Decomposition Proof Systems for Gödel-Dummett Logics *

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Abstract.
The main goal of the paper is to suggest some analytic proof systems for LC and its finite-valued counterparts which *are* suitable for proof-search. This goal is achieved through following the general Rasiowa-Sikorski methodology for constructing analytic proof systems for semantically-defined logics. All the systems presented here are terminating, contraction-free, and based on *invertible* rules, which have a *local* character and at most two premises.

Keywords: Gödel Logics, Intermediate Logics, Fuzzy Logics, Decomposition Systems, Gentzen-type systems, Hypersequents, Tableaux, Analytic rules

1. The Gödel-Dummett Logic LC

In (Gödel, 1933) Gödel introduced a sequence $\{G_n\}$ $(n \geq 2)$ of n-valued Matrices. He used these matrices to show some important properties of intuitionistic logic. An infinite-valued matrix G_ω in which all the $G_n s$ can be embedded was later introduced by Dummett in (Dummett, 1959). The logic of G_ω was axiomatized in the same paper, and has been known since then as Gödel-Dummett's LC. It is probably the most important intermediate logic, which turns up in several places, such as the provability logic of Heyting's Arithmetics (Visser, 1982), and relevance logic (Dunn et al., 1971). Recently it has again attracted a lot of attention because of its recognition as one of the three most basic fuzzy logics (Hajek, 1998).

The language of LC is that of intuitionistic logic. Semantically, it corresponds to linearly ordered Kripke structures. It also corresponds of course to the matrix $G_\omega = \langle N \cup \{t\}, \leq, \rightarrow, \neg, \vee, \wedge \rangle$, where $\leq$ is the usual order on N extended by a greatest element t, the interpretation

* This research has been partially supported by the LoSSeD workpackage of the CRIT-2 project funded by European ESPRIT and INCO programes.

of the propositional constant $\perp$ is the number 0, $a \to b$ is t if $a \leq b$ and b otherwise, $\neg a$ is simply $a \to 0$, and $\wedge$ and $\vee$ are, respectively, the *min* and *max* operations on $\langle N \cup \{t\}, \leq \rangle$.[1] The matrices of $\{G_n\}$ are similar, but the set of truth values of G_n is $\{0, ..., n-2\} \cup \{t\}$. The consequence relation $\vdash_{LC}$ is defined as follows: $\varphi_1, \ldots, \varphi_n \vdash_{LC} \psi$ iff $\min\{v(\varphi_1), \ldots, v(\varphi_n)\} \leq v(\psi)$ for every valuation v in G_ω.[2] This is equivalent [3] to taking t as the only designated element, and defining: $\varphi_1, \ldots, \varphi_n \vdash_{LC} \psi$ iff, for every v in G_ω, either $v(\psi) = t$ or $v(\varphi_i) \neq t$ for some $1 \leq i \leq n$. The consequence relation corresponding to G_n is defined similarly.

A Hilbert-type axiomatization for LC can be obtained from intuitionistic logic by adding to it the axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$ — see (Dummett, 1959).

A cut-free Gentzen-type formulation for LC was first given by Sonobe in (Sonobe, 1975). His approach was improved in (Avellone et al., 1999) and (Dyckhoff, 1999), where terminating, contraction-free (and cut-free) versions have been presented. All those systems have, however, the serious drawback of using a rule with an arbitrary number of premises, all of which contain formulas of essential importance for the inference. A cut-free formulation of LC free of this drawback, and, unlike other formulations, having exactly the same logical rules as the standard formulation of Intuitionistic Logic, was given in (Avron, 1991). However, the latter formulation is not very convenient for proof search. The main reason is that some of its rules are not invertible.

Our main goal in this paper is to suggest some analytic proof systems for LC and its finite-valued counterparts which are suitable for proof-search, and only contain rules of a strictly local character (with at most two premises). To achieve this goal, we shall follow the general Rasiowa-Sikorski methodology for constructing analytic proof systems for semantically defined logics. The main ideas are to decompose a formula φ to simpler formulas of the same vocabulary (though not necessarily to subformulas of φ) and to employ, if needed, a more extensive set of axioms (or criteria for closing branches) than is usual in standard systems (more explanations are given in the next section and in the papers cited there). These ideas are discussed in detail when we present R-S deduction systems, for which they are raison d'etre.

[1] This interpretation is not the one given by Gödel and Dummett, but its dual. We note also that for the application as a fuzzy logic it is more useful (Hajek, 1998) to use instead of $N \cup \{t\}$ the real interval [0,1], with 1 playing the role of t. This makes a difference only when we consider inferences from infinite theories, and in this paper it is convenient to clearly distinguish between t and the other truth values.

[2] As usual, if $n = 0$ the "minimal element" is taken to be t.

[3] A proof of this well-known result can be found in (Avron, 1991).

However, later we go on to show how the methodology can be used with other deduction mechanisms, like hypersequential calculi.

Like in (Dyckhoff, 1999), our systems will be terminating, contraction-free, and based on *invertible* rules.

2. R-S Deduction Systems for LC

2.1. R-S Deduction Systems — Background

A Rasiowa-Sikorski (R-S) deduction system (Rasiowa and Sikorski, 1963) is a variant of the tableau method[4], which operates on sequences of signed formulas. However, in contrast to tableaux, it is used for proving validity directly rather than as a refutational mechanism [5]. An R-S system usually has three main components:

- Decomposition rules,
- Expansion rules,
- Fundamental sequences.

A decomposition rule replaces some signed formula in a sequence Ω by certain simpler signed formulas of the same (or partial) vocabulary. A signed formula to which such a rule can be applied is called *decomposable*. Otherwise it is called *indecomposable*. A sequence of indecomposable signed formulas is called *basic*. Decomposition rules can therefore only be applied to sequences which are not basic. An expansion rule, in contrast, may be applied *only* to basic sequences. Such a rule augments a given basic sequence with some other indecomposable signed formulas of the same vocabulary (so the outcome is still basic). It is a fundamental requirement that both types of rules be *analytic*. [6] Another crucial demand is that rules of both types should also be *invertible* in the sense that the conclusion of a rule is provable in the system iff all its premises are provable. If the system is sound and complete with respect to its intended semantics [7], then this is equivalent to the rules being validity-preserving *in both directions*. Usually,

[4] Though it has originally been developed and applied independently of the tableau method.

[5] Accordingly, in this section we use the symbol "**T**" where standard tableaux use "**F**", and vice versa.

[6] There is no complete uniformity regarding the exact meaning of the term "analytic" in the literature. Here a rule is called analytic if the multiset of symbols occurring in any formula in its premises is contained in the multiset of symbols occurring in the conclusion.

[7] Which almost always exists, since the R-S methodology is semantically oriented.

however, they have the stronger property of being truth-preserving (in both directions) with respect to each intended model separately — which is also the case with the system developed here. Invertibility is a strong property symbolized by a double horizontal line in the standard notation of the rules, which is:

$$\frac{\Omega}{\Omega_1 \mid \Omega_2 \mid \ldots \mid \Omega_n}$$

where all the Ω's are sequences of signed formulas. Ω is called the *conclusion* of the rule, and $\Omega_1, \Omega_2, \ldots, \Omega_n$ — its *premises*. Note that since R-S systems are used for proving validity, the vertical bar separating individual premises is taken to correspond to a meta-conjunction on the validity level, while a sequence is understood as equivalent to a meta-disjunction of its elements.

The general idea in an R-S deduction system is to prove an ordinary formula φ by first decomposing $\mathbf{T}(\varphi)$ with help of the decomposition rules into simpler sequences which are valid iff the original φ is. If no decomposition rule is applicable, expansion rules might be applied. This process results finally in sequences, which are either fundamental, or non-fundamental, basic and closed under the expansion rules. Finally, a sequence is considered proved if the above process yields in the end only fundamental sequences, which play the role of *axioms* here. [8]

More information on R-S systems, their applications, and the general methodology of using this formalism for developing deduction systems for various kinds of logics from the analysis of their semantics, can be found in (Konikowska, 1999; Konikowska, 2000; Konikowska, 2000a).

2.2. An R-S System for LC with Simple Fundamental Sequences

In what follows we use p, q, r to denote atomic formulas (including $\perp$), and lower case Greek letters to denote arbitrary formulas.

Intuitively, the sign $\mathbf{T}$, when used in an R-S system, stands for "true",or "satisfied", while $\mathbf{F}$ stands for "false", or "not satisfied". Consistently with this intuition, satisfiability of signed formulas by a valuation v in G_ω (or G_n) is defined by:

$$v \models \mathbf{T}(\varphi) \text{ iff } v(\varphi) = t, \qquad v \models \mathbf{F}(\varphi) \text{ iff } v(\varphi) \neq t$$

The above description of R-S systems dictates then the following:

[8] In most cases, including the systems presented below, it is possible to allow only basic fundamental sequences, but in general it is more efficient not to impose this restriction.

DEFINITION 1. *A sequence* $\Omega = s_1, s_2, \ldots, s_n$ *of signed formulas in the language of LC is* satisfied *by a valuation* v *in* G_ω *or* G_n *iff* $v \models s_i$ *for some* $i, 1 \leq i \leq n$. *It is* valid *iff* $v \models \Omega$ *for every valuation* v.

In this section we develop our R-S formulation of LC. We start with some explanations how the rules of this system are obtained.

Under the R-S methodology, for each n-ary logical construct C we try to find necessary and sufficient conditions for $\mathbf{T}(C(\psi_1, \ldots, \psi_n))$ and for $\mathbf{F}(C(\psi_1, \ldots, \psi_n))$ to be satisfied in terms of some simpler formulas being satisfied. In case of LC, this can be done in a straightforward way for disjunction and conjunction. The conditions here are quite classical: e.g., $\mathbf{T}(\varphi \vee \psi)$ is satisfied iff either $\mathbf{T}(\varphi)$ or $\mathbf{T}(\psi)$ is satisfied. This gives rise to standard rules for these connectives.

However, $\rightarrow$, the basic connective of LC, cannot be handled so simply, since satisfaction of $\mathbf{T}(\varphi \rightarrow \psi)/\mathbf{F}(\varphi \rightarrow \psi)$ cannot in general be expressed just in terms of satisfaction of $\mathbf{T}(\varphi), \mathbf{F}(\varphi), \mathbf{T}(\psi)$, and $\mathbf{F}(\psi)$. In other words: the answer to the question whether $v(\varphi \rightarrow \psi)$ is t or not is not determined by the answers to the corresponding questions concerning φ and ψ. What it really depends on is the order relation between $v(\varphi)$ and $v(\psi)$:

$$v \models \mathbf{T}(\varphi \rightarrow \psi) \text{ iff } v(\varphi) \leq v(\psi), \qquad v \models \mathbf{F}(\varphi \rightarrow \psi) \text{ iff } v(\varphi) > v(\psi)$$

Therefore in the case of an implicational formula we go one level deeper: if φ or ψ is a composed formula, then it must have been obtained out of some simpler formulae using $\vee, \wedge$ or $\rightarrow$. For each of these cases, we develop a separate pair of decomposition rules.

As an example, let us show how the rule for $(\varphi_1 \rightarrow \varphi_2) \rightarrow \psi$ is developed. By the semantics of $\rightarrow$, to find when $\mathbf{T}((\varphi_1 \rightarrow \varphi_2) \rightarrow \psi)$ is true under a valuation v, we should distinguish two cases:

1. $v(\varphi_1) \leq v(\varphi_2)$. Then $v(\varphi_1 \rightarrow \varphi_2) = t$ and $v((\varphi_1 \rightarrow \varphi_2) \rightarrow \psi) = v(\psi)$. Hence we should have $v(\psi) = t$.

2. $v(\varphi_1) > v(\varphi_2)$. Then $v(\varphi_1 \rightarrow \varphi_2) \neq t$ and $v((\varphi_1 \rightarrow \varphi_2) \rightarrow \psi) = v(\varphi_2 \rightarrow \psi)$. Hence we should have $v(\varphi_2 \rightarrow \psi) = t$.

Thus the original formula is true iff either both $\mathbf{T}(\varphi_1 \rightarrow \varphi_2)$ and $\mathbf{T}(\psi)$ are true, or both $\mathbf{F}(\varphi_1 \rightarrow \varphi_2)$ and $\mathbf{T}(\varphi_2 \rightarrow \psi)$ are true. This is a condition in disjunctive normal form. Using standard classical reasoning on the metalevel (and the fact that $\mathbf{F}(\varphi)$ is true iff $\mathbf{T}(\varphi)$ is not true) we transform it to conditions in conjunctive normal form for the truth of $\mathbf{T}((\varphi_1 \rightarrow \varphi_2) \rightarrow \psi)$ and of $\mathbf{F}((\varphi_1 \rightarrow \varphi_2) \rightarrow \psi)$, respectively [9]. After

[9] This is needed since the branching | between the premises of a rule corresponds here to meta-conjunction.

some simplifications (using facts like that the truth of $\mathbf{T}(\psi)$ implies that of $\mathbf{T}(\varphi \to \psi)$) we get the rules $(\mathbf{T}(\to) \to)$ and $(\mathbf{F}(\to) \to)$ given below.

Other rules could have been developed similarly, but we have applied a shortcut, using the following equivalences valid in G_ω:

$$\begin{array}{lcl} (\varphi_1 \vee \varphi_2) \to \psi & \equiv & (\varphi_1 \to \psi) \wedge (\varphi_2 \to \psi) \\ \varphi \to (\psi_1 \vee \psi_2) & \equiv & (\varphi \to \psi_1) \vee (\varphi \to \psi_2) \\ (\varphi_1 \wedge \varphi_2) \to \psi & \equiv & (\varphi_1 \to \psi) \vee (\varphi_2 \to \psi) \\ \varphi \to (\psi_1 \wedge \psi_2) & \equiv & (\varphi \to \psi_1) \wedge (\varphi \to \psi_2) \\ \varphi \to (\psi_1 \to \psi_2) & \equiv & (\varphi \to \psi_2) \vee (\psi_1 \to \psi_2) \end{array}$$

The above equivalences should be understood in the strongest possible way, i.e. $\varphi \equiv \psi$ iff, for any valuation v, $v(\varphi) = v(\psi)$.

Note that no decomposition rule is applicable to signed formulas having one of the following forms:

$$\mathbf{T}(p), \mathbf{F}(p), \mathbf{T}(p \to q), \mathbf{F}(p \to q)$$

These are therefore the indecomposable signed formulas of LC_{RS}.

Expansion rules of an R-S system are usually discovered while attempting to prove completeness of the system (relative to its intended semantics). The expansion rules of LC_{RS} have also been obtained in this way, and they reflect properties of the order relation of G_ω.

Now we turn to precise definitions of the relevant proof-theoretical notions.

THE SYSTEM LC_{RS}.

Fundamental Sequences are those containing either:

- $\mathbf{F}(\bot)$, or
- both $\mathbf{T}(\varphi)$ and $\mathbf{F}(\varphi)$

Decomposition Rules

$$(\mathbf{T}\vee) \quad \frac{\Omega', \mathbf{T}(\varphi \vee \psi), \Omega''}{\Omega', \mathbf{T}(\varphi), \mathbf{T}(\psi), \Omega''}$$

$$(\mathbf{F}\vee) \quad \frac{\Omega', \mathbf{F}(\varphi \vee \psi), \Omega''}{\Omega', \mathbf{F}(\varphi), \Omega'' \mid \Omega', \mathbf{F}(\psi), \Omega''}$$

$$(\mathbf{T}\wedge) \quad \frac{\Omega', \mathbf{T}(\varphi \wedge \psi), \Omega''}{\Omega', \mathbf{T}(\varphi), \Omega'' \mid \Omega', \mathbf{T}(\psi), \Omega''}$$

$$(\mathbf{F}\wedge)\qquad \frac{\Omega', \mathbf{F}(\varphi \wedge \psi), \Omega''}{\Omega', \mathbf{F}(\varphi), \mathbf{F}(\psi), \Omega''}$$

$$(\mathbf{T}\vee \to)\qquad \frac{\Omega', \mathbf{T}((\varphi_1 \vee \varphi_2) \to \psi), \Omega''}{\Omega', \mathbf{T}(\varphi_1 \to \psi), \Omega'' \mid \Omega', \mathbf{T}(\varphi_2 \to \psi), \Omega''}$$

$$(\mathbf{F}\vee \to)\qquad \frac{\Omega', \mathbf{F}((\varphi_1 \vee \varphi_2) \to \psi), \Omega''}{\Omega', \mathbf{F}(\varphi_1 \to \psi), \mathbf{F}(\varphi_2 \to \psi), \Omega''}$$

$$(\mathbf{T} \to \vee)\qquad \frac{\Omega', \mathbf{T}(\varphi \to (\psi_1 \vee \psi_2)), \Omega''}{\Omega', \mathbf{T}(\varphi \to \psi_1), \mathbf{T}(\varphi \to \psi_2), \Omega''}$$

$$(\mathbf{F} \to \vee)\qquad \frac{\Omega', \mathbf{F}(\varphi \to (\psi_1 \vee \psi_2)), \Omega''}{\Omega', \mathbf{F}(\varphi \to \psi_1), \Omega'' \mid \Omega', \mathbf{F}(\varphi \to \psi_2), \Omega''}$$

$$(\mathbf{T}\wedge \to)\qquad \frac{\Omega', \mathbf{T}((\varphi_1 \wedge \varphi_2) \to \psi), \Omega''}{\Omega', \mathbf{T}(\varphi_1 \to \psi), \mathbf{T}(\varphi_2 \to \psi), \Omega''}$$

$$(\mathbf{F}\wedge \to)\qquad \frac{\Omega', \mathbf{F}((\varphi_1 \wedge \varphi_2) \to \psi), \Omega''}{\Omega', \mathbf{F}(\varphi_1 \to \psi), \Omega'' \mid \Omega', \mathbf{F}(\varphi_2 \to \psi), \Omega''}$$

$$(\mathbf{T} \to \wedge)\qquad \frac{\Omega', \mathbf{T}(\varphi \to (\psi_1 \wedge \psi_2)), \Omega''}{\Omega', \mathbf{T}(\varphi \to \psi_1), \Omega'' \mid \Omega', \mathbf{T}(\varphi \to \psi_2), \Omega''}$$

$$(\mathbf{F} \to \wedge)\qquad \frac{\Omega', \mathbf{F}(\varphi \to (\psi_1 \wedge \psi_2)), \Omega''}{\Omega', \mathbf{F}(\varphi \to \psi_1), \mathbf{F}(\varphi \to \psi_2), \Omega''}$$

$$(\mathbf{T} \to (\to))\qquad \frac{\Omega', \mathbf{T}(\varphi \to (\psi_1 \to \psi_2)), \Omega''}{\Omega', \mathbf{T}(\psi_1 \to \psi_2), \mathbf{T}(\varphi \to \psi_2), \Omega''}$$

$$(\mathbf{F} \to (\to))\qquad \frac{\Omega', \mathbf{F}(\varphi \to (\psi_1 \to \psi_2)), \Omega''}{\Omega', \mathbf{F}(\psi_1 \to \psi_2), \Omega'' \mid \Omega', \mathbf{F}(\varphi \to \psi_2), \Omega''}$$

$$(\mathbf{T}(\to) \to)\qquad \frac{\Omega', \mathbf{T}((\varphi_1 \to \varphi_2) \to \psi), \Omega''}{\Omega', \mathbf{T}(\varphi_2 \to \psi), \Omega'' \mid \Omega', \mathbf{T}(\psi), \mathbf{F}(\varphi_1 \to \varphi_2), \Omega''}$$

$$(\mathbf{F}(\to) \to)\qquad \frac{\Omega', \mathbf{F}((\varphi_1 \to \varphi_2) \to \psi), \Omega''}{\Omega', \mathbf{T}(\varphi_1 \to \varphi_2), \mathbf{F}(\varphi_2 \to \psi), \Omega'' \mid \Omega', \mathbf{F}(\psi), \Omega''}$$

Expansion Rules

Transitivity: $$\frac{\Omega', \mathbf{F}(p \to q), \mathbf{F}(q \to r), \Omega''}{\Omega', \mathbf{F}(p \to q), \mathbf{F}(q \to r), \mathbf{F}(p \to r), \Omega''}$$

Left Maximality: $$\frac{\Omega', \mathbf{F}(p \to q), \mathbf{F}(p), \Omega''}{\Omega', \mathbf{F}(p \to q), \mathbf{F}(p), \mathbf{F}(q), \Omega''}$$

Right Maximality: $$\frac{\Omega', \mathbf{T}(p \to q), \Omega''}{\Omega', \mathbf{T}(p \to q), \mathbf{T}(q), \Omega''}$$

Linearity: $$\frac{\Omega', \mathbf{T}(p \to q), \Omega''}{\Omega', \mathbf{T}(p \to q), \mathbf{F}(q \to p), \Omega''}$$

Minimality of $\bot$: $$\frac{\Omega', \mathbf{F}(p \to \bot), \Omega''}{\Omega', \mathbf{F}(p \to \bot), \mathbf{F}(\bot \to p), \Omega''}$$

DEFINITION 2.

1. *A decomposition tree for a sequence Ω is any tree T with vertices labeled by sequences of signed formulas such that:*

   a) The root of T is labeled by Ω.

   b) If l labeled by Σ is a vertex of T, then:

      i) l is a leaf iff either Σ is a fundamental sequence, or Σ is basic and no expansion rule which introduces some new signed formulas into Σ is applicable to Σ [10]*;*

      ii) If l has sons labeled by $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$, then

      $$\frac{\Sigma}{\Sigma_1 \mid \Sigma_2 \mid \ldots \mid \Sigma_n}$$

      is a rule applicable to Σ.

   c) T is a maximal tree satisfying the above conditions.

2. *A decomposition tree for a formula* [11] *φ is a decomposition tree for the one-element sequence $\mathbf{T}\varphi$.*

[10] The last condition prevents needless expansion of sequences ad infinitum by repeated applications of expansion rules.

[11] Note that by a "formula" we always mean an ordinary, unsigned formula (otherwise we explicitly write "signed formula").

Thus in order to obtain a decomposition tree of φ (or Ω) we label the root by $\mathbf{T}(\varphi)$ (resp. Ω), and then expand the tree by applying first the decomposition rules, followed by the expansion rules. If the rule R applied to the label of a vertex has n premises, then the vertex has n sons, labeled by the n premises of the rule, respectively. If in the course of expanding the branch we get a fundamental sequence, we terminate the branch — successfully. The branch is also terminated, but unsuccessfully, if we get a basic sequence to which no expansion rule that actually expands it (and hence no rule at all) can be applied. Hence every leaf of the decomposition tree is labeled by either a fundamental sequence or a non-fundamental basic sequence. Since the tree is maximal, each such basic sequence must be closed under all expansion rules.

DEFINITION 3.

A decomposition tree for a formula φ (or a sequence Ω) is called a proof of φ (Ω) if it is finite and all its leaves are labeled by fundamental sequences. A formula φ (sequence Ω) is said to be provable if it has a proof.

THEOREM 1. *A formula φ (a sequence Σ) is provable in LC_{RS} iff it is valid in G_ω.*

Proof: It is straightforward to check that all the fundamental sequences of LC_{RS} are valid with respect to G_ω, and that each of its rules is two-way sound in the sense that its conclusion is satisfied by a given valuation v in G_ω if *and only if* all its premises are satisfied by v (this is the strong semantic invertibility mentioned in 2.1). Hence if φ (Σ) is provable in LC_{RS} then it is valid in G_ω. This entails soundness.

In view of the two-way soundness and analycity of the rules of LC_{RS}, for every sequence Ω we can construct — by induction on the complexity of Ω — a finite set S of basic sequences such that:

- Every element of Ω is closed under each of the expansion rules;
- Ω is valid iff all the elements of S are valid;
- If all the elements of S are provable in LC_{RS} then so is Ω.

To prove completeness it suffices therefore to show that if Δ is a non-fundamental basic sequence closed under the expansion rules then Δ is not valid. So let Δ be such a sequence. Then Δ has the following properties:

P1 If $\mathbf{F}(p \to q) \in \Delta$ and $\mathbf{F}(q \to r) \in \Delta$, then $\mathbf{F}(p \to r) \in \Delta$

P2 If $\mathbf{F}(p \to q) \in \Delta$ and $\mathbf{F}(p) \in \Delta$, then $\mathbf{F}(q) \in \Delta$

P3 If $\mathbf{T}(p \to q) \in \Delta$, then $\mathbf{T}(q) \in \Delta$.

P4 If $\mathbf{T}(p \to q) \in \Delta$, then $\mathbf{F}(q \to p) \in \Delta$.

P5 If $\mathbf{F}(p \to \bot) \in \Delta$, then $\mathbf{F}(\bot \to p) \in \Delta$.

P6 If $\mathbf{T}(\psi) \in \Delta$, then $\mathbf{F}(\psi) \notin \Delta$

P7 $\mathbf{F}(\bot) \notin \Delta$

Note that P6 and P7 correspond to the assumption that Δ is not fundamental, whereas P1–P5 follow from the fact that Δ is closed under the five expansion rules (listed in the same order).

Let $p \prec q$ denote either $\mathbf{F}(p \to q)$ or $\mathbf{T}(q \to p)$ [12] . Then Δ has the following crucial property:

(*) There are no $p_1, \ldots, p_n$ such that $p_1 = p_n$, $(p_i \prec p_{i+1}) \in \Delta$ for $1 \leq i \leq n-1$, and $\mathbf{T}(p_j \to p_i) \in \Delta$ for some i, j.

Indeed, if such $p_1, \ldots, p_n$ exist, then by P4 $\mathbf{F}(p_i \to p_{i+1}) \in \Delta$ for $1 \leq i \leq n-1$. Therefore P1 and the equality $p_1 = p_n$ imply that $\mathbf{F}(p_j \to p_i) \in \Delta$ for all i, j. This contradicts P6.

Call now $q_1, \ldots, q_l$ "an $n-$*sequence* for p" if $q_l = p$, $(q_i \prec q_{i+1}) \in \Delta$ for $1 \leq i \leq l-1$, and for n different i's, $\mathbf{T}(q_{i+1} \to q_i) \in \Delta$. Define a valuation v in G_ω as follows: $v(p) = t$ iff $\mathbf{F}(p) \in \Delta$. Otherwise let $v(p)$ be the maximal n for which there exists an $n-$sequence for p (such a maximal n exists by (*)). The valuation v has the following properties:

- If $\mathbf{F}(p) \in \Delta$, then $v(p) = t$ by the definition of v.

- If $\mathbf{T}(p) \in \Delta$, then $v(p) \neq t$ by P6.

- If $\mathbf{F}(p \to q) \in \Delta$, then $v(p) \leq v(q)$. This is obvious if $\mathbf{F}(q) \in \Delta$. If not, then also $\mathbf{F}(p) \notin \Delta$ by P2, and any $n-$sequence for p can be turned into an $n-$sequence for q by adding $\mathbf{F}(p \to q)$ to it.

- If $\mathbf{T}(p \to q) \in \Delta$, then $v(p) > v(q)$. This follows from P3 (together with P6) and the fact that any $n-$sequence for q can be turned into an $(n+1)-$sequence for p by adding $\mathbf{T}(p \to q)$ to it.

[12] The notation reflects the fact that if $p \prec q \in \Delta$, and v refutes Δ, then $v(p) \leq v(q)$. See the list of properties of Δ below.

- $v(\bot) = 0$. Indeed, $v(\bot) \neq t$ by P7. On the other hand, by P1, P4 and P5, if $q_1 \ldots q_l$ is an $n-$sequence for $\bot$ then Δ contains $\mathbf{F}(q_i \to q_j)$ for all i, j. This contradicts (*) for $n > 0$. Hence the only n for which an $n-$sequence for $\bot$ exists is 0.

It immediately follows from these facts that v is a valuation giving a countermodel of Δ. Hence Δ is not valid.

2.3. An Equivalent Gentzen-type Formulation

In (Konikowska, 2000) there is a simple algorithm for translating a given R-S deduction system RS into an equivalent Gentzen-type calculus $G(RS)$ of sequents of ordinary formulas. The algorithm translates a sequence Ω into a sequent $\Gamma \Rightarrow \Delta$, in which Γ consists of the **F**-formulas of Ω and Δ consists of the **T**-formulas of Ω (the signs are omitted in both cases) [13]. The axioms of $G(RS)$ are the translations of the fundamental sequences of RS, and its rules of inference are the obvious translations of the rules of RS (written in "reverse", so that premises of a rule of RS are translated into premises of the corresponding rule of $G(RS)$, and the same applies to conclusions[14]). Hence every proof in RS can be transformed stepwise into a proof in $G(RS)$ of the same formula, and vice versa. In particular, a formula is provable in RS iff its translation is provable in $G(RS)$. Moreover: a sequent is provable in $G(RS)$ iff it is the translation of a provable sequence of RS (and vice versa).

The purely implicational fragment of the calculus GLC_{RS} produced by this algorithm in the case of LC_{RS} is given below.

THE SYSTEM GLC_{RS}.

Axioms: $\varphi \Rightarrow \varphi, \qquad \bot \Rightarrow$

Structural Rules: Weakening and Permutation (on both sides)

Logical Rules

$$(\Rightarrow\to(\to)) \quad \frac{\Gamma \Rightarrow \Delta, \psi_1 \to \psi_2, \varphi \to \psi_2}{\Gamma \Rightarrow \Delta, \varphi \to (\psi_1 \to \psi_2)}$$

$$(\to(\to)\Rightarrow) \quad \frac{\Gamma, \psi_1 \to \psi_2 \Rightarrow \Delta \qquad \Gamma, \varphi \to \psi_2 \Rightarrow \Delta}{\Gamma, \varphi \to (\psi_1 \to \psi_2) \Rightarrow \Delta}$$

[13] Note again that in the translation of tableau systems the signs are exactly opposite.

[14] Recall that the premises of an R-S rule are written *below* the double line, whereas its conclusion is written above it.

$$(\Rightarrow (\to) \to) \quad \frac{\Gamma \Rightarrow \Delta, \varphi_2 \to \psi \qquad \Gamma, \varphi_1 \to \varphi_2 \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, (\varphi_1 \to \varphi_2) \to \psi}$$

$$((\to) \to\Rightarrow) \quad \frac{\Gamma, \varphi_2 \to \psi \Rightarrow \Delta, \varphi_1 \to \varphi_2 \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, (\varphi_1 \to \varphi_2) \to \psi \Rightarrow \Delta}$$

Analytic Omission Rules

Transitivity: $$\frac{\Gamma, p \to q, q \to r, p \to r \Rightarrow \Delta}{\Gamma, p \to q, q \to r \Rightarrow \Delta}$$

Left Maximality: $$\frac{\Gamma, p \to q, p, q \Rightarrow \Delta}{\Gamma, p \to q, p \Rightarrow \Delta}$$

Right Maximality: $$\frac{\Gamma \Rightarrow \Delta, q, p \to q}{\Gamma \Rightarrow \Delta, p \to q}$$

Linearity: $$\frac{\Gamma, q \to p \Rightarrow \Delta, p \to q}{\Gamma \Rightarrow \Delta, p \to q}$$

Minimality of $\bot$: $$\frac{\Gamma, p \to \bot, \bot \to p \Rightarrow \Delta}{\Gamma, p \to \bot \Rightarrow \Delta}$$

THEOREM 2.

1. *GLC_{RS} is sound and complete for LC.*
2. *GLC_{RS} is closed under Contraction and Cut.*

Proof: The first part is just a reformulation of Theorem 1 (given the way GLC_{RS} has been obtained from LC_{RS}). The second part is a corollary of the first.

2.4. A Cut-free Version Without Expansion Rules

From the viewpoint of the usual methodology of Gentzen-type systems, there is a big difference between the logical rules and the omission rules of GLC_{RS}. All the logical rules have what might be called the *semi-subformula property*: written in Polish notation, every formula in their premises either appears in their conclusion or is obtained from some formula there by deleting some of its symbols. This is not very different from the usual subformula property. The omission rules, in contrast, do not have this property (although they are still analytic). These rules are close in nature to what is known in Gentzen-type systems as *analytic cut* (although they are simpler!). Systems with such rules are somewhat less natural in the Gentzen-type framework.

Instead of using omission rules, we can take as an axiom any valid basic sequent (a basic sequent here is a sequent with all elements being either atomic formulas or implications of atomic formulas). This is acceptable if a purely syntactic, constructive characterization of such sequents can be given. The completeness proof for LC_{RS} implicitly includes, in fact, such a characterization. To formulate it explicitly, we need to introduce suitable notation [15]:

DEFINITION 4. *Let* $\Gamma \Rightarrow \Delta$ *be a basic sequent.*

- *We say that* $(p \leq q) \in (\Gamma \Rightarrow \Delta)$ *iff* $(p \rightarrow q) \in \Gamma$.
- *We say that* $(t \leq q) \in (\Gamma \Rightarrow \Delta)$ *iff* $q \in \Gamma$.
- *We say that* $(p < q) \in (\Gamma \Rightarrow \Delta)$ *iff* $(q \rightarrow p) \in \Delta$.
- *We say that* $(q < t) \in (\Gamma \Rightarrow \Delta)$ *iff* $q \in \Delta$.
- *Let* p, q *be either atomic formulas or* t [16].
  We say that $(p \lhd q) \in (\Gamma \Rightarrow \Delta)$ *iff either* $(p \leq q) \in (\Gamma \Rightarrow \Delta)$ *or* $(p < q) \in (\Gamma \Rightarrow \Delta)$.
- *A sequence* $q_1, \ldots, q_l$ *(where* q_i *is either atomic or* t*) is called a* strictly increasing *sequence for* $\Gamma \Rightarrow \Delta$ *if* $(q_j \lhd q_{j+1}) \in (\Gamma \Rightarrow \Delta)$ *for* $1 \leq i \leq l-1$*, and either* $(q_i < q_{i+1}) \in (\Gamma \Rightarrow \Delta)$ *for some* $1 \leq j \leq l-1$*, or* $q_1 = t,\ q_l = \bot$.

Note: It can easily be checked that if $q_1, \ldots, q_l$ is a strictly increasing sequence for $\Gamma \Rightarrow \Delta$, and v refutes $\Gamma \Rightarrow \Delta$, then $v(q_1), \ldots, v(q_l)$ (where $v(q_i) =_{Df} t$ in case q_i is t) is monotonically increasing, but not constant.

THE SYSTEM GLC^*_{RS}.

Axioms: Every basic sequent for which there exists a strictly increasing sequence $q_1 \ldots q_l$ satisfying one of the following:

1. $q_1 = q_l$

[15] The fact that this notation is similar to the notation used in (Baaz and Fermüller, 1999) is no accident, of course. See Section 4.

[16] Note that in this paper t is *not* a symbol of the language of LC.

2. $q_1 = t$
3. $q_l = \bot$

Logical Rules: Like in GLC_{RS}.

THEOREM 3.

1. *GLC^*_{RS} is sound and complete for LC.*
2. *GLC^*_{RS} is closed under Weakening, Contraction and Cut.*

Proof: The first part easily follows from the proof of Theorem 1. The second is again a corollary of the first.

Notes.

1. To avoid the need for the permutation rule, we assume that the sequents of GLC^*_{RS} employ multisets of formulas (rather than sequences) on both sides (that the system is closed under weakening can easily be seen also by a straightforward induction on the length of proofs).

2. GLC^*_{RS} corresponds of course to an alternative R-S system for LC (LC^*_{RS}). In LC^*_{RS} no expansion rules are used. Instead there is a much more extensive set of "fundamental sequences". This is still in full coherence with the R-S methodology, which allows for a tradeoff between rules (especially expansion rules) and fundamental sequences.

3. It can easily be proved (either semantically or proof-theoretically) that GLC^*_{RS} is closed under substitutions. Hence it is possible to extend its set of axioms to include all their substitution instances.

4. Like Dyckhoff's system $G4 - LC$ in (Dyckhoff, 1999), GLC^*_{RS} is contraction-free and terminating.

An advantage of GLC^*_{RS} over GLC_{RS} and $G4 - LC$ is that it can be extended in a straightforward way to any *finite* Gödel logic G_k:

THEOREM 4. *Let GLC^{*k}_{RS} be obtained from GLC^*_{RS} by adding to it as axioms all basic sequents $\Gamma \Rightarrow \Delta$ which have a strictly increasing sequence $q_1 \dots q_l$ such that $(q_i < q_{i+1}) \in (\Gamma \Rightarrow \Delta)$ for at least k different q_i's. Then GLC^{*k}_{RS} is a cut-free, sound and complete system for G_k.*

Proof: It is easy to see that every refutation of the new axioms requires more than k different elements of G_ω. This entails soundness.

For completeness, assume that $\Gamma \Rightarrow \Delta$ is not provable. Call $q_1, \ldots, q_l$ "an $n-$*sequence* for p" if $q_l = p$, $q_1, \ldots, q_l$ is strictly increasing for $\Gamma \Rightarrow \Delta$, and for n different i's, $(q_i < q_{i+1}) \in (\Gamma \Rightarrow \Delta)$. The new axioms ensure that for no p can there be an n-sequence for p with $n \geq k$. Let $v(p) = t$ if either $p \in \Gamma$ or there is a $(k-1)$-sequence for p. Otherwise let $v(p)$ be the maximal n for which there exists an $n-$sequence for p. Following the proof of Theorem 1, it is not difficult to show that v is a refuting valuation for $\Gamma \Rightarrow \Delta$ in G_k. Details are left to the reader.

3. A More Efficient Hypersequent Calculus

In this section we show how the R-S methodology applied in this work can be combined with the use of hypersequents (the data structure used for LC in (Avron, 1991)) to improve the system GLC^*_{RS}. [17] Use of hypersequents rather than ordinary sequents will allow us to have:

- Considerably fewer logical (or "decomposition") rules;
- Fewer types of indecomposable formulas;
- Fewer types of axioms (or "fundamental sequences").

3.1. The System GLC^*

We start by recalling some definitions from (Avron, 1991).

DEFINITION 5. *A (single-succedent) hypersequent is a structure of the form:*

$$\Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2 \mid \cdots \mid \Gamma_n \Rightarrow A_n$$

where $\Gamma_i \Rightarrow A_i$ is an ordinary single-succedent sequent. [18]

Each $\Gamma_i \Rightarrow A_i$ is called a component *of the hypersequent (Note that we do not allow components with an empty succedent, although such can be easily added if so desired).*

We use G, H as variables for (possibly empty) hypersequents, S for sequents. We shall assume that the order of the components in a hypersequent and the order of formulas on the l.h.s. of a component do not matter (i.e.: we again use multisets throughout rather than sequences or sets).

[17] Hypersequents are a generalization of Gentzen-type sequents. Up to now, no work has been done on generalizing R-S systems in an analogous way - but the way towards doing this seems clear, and it will be a subject for future work.

[18] Note that while the symbol | denotes conjunction on the meta-level in R-S systems, in hypersequents it denotes disjunction on the meta-meta-level.

DEFINITION 6. *The interpretation of a standard* sequent *of the form* $A_1, A_2, \ldots, A_n \Rightarrow B$ *is* $A_1 \to (A_2 \to \cdots \to (A_n \to B)\cdots)$. *The interpretation of a hypersequent* $\Gamma_1 \Rightarrow A_1|\cdots|\Gamma_n \Rightarrow A_n$ *is in turn* $\varphi_{\Gamma_1 \Rightarrow A_1} \vee \cdots \vee \varphi_{\Gamma_n \Rightarrow A_n}$, *where* $\varphi_{\Gamma_i \Rightarrow A_i}$ *is the interpretation of* $\Gamma_i \Rightarrow A_i$.

DEFINITION 7. *A hypersequent* G *is called* valid *in* G_ω ($\models_{G_\omega} G$) *if its interpretation is valid in* G_ω.

The main advantage of using here single-succedent hypersequents rather than multiple-succedent sequents is that in LC, like in multi-succedent intuitionistic logic, the classical rule for introducing $\to$ on the right-hand side of $\Rightarrow$ is only sound in general when the conclusion has just a single succedent. It is in fact easy to demonstrate that

$$\models_{G_\omega} G|\Gamma \Rightarrow \varphi \to \psi \text{ iff } \models_{G_\omega} G|\Gamma, \varphi \Rightarrow \psi$$

As a result, the indecomposable formulas are only the atomic ones (on both sides of a component) and basic implications (i.e.: implications of atomic formulas) which occur on the l.h.s. of a component. Such occurrences will be further constrained as follows:

DEFINITION 8.

1. *A* basic sequent *is a sequent of the form:*

$$q_1, \ldots, q_k, p \to r_1, \ldots, p \to r_l \Rightarrow p$$

where $p, q_1, \ldots, q_k$ *and* $r_1, \ldots, r_l$ *are all atomic.*

2. *A hypersequent is called* basic *if each of its components is basic.*

The main properties of basic sequents and hypersequents are given in the following lemma, the easy proof of which we leave to the reader.

LEMMA 1.

1. *A basic sequent* $\Gamma \Rightarrow p$ *is refuted by a valuation* v *in* G_ω *only if* $v(p) < v(q)$ *for all atomic* $q \in \Gamma$, *and* $v(p) \leq v(q)$ *for every* q *such that* $(p \to q) \in \Gamma$. *If* $v(p) \neq t$ *for every atomic variable* p *then the converse is also true.*

2. *A hypersequent is refuted by* v *iff* v *refutes all its components.*

Like GLC^*_{RS}, our new hypersequent calculus consists of logical decomposition rules, together with a purely syntactic, constructive characterization of valid basic hypersequents. For the latter we again need some notations:

DEFINITION 9. *Let p, q be atomic formulas, and let G be a hypersequent.* [19]

- $(p \leq q) \in G$ *iff* $\Gamma, p \to q \Rightarrow p$ *is a component of* G *for some* Γ.

- $(p < q) \in G$ *iff* $\Gamma, q \Rightarrow p$ *is a component of* G *for some* Γ.

- $(p \lhd q) \in G$ *iff either* $(p \leq q) \in G$ *or* $(p < q) \in G$.

- *A sequence* $q_1 \dots q_l$ *(where* q_i *is atomic) is called a* strictly increasing *sequence for* G, *if* $(q_j \lhd q_{j+1}) \in G$ *for* $1 \leq i \leq l-1$, *and* $(q_i < q_{i+1}) \in G$ *for some* $1 \leq i \leq l-1$.

THE SYSTEM GLC^*.

Axioms: Every basic hypersequent for which there exists a strictly increasing sequence $q_1 \dots q_l$ such that one of the following holds:

1. $q_1 = q_l$
2. $q_n = \bot$

Logical Rules:

$$(\Rightarrow \wedge) \quad \frac{G|\Gamma \Rightarrow \varphi \qquad G|\Gamma \Rightarrow \psi}{G|\Gamma \Rightarrow \varphi \wedge \psi}$$

$$(\wedge \Rightarrow) \quad \frac{G|\Gamma, \varphi, \psi \Rightarrow \theta}{G|\Gamma, \varphi \wedge \psi \Rightarrow \theta}$$

$$(\Rightarrow \vee) \quad \frac{G|\Gamma \Rightarrow \varphi|\Gamma \Rightarrow \psi}{G|\Gamma \Rightarrow \varphi \vee \psi}$$

$$(\vee \Rightarrow) \quad \frac{G|\Gamma, \varphi \Rightarrow \theta \qquad G|\Gamma, \psi \Rightarrow \theta}{G|\Gamma, \varphi \vee \psi \Rightarrow \theta}$$

$$(\Rightarrow \to) \quad \frac{G|\Gamma, \varphi \Rightarrow \psi}{G|\Gamma \Rightarrow \varphi \to \psi}$$

[19] Note that t is not needed here!

$$(\to\wedge)\quad \frac{G|\Gamma, \varphi\to\psi_1, \varphi\to\psi_2 \Rightarrow \theta}{G|\Gamma, \varphi\to\psi_1\wedge\psi_2 \Rightarrow \theta}$$

$$(\wedge\to)\quad \frac{G|\Gamma, \varphi_1\to\psi \Rightarrow \theta \qquad G|\Gamma, \varphi_2\to\psi \Rightarrow \theta}{G|\Gamma, \varphi_1\wedge\varphi_2\to\psi \Rightarrow \theta}$$

$$(\to\vee)\quad \frac{G|\Gamma, \varphi\to\psi_1 \Rightarrow \theta \qquad G|\Gamma, \varphi\to\psi_2 \Rightarrow \theta}{G|\Gamma, \varphi\to\psi_1\vee\psi_2 \Rightarrow \theta}$$

$$(\vee\to)\quad \frac{G|\Gamma, \varphi_1\to\psi, \varphi_2\to\psi \Rightarrow \theta}{G|\Gamma, \varphi_1\vee\varphi_2\to\psi \Rightarrow \theta}$$

$$(\to(\to)\quad \frac{G|\Gamma, \varphi\to\psi_2 \Rightarrow \theta \qquad G|\Gamma, \psi_1\to\psi_2 \Rightarrow \theta}{G|\Gamma, \varphi\to(\psi_1\to\psi_2) \Rightarrow \theta}$$

$$(\to)\to)\quad \frac{G|\Gamma, \psi \Rightarrow \theta \qquad G|\varphi_1 \Rightarrow \varphi_2|\Gamma, \varphi_2\to\psi \Rightarrow \theta}{G|\Gamma, (\varphi_1\to\varphi_2)\to\psi \Rightarrow \theta}$$

$$(\to\Rightarrow)\quad \frac{G|\Gamma \Rightarrow r|p\to q \Rightarrow p \qquad G|\Gamma, q \Rightarrow r}{G|\Gamma, p\to q \Rightarrow r}$$

Notes:

1. All the rules of GLC^* are again easily seen to be sound and invertible with respect to G_ω. All of them are decomposition rules, except $\to\Rightarrow$, which has also the flavor of an expansion rule (in the sense of R-S systems).

2. An initial version of GLC^* together with the related theorems were presented in (Avron, 2000). Here we have simplified the characterization of the axioms as well as the proofs.

Examples of axioms: Here is the full list (up to the order of components and names of variables) of the simplest axioms which use at most 3 different variables (the rest can be obtained by repeatedly adding either a new arbitrary basic component, or some indecomposable formula to an existing component):

$$
\begin{array}{rl}
\bot \Rightarrow \bot & p \Rightarrow p \\
\bot \Rightarrow p & p \Rightarrow q | q \Rightarrow p \\
(p \to q) \Rightarrow p | p \Rightarrow q & (p \to \bot) \Rightarrow p | p \Rightarrow q \\
p \Rightarrow q | q \Rightarrow r | r \Rightarrow p & q \to p \Rightarrow q | q \Rightarrow r | r \Rightarrow p \\
(q \to p) \Rightarrow q | (r \to q) \Rightarrow r | r \Rightarrow p & (q \to \bot) \Rightarrow q | (r \to q) \Rightarrow r | r \Rightarrow p
\end{array}
$$

THEOREM 5. *A sequent G is valid iff $\vdash_{GLC^*} G$.*

Proof: Because of the soundness and the semantic invertibility of the rules of GLC^*, for every hypersequent G we can effectively find a finite set $\mathcal{B}$ of basic hypersequents such that G is derivable from $\mathcal{B}$ in GLC^*, and G is valid iff each $H \in \mathcal{B}$ is valid. On the other hand, it easily follows from Lemma 1 that the axioms of GLC^* are valid. Hence it suffices to show that each basic hypersequent which is not an axiom is refutable. Given such a hypersequent G and atomic p, call $q_1, \ldots, q_l$ "an $n-$*sequence* for p" if $q_l = p$, $(q_i \lhd q_{i+1}) \in G$ for $1 \leq i \leq l-1$, and for n different i's, $(q_i < q_{i+1}) \in G$. Let $v(p)$ be the maximal n for which there exists an $n-$sequence for p. Like in the proof of Theorem 1, it easily follows from the fact of G not being an axiom that v is a well-defined valuation in G_ω (in particular: $v(\bot) = 0$). Moreover: $v(p) \neq t$ for every atomic p. Hence we may apply Lemma 1, and an argument similar to that used in the proof of Theorem 1, to show that v indeed refutes G.

COROLLARY 1.

1. *The cut rule and all the standard structural rules are admissible in GLC^*.*

2. *A formula φ is valid in LC iff $\Rightarrow \varphi$ has a proof in GLC^*.*

Again a corresponding system for G_k can easily be obtained:

THEOREM 6. *Let GLC^{*k} be obtained from GLC^* by adding to it as axioms all hypersequents G which have a strictly increasing sequence $q_1 \ldots q_n$ such that $(q_i < q_{i+1}) \in G$ for at least k different q_i's. Then GLC^{*k} is a cut-free, sound and complete system for G_k.*

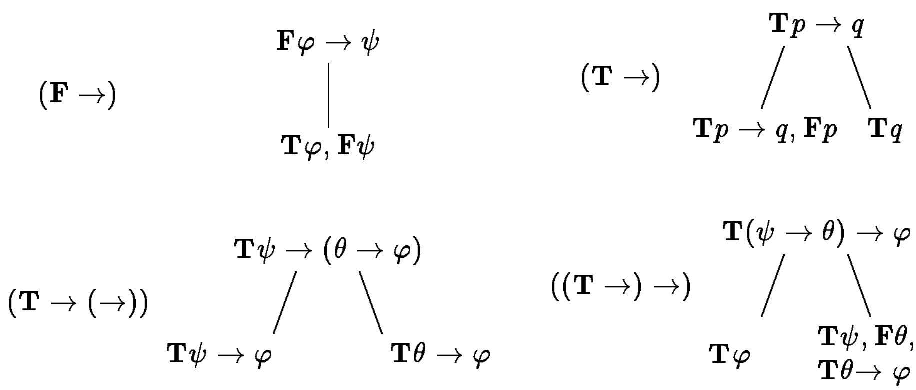
Proof: The proof is similar to that of Theorem 4.

3.2. A Tableau System for LC based on GLC^*

In order to develop a tableau system for proof search in GLC^*, we represent a hypersequent G by a set S_G of signed formulas with links between them. An occurrence of $\mathbf{T}\varphi$ in S_G means that φ occurs on the l.h.s. of at least one component of G, while an occurrence of $\mathbf{F}\varphi$ in S_G means that φ is the r.h.s of at least one component of G[20]. A link from $\mathbf{T}\varphi$ to $\mathbf{F}\psi$ means that φ occurs on the l.h.s. of a component of G in which ψ is the r.h.s. Thus every signed formula of the form $\mathbf{T}\varphi$ in S_G is linked to at least one signed formula of the form $\mathbf{F}\varphi$.

In our tableau system, the set of signed formulas on some branch, to which no reduction rule has yet been applied on that branch, represents (together with the links between its elements) a hypersequent that we need to prove. In principle, a branch is closed if it contains a set of signed formulas (together with links) which represents a substitution instance of an axiom. In practice we may apply this test only when no reduction rule can be applied on that branch (the branch represents in such a case a basic hypersequent). Before this stage is reached, we will close a branch in some simple cases only (for example: when the branch contains a formula of the form $\mathbf{T}\bot$, or a pair $\mathbf{T}\varphi, \mathbf{F}\varphi$ with a link between these two formulas).

As usual, the rules of our system replace some signed formula on a branch by other (usually simpler) ones. In addition, they also change the set of links on that branch. The figure below contains all the purely implicational rules of the tableaux system. The full list can be found in (Avron, 2000), from which the material presented in this subsection has been taken.



We next describe the changes in links that each rule causes. Using the results of the previous section, it is easy to show that the resulting tableau system is sound and complete.

[20] Note that unlike in Section 1, we use here the standard notation of tableaux.

$(\mathbf{T} \to (\to))$: Every formula that was linked to $\mathbf{T}\psi \to (\theta \to \varphi)$ before the application of the rule should be linked to the new $\mathbf{T}\psi \to \varphi$ and to the new $\mathbf{T}\theta \to \varphi$ after it.

$(\mathbf{F} \to)$: Every formula that was linked to $\mathbf{F}\varphi \to \psi$ before the application of the rule should be linked to the new $\mathbf{F}\psi$ after it. In addition, a link should be added between the new $\mathbf{T}\varphi$ and the new $\mathbf{F}\psi$.

$(\mathbf{T} \to)$: The new $\mathbf{T}p \to q$ should be linked to the new $\mathbf{F}p$. The new $\mathbf{T}q$ should be linked to all the formulas other than $\mathbf{F}p$ to which $\mathbf{T}p \to q$ was linked before the application of the rule.

$(\mathbf{T}(\to) \to)$: The new $\mathbf{T}\psi$ should be linked to the new $\mathbf{F}\theta$. The new $\mathbf{T}\varphi$ and the new $\mathbf{T}\theta \to \varphi$ should be linked to the formulas to which $\mathbf{T}(\psi \to \theta) \to \varphi$ was linked before the application of the rule.

Note. In the terminology of the previous section, the indecomposable formulas of this system are those of the form $\mathbf{T}p, \mathbf{F}p$ and $\mathbf{T}p \to q$[21]. Almost all its rules are strict decomposition rules. As noted above, the only exception is $(\mathbf{T} \to)$, which is more an expansion rule. $(\mathbf{T} \to)$ should indeed be applied only if no other rule is applicable, and only if p, q are atomic, $p \neq q$, $p \neq \bot$, and there is a formula different from $\mathbf{F}p$ to which $\mathbf{T}p \to q$ is linked on that branch. We note also that in practice no rule should be applied to signed formulas of the forms $\mathbf{T}\bot \to \psi$ and $\mathbf{T}\psi \to \psi$, and such formulas should simply be ignored.

Detailed examples of the use of the above tableaux calculus are given in (Avron, 2000).

4. Conclusion and Comparison with Other Works

In this paper we have introduced three new cut-free calculi for *LC*. In these calculi the great advantage that the hypersequent system *GLC* of (Avron, 1991) has over all other known systems for *LC* is lost: *GLC* has exactly the same *logical* rules as intuitionistic logic, and the differences between the two logics is (according to *GLC*) only with respect to the *structural rules.* This is no longer true for the calculi of this paper, since all of them employ several logical rules which are not intuitionistically valid. Another nice property of *GLC* which is partially lost here is the pure subformula property. Instead, we have only the semi-subformula property, which is a little bit less pure and elegant. For *understanding*

[21] They correspond, respectively, to $\mathbf{F}(p), \mathbf{T}(p)$ and $\mathbf{F}(p \to q)$ of the R-S formulation.

LC, and for reasoning about it, GLC is therefore better (in our opinion) than our new systems. On the other hand, these systems are much more suitable for proof search than GLC. This is due to the fact that unlike GLC, they are terminating, contraction-free, and all their rules are invertible.

Among our three systems, the hypersequent system GLC^* is definitely more efficient and economical than GLC^*_{RS}. It is less clear which of GLC_{RS} and GLC^* is superior. The advantages of GLC_{RS} are that it uses ordinary sequents (rather than hypersequents), and its axioms are much simpler. Whether this compensates for the need to use several omission (or "expansion") rules should perhaps best be judged on an experimental basis.

It is interesting to compare our systems here with two other systems that have recently (and independently) been introduced with the same purpose:

– The system $G4-LC$ of (Dyckhoff, 1999) (which is an improved version of the system in (Avellone et al., 1999)) has a lot in common with GLC_{RS}. Like GLC_{RS}, it uses ordinary, multiple-succedent sequents. The two systems have the same axioms, and both are cut-free, terminating, contraction-free, and with invertible rules only. In both cases, this is achieved by giving up the pure subformula property (and having the semi- subformula property instead). Moreover: they both use decomposition rules in an essential way (some of these rules being identical). However, the principal rule of $G4-LC$ (the one that allows inference of the characteristic axiom of LC) does not have a fixed number of premises. Moreover: the corresponding tableau rule requires analyzing several formulas *simultaneously*, so it has a global character (although it *is* local according to the formal definition of that term in (Troelstra and Schwichtenberg, 2000)). In GLC_{RS}, in contrast, all the rules are strictly local: the corresponding tableau system analyzes just one formula at a time. Another important shortcoming of $G4-LC$ is that its principal rule may be applied only if its conclusion cannot possibly be obtained by any other rule of the system (the rule will still be sound if this side condition is removed, but then it will not be invertible any more). As a result, its set of valid proofs is not closed under substitution of formulas for propositional variables. In GLC_{RS}, in contrast, all the rules are pure, with no side conditions, they are completely independent of each other and remain sound (in both directions) under substitutions. We hope that this advantage will make it easier to extend GLC_{RS} (as well as GLC^*) to the first-order case (something which is a little problematic for

$G4-LC$. See (Dyckhoff, 1999)). This, however, should be checked in the next stage of this research.

- The system $\mathbf{R}G_\infty$ of (Baaz and Fermüller, 1999) seems similar to GLC^* in that it employs hypersequents rather than ordinary sequents. Its main advantage over the systems of this work is that it has the pure subformula property. This, however, is achieved only at the cost of using *two* types of components: $A \leq B$ and $A < B$ (with the obvious semantical interpretations in G_ω), and being forced to use the constant 1 (our t) in the system (even for axiomatizing just the pure implicational fragment of LC!). Now despite of the apparent use of hypersequents, $\mathbf{R}G_\infty$ is actually equivalent to GLC^*_{RS}, since a hypersequent G of $\mathbf{R}G_\infty$ can be translated into a sequent $\Gamma \Rightarrow \Delta$ as follows:

  1. If $1 \leq B$ is a component of G then $B \in \Delta$
  2. If $B \leq 1$ is a component of G then $B \rightarrow B \in \Delta$
  3. If $1 < B$ is a component of G then $B \rightarrow B \in \Gamma$
  4. If $B < 1$ is a component of G then $B \in \Gamma$
  5. If $A \leq B$ is a component of G $(A, B \neq 1)$ then $A \rightarrow B \in \Delta$
  6. If $A < B$ is a component of G $(A, B \neq 1)$ then $B \rightarrow A \in \Gamma$

  It can easily be checked that this translation makes the axioms and rules of $\mathbf{R}G_\infty$ identical to those of GLC^*_{RS} [22]. We believe, however, that the fact that the implementation of GLC^*_{RS} does not involve implementing a new logical framework, makes that system superior to $\mathbf{R}G_\infty$ from a practical point of view (although from the viewpoint of proof search there is of course no difference between the two versions). Moreover: since GLC^* is obviously more efficient than GLC^*_{RS}, it is (in our opinion, at least) also more efficient than the equivalent $\mathbf{R}G_\infty$.

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[22] Note that one of the rules in (Baaz and Fermüller, 1999), $(\supset:<:r)$, is erroneous. This has hidden the equivalence for a while. The authors of (Baaz and Fermüller, 1999) have confirmed that the correct rule, as computed by their algorithm, should have been the translation of the rule $(\rightarrow(\rightarrow)\Rightarrow)$ of GLC^*_{RS}.

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