5-valued Non-deterministic Semantics for The Basic Paraconsistent Logic **mCi**

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Abstract

One of the most important paraconsistent logics is the logic **mCi**, which is one of the two basic logics of formal inconsistency. In this paper we present a 5-valued characteristic nondeterministic matrix for **mCi**. This provides a quite non-trivial example for the utility and effectiveness of the use of non-deterministic many-valued semantics.

1 Introduction

A *paraconsistent* logic is a logic which allows nontrivial inconsistent theories. There are several approaches to the problem of designing a useful paraconsistent logic (see e.g. [8, 13, 10, 9]). One of the oldest and best known is da Costa's approach ([14, 15]), which seeks to allow the use of classical logic whenever it is safe to do so, but behaves completely differently when contradictions are involved. da Costa's approach has led to the family of LFIs (Logics of Formal (In)consistency — see [12]). This family is based on two main ideas. The first is that propositions should be divided into two sorts: the "normal" (or consistent) propositions, and the "abnormal" (or inconsistent) ones. Classical logic can (and should) be applied freely to normal propositions, but not to abnormal ones. The second idea is to formally introduce this classification into the language. When this is done by employing a special (primitive or defined) unary connective $\circ$ (where the intuitive meaning of $\circ\varphi$ is : "φ is consistent") we get a special type of LFIs: the C-systems ([11]). The class of C-systems is the most important and useful subclass of the class of logics of formal (in)consistency.

For a long time the class of C-systems has had one major shortcoming: it lacked a corresponding intuitive semantics, which would be easy to use and would provide real insight into these logics [1]. In [3] this was remedied by providing simple, modular *non-deterministic* semantics for almost all the propositional C-systems considered in the literature. This semantics is based on the use of non-deterministic matrices (Nmatrices). These are multi-valued structures (introduced in [5, 6]) where the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. Although applicable to a much larger family of logics, the semantics of finite Nmatrices has all the advantages that the semantics of ordinary finite-valued semantics provides. In particular:

1. The semantics of finite Nmatrices is *effective* in the sense that for determining whether $T \vdash_{\mathcal{M}} \varphi$ (where $\mathcal{M}$ is an Nmatrix) it always suffices to check only *partial* valuations, defined only on subformulas of $T \cup \{\varphi\}$. It follows that a logic which has a finite characteristic Nmatrix is necessarily decidable.

2. A logic with a finite characteristic Nmatrix is finitary (i.e.: the compactness theorem obtains for it – see [6]).

3. There is a well-known uniform method ([16, 7]) for constructive cut-free calculus of n-sequents for any logic which has an n-valued characteristic matrix. This method can easily be extended to logics which have a finite characteristic Nmatrix (see [4]).

Now [3] has left one major gap: no semantics was provided in it for one of the most basic systems considered in [12]. This is Marco's system **mCi**, to which the whole of section 4 of [12] is devoted, and is the minimal C-system in which an appropriate *inconsistency* operator (dual to the consistency operator $\circ$) can be defined. The main goal of this paper is to complete the work started in [3] by closing this gap. Another goal is to give still another quite non-trivial example for the utility and effectiveness of the use of non-deterministic many-valued semantics. Both goals are achieved here by presenting a finite (in fact: 5-valued) characteristic Nmatrix for **mCi**. [2]

[1] The bivaluations semantics and the possible translations semantics described in [11, 12, 17] are not satisfactory from these points of view, since their effectiveness (in the sense explained below) is not apriorily guaranteed, and so a corresponding proposition should be proved from scratch for any useful instance of these types of semantics.

[2] A possible-translation semantics for **mCi** has been provided in [17].

2 Preliminaries

2.1 The System mCi

Let $\mathcal{L}_{\mathrm{cl}}^{+} = \{\wedge, \vee, \supset\}$, $\mathcal{L}_{\mathrm{cl}} = \{\wedge, \vee, \supset, \neg\}$, and $\mathcal{L}_{\mathrm{C}} = \{\wedge, \vee, \supset, \neg, \circ\}$. For $n \geq 0$, let $\neg^0\varphi = \varphi$, $\neg^{n+1}\varphi = \neg(\neg^n\varphi)$.

Definition 1 Let $\mathbf{HCL}^{+}$ be some Hilbert-type system which has Modus Ponens as the only inference rule, and is sound and strongly complete for the $\mathcal{L}_{\mathrm{cl}}^{+}$-fragment of CPL (classical propositional logic) [3]. The logic **mCi** is the logic in $\mathcal{L}_{\mathrm{C}}$ obtained from $\mathbf{HCL}^{+}$ by adding the schemata:

(t) $\neg\varphi \vee \varphi$

(p) $\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi)$

(i) $\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)$

(cc) $\circ\neg^n\circ\varphi$ (for every $n \geq 0$ [4]).

2.2 Non-deterministic Matrices

Our main semantic tool in what follows will be the following generalization of the concept of a matrix:

Definition 2

1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

   (a) $\mathcal{V}$ is a non-empty set of *truth values*.

   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.

   (c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.

   We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{V}$.

2. A *(legal) valuation* in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{L} \rightarrow \mathcal{V}$ (where we identify a language with its set of formulas) that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{L}$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

[3] I.e.: for every sentence φ and theory $\mathbf{T}$ in $\mathcal{L}_{\mathrm{cl}}^{+}$, $\mathbf{T} \vdash_{\mathbf{HCL}^{+}} \varphi$ iff $\mathbf{T} \vdash_{CPL} \varphi$.

[4] Actually, it suffices to take here $n \geq 1$, since $\circ\circ\varphi$ is a theorem of $\mathbf{HCL}^{+} + \{(t), (p), (i)\}$.

3. A valuation v in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model* in $\mathcal{M}$ of a set $\mathbf{T}$ of formulas (notation: $v \models^{\mathcal{M}} \mathbf{T}$) if it satisfies every formula in $\mathbf{T}$.

4. $\vdash_{\mathcal{M}}$, the consequence relation induced by the Nmatrix $\mathcal{M}$, is defined as follows: $T \vdash_{\mathcal{M}} \varphi$ if for every v such that $v \models^{\mathcal{M}} T$, also $v \models^{\mathcal{M}} \varphi$.

5. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is *sound* for an Nmatrix $\mathcal{M}$ (where $\mathcal{L}$ is the language of $\mathcal{M}$) if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{M}}$. $\mathbf{L}$ is *complete* for $\mathcal{M}$ if $\vdash_{\mathbf{L}} \supseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is *characteristic* for $\mathbf{L}$ if $\mathbf{L}$ is both sound and complete for it (i.e.: if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{M}}$). $\mathcal{M}$ is *weakly-characteristic* for $\mathbf{L}$ if for every formula φ of $\mathcal{L}$, $\vdash_{\mathbf{L}} \varphi$ iff $\vdash_{\mathcal{M}} \varphi$.

3 An Nmatrix for mCi

In our semantics for **mCi** we shall employ five truth values: T, F, t, f, and I. Intuitively, I is the truth value of inconsistent propositions. T and F are the truth values of propositions which are *necessarily* consistent, while t and f are the truth values of propositions which are *contingently* consistent.

Definition 3 The Nmatrix $\mathcal{M}_{mCi} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V} = \{I, T, F, t, f\}$
- $\mathcal{D} = \{I, T, t\}$
- $\mathcal{O}$ is defined by:

$$a \widetilde{\vee} b = \begin{cases} \{t, I\} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \{f\} & \text{if } a, b \notin \mathcal{D} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \{t, I\} & \text{if either } a \notin \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{f\} & \text{if } a \in \mathcal{D} \text{ and } b \notin \mathcal{D} \end{cases}$$

$$a \widetilde{\wedge} b = \begin{cases} \{f\} & \text{if either } a \notin \mathcal{D} \text{ or } b \notin \mathcal{D} \\ \{t, I\} & \text{otherwise} \end{cases}$$

$$\widetilde{\neg} a = \begin{cases} \{F\} & \text{if } a = T \\ \{T\} & \text{if } a = F \\ \{f\} & \text{if } a = t \\ \{t, I\} & \text{if } a \in \{f, I\} \end{cases}$$

$$\widetilde{\circ} a = \begin{cases} \{F\} & \text{if } a = I \\ \{T\} & \text{if } a \neq I \end{cases}$$

These tables reflect the fact that the only sentences which are necessarily consistent according to **mCi** are sentences of the form $\neg^n \circ\varphi$.

4 The Soundness and Completeness Theorem

Theorem 1 **mCi** *is sound for* $\mathcal{M}_{mCi}$.

Proof: Obviously, it suffices to show that if v is a legal valuation in $\mathcal{M}_{mCi}$ then $v(\varphi) \in \mathcal{D}$ whenever φ is an axiom of **mCi**, and that v respects MP (in the sense that $v(\psi) \in \mathcal{D}$ whenever $v(\varphi) \in \mathcal{D}$ and $v(\varphi \supset \psi) \in \mathcal{D}$). This is straightforward for the axioms of $\mathbf{HCL}^+$ (and in fact follows from Theorem 1 of [3]). Now we show the validity in $\mathcal{M}_{mCi}$ of the special axioms of **mCi**:

(t) From the table for negation it follows that if $v(\varphi) \notin \mathcal{D}$ then $v(\neg\varphi) \in \mathcal{D}$. Hence the validity of axiom (**t**) follows from the table for $\vee$.

(p) From the table for negation it follows that if $v(\varphi) \neq I$ then either $v(\varphi) \notin \mathcal{D}$, or $v(\neg\varphi) \notin \mathcal{D}$. Therefore it follows from the tables for $\wedge$ and $\supset$ that $v(\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi) \in \mathcal{D}$ in this case. On the other hand, if $v(\varphi) = I$ then $v(\circ\varphi) = F$ and so again $v(\circ\varphi \supset ((\varphi \wedge \neg\varphi) \supset \psi) \in \mathcal{D}$ by the table for $\supset$.

(i) The tables for $\neg$ and $\circ$ entail that if $v(\varphi) \neq I$ then $v(\neg\circ\varphi) = F$, and so $v(\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)) \in \mathcal{D}$ by the table for $\supset$. On the other hand, if $v(\varphi) = I$ then $v(\neg\circ\varphi \supset (\varphi \wedge \neg\varphi)) \in \mathcal{D}$ by the tables for $\neg$, $\wedge$, and $\supset$.

(cc) By the truth tables for $\circ$, $v(\circ\varphi) \in \{T, F\}$. By the table for $\neg$, this fact entails that for every $n \geq 0$, $v(\neg^n \circ \varphi) \in \{T, F\}$. By the table for $\circ$ it follows therefore that $v(\circ \neg^n \circ\varphi) = T \in \mathcal{D}$.

We leave the proof that v respects MP for the reader. □

Theorem 2 **mCi** *is complete for* $\mathcal{M}_{mCi}$.

Proof: Assume that $\mathbf{T}$ is a theory and φ_0 a sentence such that $\mathbf{T} \nvdash_{mCi} \varphi_0$. We construct a model of $\mathbf{T}$ in $\mathcal{M}_{mCi}$ which is not a model of φ_0. For this extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \nvdash_{mCi} \varphi_0$ (and so $\varphi_0 \notin \mathbf{T}^*$. $\mathbf{T}^*$ has the following properties:

1. $\psi \notin \mathbf{T}^*$ iff $\psi \supset \varphi_0 \in \mathbf{T}^*$.
2. If $\psi \notin \mathbf{T}^*$ then $\psi \supset \varphi \in \mathbf{T}^*$ for every sentence φ of $\mathcal{L}_C$.
3. $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.

4. $\varphi \wedge \psi \in \mathbf{T}^*$ iff both $\varphi \in \mathbf{T}^*$ and $\psi \in \mathbf{T}^*$.

5. $\varphi \supset \psi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.

6. For every sentence φ of $\mathcal{L}_C$, either $\varphi \in \mathbf{T}^*$ or $\neg\varphi \in \mathbf{T}^*$.

7. If both $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$ then $\circ\varphi \notin \mathbf{T}^*$.

8. If $\neg\circ\varphi \in \mathbf{T}^*$ then both $\varphi \in \mathbf{T}^*$ and $\neg\varphi \in \mathbf{T}^*$

9. $\circ\neg^n\circ\varphi$ for every $n \geq 0$.

Property 1 follows from the deduction theorem (which is obviously valid for **mCi**) and the maximality of $\mathbf{T}^*$. Property 2 is proved first for $\psi = \varphi_0$ as follows: by 1, if $\varphi_0 \supset \varphi \notin \mathbf{T}^*$ then $(\varphi_0 \supset \varphi) \supset \varphi_0 \in \mathbf{T}^*$. Hence $\varphi_0 \in \mathbf{T}^*$ by the positive tautology $((\varphi_0 \supset \varphi) \supset \varphi_0) \supset \varphi_0$. A contradiction. Property 2 then follows for all $\psi \notin \mathbf{T}^*$ by 1 and the transitivity of implication. Properties 3–5 are easy corollaries of 1, 2, and the closure of $\mathbf{T}^*$ under positive classical inferences (for example: suppose $\varphi \vee \psi \in \mathbf{T}^*$, but neither $\varphi \in \mathbf{T}^*$, nor $\psi \in \mathbf{T}^*$. By property 1, $\varphi \supset \varphi_0 \in \mathbf{T}^*$ and $\psi \supset \varphi_0 \in \mathbf{T}^*$. Since φ_0 follows in positive classical logic from $\varphi \vee \psi$, $\varphi \supset \varphi_0$, and $\psi \supset \varphi_0$, we get $\varphi_0 \in \mathbf{T}^*$. A contradiction). Property 6 is immediate from Property 3 and Axiom **(t)**. Property 7 follows from Axiom (**p**), while Property 8 follows from Axiom (**i**). Finally, Property 9 follows from Axiom (**cc**).

Define now a valuation v in $\mathcal{M}_{mCi}$ as follows:

$$v(\psi) = \begin{cases} I & \text{if} \quad \psi \in \mathbf{T}^*, \neg\psi \in \mathbf{T}^* \\ F & \text{if} \quad \psi \notin \mathbf{T}^* \ \text{ and } \psi \text{ is of the form } \ \neg^n\circ\varphi \\ f & \text{if} \quad \psi \notin \mathbf{T}^* \ \text{ and } \psi \text{ is not of the form } \ \neg^n\circ\varphi \\ T & \text{if} \quad \neg\psi \notin \mathbf{T}^* \ \text{ and } \psi \text{ is of the form } \ \neg^n\circ\varphi \\ t & \text{if} \quad \neg\psi \notin \mathbf{T}^* \ \text{ and } \psi \text{ is not of the form } \ \neg^n\circ\varphi \end{cases}$$

From Property 6 of $\mathbf{T}^*$ it follows that v is well-defined. From the same property it easily follows also that $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$. We use this to prove that v is a legal valuation, i.e.: it respects the interpretations of the connectives in $\mathcal{M}_{mCi}$. That this is the case for the positive connectives easily follows from Properties 3–5 of $\mathbf{T}^*$, and the fact that by definition of v, $v(\psi_1 * \psi_2) \in \{I, t, f\}$ for every ψ_1, ψ_2, and $* \in \{\vee, \wedge, \supset\}$. We prove next the cases of $\neg$ and $\circ$:

- Suppose $v(\psi) = I$. Then by the definition of v, both ψ and $\neg\psi$ are in $\mathbf{T}^*$. Hence $\circ\psi \notin \mathbf{T}^*$ by Property 7. Since $\circ\psi$ is $\neg^0\circ\psi$, this means that $v(\circ\psi) = F$.

- Suppose $v(\psi) \neq I$. Then by the definition of v, either ψ or $\neg\psi$ is not in $\mathbf{T}^*$. Hence $\neg\circ\psi \notin \mathbf{T}^*$ by Property 8. Since $\neg\circ\psi$ is $\neg^1\circ\psi$, this means that $v(\circ\psi) = T$.

- Suppose $v(\psi) = T$. Then ψ is of the form $\neg^n \circ \varphi$, and $\neg\psi \notin \mathbf{T}^*$. But then $\neg\psi$ is $\neg^{n+1} \circ \varphi$, and so the fact that $\neg\psi \notin \mathbf{T}^*$ entails that $v(\neg\psi) = F$.

- Suppose $v(\psi) = F$. Then the formula ψ is of the form $\neg^n \circ \varphi$, and $\psi \notin \mathbf{T}^*$. Therefore by Properties 9 and 6, $\circ\neg\psi \in \mathbf{T}^*$, and $\neg\psi \in \mathbf{T}^*$. It follows by Property 7 that $\neg\neg\psi \notin \mathbf{T}^*$. Since $\neg\psi$ is in this case $\neg^{n+1}\circ\varphi$, this entails that $v(\neg\psi) = T$.

- Suppose $v(\psi) = t$. Then $\neg\psi \notin \mathbf{T}^*$, and ψ is not of the form $\neg^n \circ \varphi$. Hence also $\neg\psi$ is not of the form $\neg^n \circ \varphi$, and since $\neg\psi \notin \mathbf{T}^*$, we have $v(\neg\psi) = f$.

- Suppose $v(\psi) = f$. Then $\psi \notin \mathbf{T}^*$, and so (by Property 6) $\neg\psi \in \mathbf{T}^*$. Hence $v(\neg\psi) \neq f$. Since in this case ψ and $\neg\psi$ are not of the form $\neg^n\circ\varphi$, $v(\neg\psi) \notin \{T, F\}$ as well. It follows that $v(\neg\psi) \in \{t, I\}$.

- Suppose $v(\psi) = I$. Then both ψ and $\neg\psi$ are in $\mathbf{T}^*$. Hence $\circ\psi \notin \mathbf{T}^*$ by Property 7, and so by Property 9 ψ is not of the form $\neg^n \circ \varphi$. This implies that also $\neg\psi$ is not of this form, and so $v(\neg\psi) \notin \{T, F\}$. Since $\neg\psi \in \mathbf{T}^*$, $v(\neg\psi) \neq f$ as well. Hence $v(\neg\psi) \in \{t, I\}$.

Since $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$, $v(\psi) \in \mathcal{D}$ for every $\psi \in \mathbf{T}$, while $v(\varphi_0) \notin \mathcal{D}$. Hence v is a model of $\mathbf{T}$ which is not a model of $v(\varphi_0)$. □

Together Theorems 1 and 2 provide the main result of this paper:

Theorem 3 $\mathcal{M}_{mCi}$ *is a characteristic Nmatrix for* **mCi**.

Corollary 1 **mCi** *is decidable.*

Example 1 $\vdash_{mCi} \neg\neg\circ\varphi \supset \circ\varphi$

Proof: Let v be a valuation in $\mathcal{M}_{mCi}$ and let φ be a sentence. Then $v(\circ\varphi) \in \{T, F\}$, and so $v(\neg\neg\circ\varphi) = v(\circ\varphi)$ by the table for $\neg$. It follows from the table for $\supset$ that $v(\neg\neg \circ \varphi \supset \circ\varphi) \in \{t, I\} \subseteq \mathcal{D}$. □

Example 2 $\circ p \supset \circ\neg p$ *is not a theorem of* **mCi**.

Proof: Define a (partial) valuation v by $v(p) = f$, $v(\neg p) = I$, $v(\circ p) = T$, $v(\circ\neg p) = F$, and $v(\circ p \supset \circ\neg p) = f$. Then v is a legal partial valuation, and by the effectivity of the semantics (see the introduction) it can be extended to a countermodel of $\circ p \supset \circ\neg p$. □

5 Extensions of mCi and Modularity

One of the most important advantages of the semantics of Nmatrices is its *modularity*. The idea is as follows. Let **L** be some basic logic, and suppose that $\mathcal{M}$ is a characteristic Nmatrix for **L**. Then to each natural axiom Ax that one might like to add to **L** there usually corresponds a condition that refinements of $\mathcal{M}$ should satisfy in order for Ax to be sound in them. A characteristic Nmatrices for natural extensions of **L** by a finite set of such axioms can then be produce in a modular way by refining $\mathcal{M}$ according to these conditions. Proving the soundness and completeness of such an extension of **L** for the corresponding resulting refinement of $\mathcal{M}$ usually involves only a straightforward adaptation of the proof of the soundness and completeness of **L** for $\mathcal{M}$. A lot of examples of this modularity have been given in [1, 2] and [3]. The methods of the latter can be applied to extensions of **mCi** in the most obvious way. Here are 3 examples:

- Let **(c)** be the scheme $\neg\neg\varphi \supset \varphi$. A characteristic Nmatrix for the extension of **mCi** by **(c)** is obtained from $\mathcal{M}_{mCi}$ by letting $\tilde{\neg} f = \{t\}$ (rather than $\tilde{\neg} f = \{t, I\}$).
- Let **(e)** be the scheme $\varphi \supset \neg\neg\varphi$. A characteristic Nmatrix for the extension of **mCi** by **(e)** is obtained from $\mathcal{M}_{mCi}$ by letting $\tilde{\neg} I = \{I\}$.
- A characteristic Nmatrix for the extension of **mCi** by both **(c)** and **(e)** is obtained from $\mathcal{M}_{mCi}$ by letting $\tilde{\neg} f = \{t\}$ and $\tilde{\neg} I = \{I\}$.

We leave the proofs of these claims for the reader.

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