural and logical are generalized to the framework of hypersequents in a similar way. In addition, this framework allows new types of structural rules. The simplest of these are the external structural rules. For example, external contraction has the form:

$$\frac{G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta|H}{G|\Gamma \Rightarrow \Delta|H}$$

External weakening and permutation are defined similarly. An example of a rule which is peculiar to the hypersequential framework is the following splitting rule (some versions of which are used in many cut-free formulations of known logics):

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H}{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2 \Rightarrow \Delta_2|H}$$

Given a calculus G of hypersequent we define the associated consequence relation, $\vdash_G$ as follows: $T \vdash_G \phi$ iff there exists finite lists $\Gamma_1, \dots, \Gamma_n$ of elements of T such that $\Gamma_1 \Rightarrow \phi| \dots |\Gamma_n \Rightarrow \phi$ is a theorem of G. It is not difficult to see that if G is closed under the external structural rules (as we always assume) and $A \Rightarrow A$ is provable for every A, then this indeed is a Tarskian consequence relation. (note that if G allows only the use of ordinary sequents we get the same definition as before!).

The use of hypersequents makes it possible to give cut-free formulations (with strong completeness!) to several well-known many-valued logics. Examples are LC, $S5$ and Łukasiewicz 3-valued logic L_3 (see [4] for these and others). This demonstrates that these logics are really natural. I should point out, however,

[4] see [4] for a survey.

that all the many-valued logics I know which have cut-free formulations (using either ordinary sequents or hypersequents) are either 3-valued, 4-valued or infinite-valued. Does this fact reflect something? Unfortunately, this is a question for which I have no answer.

4 An Example: Sobociński's Many-valued Logic(s)

In this section we present a case study in which the various ideas which were described above are applied, with suggestive results.

In [12] Sobociński introduced a 3-valued matrix which we shall call here (following [1]) $\mathcal{M}_3$. The elements of M_3 are 1, 0 and -1. The designated elements are 1 and 0. The negation operation is simply the arithmetical one, while the implication operation $\rightarrow$ is defined as follows:

$$a \rightarrow b = \begin{cases} 0 & a = b = 0 \\ -1 & a > b \\ 1 & \text{otherwise} \end{cases}$$

(Sobociński introduced also what we call today multiplicative disjunction and conjunction. These, however, can be defined from the above negation and implication in the usual manner). In his paper Sobociński gave a Hilbert-type axiomatization of this logic with MP for $\rightarrow$ as the only rule of inference. An equivalent cut-free Gentzen-type formulation GRM_m was found later.[5] It can be obtained from the classical calculus for this language (with the above multiplicative form of the rules for $\rightarrow$) by replacing the weakening rules (on both side) by the following structural rule, which today (following [9]) is usually called "mix":

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Equivalent descriptions of GRM_m are:

- Multiplicative Linear Logic together with contraction and mix
- Intensional Relevant Logic together with mix

In any case, the Gentzen-type formulation clearly shows that $\vdash_{GRM_m}$ is a natural substructural logic.[6] Nothing in the structures used in this formulation suggests that it is a many-valued logic!

Now we come to a very important point. The description we just have given concerning the proof theory of Sobociński's 3-valued logic is a standard one. Yet it is misleading. Actually, $\vdash_{GRM_m}$ and $\vdash_{\mathcal{M}_3}$ are *not* identical, and the former is *not* a 3-valued logic! The reason is that the correspondence between these

[5] I do not know who was the first to discover it. It can be found in [2], but was well-known much before.

[6] Personally, I have first encountered it as such, before knowing its connection to $\mathcal{M}_3$, and I am sure that I am not the only one with this experience!

two logics is only a weak one: The two logics have the same valid formulas, but not the same consequence relation (Also Sobociński has proved, in fact, only weak completeness relative to $\mathcal{M}_3$. His Hilbert-type system is indeed strongly equivalent to GRM_m). An example of the difference is the fact that $\sim (A \to B) \vdash_{\mathcal{M}_3} \sim B$ but $\sim (A \to B) \not\vdash_{GRM_m} \sim B$.

The fact just mentioned naturally leads to the following two questions:

1. Is GRM_m many-valued logic (in the strong sense) at all?
2. Is there a nice Gentzen-type proof system for $\vdash_{\mathcal{M}_3}$?

The answer to both questions is not simple, but it is positive nevertheless. It turns out that GRM_m is an *infinite*-valued logic, but not finite-valued, while $\vdash_{\mathcal{M}_3}$ does have a cut-free formulation, but only if we use hypersequents. We give next some more details.

Let us start with the first question. In the relevance logic literature (see [1] and [8]) there have been extensive investigations of the semi-relevant system RM of Dunn and McCall. This system is obtained from Sobociński's system (the Hilbert-type counterpart of GRM_m) by adding to its language extensional (or additive) conjunction and disjunction together with the corresponding axioms and rules of the relevance system R (including the distribution axiom, which is missing in Linear Logic). Now R.K.Meyer has proved ([1]) that RM has an infinite characteristic matrix $\mathcal{S}_z$, known as Sugihara Matrix. The truth-values of this matrix are the integers, and the designated values are the non-negative integers. Negation is again the arithmetical one, $\vee$ and $\wedge$ are, respectively, the operations of max and min, while $\to$ is defined as follows:

$$a \to b = \begin{cases} max(|a|, |b|) & a \leq b \\ -max(|a|, |b|) & a > b \end{cases}$$

Now $\mathcal{S}_z$ is also a characteristic matrix for GRM_m in the *strong* sense defined above (provided we limit ourselves to finite sets of assumptions. If we allow infinite theories then a little bit more complicated infinite matrix should be used.[7] Moreover: no finite-valued matrix has this property, since it is not too difficult to show that although

$$\sim ((p_{n+1} \to p_{n+1}) \to (p_n \to p_n)), \ldots, \sim ((p_2 \to p_2) \to (p_1 \to p_1)) \not\vdash_{GRM_m} p_1$$

no n-valued matrix can be used to demonstrate this fact.

It is interesting indeed that the fact, that what we have here is a strictly infinite-valued logic, is revealed already on the level of weak completeness when we pass to the stronger language of RM. On the other hand this passage forces us to use a cut free Gentzen-type calculus GRM, which is much more complicated then GRM_m. This calculus uses hypersequents, and it is obtained from GRM_m by:

1. Adding the standard additive form of the rules for the additional connectives

[7] Again, I know no place in which these results are explicitly proved, but they are implicit in the works of Meyer and Dunn, especially [7].

2. Changing all rules to their hypersequential version
3. Adding the standard external structural rules as well as the splitting rule described above

More details about this system can be found in [2]. Among other things, it is shown there that it allows us, e.g., a constructive proof of the admissibility of the disjunctive syllogism in RM. Proofs of this kind are exactly what we expect a good proof system to offer us!

To sum up: In case we consider only weak completeness, the passage from simple sequential calculus to a hypersequential one can be seen here as forced by a move from a finite-valued logic to an infinite-valued one. If, on the other hand, we look at the matter from the point of view of consequence relations and strong completeness, it seems that it is caused by a strengthening of the language without changing the semantics.

We turn next to the second question. In [3] it is shown that a strongly complete, cut-free proof system for $\vdash_{\mathcal{M}_3}$ can be obtained from the purely multiplicative fragment of the (hypersequential) system GRM if we strengthen the splitting rule to the following rule:

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H}{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'|H}$$

In this case, therefore, the passage to a calculus of hypersequents corresponds to a passage from what is really an infinite-valued logic (GRM_m) to a finite-valued one ($\vdash_{\mathcal{M}_3}$) (note that both calculi of hypersequents are here conservative extensions of the purely sequential system GRM_m!).

The outshot of these examples is that neither sequents nor hypersequents are structures that necessarily correspond to finite-valued or infinite-valued logics. They are natural structures of proof-theory, and so one aspects that appropriate cut-free calculi based on them can be used when we deal with *natural* logics (finite-valued, infinite-valued, or not many-valued at all!).

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