SOME PROPERTIES OF LINEAR LOGIC PROVED BY SEMANTIC METHODS

Arnon Avron *

Department of Computer Science
Raymond and Beverly Sackler
Faculty of Exact Sciences
Tel Aviv University
Tel Aviv, Israel

e-mail: aa@math.tau.ac.il

Abstract

We construct several simple algebraic models of the multiplicative and multiplicative-additive fragments of linear logic and demonstrate the value of such models by proving some unexpected proof-theoretical properties of these fragments.

I. Introduction

The research described below started when we considered the following simple question: Is there a sentence A of the multiplicative fragment of Linear Logic such that $\Rightarrow A, A$ is a theorem of (the Gentzen-type version of) this logic? (The motivation for considering this question will be explained below, see III.10 (3)). After failing to produce an example of such a sentence we have tried to show by a proof-theoretical argument that it cannot

* Part of this research was done while the author was visiting the Computer Science Department of Edinburgh University, and was supported by grants from the British Academy of Science and the UK Science and Engineering Research Council (Visiting Fellowship number GR/G 55471).

exist. Again, we failed (This had been due, in fact, to a lack of insight, since Danos and Schellinx did provide later such an argument[1]).

In situations like this in logic one usually tries to apply *semantical* methods. Indeed a fruitful investigation of a logical system almost always consists of two complementary parts: semantical and proof-theoretical. Ideally, there should be a perfect match between the two, but even a partial match (like having only soundness) is frequently of great value. In linear logic, however, (as far as I know), no serious attempt has been made up to now to use semantic considerations in order to solve strict proof-theoretical problems.[2] The structures that were investigated in this area were of a very abstract character (like the "phase" semantics of [Gi87]) and were never applied for solving concrete problems that are not directly related to them. We have felt, therefore, that the time has come to develop *concrete* structures that can serve as models for linear logic and be helpful for solving problems about it like the one we described above.

This paper has, accordingly, two purposes. The first is to develop classes of simple, concrete models of linear logic.[3] The second is to demonstrate their value by using them to show several somewhat unexpected features of the multiplicative and the multiplicative-additive fragments of linear logic. Among other things we show, e.g., that there is no sentence A of the multiplicative language such that $\Rightarrow A, \ldots, A$ is a theorem (except, of course, sequents of the form $\Rightarrow A$), that there is no multiset of theorems $[A_1, \ldots, A_n]$ ($n \geq 2$) of the multiplicative fragment such that $\Rightarrow A_1, \ldots, A_n$ is also a theorem, and that a sequent of the form $\Gamma \Rightarrow \Gamma$ is provable in the multiplicative-additive fragment iff the multiset Γ is a singleton (as a multiset).

It is our hope that the investigations below will finally lead to a class of concrete

---

[1] In a message to the LINEAR mbox on February 7, 1992. In that note they proved also, using their basic argument, Theorems III.4, III.6 and the "only if" part of Theorem III.8 below. They did not provide alternative proofs for the new results in section 4 (like IV.2.3.), but they were not aware of those results then. Their argument, as it is, applied only to the multiplicative fragment.

[2] Perhapse the case is put too strongly here. It depends on what one might call "strict proof-theoretical problems". A paper like [BCST], for example, certainly treats proof-theoretical problems- but not of the kind I mean. The examples below illustrate, I hope, what I have in mind.

[3] For an example of what I have in mind when I speak about "concrete models" versus "abstract models", think of the two-valued model of classical propositional logic versus the class of Boolean algebras.

models of linear logic for which we have both soundness and completeness.

II. Preliminaries

II.1 Notations. We shall usually use those which were employed in [Av88]. In particular, we shall use $\to, +, \wedge$, and $\vee$ for, respectively, linear implication, "par", "with" and "plus". While in [Av88] this was a matter of habit (following the tradition in relevance logic), in the present context these notations turn out to be particularly suggestive. We shall, in addition, treat negation as an independent connective, denoted by $\neg$. We shall, however, follow [Gi87] in using the symbols $\otimes, 1$ and $\perp$ for the multiplicative conjunction and the two multiplicative propositional constants.

The full propositional linear logic will be denoted by LL, while its multiplicative and multiplicative-additive fragments (without the propositional constants) will be denoted by LL_m and LL_a, respectively. $LL_m^{\perp}$ and $LL_a^{\perp}$ denote the corresponding fragments *with* the *multiplicative* constants.

II.2 Gentzen-type formulation $GLL_a^{\perp}$. We shall use the conventional two-sided version in which we have multisets on both sides of $\Rightarrow$. The rules are:

$$A \Rightarrow A$$

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad\qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad \text{(cut)}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \quad (\Rightarrow \neg)$$

$$(+\Rightarrow) \quad \frac{A, \Gamma_1 \Rightarrow \Delta_1 \qquad\qquad B, \Gamma_2 \Rightarrow \Delta_2}{A + B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A + B} \quad (\Rightarrow +)$$

$$(\vee \Rightarrow) \quad \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} \quad (\Rightarrow \vee)$$

$$(\perp \Rightarrow) \qquad \perp \Rightarrow \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \perp} \qquad (\Rightarrow \perp)$$

(The rules for $\otimes, \wedge$ and 1 are dual to those for, respectively $+, \vee$ and $\perp$. These three connectives can be taken as defined ones.)

II.3 The Associated Consequence Relation. The linear consequence relation (denoted $\vdash^{\ell}_{LL}$ in [Av88]) which is usually associated with linear logic (and corresponds to the sequents of GLL) is not an ordinary consequence relation in the Tarskian sense. It is not suitable, therefore, for semantical investigations of the usual type. We shall use therefore instead the "extensional" consequence relation $\vdash_{LL}$ which was introduced in [Av88]. It can be given any of the following characterizations (the equivalence between these characterizations is shown in [Av88] or easily follows from results proved there).

(1) $T \vdash_{LL} A$ iff there are a subset $\{B_1, \ldots, B_n\}$ of T and a proof (in the ordinary sense) of $\Rightarrow A$ from $\{\Rightarrow B_1, \ldots, \Rightarrow B_n\}$.

(2) $T \vdash_{LL} A$ iff A is true in every model (in the sense of the "phase semantics" of [Gi87]) of T.

(3) $T \vdash_{LL} A$ iff there exists a proof (in the ordinary sense) of A from T in the Hilbert-type system HLL of [Av88].

(4) (i) In the multiplicative fragment (or any extension of it) $T \vdash_{LL^{\perp}_m} A$ iff A belongs to any superset of T (in its language) which includes all (instances of) theorems of $LL^{\perp}_m$ and is closed under MP for $\rightarrow$. The same applies to LL_m.

(ii) In the multiplicative-additive fragment $T \vdash_{LL^{\perp}_a} A$ if A belongs to any superset of T which includes all theorems of $LL^{\perp}_a$ and is closed under MP for $\rightarrow$ and adjunction for $\wedge$ (from A and B infer $A \wedge B$). The same applies to LL_a.

The following *deduction theorem* for LL_m ($LL^{\perp}_m$) can easily be proved:

$$T, A \vdash_{LL^{\perp}_m} B \qquad \text{iff} \qquad T \vdash_{LL_m} \overbrace{A \rightarrow (A \rightarrow \ldots \rightarrow (A}^{n \text{ times}} \rightarrow B))$$

for some $n \geq 0$ (hence $A_1, \ldots, A_n \vdash_{LL_m} B$ iff there exist (possibly empty) multisets $\Gamma_1, \ldots, \Gamma_n$ s.t. Γ_i consists only of A_i's ($i = 1, \ldots, n$) and $\vdash_{GLL_m} \Gamma_1, \Gamma_2, \ldots, \Gamma_n \Rightarrow B$).

II.4 General Algebraic Semantics. In this paper we shall use algebraic semantics for various fragments of Linear Logic. By this we mean that semantics will be given by valuations in algebraic structures. These structures have operations which correspond to the connectives of the language, and they are equipped with a "truth subset" of "designated values" (and/or an order relation which reflects one of the consequence relations of the logic). Examples are Boolean algebras and classical logic, Heyting algebras and intuitionistic logic, and Modal algebras (see [BS84]) and modal logics.[4]

In [Av88] the following general algebraic structures were introduced relative to which fragments of LL are strongly complete (with respect to their *extensional* consequence relation). The main idea in the next subsections is to use *concrete* instances of these structures

Basic Relevant disjunction structures. These are structures of the form $\overline{D}=\langle D, \leq, \neg, +\rangle$ where

(i) $\langle D, \leq\rangle$ is a poset

(ii) $\neg$ is an involution on $\langle D, \leq\rangle$ (i.e. $\neg\neg a = a$ and $a \leq b \Rightarrow \neg b \leq \neg a$ for all a, b)

(iii) $+$ is an associative, commutative, order-preserving operation on $\langle D, \leq\rangle$

(iv) $a \leq b + c \Rightarrow \neg b \leq \neg a + c$ for all a, b, c

(v) $a + \neg(\neg b + b) \leq a$ for all a, b.

Basic Relevant disjunction lattices. These are relevant disjunction structures in which $\langle D, \leq\rangle$ is a lattice.

Relevant disjunction monoids. These are structures $\overline{D}=\langle D, \leq, \neg, +, \perp\rangle$ which satisfy conditions (i)-(iv) in the definition of basic relevant disjunction structures and in which $\perp$ is an identity element (with respect to $+$). It is shown in [Av90] that such structures are basic relevant disjunction structures, i.e. they also satisfy condition (v).

Additive Relevant disjunction monoids. These are relevant disjunction monoids in which $\langle D, \leq\rangle$ is a lattice.

In all these structures, if we define the subset of designated values T_D to be $\{a \in D \mid \neg a + a \leq a\}$ (in the monoids this is equivalent to $\{a \in D \mid a \geq 1\}$ where $1 = \neg\perp$) then we

[4] Kripke-style semantics can always be presented in this form as well.

have the following strong completeness results:

(1) LL_m is strongly complete w.r.t. basic relevant disjunction structures (i.e. $T \vdash_{LL_m} \varphi$ iff for any such structure $\overline{D}$ and for any valuation v in D, if $v(\psi) \in T_D$ for every $\psi \in T$ then $v(\varphi) \in T_D$ as well).

(2) LL_m and $LL_m^{\perp}$ are strongly complete w.r.t. relevance disjunction monoids.

(3) LL_a is strongly complete w.r.t. basic relevant disjunction lattices.

(4) LL_a and $LL_a^{\perp}$ are strongly complete w.r.t. additive relevant disjunction monoids.

Notes. (1) In each of the above classes the subclass of structures with a maximal and a minimal element provides an adequate semantics for the appropriate system together with the *additive* constants.

(2) Relevant disjunction monoids in which $\{D, \leq\}$ is a *complete* lattice were called in [Av88] "Girard structures". The full additive-multiplicative linear logic is strongly complete w.r.t. to them, and they can be used to provide semantics to the full first-order LL (including the "exponentials"). This semantics is equivalent to the "phase semantics" of [Gi87] (see [Av88] for more details).

(3) Another general semantical framework which seems to have close relationships to that of [Av88] but was developed independently is that of *Weakly Distributive Categories* (see [CS91] and [Ba90]). [5]

III. Semantics in the integers

We start with the following obvious observation: if we take the two truth values of classical logic to be 0 and 1 then the operations which correspond to the connectives are defined by $\neg a = 1 - a$, $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$. Moreover, A is a tautology iff $v(A) = 1$ (or $v(A) \geq 1$) for every valuation v. The idea of this section is to extend these definitions to the whole set of integers, taking ordinary addition as the operation which

[5] The works of Barr, Cockett and Seely use heavy categorical terminology, while [Av88] uses pure algebraic terminology. This seems to be the reason why the two approaches have been developed without awareness of each other. Although the similarity is by now obvious, the exact correspondence still need to be worked out.

corresponds to the connective $+$. [6]

III.1 Definition. An integral model is a function v from the set of sentences of $LL_a^{\perp}$ to the integers for which:

(i) $v(\neg A) = 1 - v(A)$

(ii) $v(A + B) = v(A) + v(B)$

(iii) $v(A \vee B) = \max\big(v(A), v(B)\big)$

(iv) $v(1) = 1$

Notes. (1) The derived equations for the other connectives are:

(v) $v(A \otimes B) = v(A) + v(B) - 1$

(vi) $v(A \rightarrow B) = 1 - v(A) + v(B)$

(vii) $v(A \wedge B) = \min\big(v(A), v(B)\big)$

(viii) $v(\perp) = 0$

(2) The definitions above (especially that of $v(A+B)$) provide the justification of our remark in the introduction that the notations in [Av88] are especially suggestive in the present context.

III.2 Definition. We say that an integral model v *validates* a proposition A if $v(A) > 0$ (i.e. $v(A) \geq 1$) and that v *strongly validates* A if $v(A) = 1$.

Note. As promised, in this definition we really use two concrete instances of the general structures described in II.4. It is easy to check that $\langle Z, =, \lambda x.1 - x, +, 0\rangle$ is a relevant disjunction monoid (where Z is the set of integers), while $\langle Z, \leq, \lambda x.1 - x, +, 0\rangle$ is an additive relevant disjunction monoid (where $\leq$ is the usual order relation of the integers). The set of designated values is $\big\{y \in Z \mid y = (\lambda x.1 - x)(0)\big\} = \{1\}$ in the first case and $\{y \in Z \mid y \geq 1\}$ in the second. Strong validity refers to valuations in the first structure, validity – to valuations in the second one.

[6] I have learnt from P. Lincoln that a similar, though less general, idea has been used by Girard with connection to the variable-free fragment of $LL_m^{\perp}$. Structures which are practically identical to the integral models which are defined below and their various generalizations (like the c-models of IV.2.1) were independently introduced also in [CS91] and in [Ba91]. In [CS91] they are called "interpretations of the initial model in shift monoids on Z."

III.3 Theorem. (1) *If $T \vdash_{LL_a^\perp} A$ and v validates all the sentences of T then v validates A.* (2) *If $T \vdash_{LL_m^\perp} A$ and v strongly validates all the sentences of T then v strongly validates A.*

Proof: This is, in fact, an easy corollary of the last note and section II.4. For the sake of being self-contained, we present here a direct proof.

Extend first any integral model v to sequents by defining $v(A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m)$ to be $\sum_{i=1}^{n} \big(1 - v(A_i)\big) + \sum_{j=1}^{m} v(B_j)$ (i.e. $v(A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m) = v(\neg A_1 + \ldots + \neg A_n + B_1 + \ldots + B_m)$) and $v(\Rightarrow) = 0$. Obviously $v(A \Rightarrow A) = 1$ for all A, and also $v(\perp \Rightarrow) = 1$. We next show that if $\Gamma \Rightarrow \Delta$ can be derived from $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ by one of the rules of $GLL_a^\perp$ and $v(\Gamma_i \Rightarrow \Delta_i) \geq 1$ $(i = 1, \ldots, n)$ then $v(\Gamma \Rightarrow \Delta) \geq 1$ and that if the rule is one of the multiplicative rules (including cut) and $v(\Gamma_1 \Rightarrow \Delta_i) = 1$ $(i = 1, \ldots, n)$ then $v(\Gamma \Rightarrow \Delta) = 1$. The cases of the introduction rules on the right are all very easy. The case of $(\neg \Rightarrow)$ follows from the identity $1 - (1 - a) = a$. The cases of $(+ \Rightarrow)$ and $(\vee \Rightarrow)$ are easy corollaries of the following 3 obvious observations:

(i) $v(\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2) = v(\Gamma_1 \Rightarrow \Delta_1) + v(\Gamma_2 \Rightarrow \Delta_2)$

(ii) $v(A, \Gamma \Rightarrow \Delta) = 1$ iff $v(A) = v(\Gamma \Rightarrow \Delta)$

(iii) $v(A, \Gamma \Rightarrow \Delta) \geq 1$ iff $v(A) \leq v(\Gamma \Rightarrow \Delta)$.

As for cut, assume, e.g., that $v(A, \Gamma_1 \Rightarrow \Delta_1) = 1$ and that $v(\Gamma_2 \Rightarrow \Delta_2, A) = 1$. Then by (ii), $v(A) = v(\Gamma_1 \Rightarrow \Delta_1)$, and by (i) $v(\Gamma_2 \Rightarrow \Delta_2) + v(A) = 1$. It follows that $v(\Gamma_1 \Rightarrow \Delta_1) + v(\Gamma_2 \Rightarrow \Delta_2) = 1$ and so $v(\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2) = 1$, by (i) again. The proof that if $v(A, \Gamma_1 \Rightarrow \Delta_1) \geq 1$ and $v(\Gamma_2 \Rightarrow \Delta_2, A) \geq 1$ then $v(\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2) \geq 1$ is similar.

What we show so far shows that if $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n \vdash_{GLL_a^\perp} \Gamma \Rightarrow \Delta$ and v validates $\Gamma_i \Rightarrow \Delta_i$ $(i = 1, \ldots, n)$ then v validates $\Gamma \Rightarrow \Delta$ and that if $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n \vdash_{GLL_m^\perp} \Gamma \Rightarrow \Delta$ and v strongly validates $\Gamma_i \Rightarrow \Delta_i$ $(i = 1, \ldots, n)$ then v strongly validates $\Gamma \Rightarrow \Delta$. This immediately entails the theorem, by the first characterization of $\vdash_{LL}$ in II.3.

We present now several (somewhat unexpected) applications of the last theorem.

III.4 Theorem. *There are no sentences A and B such that A, B and $A + B$ are all theorems of $LL_m^\perp$. Moreover, if $n \geq 2$ and $\Gamma_i \Rightarrow \Delta_i$ is provable in $GLL_m^\perp$ $(i = 1, \ldots, n)$ then $\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n$ is not.*

Proof: By part (2) of Theorem III.3, $v(\Gamma_i \Rightarrow \Delta_i) = 1$ for $1 \leq i \leq n$ and so $v(\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n) = \sum_{i=1}^{n} v(\Gamma_i \Rightarrow \Delta_i) = n \geq 2 \neq 1$. Hence $\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n$ is not provable. This shows the second part. The first part is an immediate corollary.

III.5 Note. In [Gi87] the following "mix" rule (which we call "combining" below) was considered (and rejected) as a sensible addition to linear logic: from $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. The last corollary means that not even a single instance of this rule is admissible in $LL_m^{\perp}$. This is not true for LL_a. If $\vdash_{LL_a} A$ then both $\Rightarrow A \vee \neg A$ and $\Rightarrow A \vee \neg A, A \vee \neg A$ are provable in GLL_a (the first from $\Rightarrow A$ by $\Rightarrow \vee$. The second is provable for every A). This is also a counterexample to the first part of III.4.

Still, there are interesting special cases in which combining is never admissible even in LL_a. See IV.2.3 below.

III.6 Theorem. *There is no A in the language of $LL_m^{\perp}$ such that $A + A + \ldots + A$ is provable in linear logic (unless, of course, the number of summands is 1).*

Proof: Let v be any integral model. Then $v(A + A + \ldots + A) = n \cdot v(A)$ and this cannot be 1 unless $n = 1$. Hence the theorem follows from III.3(2).

III.7 Note. Again, the claim is valid only for the multiplicative fragment. $B + B$ is provable in LL_a for any B of the form $A \vee \neg A$.

Theorem III.6 can be strengthened as follows:

III.8 Theorem. *Let $[A]_n$ denote the multiset which consists of n copies of A. Given m and k there exists a sentence A of $LL_m^{\perp}$ such that the sequent $[A]_m \Rightarrow [A]_k$ is provable in $GLL_m^{\perp}$ (or just GLL_m) iff $m - 1$ is divisible by $m - k$ (we take here 0 to be divisible by any number, including 0).*

Proof: Suppose such an A exists. Let v be any valuation, and let $v(A) = a$. By Theorem III.3(2) it follows that $m(1 - a) + ka = 1$. Hence $m - 1 = a(m - k)$.

For the converse, assume first that $m \geq k$ and that there exists an integer a such that $m - 1 = a(m - k)$. Let $a(P \to P)$ denote $(P \to P) + \ldots + (P \to P)$ where the number of summands is a (the case $a = 0$, i.e.: $m = k = 1$ or $m = 1$ and $k = 0$, is easy).

Then $\vdash_{GLL_m} \left[a(A \to A)\right]_m \Rightarrow \left[a(A \to A)\right]_k$. Indeed, starting with the provable sequents $P, [P \to P]_m \Rightarrow P$ and using $P, P \to P \Rightarrow P$ in $(a-1)m$ applications of $(+\Rightarrow)$ we get a proof of:

$$[P]_{m(a-1)+1}, \left[a(P \to P)\right]_m \Rightarrow [P]_{m(a-1)+1} \ .$$

But from the definition of a it follows that $m(a-1)+1 = ka$. Hence we have a proof of $[P]_{ka}$, $\left[a(P \to P)\right]_m \Rightarrow [P]_{ka}$. From this it is easy to get a proof of $\left[a(P \to P)\right]_m \Rightarrow \left[a(P \to P)\right]_k$.

If $k > m$ then we use the fact that $m-1$ is divisible by $m-k$ iff $k-1$ is divisible by $k-m$. The first case implies then that $\vdash_{GLL_m} \left[a(P \to P)\right]_k \Rightarrow \left[a(P \to P)\right]_m$ (where $a = (k-1)/(k-m)$) and so $\vdash_{GLL_,} \left[\neg a(P \to P)\right]_m \Rightarrow \left[\neg a(P \to P)\right]_k$.

III.9 Theorem. $\vdash_{LL_m^\perp} A$ *iff* $\{\neg A\}$ *is an inconsistent theory (of* $LL_m^\perp$*).*

Proof: The "only if" part is trivial. Suppose now that $\{\neg A\}$ is inconsistent. Then for some B, $\{\neg A\} \vdash_{LL_m^\perp} B$ and $\{\neg A\} \vdash_{LL_m^\perp} \neg B$. This entails, by the deduction theorem (see II.3) that there are provable sequents in $GLL_m^\perp$ of the forms: $\neg A, \neg A, \ldots, \neg A \Rightarrow B$ and $\neg A, \ldots, \neg A \Rightarrow \neg B$. Using cut it follows that there is a provable sequent in $GLL_m^\perp$ of the form $\neg A, \ldots, \neg A \Rightarrow$ and so also of the form $\Rightarrow A, \ldots, A$. But by III.8 (or just III.6) the only provable sequent of this form in $GLL_m^\perp$ can be $\Rightarrow A$. Hence $\vdash_{LL_m^\perp} A$.

III.10 Notes. 1) Again, the corollary fails in LL_a: $\left\{\neg(\neg p \vee p)\right\}$ is not consistent (it implies both $\neg p \vee p$ and its negation) but $\neg p \vee p$ is not provable.

2) In classical logic we have, of course, that for *every* theory T, $T \vdash A$ iff $T \cup \{\neg A\}$ is inconsistent. In the intentional (= multiplicative) relevance logics this is true in case T is *consistent*. The last theorem means that in multiplicative linear logic the *empty* theory still has this property. This cannot be improved, though, since even consistent theories which are singletons do not necessarily have this property. Thus if we take $T = \{p + p\}$ (i.e., $T = \{\neg p \to p\}$), then $T \cup \{\neg p\}$ is not consistent, although T is (even classically), and $T \nvdash_{LL_m^\perp} p$. This last claim can be shown using the deduction theorem: if $T \vdash p$ then there would be a provable sequent of the form $p+p, p+p, \ldots, p+p \Rightarrow p$. It is not difficult to show, however that in every provable sequent of $GLL_m^\perp$ every atomic formula occurs an even number of times (an easier semantical proof is given in IV.1).

3) An attempt to show III.9 provided the original motivation for considering the question: when can sentences of the form $A + A + \ldots + A$ be provable in LL_m?

IV. Variations and Further Constructions

IV.1 Using the real numbers. Instead of using the integers in definitions III.1 and III.2, we can use the rationals or the reals, provided we choose the condition $v(A) \geq 1$ (rather than $v(A) > 0$) in the definition of validation. Theorem III.3 still obtains in this case. This possibility can be used, e.g., for an easy proof that $p + p \not\vdash_{L_m^\perp} p$ (compare III.10(2)). Simply take $v(p) = 1/2$.

IV.2 Using other designated values. In the previous section we choose 1 (following classical logic) as the main designated value (the only one for $LL_m^\perp$, the minimal one for LL_m). This, however, is not the only possibility.

IV.2.1 Definition. Let c be an integer. A c-model is a function v as in III.1, except that we replace condition (i) and (iv) by:

(i)* $v(\neg A) = c - v(A)$

(iv)* $v(1) = c$.

We say that a c-model v c-validates A if $v(A) \geq c$ and strongly c-validates it if $v(A) = c$.

Notes. (1) the integral models of III.1 are just 1-models.

(2) Again we could use the rationals or the reals instead of the integers, but the integers are the most fruitful choice.

(3) c-models are again simple instances of the general structures of II.4.

IV.2.2 Theorem. *Parts (1) and (2) of Theorem III.3 are true if we read everywhere "(strongly) c-validates" for "(strongly) validates".*

Proof: Similar to that of III.3.

We present now applications of the last theorem which, for the first time, use part (1) and are about $LL_a^\perp$:

IV.2.3 Theorem. *A sequent of the form* $A_1, \ldots, A_n \Rightarrow A_1, \ldots, A_n$ *(i.e.,* $\Gamma \Rightarrow \Gamma$*) is provable in* $GLL_a^{\perp}$ *iff* $n = 1$*. More general: If* $\Gamma_i \Rightarrow \Delta_i$ $(i = 1, \ldots, n)$ *are all instances of provable sequents of* $GLL_m^{\perp}$ *then* $\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n$ *is provable in* $GLL_a^{\perp}$ *iff* $n = 1$ *(in particular: no sentence of the form* $(A \otimes B) \to (A + B)$ *is provable in* $GLL_a^{\perp}$*).*

Proof: The assumptions imply that in any c-model $v(\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n)$ is nc for every c. But $nc < c$ for $c < 0$.

IV.2.4 Theorem. *Let* S *be a set of atomic formulas or negations of atomic formulas such that* $\{\neg p, p\} \subseteq S$ *for no* p*. Assume that* Γ *is a set of sentences such that for any* $A \in \Gamma$*, either* A *is an instance of a theorem of* $LL_m^{\perp}$*, or* $\neg A$ *is such an instance or* A *is built from the formulas in* S *using* $+, \otimes, \vee$ *and* $\wedge$*. Assume further that* Γ *includes at least one sentence of the third type. Then* $\Rightarrow \Gamma$ *is not provable in* $GLL_a^{\perp}$*.*

Proof: Take $c = 0$, and let $v(p) = -1$ if $p \in S$, $v(p) = 1$ otherwise. It is easy to see that $v(A) < 0$ for every A in Γ of the third type. On the other hand $v(A) = 0$ $(= c)$ if A or $\neg A$ are instances of theorems of $LL_m^{\perp}$. It follows that $v(\Gamma) < 0$ and so $\Rightarrow \Gamma$ is not provable, by part (1) of IV.2.2.

IV.2.5 Notes. The case $c = 0$ is somewhat strange, since in this case $v(1) = v(\perp)$ $(= 0)$ and $v(A + B) = v(A \otimes B)$. Still, it was useful in the last theorem. Another application is a very easy proof that $\{\neg p, p\} \not\vdash_{LL_a^{\perp}} q$: take $c = 0$, $v(p) = 0$, $v(q) = -1$. Taking $c < 0$ allows, on the other hand, a short proof that $p \not\vdash_{GLL_a^{\perp}} p + p$ (recall that this is not equivalent to $\not\vdash p \to (p + p)$!). Take $c = -1$, $v(p) = -1$. Finally, taking $c = 2$ and $v(p) = 1$ provides an alternative proof (using only the integers) that $p + p \not\vdash_{GLL_a^{\perp}} p$.

IV.3 Finite models in the multiplicative case. Instead of considering the integers we could have considered just $\{0, 1, \ldots, n - 1\}$ with operations $+$ and $\neg$ defined modulo n. It is easy to see that the proof of part (2) of IV.2.2 goes through without a change. The various theorems about the multiplicative fragments could then be proved using these models. In fact, they could be proved to show something stronger. Thus not only can we construct a model in which $A + A + \ldots A$ ($n \geq 2$ summands) is not true, but a model in which it is *false*, i.e., $\neg(A + \ldots + A)$ is true. For this, we just take $c = 1$ (say) and the integers modulo n. In this model $v\big(\neg(A + \ldots + A)\big) = 1$.

An interesting special case of this class of finite models is when $n = 2$. We may view the resulting structure as a new interpretation of the multiplicative linear connectives in classical logic. $+$ is interpreted as *excluded or*, while linear implication as classical *equivalence*. It follows that as long as only the multiplicative fragment is considered, linear implication can be understood as either classical implication or classical equivalence and "par", respectively, as either inclusive or exclusive "or" – and in both cases we get classical tautologies.

IV.4 More complicated models. None of the models we considered so far (or all of them together) is complete for LL_m or LL_a. In particular: $(A \to A) \to (B \to B)$ is valid in all of them, although not provable in linear (or relevance) logic. Using the method of [Av90] we can easily construct appropriate countermodels (even with the *additive* constants, but not with the multiplicative ones) as follows:

IV.4.1 Theorem. *Let* $\big\{ \langle D_\alpha, \leq_\alpha, \neg_\alpha, +_\alpha \rangle \big\}_{\alpha \in I}$ *be a set of basic relevant disjunction structures (lattices) such that* $D_\alpha \cap D_\beta = \emptyset$ *if* $\alpha \neq \beta$. *Define a new structure* $\mathcal{D} = \langle D, \leq, \neg, + \rangle$ *as follows:*

1) $D = \bigcup\limits_{\alpha \in x} D_\alpha \cup \{T, F\}$ *where* T, F *are two new objects not belonging to* $\bigcup\limits_{\alpha \in I} D_\alpha$.
2) *If* $a, b \in D$ *then* $a \leq b$ *iff either* $b = T$ *or* $a = F$ *or* a, b *belong to the same* D_α *and* $a \leq_\alpha b$.
3) $\neg T = F$, $\neg F = T$, $\neg a = \neg_\alpha a$ *if* $a \in D_\alpha$.
4) $a + T = T + a = T$, $a + F = F + a = F$ *if* $a \neq T$, $a + b = F$ *if* $a \in D_\alpha$, $b \in D_\beta$ *and* $\alpha \neq \beta$, $a + b = a +_\alpha b$ *if* $a, b \in D_\alpha$.

Then $\mathcal{D}$ is also a basic relevant disjunction structure (lattice). Moreover, $\mathcal{D}$ has a maximal and a minimal element, and $T_D = \{T\} \cup \bigcup\limits_{\alpha \in I} T_{D_\alpha}$. We leave the easy (though tedious) proof to the reader.

IV.4.2 Corollary. *If* A *and* B *have no propositional variable in common then* $A + B$ *is not provable in* LL_a. *In particular,* $(p \to p) \to (q \to q)$ *is not provable.*

Proof: Take any two disjoint basic relevant disjunction lattices $\mathcal{D}_1$ and $\mathcal{D}_2$, and combine them using the construction of the previous theorem. By that theorem and II.4, we get a model $\mathcal{D}$ of LL_a together with the additive constants. Now let v be any valuation in $\mathcal{D}$ which associates with each propositional variable of A a value in $\mathcal{D}_1$, and each propositional variable of B a value of $\mathcal{D}_2$. Then $v(A+B) = F$ and $F \notin T_D$.

V. Conclusion

IV.4.2 can, of course, easily be shown by proof-theoretical methods. In [AB] one can also find a semantic proof (using a matrix) of IV.4.2 for the system R, which is stronger than $LL_a^\perp$. The main importance of IV.4.1 is that it provides a general method of constructing new concrete models of LL_a from given ones which might have new properties. IV.4.2 is just a demonstration of the potential power of such a construction. It is our hope that using constructions of this sort we shall finally be able to provide a *useful*, concrete model (or set of concrete models) which will *characterize* Linear Logic. So far, however, this goal has not been achieved.

References

[AB75] Anderson A.R. and Belnap N.D. **Entailment**, vol. 1, Princeton University Press, Princeton, NJ, 1975.

[Av88] Avron A, *The Semantics and Proof Theory of Linear Logic*, Theoretical Computer Science, vol. 57 (1988), pp. 161-189.

[Av90] Avron A, *Relevance and Paraconsistency – A New Approach*, Journal of Symbolic Logic, vol. 55 (1990), pp. 707-732.

[Ba90] Barr M. *Accessible categories and Models of Linear Logic*, Journal of Pure and Applied Algebra, vol. 69 (1990), pp. 219-232.

[Ba91] Barr M. *Fuzzy Models of Linear Logic*, preprint, McGill University, 1991.

[BCST] Blute R.F., Cockett J.R.B., Seely R.A.G. and Trimble T.H., *Natural Deduction and Coherence for Weakly Distributive Categories*, preprint, McGill University, 1992.

[BS84] Bull R. and Segerberg K. *Basic Modal Logic* in **Handbook of Philosophical Logic**, ed. by D. Gabbay and F. Guenthner, vol. II, Reidel Publishing Company (1984), pp. 1-88.

[CS91] Cockett, J.R.B. and Seely R.A.G. *Weakly distributive categories*, in Fourman M.P., Johnstone P.T., Pitts A.M., eds: **Applications of Categories to Computer Science**, London Mathematical Society Lecture Note Series 177 (1992), pp. 45-65

[Gi87] Girard J.Y. *Linear Logic*, Theoretical Computer Science, vol. 50 (1987), pp. 1-102.