The Semantics and Proof Theory of Linear Logic

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Abstract

Linear logic is a new logic which was recently developed by Girard in order to provide a logical basis for the study of parallelism. It is described and investigated in [Gi]. Girard's presentation of his logic is not so standard. In this paper we shall provide more standard proof systems and semantics. We shall also extend part of Girard's results by investigating the consequence relations associated with Linear Logic and by proving corresponding *strong* completeness theorems. Finally, we shall investigate the relation between Linear Logic and previously known systems, especially Relevance logics.

1 Introduction

Linear logic is a new logic which was recently developed by Girard in order to provide a logical basis for the study of parallelism. It is described and investigated in [Gi]. As we shall see, it has strong connections with Relevance Logics. However, the terminology and notation used by Girard completely differs from that used in the relevance logic literature. In the present paper we shall use the terminology and notations of the latter. The main reason for this choice is that this terminology has already been in use for many years and is well established in books and papers. Another reason is that the symbols used in the relevantists work are more convenient from the point of view of typing.

The following table can be used for translations between these two systems of names and notations:

Girard	*Relevance logic*
Multiplicative	*Intensional, Relevant*
Additive	*Extensional*
Exponential	*Modal*
With (&)	*And* ($\wedge$)
Plus ($\oplus$)	*Or* ($\vee$)
Entailment ( )	*Entailment* ($\rightarrow$)
Par ( )	*Plus* ($+$)
Times ($\otimes$)	*Cotenability* ($\circ$)
$1, \bot$	t, f
$!, ?$	$\Box, \Diamond$

2 Proof theory

2.1 Gentzen systems and consequence relations

The proof-theoretical study of linear logic in [Gi] concentrates on a Gentzen-type presentation and on the notion of a Proof-net which is directly derivable from it. This Gentzen-type formulation is obtained from the system for classical logic by deleting the structural rules of contraction and weakening. However, there are many versions in the literature of the Gentzen rules for the conjunction and disjunction. In the presence of the structural rules all these versions are equivalent. When one of them is omitted they are not. Accordingly, two kinds of these connectives are available in Linear Logic (as well as in Relevance Logic):

- The intensional ones ($+$ and $\circ$), which can be characterized as follows:

$$? \vdash \Delta, A, B \quad \text{iff} \quad ? \vdash \Delta, A + B$$

$$?, A, B \vdash \Delta \quad \text{iff} \quad ?, A \circ B \vdash \Delta$$

- The extensional ones ($\vee$ and $\wedge$), which can be characterized as follows:

$$? \vdash \Delta, A \wedge B \quad \text{iff} \quad ? \vdash \Delta, A \;\text{ and }\; ? \vdash \Delta, B$$

$$A \vee B, ? \vdash \Delta \quad \text{iff} \quad A, ? \vdash \Delta \;\text{ and }\; B, ? \vdash \Delta$$

In [Av2] we show how the standard Gentzen-type rules for these connectives are easily derivable from this characterization. We characterize there the rules for the intensional connectives as *pure* (no side-conditions) and those for the extensional ones as impure. [1] The same rules, essentially, were used also by Girard. He preferred, however, to use a variant in which only one-side sequents are employed, and in which the negation connective can directly be applied only to atomic formulas (the negation of other formulas being defined by De-Morgan rules, including double-negation).[2] This is convenient

[1]As explained in [Av2], this distinction is crucial from the implementation point of view. It explains, e.g. why Girard has found the intensionals (or multiplicatives) much easier to handle than the extensionals (additives).

[2]This variant is used also in [Sw] for the classical system.

for introducing the proof-nets that he has invented as an economical tool for developing Gentzen-type proofs in which only the active formulas in an application of a rule are displayed. For the purposes of the present paper it is better however to use the more usual presentation.

Girard noted in [Gi] that he had given absolutely no meaning to the concept of a "linear logical theory" (or any kind of an associated consequence relation). Hence the completeness theorem he gave in his paper is of the weak kind. It is one of our main goals here to remedy this. For this we can employ two methods that are traditionally used for associating a consequence relation with a Gentzen-type formalism. In classical and intuitionistic logics the two methods define the same consequence relation. In Linear Logic they give rise to two different ones:

The internal consequence relation ($\vdash^{LL}_{\mathrm{KI}}$) : $A_1, \ldots, A_n \vdash^{LL}_{\mathrm{KI}} B$ iff the corresponding sequent is derivable in the linear Gentzen-type formalism.[3]

The external consequence relation ($\vdash_{LL}$) : $A_1, \ldots, A_n \vdash_{LL} B$ iff the sequent $\Rightarrow B$ [4] is derivable in the Gentzen-type system which is obtained from the linear one by the addition of $\Rightarrow A_1, \ldots, \Rightarrow A_n$ as axioms (and taking cut as a primitive rule).

It can easily be seen that these two consequence relations can be characterized also as follows:

- $A_1, \ldots, A_n \vdash^{LL}_{\mathrm{KI}} B$ iff $A_1 \to (A_2 \to (\cdots(A_n \to B)\ldots))$ is a theorem of Linear Logic.
- $A_1, \ldots, A_n \vdash_{LL} B$ iff $? \vdash^{LL}_{\mathrm{KI}} B$, for some (possibly empty) multiset ? of formulas each element of which is:

  Intensional (multiplicative) fragment: identical to one of the A_i's.

  The full propositional fragment: identical to $A_1 \wedge A_2 \wedge \cdots \wedge A_n \wedge t$.

In what follows we shall use both consequence relations. We start by developing a natural deduction presentation for the first and a Hilbert-type presentation for the second.

2.2 Natural deduction for Linear Logic

Prawitz-style rules:

$$\frac{\begin{array}{c}[A]\\ \sim B \quad B\end{array}}{\sim A} \qquad \frac{\sim\sim A}{A}$$

[3] Since linear logic has the internal disjunction +, it suffices to consider only single-conclusioned consequence relations.

[4] We use $\Rightarrow$ as the formal symbol which separates the two sides of a sequent in a Gentzen type calculus and $\vdash$ to denote (abstract) consequence relations.

$$\frac{\begin{array}{c}[A]\\ B\end{array}}{A \to B} \qquad \frac{A \quad A \to B}{B}$$

$$\frac{A \quad B}{A \circ B} \qquad \frac{A \circ B \quad \begin{array}{c}[A, B]\\ C\end{array}}{C}$$

$$t \qquad \frac{A \quad t}{A}$$

$$\frac{A \quad B}{A \wedge B}(*) \qquad \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B}$$

$$\frac{A}{A \vee B} \quad \frac{B}{A \vee B} \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ C\end{array} \quad \begin{array}{c}[B]\\ C\end{array}}{C}(**)$$

Most of the above rules look almost the same as those for classical logic. The difference is due to the *interpretation of what is written.* For Linear logic we have:

1. We take the assumptions as coming in *multisets*. Accordingly, *exactly one* occurrence of a formula occurring inside [ ] is discharged in applications of $\sim Int$, $\to Int$, $\circ Elim$ and $\vee Elim$. The consequences of these rules may still depend on other occurrences of the discharged formula!

2. Discharging the formulas in [ ] is *not* optional but *compulsory.* Moreover: the discharged occurrences should *actually be used* in deriving the corresponding premiss. (In texts of relevance logics it is customary to use "relevance indices" to keep track of the (occurrences of) formulas that are really used for deriving each item in a proof.)

3. For $\wedge Int$ we have the side condition that A and B should depend on exactly the same multi-set of assumptions (condition (*)). Moreover, the shared hypothesis are considered as appearing *once*, although they seem to occur *twice*.

4. For $\vee Elim$ we have the side condition that apart from the discharged A and B the two C's should depend on the same multiset of assumptions ((**)). Again, the shared hypothesis are considered as appearing *once*.

5. The elimination rule for t might look strange for one who is accustomed to usual N.D. systems. One should then realize that the premiss A and the conclusion A might differ in the multiset of assumptions on which they depend!

Notes:

1. Again we see that the rules for the extensional connectives are *impure*, while those for the intensional ones are pure (no side-conditions!)

2. the rule for $\sim Int$ is different from the classical (or intuitionistic) one, since no occurrence of A on which B depends is discharged. In fact we have that the dual:

$$\frac{\sim B \quad \begin{matrix}[A]\\ B\end{matrix}}{\sim A}$$

   is derivable, but the classical version

$$\frac{\begin{matrix}[A]\\ \sim B\end{matrix} \quad \begin{matrix}[A]\\ B\end{matrix}}{\sim A}$$

   is *not valid!*

3. It is not difficult to prove a normalization theorem for the positive fragment of this system. As usual, this is more problematic when negation is included. This case might be handled by adding the above derived introduction rule as primitive and then replacing the given elimination rule with the two rules which are obtained from the introduction rules by interchanging the roles of A and $\sim A$.

It is easier to see what's going on if the N.D. system is formulated in **sequential form**:

$$A \vdash A$$

$$\frac{?_1, A \vdash \sim B \quad ?_2 \vdash B}{?_1, ?_2 \vdash \sim A} \qquad \frac{? \vdash \sim\sim A}{? \vdash A}$$

$$\frac{?_1 \vdash A \quad ?_2 \vdash B}{?_1, ?_2 \vdash A \circ B} \qquad \frac{?_1 \vdash A \circ B \quad ?_2, A, B \vdash C}{?_1, ?_2 \vdash C}$$

$$\frac{?, A \vdash B}{? \vdash A \to B} \qquad \frac{?_1 \vdash A \quad ?_2 \vdash A \to B}{?_1, ?_2 \vdash B}$$

$$\frac{? \vdash A \quad ? \vdash B}{? \vdash A \wedge B} \qquad \frac{? \vdash A \wedge B}{? \vdash A} \quad \frac{? \vdash A \wedge B}{? \vdash B}$$

$$\frac{? \vdash A}{? \vdash A \vee B} \quad \frac{? \vdash B}{? \vdash A \vee B} \qquad \frac{A, ? \vdash C \quad B, ? \vdash C \quad \Delta \vdash A \vee B}{?, \Delta \vdash C}$$

$$\vdash t \qquad \frac{?_1 \vdash A \quad ?_2 \vdash t}{?_1, ?_2 \vdash A}$$

Again $?, \Delta$ denote *multisets* of formulas. Note also that *weakening is not allowed*, i.e $? \vdash A$ does not imply $?, \Delta \vdash A$.

Lemma: The N.D system has the following properties:

1. If $?_1 \vdash A$ and $A, ?_2 \vdash B$ then $?_1, ?_2 \vdash B$
2. If $?_1, A \vdash B$ and $?_2 \vdash\sim B$ then $?_1, ?_2 \vdash\sim A$ (but from $?_1, A \vdash B$ and $?_2, A \vdash\sim B$ does *not* follow that $?_1, ?_2 \vdash\sim A$!)
3. If $? \vdash A$ then $? \vdash\sim\sim A$
4. If $?, A \vdash B$ then $?, \sim B \vdash\sim A$

Definition: Let $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ be a sequent. An *interpretation* of it is any single-conclusion sequent of one of the following forms:

$$A_1, \ldots, A_n, \sim B_1, \ldots, \sim B_{i-1}, \sim B_{i+1}, \ldots \sim B_m \vdash B_i \quad (1 \leq i \leq m)$$

$$A, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n, \sim B_1, \ldots, \sim B_m \vdash\sim A_i \quad (1 \leq i \leq n)$$

Theorem: $? \Rightarrow \Delta$ is provable in the Gentzen-type system iff any interpretation of it is provable in the N.D system. Moreover, $? \vdash^{LL}_{\mathrm{Kl}} A$ iff there is a proof of A from ? in this N.D. system.

We leave the proof of both the last theorem and the lemma above to the reader.

2.3 Hilbert systems and deduction theorems for the intensional fragment

There is a standard method for obtaining from a given *pure* N.D. formalism an equivalent Hilbert-type system with $M.P$ as the only rule of inference: [5] One needs first to introduce some purely implicational axioms which suffice for proving an appropriate deduction theorem. The second step is then to replace the various rules by axioms in the obvious way. For example, a rule of the form:

$$\frac{\begin{matrix}[A_{1,1}, A_{1,2}] & [A_2] & \\ B_1 & B_2 & B_3\end{matrix}}{C}$$

will be translated into the axiom:

$$(A_{1,1} \rightarrow (A_{1,2} \rightarrow B_1)) \rightarrow ((A_2 \rightarrow B_2) \rightarrow (B_3 \rightarrow C)))$$

Obviously the first part of this procedure is the more difficult one. This is especially true when a non-standard logic is treated. Accordingly, we start by formulating some intuitive versions of the deduction theorem: [6]

Classical-intuitionistic: There is a proof of $A \rightarrow B$ from the *set* ? iff there is a proof of B from $? \cup \{A\}$.

$RMI_{\rightarrow}$**:** There is a proof of $A \rightarrow B$ from the *set* ? which *uses all* the formulas in ? iff there is a proof of B from $? \cup \{A\}$ which uses all formulas in $? \cup \{A\}$.

[5] See e.g. [Ho], pp. 32

[6] The names $R_{\rightarrow}$ and $RMI_{\rightarrow}$ below are taken from [AB] and [Av1].

$R_{\rightarrow}$ **(Implicational fragment of R):** There is a proof of $A \rightarrow B$ from the *multiset* ? in which every (occurrence of) formula in ? is *used at least once* iff there is such a proof of B from the *multiset* ?, A.

$HL_{\rightarrow}$ **(Implicational fragment of Linear logic):** There is a proof of $A \rightarrow B$ from the multiset ? in which every formula of ? is *used exactly once* iff there is such a proof of B from the *multiset* ?, A.

As we said above, these are intuitive formulations. They involved references to "the number of times (an occurrence of) a formula is used in a given proof". This notion can be made precise, but it easier (and more illuminating) to take the different versions as referring to stricter and stricter notions of a "proof".

In the following assume Hilbert-type systems with M.P as the only rule of inference:

A Classical (or intuitionistic) proof is a sequence (or directed graph) of formulas such that each formula in it is either an axiom of the system , or an assumption, or follows from previous ones by M.P..

A S-strict (M-strict) proof is a classical proof in which every (occurrence of a) formula other than the last is used *at least once* as a premiss of M.P..

A linear proof is a classical proof in which every *occurrence* of formula other than the last is used *exactly once* as a premiss of M.P..

Examples:

- A classical but not strict proof of A from $\{A, B\}$:
  1. B (ass.).
  2. A (ass.).
- S-strict proof which is not M-strict:
  1. A (ass.).
  2. A (ass.).
  3. $A \rightarrow A$ (axiom) [7]
  4. A ($\{1, 2\}, \{3\}$, M.P.).
- A M-strict proof which is not linear:
  1. A (ass.)
  2. $A \rightarrow B$ (ass.)

[7] $A \vdash A$ according to all notions of proof. Hence $A \rightarrow A$ should be a theorem according to all the versions of the deduction theorem.

3. $A \to (B \to C)$ (ass.)
4. B (1,2, M.P.)
5. $B \to C$ (1,3, M.P.)
6. C (4,5, M.P.)

- A linear proof of C from $A \to B,\ B \to C,\ A$:

  1. $A \to B$ (ass.)
  2. $B \to C$ (ass.)
  3. A (ass.)
  4. B (1,3 M.P.)
  5. C (2,4 M.P.)

- A linear proof of A from {A,B} in classical logic:

  1. $A \to (B \to A)$ (axiom)
  2. A (ass.)
  3. $B \to A$ (1,2 M.P.)
  4. B (ass.)
  5. A (3,4 M.P.)

Definition: We say that B is *classically (M-strictly, S-strictly, linearly) provable from* $A_1 \ldots, A_n$ iff there is a classical (M-strict, S-strict, linear) proof in which B is the last formula and $A_1, \ldots, A_n$ are (exactly) the assumptions.

Alternatively, these consequence relations may be characterized as follows:

Classical-intuitionistic:

1. $? \vdash A$ whenever A is an axiom or $A \in ?$.
2. If $?_1 \vdash A \to B$ and $?_2 \vdash A$ then $?_1 \cup ?_2 \vdash B$ (here $?, ?_1, ?_2$ are sets of formulas).

S-strict:

1. $\{A\} \vdash A$.
2. $\emptyset \vdash A$ if A is an axiom.
3. If $?_1 \vdash A \to B$ and $?_2 \vdash A$ then $?_1 \cup ?_2 \vdash B$.

M-strict:

1. $A \vdash A$.
2. $\emptyset \vdash A$ if A is an axiom.

3. If $?_1 \vdash A \to B$, $?_2 \vdash A$ and $?$ is a *contraction* of $?_1, ?_2$ then $? \vdash B$. (Here $?_1, ?_2, ?$ are multisets)

Linear:

1. $A \vdash A$.
2. $\emptyset \vdash A$ if A is an axiom.
3. If $?_1 \vdash A \to B$, $?_2 \vdash A$ then $?_1, ?_2 \vdash B$ (Again $?_1, ?_2$ are multisets).

The last example above indicates that with enough axioms, every classical proof can be converted into a linear one. What is really important concerning each notion of a proof is (therefore) to find a *minimal* system for which the corresponding deduction theorem obtains (with the obvious correspondence between the various notions of a proof and the various notions of the deduction theorem). *Accordingly we define:*

$H_\to$ **(Intuitionistic Implicational calculus):** The minimal system for which the classical deduction theorem obtains. It corresponds to the notion of a classical proof.

$RMI_\to$ **(Dunn-McColl):** The minimal system corresponding to S-strict proofs (i.e. it is the minimal system for which there is a S-strict proof of $A \to B$ from $?$ iff there is a S-strict proof of B from $? \cup \{A\}$.)

$R_\to$ **(Church):** The minimal system corresponding to M-strict proofs.

$HL_\to$: The minimal system corresponding to linear proofs.

The first example shows that $A \to (B \to A)$ should be a theorem of $H_\to$, the second that $A \to (A \to A)$ should be a theorem of $RMI_\to$, the third that $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ should be a theorem of $R_\to$, the forth that $(A \to B) \to (B \to C) \to (A \to C)$ should be a theorem of $HL_\to$. It is also possible to show, using Gentzen-type formulations, that $\nvdash_{RMI_\to} A \to (B \to A)$, $\nvdash_{R_\to} A \to (A \to A)$ and $\nvdash_{HL_\to} A \to (B \to C) \to (A \to B) \to (A \to C)$.

Our next step is to present formal systems for these four logics. In all of them *M.P. is the only rule of inference*:

$HL_\to$ **(linear):**

I. $A \to A$ (reflexivity)

B. $(B \to C) \to ((A \to B) \to (A \to C))$ (transitivity)

C. $(A \to (B \to C)) \to (B \to (A \to C))$ (permutation)

$R_\to$ **(M-strict):** I., B., C., and either of:

S. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

W. $(A \to (A \to B)) \to (A \to B)$ (contraction)

$RMI_{\rightarrow}$ **(S-strict):** Replace axiom I. of $R_{\rightarrow}$ by:

Mingle. $A \rightarrow (A \rightarrow A)$

$H_{\rightarrow}$ **(intuitionistic):** Replace axiom I. of $R_{\rightarrow}$ by:

K. $A \rightarrow (B \rightarrow A)$ (weakening)

or just take **K.** and **S.** as axioms.

The proofs that these systems really are the minimal systems required are all similar and not very difficult. For the case of $R_{\rightarrow}$ and $RMI_{\rightarrow}$ they essentially can be found in [AB] or [Du]. [8] The deduction-theorem parts are provable by induction on the length of the various types of proof. The minimality- by providing a proof of the needed type of B from $A_1, \ldots, A_n$ whenever an axiom has the form $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$. (Some of these proofs were given in the examples above).

Note. The names I, B, C, S, W and K are taken from combinatory logic. It is well known that $H_{\rightarrow}$ corresponds to the typed λ-calculus (which in turn, can be defined in terms of the combinators K and S) while $R_{\rightarrow}$ corresponds to the typed λI-calculus. $HL_{\rightarrow}$ may be described as corresponding to "Linear λ-calculus", based on the combinators I, B and C. It is not difficult also to directly translate the notion of a "Linear proof" into a corresponding notion of a "Linear λ-term".

Once we have the system $HL_{\rightarrow}$ at our disposal we can produce a Hilbert-type formulation of the intensional fragment of Linear logic exactly as described above. All we have to do is to add to $HL_{\rightarrow}$ the axioms:

N1 $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$

N2 $\sim\sim A \rightarrow A$

∘1 $A \rightarrow (B \rightarrow A \circ B)$

∘2 $(A \circ B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C)$

t1 t

t2 $t \rightarrow (A \rightarrow A)$

We call the system which corresponds to the $\{\rightarrow, \sim, \circ, +\}$ fragment of Linear logic HL_m. It can be axiomatized by adding N1 and N2 to $HL_{\rightarrow}$. ∘1 and ∘2 are derivable in the resulting system if we define $A \circ B$ as $\sim (A \rightarrow \sim B)$. Alternatively we can (conservatively) add [∘1] and [∘2] to HL_m and prove that $A \circ B$ is equivalent to $\sim (A \rightarrow \sim B)$ in the resulting system. As for $+$ —it is definable in these systems as $\sim A \rightarrow B$, but it is difficult to treat it independently of $\sim$ in the N.D. and Hilbert-type contexts. (By this we

[8] The notions of S-strict and M-strict proofs were not explicitly formulated there, though, but I believe that they provide the best interpretation of what *is* done there.

mean that it is difficult to characterize it by axioms and rules in which only $+$ and $\rightarrow$ occurre). On the other hand t is not definable in HL_m, but $t1$ and $t2$ can conservatively be added to HL_m to produce HL_m^t. (The other intensional constant, f, is of course equivalent to $\sim t$). Once we have HL_m and HL_m^t, it is easy to prove that A is a theorem of either iff it is derivable in the corresponding $N.D.$ system. Moreover, using the characterizations given to $\vdash_{LL}$ and $\vdash_{\mathrm{KI}}^{LL}$ in the previous section it is straightforward to prove:

Theorem:

1. Let ? be a *multiset* of formulas in the language of HL_m (HL_m^t). Then ? $\vdash_{\mathrm{KI}}^{LL} A$ iff there is a *linear* proof of A from ? in HL_m (HL_m^t), (in which ? is exactly the multiset of assumptions).

2. Let ? be a set of formulas in the language of HL_m (HL_m^t). Then ? $\vdash_{LL} A$ iff there is a classical proof of A from ? in HL_m (HL_m^t) in which ? is the set of assumptions used. (This is equivalent to saying that ? $\vdash_{HL_m} A$ in the usual sense).

Note: The last theorem provides alternative characterizations for the external and internal consequence relations which correspond to the intensional (mulplicative) fragment of Linear Logic. While those which were given in 2.1 are rather general, the one given here for the internal consequence relation is peculiar to linear logic. As a matter of fact, one can define a corresponding notion of a linear consequence relation for *every* Hilbert-type system. What is remarkable here is that for the present fragment the linear and the internal consequence relations are identical. (The internal consequence relation was defined in 2.1 relative to gentzen-type systems. It can independently be defined also for Hilbert-type systems which have an appropriate implication connective).

The intensional fragments of the Relevance logic R $(R_{\underset{\rightarrow}{\sim}}$ and $R^t_{\underset{\rightarrow}{\sim}})$ are obtained from $R_{\underset{\rightarrow}{\sim}}$ exactly as the corresponding fragments of Linear logic are obtained from $HL_{\rightarrow}$. The corresponding (cut free) Gentzen-type formulations are obtained from those for Linear logic by adding the contraction rule (on both sides). *All* the facts that we have stated about the Linear systems are true (and were essentially known long ago) also for these fragments of R, provided we substitute "M-strict" for "Linear". Similarly, if we add to $RMI_{\rightarrow}$ the axioms N1-N2 (and if desired also $\circ$1 and $\circ$2) we get $RMI_{\underset{\rightarrow}{\sim}}$. This system corresponds to the Gentzen-type system in which also the converse of contraction is allowed, so the two sides of a sequent can be taken as *sets* of formulas. However, exactly as the addition of N1 and N2 to $H_{\rightarrow}$ is not a conservative extension, so $RMI_{\underset{\rightarrow}{\sim}}$ is not a conservative extension of $RMI_{\rightarrow}$. Moreover, the addition of $t1$ and $t2$ to $RMI_{\underset{\rightarrow}{\sim}}$ is not a conservative extension of the latter either, so $RMI^t_{\underset{\rightarrow}{\sim}}$ is significantly stronger than $RMI_{\underset{\rightarrow}{\sim}}$. (For more details see, e.g., [Av3]).

2.4 The extensional fragment

The method of the previous section works nicely for the intensional (multiplicative) fragment of Linear Logic. It cannot be applied as it is to the other fragments, though. The problem is well known to relevant logicians and is best exemplified by the extensional (additive) conjunction. If we follow the procedure of the previous section we should add to HL_m the following three axioms for $\wedge$:

$\wedge$ **Elim1:** $A \wedge B \rightarrow A$

$\wedge$ **Elim2:** $A \wedge B \rightarrow B$

$\wedge$ **Int:** $A \rightarrow (B \rightarrow A \wedge B)$

Once we do this, however, K becomes provable and we get the full effect of weakening.

The source of this problem is of course the impurity of the introduction rule for $\wedge$ in the N.D. system for Linear Logic. The side condition there is not reflected in $\wedge$ **Int** (and in fact, this axiom is not derivable in Linear logic). The situation here resembles that concerning the introduction rule for the $\Box$ in Prawitz system for S4 (see [Pra]). The relevantists standard solution is really very similar to the treatment of $\Box$ in modal logic: Instead of the axiom $\wedge Int$ they first introduce a new *rule of proof* (besides M.P.):

Adjunction (adj):

$$\frac{A \quad B}{A \wedge B}$$

This rule suffices for simulating an application of $\wedge Int$ (in a N.D. proof) in which the common multiset of assumptions on which A and B depends is empty. In order to simulate other cases as well it should be possible to derive a proof in the Hilbert system of $A_1 \rightarrow (\ldots \rightarrow (A_n \rightarrow B \wedge C)\ldots)$ from proofs of $A_1 \rightarrow (\ldots \rightarrow (A_n \rightarrow B)\ldots)$ and $A_1 \rightarrow (\ldots \rightarrow (A_n \rightarrow C)$. The last two formulas are equivalent to $A \rightarrow B$ and $A \rightarrow C$ respectively , where $A = A_1 \circ A_2 \circ \ldots \circ A_n$. With the help of the adjunction rule it suffices therefore to add the following axiom (which is a theorem of Linear logic):

$\wedge$ **Int':** $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$.

Once we incorporate $\wedge$ we can introduce $\vee$ either as a defined connective or by some analogous axioms (see below). The extensional constants T and $\underline{0}$ can then easily be introduced as well.

The above procedure provides a Hibert-type system HL with has exactly the same theorems as the corresponding fragment of Linear Logic. Moreover, it is not difficult to prove also the following stronger result:

Theorem: $T \vdash_{HL} A$ (in the ordinary, classical sense) iff $T \vdash_{LL} A$. [9]

[9] The various propositional constants are optional for this theorem. The use of t in every particular case can be replaced by the use of a theorem of the form: $(A_1 \rightarrow A_1) \wedge \ldots \wedge (A_n \rightarrow A_n)$.

If we try to characterize also the internal consequence relation in terms of HL we run into a new difficulty: The natural extension of the notion of a “Linear proof” (a notion which works so nicely and was so natural in the intensional case!) fails to apply (as it is) when the extensionals are added. Although it is possible to extend it in a less intuitive way, it is easier to directly characterize a new “Linear consequence relation”. For this we just need to add one clause to the characterization which was given above for the previous case:

If $? \vdash A$ and $? \vdash B$ then $? \vdash A \wedge B$.

Denote by $\vdash^{HL}_{\mathrm{Kl}}$ the resulting consequence relation. It is easy to prove that the deduction theorem for $\rightarrow$ obtains relative to it and that it is in fact equivalent to $\vdash^{LL}_{\mathrm{Kl}}$.

An example: $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow A \rightarrow (B \wedge C)$ is a theorem of linear logic. Hence by the linear deduction theorem we should have:

$$(A \rightarrow B) \wedge (A \rightarrow C), A \vdash^{HL}_{\mathrm{Kl}} B \wedge C.$$

Below there is a proof of this fact. It is important to realize that this is *not* a linear proof according to the simple-minded concept of linearity which we use for $HL_{\rightarrow}$ and HL_m, but it is a “linear proof” in the sense defined by the above “linear consequence relation” of HL.

1. $(A \rightarrow B) \wedge (A \rightarrow C)$ (ass.)
2. A (ass.)
3. $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B)$ (axiom)
4. $A \rightarrow B$ (1,3 M.p.)
5. $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow C)$ (axiom)
6. $A \rightarrow C$ (1,5 M.P.)
7. B (2,4 M.P.)
8. C (2,6 M.P.)
9. $B \wedge C$ (7,8 Adj.)

For the reader’s convenience we display now:

The full system HL.

Axioms:

1. $A \rightarrow A$

2. $(A \to B) \to ((B \to C) \to (A \to C))$

3. $(A \to (B \to C)) \to (B \to (A \to C))$

4. $\sim\sim A \to A$

5. $(A \to\sim B) \to (B \to\sim A)$

6. $A \to (B \to A \circ B)$

7. $A \to (B \to C) \to (A \circ B \to C)$

8. t

9. $t \to (A \to A)$

10. $A \to (\sim A \to f)$

11. $\sim f$

12. $(A + B) \to (\sim A \to B)$

13. $(\sim A \to B) \to (A + B)$

14. $A \wedge B \to A$

15. $A \wedge B \to B$

16. $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

17. $A \to A \vee B$

18. $B \to A \vee C$

19. $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

20. $A \to T$

21. $\underline{0} \to A$

Rules:

M.P.:

$$\frac{A \quad A \to B}{B}$$

Adj:

$$\frac{A \quad B}{A \wedge B}$$

Notes:

1. HL was constructed here by imitating the way the principal relevance logic R is presented in the relevantists work (see, e.g., [Du]). The relation between these two systems can be summerized as follow:

   Linear logic + contraction = R without distribution. [10]

2. Exactly as in first-order R, the Hilbert-type presentation of first-order linear logic is obtained from the propositional system by adding Kleene's standard two axioms and two rules for the quantifiers (See [Kl]).

3 Semantics

We start by reviewing some basic notions concerning algebraic semantics of propositional logics:

Definition: An *Algebraic Structure* $\underline{D}$ for a propositional logic L consists of:

1. A set D of values.
2. A subset T_D of D of the "designated values".
3. For each connective of L a corresponding operation on D.

Definition: Let AS be a set of algebraic structures for a logic L. Define:

Weak completeness(of L relative to AS): This means that $\vdash_L A$ iff $v(A) \in T_D$ for every valuation v in any $\underline{D}\in$AS.

(Finite) Strong Completeness: (of L relative to AS): This means that for every (finite) theory T and every sentence A, $T \vdash_L A$ iff $v(A) \in T_D$ for every $\underline{D}\in$AS and every valuation v such that $\{v(B)|B \in T\} \subset T_D$.

Internal weak completeness: Suppose that instead of (or in addition to) T_D each $\underline{D}\in$AS is equipped with a partial order $\leq_D$ on D. Then L is internally weakly complete (relative to this family of structures) if:

$A \vdash_L B$ iff $v(A) \leq_D v(B)$ for every v and $\underline{D}$.

Notes:

- If $\vdash_L$ has all the needed internal connectives [11] (as $\vdash^l_{LL}$ does) then every sequent $\Gamma \vdash \Delta$ is equivalent to one of the form $A \vdash B$.
- It is important to note that the various notions of completeness depend on $\vdash_L$, the consequence relation which we take as corresponding to L.

[10] The most known systems of relevance logic include as an axiom the distribution of $\wedge$ over $\vee$. As a result they are undecidable (see [Ur]) and lack cut-free Gentzen-type formulation.

[11] See [Av2] for the meaning of this.

We next introduce the basic algebraic structures which correspond to Linear Logic:

Definition: *Basic relevant disjunction structures* are structures $\underline{\text{D}}=< D, \leq, \sim, + >$ s.t.:

1. $< D, \leq >$ is a poset.
2. $\sim$ is an involution on $< D, \sim >$.[12]
3. $+$ is an associative, commutative and order-preserving operation on $< D, \leq >$.

Definition: *Basic relevant disjunction structures with truth subset* [13] are structures $< \underline{\text{D}}, T_D >$ s.t.:

1. $\underline{\text{D}}$ is a basic relevant disjunction structure.
2. $T_D \subset D$.
3. $a \in T_D, a \leq b \Rightarrow b \in T_D$.
4. $a \leq b$ iff $\sim a + b \in T_D$.

Let HL_m be the pure intentional (or "multiplicative") fragment of HL $(\sim, \rightarrow, +, \circ)$. The correspondence between HL_m and the above structures is given by the following:

Strong completeness theorem: HL_m is *strongly complete* relative to basic relevant disjunction structures with truth subset (where the designated values are the elements of the truth subset).

This is true for the *external* (pure) C.R.. For the internal one we have the following easy corollary:

Weak internal completeness theorem: $A \vdash^l_{LL} B$ iff $v(A) \leq v(B)$ for every valuation v in the above structures. Hence the linear consequence relation is internally complete relative to the above structures.

Outline of the proof of strong completeness: For the less easy part, define the *Lindenbaum algebra* of a HL_m-theory T as follows: Let $A \equiv B$ iff $T \vdash_{HL_m} A \rightarrow B$ and $T \vdash_{HL_m} B \rightarrow A$. This is a *congruence relation.* [14] Denote by [A] the equivalence class of A. Let D be the set of equivalence classes. Define: $[A] \leq [B]$ iff $T \vdash_{HL_m} A \rightarrow B$, $\sim [A] = [\sim A]$, $[A] + [B] = [A + B]$.

[12] This means that for all a, b: $a = \sim\sim a, a \leq b \Rightarrow \sim b \leq \sim a$.

[13] These structures were first introduce in [Av3].

[14] If we consider only the implicational fragment then $HL_{\rightarrow}$ is a *minimal* logic for which this is the case!

These are all well defined. The resulting structure is a basic relevant disjunction structure with a truth subset: $T_D = \{[A] | T \vdash_{HL_m} A\}$. By defining $v(A) = [A]$ we get a valuation for which *exactly* the theorems of T get designated values.

The following proposition provides an alternative characterization of the above structures:

Proposition:[15] Let $< D, \leq, \sim, + >$ be a basic relevance structure . The following are necessary and sufficient conditions for the existence of a truth-subset of D:

D.S. $a \leq b + c \Rightarrow \sim b \leq \sim a + c$

R.A. $a + \sim (\sim b + b) \leq a$

Moreover, if a truth-subset exists it is uniquely defined by:

$$T_D = \{a | \sim a + a \leq a\}.$$

(R.A) is not a convenient condition from an algebraic point of view. Fortunately we have:

Proposition: *Each* of the following two conditions implies (R.A) in every basic relevant disjunction structure in which D.S. is satisfied:

1. Idempotency of $+$: $\forall a \; a + a = a$.
2. Existence of an identity element f for $+$ ($\forall a \; a + f = a$).

In the second case we have: $T_D = \{a | a \geq t\}$ where $t = \sim f$.

Implicitly, *Girard has chosen the second possibility.* So did before him most of the relevantists. [16] Accordingly we define:

Definition: *Relevant disjunction monoids* are relevant disjunction structures which satisfy (D.S.) and in which $+$ has an identity element.

It is easy now to formulate and prove completeness theorems as above for the full intensional fragment of Linear Logic (including the propositional constants) relative to relevant disjunction monoids. Since this fragment is a strongly conservative extension of that treated above, these completeness results will hold also for the more restricted fragment. It is worth noting also that in relevant disjunction monoids condition (D.S.) is equivalent to:

$$a \leq b \quad iff \quad t \leq \sim a + b$$

[15] This proposition as well as the next one are again taken from [Av3].

[16] Compare Dunn's work on the algebraic semantics of R and other relevance systems. For more information and references—see [Du] or [AB].

In order to get a similar characterization for the full propositional fragment of Linear logic we have to deal with *Lattices* rather than just posets. The operations of g.l.b and l.u.b provide then an obvious interpretation for the extensional ("additive") connectives $\vee$ and $\wedge$. All other definitions and conditions remain the same. (This is a standard procedure in a semantical research on relevance logics). Completeness theorems analogous to those presented above can then be formulated and similarly proved. (If we wish to incorporate also Girard's $\top$ and $\underline{0}$ then the lattices should include maximal and minimal elements.)

Again, the standard way of characterizing linear *predicate* calculus is to work with *complete* rather than ordinary lattices. We can then define :

$$v(\forall x\varphi(x)) = \inf\{v(\varphi(a))|a \in \iota\}$$

$$v(\exists x\varphi(x)) = \sup\{v(\varphi(a))|a \in \iota\}$$

(Where ι is the domain of quantification).

From now on it will be more convenient to take $\circ$ instead of $+$ as primitive and to reformulate the varios definitions accordingly. (The two operations are definable from one another by DeMorgan's connections). Our last observation leads us accordingly to consider the following structures (which will be shown to be equivalent to Girard's "phase spaces"):

Girard structures: These are structures $< D, \leq, \sim, \circ >$ s.t:

1. $< D, \leq >$ is a *complete lattice*.
2. $\sim$ is an involution on $< D, \leq >$.
3. $\circ$ is a commutative, associative, order-preserving operation on D with an identity element t.
4. $a \leq b$ iff $a \circ \sim b \leq f \;\; (f = \sim t)$.

Note: If we demand: $a \leq a \circ a$ we get Dunn's algebraic semantic for QR.

The embedding theorem: Let $\underline{D}$=$< D, \leq, \sim, \circ, t >$ be a basic relevant disjunction structure with identity t such that $\{a|a \geq t\}$ is a truth-subset. (The last condition is equivalent here to (D.S.) or to (4.) above). Then $\underline{D}$ can be embedded in a Girard structure so that existing infima and suprema of subsets of D are preserved.

Corollaries:

1. The various propositional fragments of HL are strongly complete for Girard structures.
2. The various fragments of Linear Predicate calculus are strongly conservative extensions of each other.(e,g: in case A and all sentences of T are in the language of HL_m then $T \vdash_{HL_m} A$ iff $T \vdash_{HL} A$).

3. Linear predicate calculus is strongly complete relative to Girard structures. (The embedding theorem is essential in this case since the direct construction of the Lindenbaum Algebra contains all the necessary inf and sup, but is not yet complete!)

The proof of the embedding theorem, as well as the proof of the equivalence of the notions of "Girard structures" and the "phase semantics" of Girard, depend on some general principles from the theory of complete lattices (see the relevant chapter in Birkhoff's book: "Lattice theory"):

Definition: Let I be a set. $C : P(I) \to P(I)$ is a *closure operation* on $P(I)$ if:

1. $X \subset C(X)$
2. $C(C(X)) \subset C(X)$
3. $X \subset Y \Rightarrow C(X) \subset C(Y)$

X is called *closed* if $C(X) = X$. (Obviously, X is closed iff $X = C(Y)$ for some Y).

Theorem: For I, C as above the set of closed subsets of I is a complete lattice under the $\subset$ order. Moreover, we have:

$$inf\{A_j\} = \bigcap_j A_j$$

$$sup\{A_j\} = C(\bigcup_j A_j)$$

Standard embeddings: Let C be a closure operation on I such that $C(\{x\} \neq C(\{y\})$ whenever $x \neq y$. Then $x \to C(\{x\})$ is the standard embedding of I in the resulting complete lattice of closed subsets.

In case I is a structure with, say, an operation $\circ$ we usually extend it to this complete Lattice by defining

$$X \circ Y = C(XY)$$

$$\text{where} \quad XY = \{x \circ y | x \in X, y \in Y\}$$

.

The most usual way of obtaining closure operations is given in the following:

Definition: *("Galois connections")*: Let R be a binary relation on I, $X \subset I$. Define:

$$X^* = \{y \in I | \forall x \in X \;\; xRy\}$$

$$X^+ = \{x \in I | \forall y \in X \;\; xRy\}$$

Lemma:

1. $X \subset Y \Rightarrow Y^* \subset X^*, Y^+ \subset X^+$

2. $X \subset X^{*+}, Y \subset Y^{+*}, X^{*+*} = X^*, Y^{+*+} = Y^+$

3. $*+$ and $+*$ are closure operations on $P(I)$.

An example: Take R to be $\leq$, where $< I, \leq >$ is a poset, and take C to be $*+$. We get (for $X \subset I,\ x \in I$):

X^*=set of upper bounds of X
X^+=set of lower bounds of X
$C(\{x\}) = \{x\}^{*+} = \{y | y \leq x\}$.

It is easy to check then that $x \to \{x\}^{*+}$ is an embedding of $(I, \leq)$ in the complete lattice of closed subsets of I. This embedding preserves all existing suprema and infima of subsets of I. (I is dense, in fact, in this lattice.) Moreover, if $\sim$ is an involution on $< I, \leq >$ then by defining:

$$\sim X = \{\sim y |\ y \in X^*\}$$

we get an involution on this complete lattice which is an extension of the original involution.

Proof of the embedding theorem: It is straightforward to check that the combination of the constructions described in the last example with the standard way to extend $\circ$ which was described above (applied to relevant disjunction structures with identity) suffices for the embedding theorem.

Another use of the method of Galois connections is the following:

Girard's construction (The "phase semantics"): We start with a triple $< P, \circ, \perp >$, where P is a set (the set of "*phases*",) $\perp \subset P$ and $\circ$ is an associative, commutative operation on P with an identity element. Define:

$$xRy \equiv_{df} \quad x \circ y \in \perp$$

then

$$X^* = X^+ = \{y | \forall x \in X \ \ x \circ y \in \perp\} =_{df} X^-.$$

It follows that $X \to X^{--}$ is a closure operation. The closed subsets ($X = X^{--}$) are called by Girard *"Facts"*. By the general result cited above they form a complete lattice. Define on it:

$$\sim X = X^-$$

$$X \circ Y = (XY)^{--}$$

what Girard does in ch.1 of [Gi] is, essentially, to prove the following:

Characterization theorem: The construction above provides a Girard's structure. Conversely, any Girard structure is isomorphic to a structure constructed as above (since by starting with a given Girard's structure, and taking $\perp$ to be $\{x|x \leq f\}$, the above construction returns an isomorphic Girard's structure).

Corollary: HL is *strongly complete* relative to the "phase semantics" of Girard.

Notes:

1. In the original paper Girard is proving only weak completeness of Linear logic relative to this "phase semantics".
2. It can be checked that Girard's construction works even if originally we do not have an identity element, provided we use a subset $\perp$ with the property $\perp^{--} = \perp$ (the existence of an identity element guarantees this for every potential $\perp$).

Summary: Points in Girard structures and "facts" in "phase spaces" are two equivalent notions.

4 The modal operators

4.1 Proof theory

In this section we examine how the modal operators of Girard can be treated in the above framework. It is enough to consider only the $\Box$, since the other operator is definable in terms of $\Box$ and $\sim$. Starting with the proof-theoretic part, we list in the following the rules and axioms that should be added to the various formal systems. It is not difficult then to extend the former proof-theoretic equivalence results to the resulting systems. (Again the Gentzen-type rules were given already in [Gi] in a different form).

Gentzen-type rules:

$$\frac{? \vdash \Delta}{\Box A, ? \vdash \Delta} \qquad \frac{\Box A, \Box A, ? \vdash \Delta}{\Box A, ? \vdash \Delta}$$

$$\frac{A, ? \vdash \Delta}{\Box A, ? \vdash \Delta} \qquad \frac{\Box A_1, \ldots, \Box A_n \vdash B}{\Box A_1, \ldots, \Box A_n \vdash \Box B}$$

Here the second pair of rules is exactly as in the standard Gentzen-type presentation of S4, while the first pair allows the left-hand structural rules to be applied to boxed formulas.

N.D. rules: Essentially, we need to add here Prawitz' two rules for S4:

$$\frac{\Box A}{A} \qquad \frac{A}{\Box A}(*)$$

(where, as in [Pra], for the $\Box$-introduction rule we have the side condition that all formulas on which A depends are boxed). In addition, the rules for the other connectives should be classically re-interpreted (or formulated) as far as boxed formulas are concerned. Thus, e.g, the multisets of assumptions on which A and B depends in the $\wedge$-introduction rule should be the same only *up to boxed formulas,* while for negation and implication we have:

$$\frac{\begin{array}{c}[\Box A]\\ B\end{array}}{\Box A \rightarrow B} \qquad \frac{\begin{array}{cc}[\Box A] & [\Box A]\\ B & \sim B\end{array}}{\sim \Box A}$$

Where the interpretation of these rules are in this case *exactly as in classical logic.* (It suffices, in fact, to add, besides the basic $\Box$-rules, only the boxed version of $\rightarrow Int$ to make all other additions derivable).

$HL^{\Box}$**—the Hilbert-type system:** we add to HL:

$K\Box$. $B \rightarrow (\Box A \rightarrow B)$

$W\Box$. $(\Box A \rightarrow (\Box A \rightarrow B)) \rightarrow (\Box A \rightarrow B)$

$\Box K$. $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

$\Box T_1$ $\Box A \rightarrow A$

$\Box 4$. $\Box A \rightarrow \Box\Box A$

Nec.

$$\frac{A}{\Box A}$$

Again we see that the added axioms and rules for the Hilbert-type system naturally divided into two: the first two axioms are instances of the schemes that one needs to add to the implicational fragment of Linear Logic in order to get the corresponding fragment of the intuitionistic calculus. The other axioms and rules are exactly what one adds to classical propositional calculus in order to get the modal S4. The main property of the resulting system is given in the following:

Modal deduction theorem: For every theory T and formulas A, B we have:

$$T, A \vdash_{HL^{\Box}} B \text{ iff } T \vdash_{HL^{\Box}} \Box A \rightarrow B$$

The same is true for all the systems which are obtained from the various fragments studied above by the addition of the above axioms and rule for the $\Box$.

The proof of this theorem is by a standard induction. It is also easily seen that except for $W\Box$ the provability of the other axioms and the derivability of $\Box A$ from A are all consequences of the modal deduction theorem.

Notes:

1. It is important again to emphasize that the last theorem is true for the *external* consequence relation.

2. The deduction theorem for $HL^{\Box}$ is identical to the deduction theorem for the pure consequence relation defined by S4 (in which Nec is taken as a rule of derivation, not only as a rule of proof). [17] S4 may be characterized as the *minimal modal system* for which this deduction theorem obtains. [18]

4.2 Semantics

From an algebraic point of view the most natural way to extend a Girard structure (and also the other structures which were considered in the previous section) in order to get a semantics for the modal operators is to add to the structure an operation B, corresponding to the $\Box$, with the needed properties. Accordingly we define:

Definition: *A modal Girard structure* is a Girard structure equipped with an operation B having the following properties:

1. $B(t) = t$
2. $B(x) \leq x$
3. $B(B(x)) = B(x)$
4. $B(x) \circ B(y) = B(x \wedge y)$

Lemma: In every modal Girard structure we have:

1. $B(a) \leq t$
2. $B(a) \leq b \rightarrow b$
3. $B(a) \circ b \leq b$
4. $B(a) \circ B(a) = B(a)$
5. $a \leq b \Rightarrow B(a) \leq B(b)$
6. $a \geq t \Rightarrow B(a) = t$
7. If $B(a) \leq b$ then $B(a) \leq B(b)$
8. If $a_1 \circ a_2 \circ \ldots \circ a_n \leq b$ then $B(a_1) \circ B(a_2) \circ \ldots \circ B(a_n) \leq B(b)$.

Proof

1. $B(a) = B(a) \circ t = B(a) \circ B(t) = B(a \wedge t) \leq a \wedge t \leq t$
2. Immediate from 1.

[17] This consequence relation corresponds to *validity* in Kripke models. See [Av2].

[18] This modal deduction theorem was independently used by the author as the main tool for implementing S4 in the Edinburgh LF—see [AM].

3. Equivalent to 2.

4. $B(a) \circ B(a) = B(a \wedge a) = B(a)$.

5. $a \leq b \Rightarrow a = a \wedge b$ and so:
$a \leq b \Rightarrow B(a) = B(a) \circ B(b) \leq B(b)$ (by 3.)

6. Immediate from 5. and 1.

7. $B(a) \leq b \Rightarrow \quad B(a) = B(B(a)) \leq B(b)$ (by 5.)

8. since $B(a) \leq a$, we have that $B(a_1) \circ B(a_2) \circ \ldots \circ B(a_n) \leq b$ whenever $a_1 \circ a_2 \circ \ldots \circ a_n \leq b$. But $B(a_1) \circ B(a_2) \circ \ldots B(a_n) = B(a_1 \wedge \ldots \wedge a_n)$. Hence 8. Follows from 7.

Theorem: $HL^{\Box}$ is sound and strongly complete relative to modal Girard structures.

Proof: The soundness follows immediately from the previous lemma. The proof of completeness is similar to the previous ones. The only extra step needed is to show that we can extend the operation B (defined from the $\Box$) of the Lindenbaum Algebra LA to the completion of this algebra. This is done by defining:

$$B^{'}(x^{'}) = sup\{B(x) | x \leq x^{'} | x \in LA\}$$

Using the fact that in Girard structures $\circ$ distributes over sup, it is not difficult to show that $B^{'}$ is an operation as required. (The fact is needed for establishing that $B^{'}(a) \circ B^{'}(b) = B^{'}(a \wedge b)$. It was used also in [Gi] for quite similar purposes).

Corollary: $HL^{\Box}$ is a strongly conservative extension of HL.

Proof: Let A be a sentence in the Language of HL, and let T be a theory in this Language. Suppose $T \not\vdash_{HL} A$. We show that $T \not\vdash_{HL^{\Box}} A$. The completeness theorem for HL provides us with a Girard structures and a valuation in it for which all the theorems of T are true and A is false. To show that $T \not\vdash_{HL^{\Box}} A$ it is enough therefore to turn this Girard structure into a modal one. That this can always be done is the content of the next theorem.

Theorem: In every Girard structure we can define at least one operator B which makes it modal.

Proof: Define: $B(a) = sup\{x \leq a \wedge t | x \circ x = x\}$. The theorem is a consequence of the following facts:

1. $B(a) \leq a$

2. $B(a) \leq t$

3. $B(t) = t$

4. $B(a) \circ B(b) \leq B(a)$

5. $B(a) \circ B(b) \leq a \wedge b \wedge t$

6. $B(a) \circ B(b) \geq B(a \wedge b)$

7. $B(a) \circ B(a) = B(a)$

8. $B(B(a)) = B(a)$

9. $B(a) \circ B(b) \leq B(a \wedge b)$

10. $B(a) \circ B(b) = B(a \wedge b)$

(1)-(3) are immediate from the definition of B. It follows that $B(a) \circ B(b) \leq B(a) \circ t = B(a)$. Hence (4). (5) follows then from (1) and (2). (7) is immediate from (6) and (4), while (8) follows from (7),(2) and the definition of $B(B(a))$. (10) is just a combination of (6) an (9). It remains therefore to prove (6) and (9).

Proof of (6): suppose that $z \leq (a \wedge b) \wedge t$ and $z \circ z = z$. Then $z \leq B(a)$ and $z \leq B(b)$ by definition of $B(a), B(b)$. Hence $z = z \circ z \leq B(a) \circ B(b)$. This is true for every such z and so $B(a \wedge b) \leq B(a) \circ B(b)$ by definition of $B(a \wedge b)$.

Proof of (9): By (7) $(B(a) \circ B(b)) \circ (B(a) \circ B(b)) = B(a) \circ B(b)$. This, (5) and the definition of $B(a \wedge b)$ imply (9).

The last construction is a special case of a more general construction. This general construction is strong enough for obtaining any possible modal operation, and is, in fact, what Girard is using in [Gi] for providing a semantics for the modal operators. The construction is described in the following theorem, the proof of which we leave to the reader:

Theorem: Suppose G is a Girard structure and suppose that F is a subset of G which has the following properties:

1. F is closed under arbitrary sup.

2. F is closed under $\circ$.

3. $x \circ x = x$ for every $x \in F$.

4. t is the maximal element of F.

Then $B(a) =_{Df} sup\{x \in F | x \leq a\}$ is a modal operation on G.

Conversely, if B is a modal operation on G then the set $F = \{x \in G | x = B(x)\}$ has the above properties and for every a: $B(a) = sup\{x \in F | x \leq a\}$.

In [Gi] Girard (essentially) defines topolinear spaces to be Girard structure together with a subset F having the above properties. Only he has formulated this in terms of the "phase" semantics. His definition of the interior of a fact is an exact equivalent of the definition of B which was given in the last theorem. The subset F corresponds (in fact, is identical) to the collection of open facts of his topolinear spaces. (Note, finally, that in every Girard structure the subset $\{x \in G | x = x \circ x \ \ and \ \ x \leq t\}$ has the 4 properties described above. By using it we get the construction described in the previous theorem).

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