Modular Semantics for Some Basic Logics of Formal Inconsistency

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ABSTRACT. We construct a modular semantic framework for LFIs (logics of formal (in)consistency) which extends the framework developed in previous papers, so it now includes all the basic axioms considered in the literature on LFIs, plus a few more. In addition, the paper provides another demonstration of the power of the idea of non-deterministic semantics, especially when it is combined with the idea of using truth-values to encode data concerning propositions.

1 Introduction

One of the oldest and best known approaches to the problem of designing useful paraconsistent logics (i.e. logics which allows nontrivial inconsistent theories — see [8; 14; 10; 9]) is da Costa's approach ([15; 16]), which seeks to allow the use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. da Costa's approach has led to the family of LFIs (Logics of Formal (In)consistency — see [13]). In [3] and [1] we developed a semantic framework for this family, in which it is possible to provide simple semantics for almost all the propositional LFIs considered in the literature. This semantics is based on the use of non-deterministic matrices (Nmatrices). These are multi-valued structures (introduced in [6; 7]) where the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options.

The semantic framework for LFIs which is based on Nmatrices has two crucial properties that previous semantic frameworks used for this task (the bivaluations semantics and the possible translations semantics described in [12; 13; 17]) in general lack: [1]

[1] In [11] it was argued that the semantics of Nmatrices is a particular case of possible translations semantics. This observation is irrelevant to our claims concerning it, because it is precisely this generality which is the source of weakness of the possible translations semantics, and the reason why *in general* it lacks the two properties described below. Similarly, possible translations semantics (as well as practically any other type of seman-

- It is *analytic*[2] in the sense that for determining whether $T \vdash_{\mathcal{M}} \varphi$ (where $\mathcal{M}$ is an Nmatrix) it always suffices to check only *partial* valuations, defined only on subformulas of $T \cup \{\varphi\}$. It follows that a logic which has a finite characteristic Nmatrix is necessarily decidable.

- It is *modular*: each axiom has its own semantics, and the semantics of a system is obtained by joining the semantics of its axioms. As demonstrated in [1; 2; 3; 4], this fact makes it possible to simultaneously prove soundness and completeness theorems for thousands of systems (this paper includes another striking example of this phenomenon).

Now [1; 3] have left one major gap: no semantics was provided there to Marcos' axiom (denoted below by (**m**)). [3] This axiom is crucial in one of the two LFIs which are considered as basic in [13]: Marcos' system **mCi**, to which the whole of section 4 of [13] is devoted. This gap was partially closed in [5], where a 5-valued Nmatrix which is characteristic for **mCi** has been given. [4] However, this Nmatrix does not provide an independent semantics for Marcos' axiom (in the form of a semantic condition that corresponds specifically to this axiom), but only to its *combination with another axiom* (axiom (**i**) below, to which, in contrast, an independent semantics *has* been provided in [1; 3]). As a result, the full modularity of the semantics was lost in [5]. The main goal of this paper is to restore it by extending the framework developed in [1; 3], so it includes systems with Marcos' axiom too (not only those that have already been investigated in [13], but also some new ones that naturally arise). The extended framework provides semantics in a modular way to practically *all* the axioms and systems considered in [13] — plus a few more. In addition, the paper provides another demonstration of the power of the idea of non-deterministic semantics, especially when it is combined with the idea of using truth-values to encode data concerning propositions (see section 3).

tics) can be viewed as a particular (but superior) case of the bivaluations semantics, and the two-valued semantics of classical logic is a particular (but superior) case of the more general semantics of Boolean algebras. In all these cases, as well as in many others, the isolation of a useful particular case of a more general framework is of crucial importance.

[2]In previous papers we use the term "*effective*" for this property, but now we believe that "analytic" is more appropriate.

[3]We have chosen the name "Marcos' axiom" for this axiom, because it was first introduced in Marcos' paper [17].

[4]A possible-translation semantics for **mCi** has been provided in [17].

2 Preliminaries

2.1 A Taxonomy of LFIs

Let $\mathcal{L}_{\mathrm{cl}}^{+} = \{\wedge, \vee, \supset\}$, $\mathcal{L}_{\mathrm{cl}} = \{\wedge, \vee, \supset, \neg\}$, and $\mathcal{L}_{\mathrm{C}} = \{\wedge, \vee, \supset, \neg, \circ\}$. [5]

DEFINITION 1. Let $\mathbf{HCL}^{+}$ (Hilbert-style positive Classical Logic) be some Hilbert-type system which has Modus Ponens as the only inference rule, and is sound and strongly complete for the $\mathcal{L}_{\mathrm{cl}}^{+}$-fragment of CPL (classical propositional logic) [6]. The logic $\mathbf{B}$ [7] is the logic in $\mathcal{L}_{\mathrm{C}}$ obtained from $\mathbf{HCL}^{+}$ by adding the following schemata (where φ and ψ vary through formulas):

(t) $\neg\varphi \vee \varphi$

(p) $\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi)$

DEFINITION 2. For $n \geq 0$, let $\neg^{0}\varphi = \varphi$, $\neg^{n+1}\varphi = \neg(\neg^{n}\varphi)$.

DEFINITION 3. Let Ax be the set consisting of the following schemata:

(m) $\circ\neg^{n}\circ\varphi$ (for every $n \geq 0$)

(c) $\neg\neg\varphi \supset \varphi$

(e) $\varphi \supset \neg\neg\varphi$

(k1) $\circ\varphi \vee \varphi$

(k2) $\circ\varphi \vee \neg\varphi$

(k) $\circ\varphi \vee (\varphi \wedge \neg\varphi)$

(i1) $\neg\circ\varphi \supset \varphi$

(i2) $\neg\circ\varphi \supset \neg\varphi$

(i) $\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)$

For $S \subseteq Ax$, $\mathbf{B}[S]$ is the extension of $\mathbf{B}$ by the axioms in S.

Notation: We'll usually denote $\mathbf{B}[S]$ by $\mathbf{B}s$, where s is a string consisting of the names of the axioms in S (thus we denote $\mathbf{B}[\{(\mathbf{i}), (\mathbf{e})\}]$ by $\mathbf{Bie}$). [8]

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[5] The intuitive meaning of $\circ\varphi$ is "φ is a consistent formula" or "φ behaves classically".

[6] I.e.: for every sentence φ and theory $\mathbf{T}$ in $\mathcal{L}_{\mathrm{cl}}^{+}$, $\mathbf{T} \vdash_{\mathbf{HCL}^{+}} \varphi$ iff $\mathbf{T} \vdash_{CPL} \varphi$.

[7] The logic $\mathbf{B}$ is called mbC in [13]

[8] In the literature on LFIs one usually writes $\mathbf{C}s$ instead of our $\mathbf{Bc}s$ when S includes the axiom **(c)**. Note also that what we call $(\mathbf{m})$ is called $(\mathbf{cc})$ in [13].

2.2 Non-deterministic Matrices

Our main semantic tool in what follows is the following generalization (from [6; 7; 1; 2]) of the concept of a matrix:

DEFINITION 4.

1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:
   (a) $\mathcal{V}$ is a non-empty set of *truth values*.
   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.
   (c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.

   We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{V}$.

2. A *valuation* in an Nmatrix $\mathcal{M}$ is a function v from the set of formulas of $\mathcal{L}$ to $\mathcal{V}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

3. A valuation v in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model* in $\mathcal{M}$ of a set $\mathbf{T}$ of formulas (notation: $v \models^{\mathcal{M}} \mathbf{T}$) if it satisfies every formula in $\mathbf{T}$.

4. $\vdash_{\mathcal{M}}$, the consequence relation induced by the Nmatrix $\mathcal{M}$, is defined as follows: $\mathbf{T} \vdash_{\mathcal{M}} \varphi$ if for every v such that $v \models^{\mathcal{M}} \mathbf{T}$, also $v \models^{\mathcal{M}} \varphi$.

5. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is *sound* for an Nmatrix $\mathcal{M}$ (where $\mathcal{L}$ is the language of $\mathcal{M}$) if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{M}}$. $\mathbf{L}$ is *complete* for $\mathcal{M}$ if $\vdash_{\mathbf{L}} \supseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is *characteristic* for $\mathbf{L}$ if $\mathbf{L}$ is both sound and complete for it (i.e.: if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{M}}$). $\mathcal{M}$ is *weakly-characteristic* for $\mathbf{L}$ if for every formula φ of $\mathcal{L}$, $\vdash_{\mathbf{L}} \varphi$ iff $\vdash_{\mathcal{M}} \varphi$.

DEFINITION 5. Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$. $\mathcal{M}_2$ is called a *simple refinement*[9] of $\mathcal{M}_1$ if $\mathcal{V}_2 \subseteq \mathcal{V}_1$, $\mathcal{D}_2 = \mathcal{D}_1 \cap \mathcal{V}_2$, and $\widetilde{\diamond}_{\mathcal{M}_2}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_1}(\vec{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{V}_2^n$.

The following proposition can easily be proved:

PROPOSITION 6. *If $\mathcal{M}_2$ is a simple refinement of $\mathcal{M}_1$ then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$. Hence if $\mathbf{L}$ is sound for $\mathcal{M}_1$ then $\mathbf{L}$ is also sound for $\mathcal{M}_2$.*

[9] A more general notion of a refinement was used in [2]. However, here we shall need only simple refinements.

2.3 General Non-deterministic Semantics for extensions of B

DEFINITION 7.

- A basic **B**-Nmatrix is an Nmatrix for the language $\mathcal{L}_C$ such that:
  1. $\mathcal{V} = \mathcal{T} \uplus \mathcal{I} \uplus \mathcal{F}$, where $\mathcal{T}$, $\mathcal{I}$, and $\mathcal{F}$ are disjoint nonempty sets.
  2. $\mathcal{D} = \mathcal{T} \cup \mathcal{I}$
  3. $\mathcal{O}$ is defined by:

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \widetilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

$$\widetilde{\circ} a = \begin{cases} \mathcal{V} & \text{if } a \in \mathcal{F} \cup \mathcal{T} \\ \mathcal{F} & \text{if } a \in \mathcal{I} \end{cases}$$

- A **B**-Nmatrix is an Nmatrix for $\mathcal{L}_C$ which is a simple refinement of some basic **B**-Nmatrix.

The following theorem from [3] can easily be proved:

THEOREM 8. **B** *is sound for any* **B**-*Nmatrix.*

3 The Nmatrix $\mathcal{M}_{10}^{B}$ and Its Simple Refinements

The main semantic idea used in [1] is that truth-values can be used to encode the data concerning sentences which determine the consequence relation of a given logic. For all the LFIs considered in [1] and [3], the data needed about a sentence φ was whether φ is true or false, whether $\neg\varphi$ is true or false, and whether $\circ\varphi$ is true or false. Accordingly, for most of the systems considered there we used as truth-values triples in $\{0,1\}^3$ (or sometimes $\{0,1\}^2$, in case this sufficed). Now Marcos' axiom (**m**) is concerned with formulas of a certain particular *syntactic* form. Therefore the key idea in handling it is to add one more bit to the truth-values, indicating whether the sentence has this particular form, or at least "behaves" as if it has such a form (this modification is needed because atomic formulas may be assigned any truth value, even if they do not have the special syntactic form that this truth value is meant to signify). Accordingly, we shall use in what follows elements of $\{0,1\}^4$ as our truth-values. The intuitive meaning of the forth bit is: the sentence belong to a certain class A of sentences which is closed under negation and includes every formula of the form $\circ\psi$ (the identity of the class A is application-dependent, but here it is usually the class of sentences of the form $\neg^n \circ\psi$).

Notation For $1 \leq i \leq 4$ we let $P_i(\langle x_1, x_2, x_3, x_4\rangle) = x_i$. We shall usually write $x_1x_2x_3x_4$ instead of $\langle x_1, x_2, x_3, x_4\rangle$ when the latter is in $\{0,1\}^4$.

DEFINITION 9. The Nmatrix $\mathcal{M}_{10}^B = \langle \mathcal{V}_{10}, \mathcal{D}_{10}, \mathcal{O}_{10}\rangle$ is defined as follows:

- $\mathcal{V}_{10} = \{1101, 1100, 1011, 1010, 1001, 1000, 0111, 0110, 0101, 0100\}$. In other words: $\mathcal{V}_{10}$ is the set of tuples in $\{0,1\}^4$ which satisfy the following two conditions:

  C(**t**): If $P_1(a) = 0$ then $P_2(a) = 1$

  C(**b**): If $P_1(a) = 1$ and $P_2(a) = 1$ then $P_3(a) = 0$

- $\mathcal{D}_{10} = \{a \in \mathcal{V}_{10} \mid P_1(a) = 1\}$

- Let $\mathcal{V} = \mathcal{V}_{10}$, $\mathcal{D} = \mathcal{D}_{10}$, $\mathcal{F} = \mathcal{V}_{10} - \mathcal{D}$. The operations in $\mathcal{O}_{10}$ are:

$$\tilde{\neg} a = \{b \in \mathcal{V} \mid P_1(b) = P_2(a), \text{ and if } P_4(a) = 1 \text{ then } \ P_4(b) = 1\}$$

$$\tilde{\circ} a = \{b \in \mathcal{V} \mid P_1(b) = P_3(a) \text{ and } P_4(b) = 1\}$$

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{D} & \text{otherwise} \end{cases}$$

Note. It can easily be checked that

$$\tilde{\circ} a = \begin{cases} \{0101, 0111\} & \text{if } P_3(a) = 0 \\ \{1101, 1011, 1001\} & \text{if } P_3(a) = 1 \end{cases}$$

DEFINITION 10.

1. The general refining conditions induced by the conditions in Ax are:

C(**m**): $\tilde{\neg}\, 0111 = \{1011\}$, $\tilde{\neg} 1011 = \{0111\}$, and:

$$\tilde{\circ} a = \begin{cases} \{0111\} & \text{if } P_3(a) = 0 \\ \{1011\} & \text{if } P_3(a) = 1 \end{cases}$$

C(**c**): If $P_1(a) = 0$ then $\tilde{\neg} a \subseteq \{x \mid P_1(x) = 1 \text{ and } P_2(x) = 0\}$.

C(**e**): If $P_1(a) = P_2(a) = 1$ then $\tilde{\neg} a \subseteq \{x \mid P_1(x) = 1 \text{ and } P_2(x) = 1\}$.

C(**k1**): If $P_1(a) = 0$ then $P_3(a) = 1$ (equivalently: 0101 and 0100 should be deleted).

C(**k2**): If $P_2(a) = 0$ then $P_3(a) = 1$ (equivalently: 1001 and 1000 should be deleted).

C(**k**): Both C(**k1**) and C(**k2**) should be satisfied.

C(**i1**): If $P_1(a) = 0$ then $\tilde{\circ} a \subseteq \{x \mid P_2(x) = 0\}$ (equivalently: C(**k1**) should be satisfied, and $\tilde{\circ}(a) \subseteq \{1011, 1001\}$ for $a \in \{0111, 0110\}$).

C(**i2**): If $P_2(a) = 0$ then $\tilde{\circ} a \subseteq \{x \mid P_2(x) = 0\}$ (equivalently: C(**k2**) should be satisfied, and $\tilde{\circ}(1011) = \tilde{\circ}(1010) = \{1011\}$).

C(**i**): Both C(**i1**) and C(**i2**) should be satisfied.

2. For $S \subseteq Ax$, let $C(S) = \{\mathrm{C}r \mid r \in S\}$, and let $\mathcal{M}_S$ be the weakest simple refinement of $\mathcal{M}_{10}^B$ in which the conditions in $C(S)$ are all satisfied (it is easy to check that this is well-defined for every $S \subseteq Ax$).

4 The Soundness and Completeness Theorem

The following is the main result of this paper:

THEOREM 11. *For $S \subseteq Ax$, $\mathcal{M}_S$ is characteristic for* $\mathbf{B}[S]$.

Proof. *Soundness:* Obviously, for each $S \subseteq Ax$, $\mathcal{M}_S$ is simple refinement of the basic **B**-Nmatrix in which $\mathcal{V} = \mathcal{V}_{10}$, $\mathcal{T} = \{a \in \mathcal{V} \mid P_2(a) = 0\}$, $\mathcal{F} = \{a \in \mathcal{V} \mid P_1(a) = 0\}$, and $\mathcal{I} = \{a \in \mathcal{V} \mid P_1(a) = P_2(a) = 1\}$. Therefore by Theorem 8 it follows that **B** is sound for $\mathcal{M}_S$. It remains to show that if $\mathbf{s} \in S$ then the axiom **s** is valid in $\mathcal{M}_S$. We do here the case of (**m**) (handling the other cases is straightforward, and is very similar to the way they were handled in [1; 2] and [3]). So let φ be a sentence, and v an assignment in $\mathcal{M}_S$ (where $(\mathbf{m}) \in S$). Then by the third part of $C(\mathbf{m})$, $v(\circ\varphi) \in \{0111, 1011\}$. Accordingly, the first two parts of $C(\mathbf{m})$ entail that $v(\neg^n \circ \varphi) \in \{0111, 1011\}$ for every $n \geq 0$. Again by the third part of $C(\mathbf{m})$, $v(\circ\neg^n \circ \varphi) = 1011$. Since 1011 is designated, this means that $\circ\neg^n \circ \varphi$ is valid in $\mathcal{M}_S$.

Completeness: Assume that $\mathbf{T}$ is a theory and φ_0 a sentence such that $\mathbf{T} \nvdash_{\mathbf{B}[S]} \varphi_0$. We construct a model of $\mathbf{T}$ in $\mathcal{M}_S$ which is not a model of φ_0. For this extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \nvdash_{\mathbf{B}[S]} \varphi_0$. $\mathbf{T}^*$ has the following properties:

1. $\psi \notin \mathbf{T}^*$ iff $\psi \supset \varphi_0 \in \mathbf{T}^*$.
2. If $\psi \notin \mathbf{T}^*$ then $\psi \supset \varphi \in \mathbf{T}^*$ for every sentence φ of $\mathcal{L}_C$.
3. $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.
4. $\varphi \wedge \psi \in \mathbf{T}^*$ iff both $\varphi \in \mathbf{T}^*$ and $\psi \in \mathbf{T}^*$.
5. $\varphi \supset \psi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.
6. For every sentence φ of $\mathcal{L}_C$, either $\varphi \in \mathbf{T}^*$ or $\neg\varphi \in \mathbf{T}^*$.
7. If both $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$ then $\circ\varphi \notin \mathbf{T}^*$.

The proofs of Properties 1–7 are *exactly* as in the proof of Theorem 1 of [3]: Property 1 follows from the deduction theorem (which is obviously valid for $\mathbf{B}[S]$) and the maximality of $\mathbf{T}^*$. Property 2 is proved first for $\psi = \varphi_0$ as follows: by 1, if $\varphi_0 \supset \varphi \notin \mathbf{T}^*$ then $(\varphi_0 \supset \varphi) \supset \varphi_0 \in \mathbf{T}^*$. Hence $\varphi_0 \in \mathbf{T}^*$ by the tautology $((\varphi_0 \supset \varphi) \supset \varphi_0) \supset \varphi_0$. A contradiction. Property 2 then follows for all $\psi \notin \mathbf{T}^*$ by 1 and the transitivity of implication. Properties 3–5 are easy corollaries of 1, 2, and the closure of $\mathbf{T}^*$ under positive classical inferences (for example: suppose $\varphi \vee \psi \in \mathbf{T}^*$, but neither $\varphi \in \mathbf{T}^*$, nor $\psi \in \mathbf{T}^*$. By property 1, $\varphi \supset \varphi_0 \in \mathbf{T}^*$ and $\psi \supset \varphi_0 \in \mathbf{T}^*$. Since φ_0 follows in positive classical logic from $\varphi \vee \psi$, $\varphi \supset \varphi_0$, and $\psi \supset \varphi_0$, we get $\varphi_0 \in \mathbf{T}^*$. A contradiction). Finally, Property 6 is immediate from Property 3 and Axiom **(t)**, and Property 7 follows from Axiom (**p**).

define a valuation v in $\mathcal{M}_S$ by $v(\varphi) = \langle x_1(\varphi), x_2(\varphi), x_3(\varphi), x_4(\varphi)\rangle$, where:

$$x_1(\varphi) = \begin{cases} 1 & \varphi \in \mathcal{T}^* \\ 0 & \varphi \notin \mathcal{T}^* \end{cases}$$

$$x_2(\varphi) = \begin{cases} 1 & \neg\varphi \in \mathcal{T}^* \\ 0 & \neg\varphi \notin \mathcal{T}^* \end{cases}$$

$$x_3(\varphi) = \begin{cases} 1 & \circ\varphi \in \mathcal{T}^* \\ 0 & \circ\varphi \notin \mathcal{T}^* \end{cases}$$

$$x_4(\varphi) = \begin{cases} 1 & \varphi \text{ is of the form } \neg^n \circ\psi \\ 0 & \text{otherwise} \end{cases}$$

Now we show that v is a valuation in $\mathcal{M}_S$. Properties 6 and 7 of $\mathcal{T}^*$ together ensure that v takes values in $\mathcal{V}_{10}$. From the definition of v it immediately follows that v is a $\mathcal{M}_{\mathbf{B}}$-valuation (i.e. $P_1(v(\circ\psi)) = P_3(v(\psi))$, $P_4(v(\circ\psi)) = 1$, $P_1(v(\neg\psi)) = P_2(v(\psi))$, and if $P_4(v(\psi)) = 1$ then $P_4(v(\neg\psi)) = 1$). It remains to show that v respects the conditions corresponding to the axioms in S.

- Suppose $(\mathbf{m}) \in S$. Then $\circ\circ\varphi \in \mathcal{T}^*$ for every φ. Hence the definition of v entails that $P_3(v(\circ\varphi)) = P_4(v(\circ\varphi)) = 1$. This, the definition of v, and the fact that if $P_3(v(\varphi)) = 1$ then either $P_2(v(\varphi)) = 0$ or $P_2(v(\varphi)) = 0$ (but not both), together imply that v satisfies the part concerning $\widetilde{\circ}$ in $C(\mathbf{m})$. Now assume that $v(\varphi) = 0111$. Then φ is of the form $\neg^n \circ \psi$. Hence so is $\neg\varphi$. This, axiom $(\mathbf{m})$, and the fact that $P_2(v(\varphi)) = 1$, imply that $P_1(v(\neg\varphi)) = P_3(v(\neg\varphi)) = P_4(v(\neg\varphi)) = 1$. Hence $v(\neg\varphi) = 1011$. That if $v(\varphi)) = 1011$ then $v(\neg\varphi) = 0111$ is proved similarly. It follows that v respects $C(\mathbf{m})$.

- Suppose $(\mathbf{c}) \in S$, and that $P_1(v(\varphi)) = 0$. Then $\varphi \notin \mathcal{T}^*$. By property 6 of $\mathcal{T}^*$ and axiom $(\mathbf{c})$, this implies that $\neg\varphi \in \mathcal{T}^*$, while $\neg\neg\varphi \notin \mathcal{T}^*$. Hence $P_1(v(\neg\varphi)) = 1$ and $P_2(v(\neg\varphi)) = 0$, and so v respects $C(\mathbf{c})$.

- Suppose $(\mathbf{k1}) \in S$, and that $P_1(v(\varphi)) = 0$. Then $\varphi \notin \mathcal{T}^*$. It follows that $\circ\varphi \in \mathcal{T}^*$, by $(\mathbf{k1})$ and property 3 of $\mathcal{T}^*$. Hence $P_3(v(\varphi)) = 1$, and so v respects $C(\mathbf{k1})$.

- Suppose $(\mathbf{i2}) \in S$, and that $P_2(v(\varphi)) = 0$. Then $\neg\varphi \notin \mathcal{T}^*$. It follows by $(\mathbf{i2})$ that $\neg\circ\varphi \notin \mathcal{T}^*$. Hence $P_2(v(\circ\varphi)) = 0$, and so v respects $C(\mathbf{i2})$.

We leave the other cases to the reader.

Obviously, $v(\psi) \in \mathcal{D}_S$ for every $\psi \in \mathcal{T}^*$, while $v(\varphi_0) \notin \mathcal{D}_S$. Hence v is a model of $\mathcal{T}$ in $\mathcal{M}_S$ which is not a model of φ_0. ■

5 Examples and Applications

Without the axiom $(\mathbf{m})$ all the systems considered so far have characteristic Nmatrices with at most 5 truth-values (see [1; 3]) in which the truth-values are just triples (or sometimes even pairs) of 0's and 1's. Accordingly, we concentrate in our examples below on systems which contain $(\mathbf{m})$, and in which this axiom is not derivable from the other axioms of the system.[10]

5.1 The System Bm

Theorem 11 provides a ten-valued characteristic Nmatrix for **Bm**. However, its proof actually uses only seven of them, since the valuation v constructed there does not use the values 1101,1001, and 0101. Hence we get a 7-valued refinement of $\mathcal{M}_{\mathbf{m}}$ which (by Proposition 6) is characteristic for **Bm**. In this 7-valued Nmatrix the interpretation of $\circ$ is fully deterministic (as dictated by $C(\mathbf{m})$), but the other operations are not. It is easy to see that none

[10] Using the semantics introduced here or the simpler ones introduced in [1; 3], it can easily be seen that if $S \subseteq Ax$ and $(\mathbf{m}) \notin S$, then $(\mathbf{m})$ is provable in $\mathbf{B}[S]$ iff the latter is an extension of **Bci**.

of the other axioms is valid in that Nmatrix (or in $\mathcal{M}_{\mathbf{m}}$). Hence none of these axioms is provable in **Bm** (this can also be seen directly from the corresponding conditions).

5.2 The System mCi

The fundamental system which is called **mCi** in [13] is what is called here **Bmi**. Theorem 11 provides a characteristic 6-valued Nmatrix for this logic, whose set of the truth-values is $\{1101, 1100, 1011, 1010, 0111, 0110\}$ (of which the first four are designated). In this Nmatrix the interpretation of $\circ$ is again the one dictated by $C(\mathbf{m})$, the interpretations of the classical positive connectives are like in Definition 7, and the interpretation of $\neg$ is as follows:

$$\tilde{\neg}a = \begin{cases} \{1101, 1011\} & a = 1101 \\ \{1101, 1100, 1011, 1010\} & a = 1100 \\ \{0111\} & a = 1011 \\ \{0111, 0110\} & a = 1010 \\ \{1011\} & a = 0111 \\ \{1101, 1100, 1011, 1010\} & a = 0110 \end{cases}$$

Now it can easily be checked that Theorem 11 provides exactly the same characteristic Nmatrix for **Bmk**. It followed that **Bmi** = **Bmk** = **mCi**. Similarly, it can easily be seen from the corresponding conditions that in the presence of axiom (**m**), axioms (**k1**) and (**i1**) are equivalent, and axioms (**k2**) and (**i2**) are equivalent (none of these facts is true relative to **B**).

Again a close examination of the proof of Theorem 11 reveals that 1101 is not really used in the refutations it provides in $\mathcal{M}_{\mathbf{mi}}$ of formulas not provable in **mCi**. Hence this proof actually provides a 5-valued Nmatrix which is characteristic for this system (and is a simple refinement of the official $\mathcal{M}_{\mathbf{mi}}$). This 5-valued Nmatrix is isomorphic to the characteristic 5-valued Nmatrix which was provided for **mCi** in [5].

5.3 The System Bmce

The values 1101,1001, and 0101 are actually not used in the proof of Theorem 11 for *any* of the systems which includes axiom (**m**). [11] Hence this proof provides characteristic Nmatrices with at most seven values for *all* extensions of **Bm**. As an example we take **Bmce**. In this Nmatrix the interpretation of $\circ$ and the interpretations of the classical positive connectives are defined like in the case of **Bmi** (with different $\mathcal{D}$ and $\mathcal{F}$, of course). The

[11] However, they *are* needed for providing a comprehensive framework that can *modularly* handle both extensions of **Bm**, and extensions of **B** in which (**m**) is not derivable.

interpretation of $\neg$ is this time as follows:

$$\tilde{\neg} a = \begin{cases} \{1100\} & a = 1100 \\ \{0111\} & a = 1011 \\ \{0111, 0110, 0100\} & a = 1010 \\ \{0111, 0110, 0100\} & a = 1000 \\ \{1011\} & a = 0111 \\ \{1011, 1010, 1000\} & a = 0110 \\ \{1011, 1010, 1000\} & a = 0100 \end{cases}$$

Note that the presence of 0100 and 1000 respectively mean that (**k1**) and (**k2**) are not provable in **Bmce**.

6 Other Axioms

We concentrated above on the set of axioms Ax for the sake of illustration and because of the particular importance of these axioms (especially with connection to Marcos' axiom). However, it is easy to apply the 10-valued framework developed here to handle in a modular way systems which are constructed from a much bigger set of axioms, practically using the same conditions that have been used in [1; 3]. Here is a list of 22 other axioms that we could easily have included in Ax:

($\mathbf{a}_\neg$) $\circ\varphi \supset \circ\neg\varphi$

($\mathbf{a}_\diamond$) $\circ\varphi \supset (\circ\psi \supset \circ(\varphi \diamond \psi))$ $(\diamond \in \{\wedge, \vee, \supset\})$

($\mathbf{o}^1_\diamond$) $\circ\varphi \supset \circ(\varphi \diamond \psi)$ $(\diamond \in \{\wedge, \vee, \supset\})$

($\mathbf{o}^2_\diamond$) $\circ\psi \supset \circ(\varphi \diamond \psi)$ $(\diamond \in \{\wedge, \vee, \supset\})$

$(\neg \supset)_\mathbf{1}$ $\neg(\varphi \supset \psi) \supset \varphi$

$(\neg \supset)_\mathbf{2}$ $\neg(\varphi \supset \psi) \supset \psi$

$(\neg \supset)_\mathbf{3}$ $\varphi \supset (\neg\psi \supset \neg(\varphi \supset \psi))$

$(\neg\vee)_\mathbf{1}$ $\neg(\varphi \vee \psi) \supset \neg\varphi$

$(\neg\vee)_\mathbf{2}$ $\neg(\varphi \vee \psi) \supset \neg\psi$

$(\neg\vee)_\mathbf{3}$ $(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$

$(\neg\wedge)_\mathbf{3}$ $\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$

$(\neg\wedge)_\mathbf{1}$ $\neg\varphi \supset \neg(\varphi \wedge \psi)$

$(\neg\wedge)_\mathbf{2}$ $\neg\psi \supset \neg(\varphi \wedge \psi)$

(K) $\circ(\varphi \supset \psi) \supset (\circ\varphi \supset \circ\psi)$

(4) $\circ\varphi \supset \circ\circ\varphi$

(T) $\circ\varphi \supset \varphi$

Note. There are 3 more axioms which have central role in LFIs (see [13]):

(l) $\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$

(d) $\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$

(b) $(\neg(\varphi \wedge \neg\varphi) \vee \neg(\neg\varphi \wedge \varphi)) \supset \circ\varphi$

As shown in [3], these axioms almost never can be handled in a finite-valued framework. However, it is not difficult to combine the method used in [3] (for providing semantics for LFIs which include one or more of these three axioms, but not axiom (**m**)) with the method used in this paper (for providing semantics for LFIs which do include axiom (**m**)) to get an effective infinite-valued semantic framework in which characteristic Nmatrices can modularly be constructed for *any* LFI which is based on some subset of the set of axioms that have been mentioned in this paper.

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