Encoding Modal Logics in Logical Frameworks*

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August 14, 1996

Abstract

We present and discuss various formalizations of Modal Logics in Logical Frameworks based on Type Theories. We consider both Hilbert- and Natural Deduction-style proof systems for representing both truth (local) and validity (global) consequence relations for various Modal Logics. We introduce several techniques for encoding the structural peculiarities of necessitation rules, in the typed λ-calculus metalanguage of the Logical Frameworks. These formalizations yield readily proof-editors for Modal Logics when implemented in Proof Development Environments, such as Coq or LEGO.

Keywords: Hilbert and Natural-Deduction proof systems for Modal Logics, Logical Frameworks, Typed λ-calculus, Proof Assistants.

Introduction

In this paper we address the issue of designing proof development environments (i.e. "proof editors" or, even better, "proof assistants") for Modal Logics, in the style of [11, 12]. To this end, we explore the possibility of using Logical Frameworks (LF's) based on Type Theory, such as the Edinburgh Logical Framework, the Calculus of Inductive Constructions or Martin-Löf predicative Type Theory [7, 4, 22, 16]. Logical Frameworks can be viewed as general "logic specification" languages. They are based on the notions of *hypothetico-general judgement* [13] and the *judgements-as-types, λ-terms-as-proofs* paradigm [7].

According to the LF methodology the crucial step, in the development of a proof editor for a given logic is the encoding (or formalization, or representation,...) of a particular presentation of the logic in the typed metalanguage of LF.

In this paper, we introduce and study various encodings in dependent typed λ-calculus of Hilbert- and Natural Deduction-style (ND-style) systems for both the consequence relations of *validity* and *truth* of **K**, **KT**, **K4**, **KT4** (**S4**), **KT45** (**S5**), **KL**. In particular, we extend and generalize the methodology developed in [2]. For each encoding we state the appropriate faithfulness and adequacy theorem.

The main challenge in encoding Modal Logics in Logical Frameworks is that of enforcing the side conditions on the application of the proper modal rules, i.e. *rules of proof* or "impure rules" in the sense of [1]. Such rules, in fact, cannot be applied uniformly to any set of premises, but are subject to various forms of restrictions, e.g.: the premises depend on no

*Partially supported by EC HCM Project No. CHRX-CT92.0046, *Lambda-Calcul Typé*, MURST 40%, CNR CC 94.0073.CT01.

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assumption; or depend only on assumptions of a certain shape (boxed, essentially boxed, etc.); or even, the premises have been derived only by proofs of a certain special shape (see Prawitz's *third version* of **S4**). This issue was dealt with in [2] using systems with multiple judgements. In this paper, we expand this solution and present new alternatives, using judgements on proofs or exploiting the underlying λ-calculus structure of the metalanguage.

Our objective is not that of extending to modal logics the "proposition-as-types", "generalized λ-terms-as-proofs" paradigm, as is the case in [14, 17]. We explore, rather, the possibility of extending to modal logics the "*judgements*-as-*dependent* types", "λ-terms-as-ND-proofs" paradigm of [7]. To this end we do not try to invent radically new deductive systems or new proof figures as in [14, 17], possibly using special extensions of the λ-calculus. These systems, albeit very interesting for the new insights that they can provide in the conceptual understanding of modality, are beyond the scope of this paper, because they cannot be used as the basis of an encoding of modal logics in existing general proof assistants. In this paper we try to provide natural encodings of *existing* and *classical* systems of modal logic (or very slight extensions of them). We want to produce *natural* editors, which do not force upon the user the overhead of unfamiliar, indirect encodings, or the burden of learning an altogether new system. A user of the original logic should transfer immediately to an editor, based on our encodings, his practical experience and "trade tricks". The only possible novelty, w.r.t. the original system, that he should experience, while using the editor, should arise only from the fact that the specification methodology of Logical Frameworks forces him to make precise and explicit all tacit conventions. Our approach therefore differs substantially from that of [14, 17], e.g. β-reductions of λ-terms, encoding proofs in our systems, are not intended to represent normalization of proofs, but only instantiation and application of Lemmata, i.e. the structural rule of Cut.

Nevertheless, in our view, the interest of this paper goes beyond that of merely tailoring Logical Frameworks to the peculiarities and idiosyncrasies of Modal Logics. LF's naturally suggest systems based on the natural deduction mechanism of assuming-discharging assumptions. Moreover, LF's allow to conceive systems which manipulate multiple judgements on formulæ and, exploiting the *judgements-as-types* paradigm, allow also to reason directly on proofs and not only on assertions. Some of the systems and encodings that we introduce and analyze, are interesting also from the purely logical point of view in that they suggest naturally alternative presentations of Modal Logics. In particular, the ND-style systems with multiple consequence relations that we introduce are new, as we know, and probably they can compete with classical systems as far as naturality or easy of use.

The paper is organized as follows. In Sect.1 we recall the basic syntactical and semantical definitions of Modal Logic and we present the classical Hilbert systems and the classical (together with some not so classical) ND-style systems for **K**, **KT**, **K4**, **KT4** (**S4**), **KT45** (**S5**), **KL**. In Sect.2 we present briefly the main features and applications of Logical Frameworks. The encoding of the syntax of Modal Logic appears in Sect.3. The encodings of the Hilbert-system systems and the ND-style systems in LF appear in Sect.4, and 5 respectively. In each section we discuss first systems for validity, then systems for truth. On several occasions we discuss more than one technique for implementing a given system; in Sect.6 we relate formally these different approaches. Final remarks, applications, and related work are discussed in Sec.7. Proofs of theorems appear in the Appendix A.

1 Modal Logics

In this section, we briefly recall the basic notions of Modal Logics (see e.g. [10, 20]); we present Hilbert- and ND-style systems for representing truth and validity consequence relations for various modal logics.

1.1 Syntax and Semantics

The formulæ of the basic modal propositional language Φ are defined by the following abstract syntax: $\varphi ::= p \mid \neg\varphi \mid \varphi \supset \psi \mid \Box\varphi$, where p ranges over the set of *atomic proposition*, denoted by Φ_a. The constant $\mathit{ff} \in \Phi_a$ denotes the always false proposition. Given $\varphi \in \Phi$, we denote by $\mathrm{FV}(\varphi)$ the set of *(free) atomic predicate variables*, defined as usual; the notion of FV is extended to sets of formulæ: $\mathrm{FV}(\Gamma) = \cup_{\varphi\in\Gamma}\mathrm{FV}(\varphi)$. By $\varphi[x_1,\ldots,x_n]$ we denote a formula φ such that $\mathrm{FV}(\varphi) \subseteq \{x_1,\ldots,x_n\}$; we define $\Phi_X \stackrel{\text{def}}{=} \{\varphi \in \Phi \mid \mathrm{FV}(\varphi) \subseteq X\}$. Finally, we take $\Diamond\varphi$ as a syntactic shorthand for $\neg\Box\neg\varphi$.

The interpretation of modal formulæ is based on Kripke's frames and models. A *frame* is a pair $\mathcal{F} = \langle W, \rightarrow\rangle$ where W is the *domain* and $\rightarrow \subseteq W \times W$ is the *accessibility relation*. Elements of W are called *states*, and are denoted by s. A *model* is a triple $\mathcal{M} = \langle W, \rightarrow, \rho\rangle$ where $\langle W, \rightarrow\rangle$ is a frame, and $\rho : \Phi_a \rightarrow \mathcal{P}(W)$ is a *valuation*.

Given a formula φ, a model $\mathcal{M}$ and a state s, we define when φ is true in s ($s \models_{\mathcal{M}} \varphi$) inductively on the structure of the formula, as usual. In particular, $s \models_{\mathcal{M}} \Box\varphi \iff \forall s'.s \rightarrow s' \Rightarrow s' \models_{\mathcal{M}} \varphi$. If φ is true in every state of a model $\mathcal{M}$, we say that φ *is valid in* $\mathcal{M}$ ($\models_{\mathcal{M}} \varphi$).

1.2 Consequence Relations

According to [1, 20], the semantic interpretation of formulæ gives rise to (at least) two (logical) consequence relations (CR's).

Definition 1.1 (Truth and Validity Consequence Relations) *Given $\Gamma \subseteq \Phi$, $\varphi \in \Phi$, and M class of models, we say that*

- φ is true in Γ *w.r.t.* M *($\Gamma \models_M \varphi$) if* $\forall\mathcal{M} \in M.\forall s \in \mathcal{M}.s \models_{\mathcal{M}} \Gamma \Rightarrow s \models_{\mathcal{M}} \varphi$*;*
- φ is true in Γ *($\Gamma \models \varphi$) if* $\forall\mathcal{M}\forall s \in \mathcal{M}.s \models_{\mathcal{M}} \Gamma \Rightarrow s \models_{\mathcal{M}} \varphi$*;*
- φ is valid in Γ *w.r.t.* M *($\Gamma \Vdash_M \varphi$) if* $\forall\mathcal{M} \in M. \models_{\mathcal{M}} \Gamma \Rightarrow \models_{\mathcal{M}} \varphi$*;*
- φ is valid in Γ *($\Gamma \Vdash \varphi$) if* $\forall\mathcal{M}. \models_{\mathcal{M}} \Gamma \Rightarrow \models_{\mathcal{M}} \varphi$*.*

These definitions are straightforwardly extended to sets of formulæ, and subclasses of models: given M a set of models, we define $\models_M = \bigcap_{\mathcal{M}\in M} \models_{\mathcal{M}}$, $\Vdash_M = \bigcap_{\mathcal{M}\in M} \Vdash_{\mathcal{M}}$.

These CR's correspond to the *(model) global relation* and the *(model) local relation* of [20], respectively. They differ on the releavance given to assumptions: in the validity CR, formulæ of Γ are seen as *theorems*, true in every state, while in the truth CR they are *assumptions*, locally true in each state we consider. This difference is made appearant in

Theorem 1.2 ([20]) *For $\Gamma \subseteq \Phi$, $\varphi \in \Phi$:* $\Gamma \Vdash \varphi \iff \{\Box^n\psi \mid \psi \in \Gamma, n \in \mathbb{N}\} \models \varphi$.

Moreover, the usual "deduction theorem" ("$\Gamma, \varphi \models \psi \iff \Gamma \models \varphi \supset \psi$") holds only for the true CR's: it is easy to see that $p \Vdash \Box p$, but of course $\not\Vdash p \supset \Box p$.

1.3 Hilbert-style systems

Hilbert-style systems have been (and still are) very important tools in investigating axiomatizations of Modal Logics. Several kinds of such systems have been proposed; they differ

K	$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$
T	$\Box\varphi \supset \varphi$
4	$\Box\varphi \supset \Box\Box\varphi$
5	$\Diamond\varphi \supset \Box\Diamond\varphi$
L	$\Box(\Box\varphi \supset \varphi) \supset \Box\varphi$
NEC $\dfrac{\varphi}{\Box\varphi}$	φ does not depend on any assumption
NEC' $\dfrac{\varphi}{\Box\varphi}$	

Validity		Truth	
K′	$= \mathbf{C} + K +$ NEC'	**K**	$= \mathbf{C} + K +$ NEC
KT′	$= \mathbf{K'} + T$	**KT**	$= \mathbf{K} + T$
K4′	$= \mathbf{K'} + 4$	**K4**	$= \mathbf{K} + 4$
KT4′	$= \mathbf{KT'} + 4$	**KT4**	$= \mathbf{KT} + 4$
KT45′	$= \mathbf{KT4'} + 5$	**KT45**	$= \mathbf{KT4} + 5$
KL′	$= \mathbf{K'} + L$	**KL**	$= \mathbf{K} + L$

Figure 1: Axioms, rules and Hilbert-style systems for Modal Logics.

essentially on the class of Kripke models they axiomatize implicitly, and on the represented CR. All of them extend the following basic propositional calculus, which we denote by **C**:

$$\mathbf{C} \stackrel{\text{def}}{=} \boxed{\begin{array}{l} A_1 : \varphi \supset (\psi \supset \varphi) \\ A_2 : (\varphi \supset (\psi \supset \vartheta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \vartheta)) \\ A_3 : (\neg\psi \supset \neg\varphi) \supset ((\neg\psi \supset \varphi) \supset \psi) \end{array}} + \boxed{\text{MP}\ \dfrac{\varphi \quad \varphi \supset \psi}{\psi}}$$

In Fig.1 we list the axioms and rules which can be added to **C** in order to obtain the Modal Logic we shall focus on, namely **K**, **KT**, **K4**, **KT4** (**S4**), **KT45** (**S5**), **KL**. In naming the systems we follow Lemmon's convention.

These systems fall into two categories, depending on which CR is represented. These correspond to adopting different necessitation rules: the pure rule NEC' yields systems which are sound and complete only w.r.t. the validity CR's. If we are interested in the truth CR's, we need the impure rule NEC.

We denote by $\pi : \Gamma \vdash_S \varphi$ the proof π of φ from the set of assumptions Γ, using the axioms and rules of system S. The set of free variables in π is denoted by $\mathrm{FV}(\pi)$.

Definition 1.3 (Valid Proofs) *Given $X \subseteq \Phi_a, \Delta \subseteq \Phi_X, \varphi \in \Phi_X$ we say that π is a* valid proof (in the system S) of φ w.r.t. (X, Δ) *(denoted by $(X, \Delta) \models_S \pi : \varphi$) if $\pi : \Delta \vdash_S \varphi$ and $\mathrm{FV}(\pi) \subseteq X$.*

Theorem 1.4 (Completeness of Hilbert-style systems) *For $\Gamma \subseteq \Phi,\ \varphi \in \Phi$:*
– For $S \in \{\mathbf{K}, \mathbf{KT}, \mathbf{K4}, \mathbf{KT4}, \mathbf{KT45}, \mathbf{K4L}\} : \Gamma \vdash_S \varphi \iff \Gamma \models_{M(S)} \varphi$;
– For $S \in \{\mathbf{K'}, \mathbf{KT'}, \mathbf{K4'}, \mathbf{KT4'}, \mathbf{KT45'}, \mathbf{K4L'}\} : \Gamma \vdash_S \varphi \iff \Gamma \Vdash_{M(S)} \varphi$;
where $M(S)$ denotes the class of models corresponding to the axioms characterizing S.

1.4 Natural Deduction-style systems

In this subsection we introduce ND-style systems for both validity and truth CR's. All these systems extend the usual ND-style system for propositional classic logic **NC**:

$$\mathbf{NC} \stackrel{\text{def}}{=} \boxed{\begin{array}{lll} \Gamma, \varphi \vdash \varphi & \supset\text{-I}\ \dfrac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \supset \psi} & \supset\text{-E}\ \dfrac{\Gamma \vdash \varphi \supset \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \\ \text{RAA}\ \dfrac{\Gamma, \neg\varphi \vdash ff}{\Gamma \vdash \varphi} & ff\text{-I}\ \dfrac{\Gamma \vdash \varphi \quad \Gamma \vdash \neg\varphi}{\Gamma \vdash ff} & ff\text{-E}\ \dfrac{\Gamma \vdash ff}{\Gamma \vdash \varphi} \end{array}}$$

$\Box$-I $\dfrac{\Box\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi}$	$\supset_\Box$-E $\dfrac{\Gamma \vdash \Box(\varphi \supset \psi) \quad \Gamma \vdash \Box\varphi}{\Gamma \vdash \Box\psi}$	$\Box$-E $\dfrac{\Gamma \vdash \Box\varphi}{\Gamma \vdash \varphi}$
$\Box'$-I $\dfrac{\emptyset \vdash \varphi}{\emptyset \vdash \Box\varphi}$	$\supset'$-E $\dfrac{\Gamma \Vdash \varphi \supset \psi \quad \Gamma \Vdash \varphi}{\Gamma \Vdash \psi}$	$\Box_\Box$-I $\dfrac{\Gamma \vdash \Box\varphi}{\Gamma \vdash \Box\Box\varphi}$
$\Box''$-I $\dfrac{\Gamma \vdash \varphi}{\Gamma \Vdash \Box\varphi}$	$\supset''$-E $\dfrac{\Gamma \vdash \varphi \supset \psi \quad \Gamma \Vdash \varphi}{\Gamma \Vdash \psi}$	$\Box_\Diamond$-I $\dfrac{\Gamma \vdash \Diamond\varphi}{\Gamma \vdash \Box\Diamond\varphi}$
$\Box'''$-I $\dfrac{\Gamma \Vdash \varphi}{\Gamma \Vdash \Box\varphi}$	$\supset'''$-E $\dfrac{\Gamma \Vdash \varphi \supset \psi \quad \Gamma \vdash \varphi}{\Gamma \Vdash \psi}$	$\Box_\supset$-I $\dfrac{\Gamma \vdash \Box(\Box\varphi \supset \varphi)}{\Gamma \vdash \Box\varphi}$

Validity		**Truth**	
NK′	$=$ **NC** $+ \supset_\Box$-E $+ \Box''$-I $+ \Box'''$-I $+ \supset'$-E $+ \supset''$-E $+ \supset'''$-E	**NS4**	$=$ **NC** $+ \Box$-I $+ \Box$-E
		NK	$=$ **NC** $+ \supset_\Box$-E $+ \Box'$-I
NKT′	$=$ **NK′** $+ \Box$-E	**NKT**	$=$ **NK** $+ \Box$-E
NK4′	$=$ **NK′** $+ \Box_\Box$-I	**NK4**	$=$ **NK** $+ \Box_\Box$-I
NKT4′	$=$ **NKT′** $+ \Box_\Box$-I	**NKT4**	$=$ **NKT** $+ \Box_\Box$-I
NKT45′	$=$ **NKT4′** $+ \Box_\Diamond$-I	**NKT45**	$=$ **NKT4** $+ \Box_\Diamond$-I
NKL′	$=$ **NK′** $+ \Box_\supset$-I	**NKL**	$=$ **NK** $+ \Box_\supset$-I

Figure 2: Rules and ND-style systems for Modal Logics.

We make extensive use of systems with multiple consequence relations. Multiple CR systems of Natural Deduction are probably not very well-known, but we do not give here a detailed presentation of them, because we feel that their "working" is self-evident. We introduce, in this paper, multiple CR systems especially in relation with Natural Deduction systems for validity. Such systems allow to achieve a sharpening of the adequacy theorems appearing in [2] and a generalization of the encodings of logics weaker than **S4**. In Section 5.2.5 we briefly outline how to introduce multiple CR systems for truth, extending those for validity. All the systems for truth appearing elsewhere in the paper are classical.

Systems are displayed in a linearized sequent-like fashion. We denote by $\pi : \Gamma \vdash^i_S \varphi$ the proof π of the fact tha φ is entailed by the assumptions Γ, accordingly to the i-th CR of the system S.

In Fig.2 we display the rules which can be added to **NC** in order to obtain ND-style versions of the Modal Logics **K**, **KT**, **K4**, **KT4** (**S4**), **KT45** (**S5**), **KL**. In naming these systems we extend Lemmon's convention for Hilbert-style systems.

These systems count as ND-style systems, in that their rules follow the general schema

$$\forall \Gamma_1, \ldots, \Gamma_n \frac{\Gamma_1, \Delta_1 \vdash^{i_1} p_1 \ldots \Gamma_n, \Delta_n \vdash^{i_n} p_n}{\Gamma_1, \ldots, \Gamma_n \vdash^i p} C$$

where C is a possible *side condition*, that is a restriction (max. level 2, in the terminology of [1]) on the applicability of the schemata; and $i_1, \ldots, i_n, i \in \{1, \ldots, m\}$ where $\vdash^1, \ldots, \vdash^m$ are the m CR of the system S. In this view, ND-style systems are characterized by the fact that one does not focus only on theorems but rather on assumption-conclusion dependencies. Rules are monotone with respect to sets of assumptions and possibly exploit assumption-discharging mechanisms. Hence, we assume the structural rules of weakening and contraction.

Strictly speaking, rules of ND-style systems should exhibit also an internal symmetry, but it is well-known that this proof-theoretic property is problematic for modalities. Such a symmetry can be recovered by substantially modifying the notion of sequents, which is out of the scope of this work; see e.g. [15, 21].

The systems in Fig.2 fall into two categories, depending on which CR is represented. **NK**,…, **NS4** represent the truth CR's while **NK'**,…, **NKL'** represent the validity CR's. ND-style systems are best suited to represent the truth consequence relation, since the $\supset$-I rule wraps up the deduction theorem in the system. Prawitz' system **NS4** is a good example of how to take full advantage of this [18].

On the other hand, ND-style systems for validity are cumbersome: since the deduction theorem does not hold for $\models\!\!\!\models$, we can no longer adopt the usual introduction rule for implication. A possible solution for overcoming this problem appears in the system **NK'** that we introduce here. This system uses two different CR's, i.e. $\vdash, \Vdash$, whose intended meaning is:

- $\Gamma \vdash \varphi$ iff "there is a proof of φ from Γ which does not use the $\Box'$-I, $\Box''$-I rules" (these derivations are said *box-intro free*);
- $\Gamma \Vdash \varphi$ iff "there is a proof of φ from Γ which does use the $\Box'$-I, $\Box''$-I rules".

Box-intro free proofs can be used in deriving valid consequences, but not the converse. The connection between these two notions of derivation is clear in the box introduction rules: we can "box" a valid formula still obtaining a valid formula (rule $\Box''$-I), but if we "box" a formula obtained on the $\vdash$ level, we obtain a valid formula ($\Box'$-I). The rules $\supset'$-E, $\supset''$-E, $\supset'''$-E allow for the "modus ponens" between valid and box-intro free derived formulæ. The rule Embed $\dfrac{\Gamma \vdash \varphi}{\Gamma \Vdash \varphi}$ is however derivable:

$$\supset''\text{-E}\;\dfrac{\Box''\text{-I}\;\dfrac{\Gamma \vdash \mathit{true}}{\Gamma \Vdash \Box \mathit{true}} \qquad \supset\text{-I}\;\dfrac{\Gamma \vdash \varphi}{\Gamma \vdash \Box \mathit{true} \supset \varphi}}{\Gamma \Vdash \varphi}$$

where *true* denotes any propositional tautology, e.g. $\varphi \supset \varphi$ (its derivation is omitted).

The rule $\supset_\Box$-E corresponds to the K axiom of Hilbert-style systems. The other rules for $\Vdash$ ($\supset'$-E, $\Box'$-I) correspond to the modus ponens and the necessitation rules, respectively. Rules corresponding to the axioms of the extensions of **NK'**, are added at the level of $\vdash$.

Notation for proofs and free variables of proofs are the same of Hibert-style systems.

Theorem 1.5 (Completeness of ND-style systems) *For* $\Gamma \subseteq \Phi,\ \varphi \in \Phi$*:*
– *For* $S \in \{\mathbf{NK}, \mathbf{NKT}, \mathbf{NK4}, \mathbf{NKT4}, \mathbf{NKT45}, \mathbf{NKL}, \mathbf{NS4}\} : \Gamma \vdash_S \varphi \iff \Gamma \models_{M(S)} \varphi$*;*
– *For* $S \in \{\mathbf{NK}', \mathbf{NKT}', \mathbf{NK4}', \mathbf{NKT4}', \mathbf{NKT45}', \mathbf{NKL}'\} : \Gamma \Vdash_S \varphi \iff \Gamma \models\!\!\!\models_{M(S)} \varphi$*;*
where $M(S)$ *denotes the class of models corresponding to the rules characterizing* S*.*

2 Logical Frameworks

Type Theories, such as the Edinburgh Logical Framework [7, 2] or the Calculus of Inductive Constructions [4, 22] were especially designed, or can be fruitfully used, as a general logic specification language, i.e. as a Logical Framework. In an LF, we can represent faithfully and uniformly all the relevant concepts of the inference process in a logical system: syntactic categories, terms, assertions, axiom schemata, rule schemata, tactics, etc. via the "judgements-as-types λ-terms-as-proofs" paradigm. The key concept is that of *hypothetico-general* judgement [13], which is rendered as a type of the dependent typed λ-calculus of the Logical Framework. The λ-calculus metalanguage of an LF supports *higher order* syntax. *Substitution*, α-conversion of bound variables and *instatiation of schemata* are also taken care of uniformly by the metalanguage. Since LF's allow for higher order assertions (*judgements*) one can treat on a par axioms and rules, theorems and derived rules.

Encodings in LF's often provide the "normative" formalization of logic under consideration. The specification methodology of LF's, in fact, forces the user to make precise all tacit, or informal, conventions, which always accompany any presentation of a logic.

Any interactive proof development environment for the type theoretic metalanguage of an LF (e.g. Coq [11], LEGO [12]), can be readily turned into one for a specific logic. We need only to fix a suitable environment (the *signature*), i.e. a declaration of typed constants corresponding to the syntactic categories, term constructors, judgements, and rule schemata. Such an LF-generated editor allows the user to reason "under assumptions" and go about in developing a proof the way mathematicians normally reason: using hypotheses, formulating conjectures, storing and retrieving lemmata, often in top-down, goal-directed fashion.

LF provide a common medium for integrating different systems. Hence LF-derived editors rival special purpose editors when efficiency can be increased by integrating independent logical systems. LF-generated editors are *natural*. A user of the original logic can transfer immediately to them his practical experience and "trade tricks." They do not force upon the user the overhead of unfamiliar indirect encodings, as would editors, say derived from FOL editors, via an encoding.

The wide conceptual universe provided by LF allows, on various occasions, to device genuinely new presentations of the logics. This will be the case for some of the encodings for Modal Logics in this paper. In particular, we shall capitalize on the feature of LF's of treating simultaneously different judgements and of treating proofs as first-class objects.

In this paper, we work in the Edinburgh Logical Framework, as presented in [7].

3 Encoding of the Syntax

In encoding the language of Modal Logic we follow the LF paradigm [7, Sect.3]: the syntactic category Φ is represented by the type o of propositions; for each syntactic constructor, we introduce a corresponding constructor over o. The signature $\Sigma(\Phi)$ for the language and the encoding function $\varepsilon_X : \Phi_X \to o$ appear below:

• **Syntactic Categories** $o : \mathbf{Type}$, • **Operations** $\neg : o \to o, \quad \Box : o \to o, \quad \supset : o \to o \to o.$	$\varepsilon_X(\varphi) \stackrel{\text{def}}{=} \varphi \text{ , } if \varphi \in X \quad \varepsilon_X(\neg\varphi) \stackrel{\text{def}}{=} (\neg\ \varepsilon_X(\varphi))$ $\varepsilon_X(\Box\varphi) \stackrel{\text{def}}{=} \Box\ \varepsilon_X(\varphi) \quad \varepsilon_X(\varphi \supset \psi) \stackrel{\text{def}}{=} \supset\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)$

Given a set $X = \{x_1, \ldots, x_n\} \subset \Phi_a$, we denote by Γ_X the context $\langle x_1 : o, \ldots, x_n : o\rangle$.

Theorem 3.1 *Given $X \subseteq \Phi_a$, the function ε_X is a compositional bijection between Φ_X and the canonical forms[1] of type o in $\Sigma(\Phi), \Gamma_X$. Morover, the encoding is compositional in the sense that for $X = \{x_1, \ldots, x_n\}, Y \subseteq \Phi_a$, $\varphi \in \Phi_X$ and $\varphi_1, \ldots, \varphi_n \in \Phi_Y$: $\varepsilon_Y(\varphi[x_1 := \varphi_1, \ldots, x_n := \varphi_n]) = \varepsilon_X(\varphi)[x_1 := \varepsilon_Y(\varphi_1), \ldots, x_n := \varepsilon_Y(\varphi_n)]$.*

All the systems we shall deal with have the same language. Hence, the signatures, that we will introduce in the rest of the paper, will include $\Sigma(\Phi)$ without explicit mention.

4 Encodings of Hilbert-style systems

4.1 Systems for validity

The encodings of these systems follow the LF paradigm for specifying a logical system [7, Sect. 4]. In Fig.3 we give the signature $\Sigma(\mathbf{K}')$ for the Hilbert-style system $\mathbf{K}$, and its

[1] The notion of *canonical form* is very close to that of long $\beta\eta$-normal form; see [7] for details.

• **Judgements** $V : o \to \mathbf{Type}$,
• **Axioms and Rules**
A_1: $\prod_{\varphi,\psi:o} (V\ \varepsilon_X(A_{1_{\varphi,\psi}}))$, *with* $\mathrm{FV}(\varphi,\psi) \subseteq X$. *Similarly for* $A_{2_{\varphi,\psi,\vartheta}}, A_{3_{\varphi,\psi}}, K_{\varphi,\psi}$.
$MP : \prod_{\varphi,\psi:o} (V\ \varphi) \to (V(\supset \varphi\psi)) \to (V\ \psi)$, $NEC : \prod_{\varphi:o} (V\ \varphi) \to (V\ (\Box\varphi))$

L: $\prod_{\varphi:o} (V\ \varepsilon_X(L_\varphi))$, *with* $\mathrm{FV}(\varphi) \subseteq X$. *Similarly for* $T_\varphi, 4_\varphi, 5_\varphi$.

Figure 3: $\Sigma(\mathbf{K}')$ and its extensions for $\mathbf{KL}'$,....

extensions for other systems ($\mathbf{K4}'$, ...).

Given $\Delta \subseteq \Phi_X$, we define the LF context $\gamma_V(\Delta)$ as follows:

$$\gamma_V(\Delta) \stackrel{\mathrm{def}}{=} \begin{cases} \langle\rangle & \text{if } \Delta \equiv \emptyset \\ \gamma_V(\Delta'), v_\varphi : (V\ \varepsilon_X(\varphi)) & \text{if } \Delta \equiv \Delta', \varphi \text{ and } v_\varphi \text{ fresh for } \gamma_V(\Delta') \end{cases}$$

We can then define the *encoding function* $\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}$, where $X \subseteq \Phi_a,\ \Delta \subseteq \Phi_X$; such function maps proofs π of $\mathbf{K}'$ such that $\mathrm{FV}(\pi) \subseteq X$ to canonical forms of type $(V\ \varepsilon_X(\varphi))$, for $\varphi \in \Phi_X$, in the environment $\Sigma(\mathbf{K}'), \Gamma_X, \gamma_V(\Delta)$:

$\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta} : \{\pi \mid (X,\Delta) \models_{\mathbf{K}'} \pi : \varphi, \varphi \in \Phi_X\} \to \{t \mid \Gamma_X, \gamma_V(\Delta) \vdash_{\Sigma(\mathbf{K}')} t : (V\ \varepsilon_X(\varphi)), \varphi \in \Phi_X\}$	
$\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(\varphi) \stackrel{\mathrm{def}}{=} v_\varphi$	$, if\ \varphi \in \Delta$
$\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(A_{1\varphi,\psi}) \stackrel{\mathrm{def}}{=} A_1\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)$	, *analogously for* $A_{2_{\varphi,\psi,\vartheta}}, A_{3_{\varphi,\psi}}, K_{\varphi,\psi}$
$\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(\mathrm{NEC}_\varphi(\pi)) \stackrel{\mathrm{def}}{=} NEC\ \varepsilon_X(\varphi)\ \varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(\pi)$	
$\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(\mathrm{MP}_{\varphi,\psi}(\pi,\pi')) \stackrel{\mathrm{def}}{=} MP\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(\pi)\ \varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(\pi')$	

Theorem 4.1 *The function* $\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}$ *is a compositional bijection between proofs* π*, such that* $(X,\Delta) \models_{\mathbf{K}'} \pi : \varphi$*, and canonical terms* p*,*[2] *such that* $\Gamma_X, \gamma_V(\Delta) \vdash_{\Sigma(\mathbf{K}')} p : (V\ \varepsilon_X(\varphi))$.

4.2 Systems for truth

In encoding these systems, we have to deal with the problematic issue of enforcing the side condition of the necessitation rule. Hence, we have to extend accordingly the LF methodology for encoding assertions. Here we consider three solutions. In the first, we add a new parameter to the basic judgement, i.e. $T : U \to o \to \mathbf{Type}$, where U is a type on which no constructor is defined. In the second, we introduce a new judgement on proof terms, corresponding to the metatheoretic notion that "the proof depends on no assumption." The third solution makes use of two judgements over formulæ, $Ta, V : o \to \mathbf{Type}$. It follows closely the one in [2, Sect.4.1].

[2]In the following, we denote generic terms by t, proof forms by p, proofs of no-assumption judgement by n, proofs of closed judgement by c,...

• **Syntactic Categories** U : **Type**,	• **Judgements** $T : U \to o \to$ **Type**,

• **Axioms and Rules**

A_1: $\prod_{\varphi,\psi:o,\omega:U} (T\ \omega\ (\varepsilon_X(A_{1_{\varphi,\psi}}))$, *with* $\mathrm{FV}(\varphi,\psi) \subseteq X$. *Similarly for* $A_{2_{\varphi,\psi,\vartheta}}, A_{3_{\varphi,\psi}}, K_{\varphi,\psi}$.

MP: $\prod_{\varphi,\psi:o,\omega:U} (T\ \omega\ \varphi) \to (T\ \omega(\supset \varphi\psi)) \to (T\ \omega\ \psi)$, NEC: $\prod_{\varphi:o} \left(\prod_{\omega:U}(T\ \omega\ \varphi)\right) \to \prod_{\omega:U}(T\ \omega\ (\Box\varphi))$

L: $\prod_{\varphi:o}\prod_{\omega:U}(T\ \omega\ \varepsilon_X(L_\varphi))$, *with* $\varphi \subseteq \mathrm{FV}(X)$. *Similarly for* $T_\varphi, 4_\varphi, 5_\varphi$.

Figure 4: $\Sigma_w(\mathbf{K})$, and its extensions for **K4**,...

4.2.1 World parameters

In Fig.4 we give the signature $\Sigma_w(\mathbf{K})$ for the Hilbert-style system $\mathbf{K}$, and its extensions for other systems (**K4**, **KT**, ...). The encoding function $\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}$ is inductively defined on the structure of proofs: given a proof $\pi : \Delta \vdash_{\mathbf{K}} \varphi$, $\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi)$ is the proof term corresponding to π, where $X = \mathrm{FV}(\pi)$.

$$\begin{array}{l}
\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\varphi) \stackrel{\mathrm{def}}{=} v_\varphi \text{ , } if\ \varphi \in \Delta \\
\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(A_{1,\varphi\psi}) \stackrel{\mathrm{def}}{=} A_1\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \omega \text{ , } similarly\ for\ A_2, A_3, K \\
\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(NEC_\varphi(\pi)) \stackrel{\mathrm{def}}{=} NEC\ \varepsilon_X(\varphi)(\lambda\omega' : U.\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\emptyset,\omega'}(\pi))\ \omega \\
\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(MP_{\varphi\psi}(\pi,\pi')) \stackrel{\mathrm{def}}{=} MP\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \omega\ \varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi)\ \varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi')
\end{array}$$

Given a variable ω of type U, $\Delta \subseteq \Phi$ with $\mathrm{FV}(\Delta) \subseteq X$, we define the LF context $\gamma_\omega(\Delta)$ as follows:

$$\gamma_\omega(\Delta) \stackrel{\mathrm{def}}{=} \begin{cases} \omega : U & \text{if } \Delta \equiv \emptyset \\ \gamma_\omega(\Delta'), v_\varphi : (T\ \omega\ \varepsilon_X(\varphi)) & \text{if } \Delta \equiv \Delta', \varphi \text{ and } v_\varphi \text{ fresh for } \gamma_\omega(\Delta') \end{cases}$$

Theorem 4.2 *The function $\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}$ is a compositional bijection between proofs π, such that $(X,\Delta) \models_{\mathbf{K}} \pi : \varphi$, and canonical terms t, such that $\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} t : (T\ \omega\ \varepsilon_X(\varphi))$.*

The idea behind the use of the extra parameter is that in making an assumption, we are forced to assume the existence of a world, say w, and to instantiate the judgement also on w. This judgement then appears as an hypothesis on w. Hence, deriving as premise a judgement, which is universally quantified with respect to U, amounts to establishing the judgement for a generic world on which no assumptions are made, i.e. on no assumptions.

4.2.2 "No Assumptions"-judgement

In Fig.5 we give the signature $\Sigma_{Na}(\mathbf{K})$ and its extensions for the systems **K4**, **KT**, ...

Given $\Delta \subseteq \Phi$ with $\mathrm{FV}(\Delta) \subseteq X$, we define the LF context $\gamma_T(\Delta)$ as follows:

$$\gamma_T(\Delta) \stackrel{\mathrm{def}}{=} \begin{cases} \langle\rangle & \text{if } \Delta \equiv \emptyset \\ \gamma_T(\Delta'), v_\varphi : (T\ \varepsilon_X(\varphi)) & \text{if } \Delta \equiv \Delta', \varphi \text{ and } v_\varphi \text{ fresh for } \gamma_T(\Delta') \end{cases}$$

The adequacy theorem relies on two technical lemmata (the second is in Sec.A.2.4):

Lemma 4.3 *$\forall t, p$ canonical forms: $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} p{:}(T\ t) \Rightarrow \exists n.\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} n{:}(Na\ p\ t)$.*

• **Judgements** $T : o \to \textbf{Type}$, $Na : \prod_{\varphi:o} T\varphi \to \textbf{Type}$,

• **Axioms and Rules**

A_1: $\prod_{\varphi,\psi:o} (T\ (\varepsilon_X(A_{1_{\varphi,\psi}}))$, *with* $\mathrm{FV}(\varphi,\psi) \subseteq X$. *Similarly for* $A_{2_{\varphi,\psi,\vartheta}}, A_{3_{\varphi,\psi}}, K_{\varphi,\psi}$.

MP: $\prod_{\varphi,\psi:o} (T(\supset \varphi\psi)) \to (T\ \varphi) \to (T\ \psi)$, NEC: $\prod_{\varphi:o} \prod_{d:(T\ \varphi)} (Na\ \varphi\ d) \to (T\ \Box\varphi)$

Na_{A_1}: $\prod_{\varphi,\psi:o} (Na\ \varepsilon_X(A_{1_{\varphi,\psi}})\ (A_1\ \varphi\ \psi))$, *with* $\mathrm{FV}(\varphi,\psi) \subseteq X$. *Similarly for* $A_{2_{\varphi,\psi,\vartheta}}, A_{3_{\varphi,\psi}}, K_{\varphi,\psi}$.

Na_{NEC}: $\prod_{\varphi:o} \prod_{d:(T\varphi)} \prod_{n:(Na\ \varphi\ d)} (Na\ \Box\varphi\ (NEC\ \varphi\ d\ n))$

Na_{MP}: $\prod_{\varphi,\psi:o} \prod_{d_1:(T\ \varphi)} \prod_{d_2:(T(\supset\varphi\psi))} (Na\ \varphi\ d_1) \to (Na\ (\supset \varphi\psi)d_2) \to (Na\ \psi\ (MP\ \varphi\ \psi\ d_2\ d_1))$,

L: $\prod_{\varphi:o}(T\ \varepsilon_X(L_\varphi))$ Na_L: $\prod_{\varphi:o}(Na\ \varepsilon_X(L_\varphi)(L\ \varphi))$, *with* $\mathrm{FV}(\varphi) \subseteq X$. *Similarly for* $T_\varphi, 4_\varphi, 5_\varphi$.

Figure 5: $\Sigma_{Na}(\mathbf{K})$ and its extensions for **K4**,....

Following the steps of the proof of Lemma 4.3, it is easy to define a function α which maps each canonical form p, such that $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} p : (T\ t)$ to the corresponding proof term n such that $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} n : (Na\ p\ t)$. Then we can define the encoding function for $\Sigma_{Na}(\mathbf{K})$ as follows:

$$\begin{array}{ll}
\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}(\varphi) \stackrel{\text{def}}{=} v_\varphi & , if\ \varphi \in \Delta \\
\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}(A_{1\,\varphi,\psi}) \stackrel{\text{def}}{=} A_1\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi) & , similarly\ for\ A_{2_{\varphi,\psi,\vartheta}}, A_{3_{\varphi,\psi}}, K_{\varphi,\psi} \\
\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}(\mathrm{NEC}_\varphi(\pi)) \stackrel{\text{def}}{=} NEC\ \varepsilon_X(\varphi)\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\emptyset}(\pi)\ \alpha\left(\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\emptyset}(\pi)\right) & \\
\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}(\mathrm{MP}_{\varphi,\psi}(\pi,\pi')) \stackrel{\text{def}}{=} MP\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}(\pi)\ \varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}(\pi') &
\end{array}$$

Theorem 4.4 *The function* $\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}$ *is a compositional bijection between valid proofs* π*, such that* $(X,\Delta) \models_{\mathbf{K}} \pi : \varphi$*, and canonical terms* p*, such that* $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} p : (T\ \varepsilon_X(\varphi))$.

4.2.3 Two-judgements systems

We next describe a method in which the two consequence relations, validity and truth, are handled together, in one comprehensive system. The method is rather general, and can be used for every Hilbert-type system in which the rules are divided into rules of derivation and rules of proof.

We start with the following observation. The basic concept of a proof of a formula A in a Hilbert-type system $\mathcal{H}$ is that of a labelled *tree*. The labels are formulæ of $\mathcal{H}$, and the following condition should be satisfied:

- The formula which labels a node which is not a leaf should follow from the formulæ which label its successors by one of the rules of $\mathcal{H}$.

A formula A follows in $\mathcal{H}$ from a set of formulæ Δ iff there is a proof-tree (of the kind just described) in which every leaf is labelled by an axiom of $\mathcal{H}$ or by an element of Δ, and the root is labelled by A. Now the main property of a *pure* Hilbert-type system is that for such a system the condition above has a *local* character. By this we mean that all we need to know in order to check it at a certain node, are the formulæ which label that node and its successors. This is *not* the case, e.g. if one of the rule is a rule of proof. Checking validity of a node which is justified by such a rule requires (among other things) checking the leaves of all the branches which pass through that node and see that they all are labelled by axioms. This is a *global* condition on the subtree of which that node is the root!

The solution to this problem is to arrange things so that all the data which is needed for checking validity of a node would be found at that node and its successors. For rules of proof this can be achieved rather easily by adding to each node a second label. This second label is either the word *valid* or the word *true*. Officially, therefore, each node is labelled by a pair $\langle A, l\rangle$, where A is a formula and $l \in \{true, valid\}$. Let us call a tree of such pairs a *generalized $\mathcal{H}$-proof* if the following conditions are satisfied:

- As a tree of formulæ, the tree is a legitimate proof-tree of the system $\mathcal{H}'$, which is obtained from $\mathcal{H}$ by turning any rule of proof into a rule of derivation.
- A node which is not a leaf is labelled *valid* iff all its successors are so labelled.
- A node which is derived by a rule of proof of $\mathcal{H}$ should be labelled *valid* (hence so should also be the case for every node in the subtree which is generated by it).
- A leaf which is labelled by an axiom of $\mathcal{H}$ is labelled *valid*.

It is a straightforward task now to prove the following

Lemma 4.5 *The erasing of the second label is a compositional bijection between:*

1. *proofs in $\mathcal{H}'$ and generalized $\mathcal{H}$-proofs, in which all nodes are labelled* valid.
2. *(ordinary) proofs in $\mathcal{H}$ and generalized $\mathcal{H}$-proofs, in which all leaves which are not labelled by axioms are labelled* true.

It is obvious, therefore, that generalized $\mathcal{H}$-proofs subsume ordinary proofs in both $\mathcal{H}$ and $\mathcal{H}'$. On the other hand they behave nicely from the LF point of view, and so can easily be represented. One possibility is to view generalized $\mathcal{H}$-proofs as ordinary proofs of a pure Hilbert-type system of *signed* formulæ (where the signs are *true* and *valid*). An equivalent approach which is perhaps more intuitive is to introduce *two* judgements, "T" (for "truth") and "V" (for "validity"). The corresponding obvious representation in the case of the modal logics treated above is given in figure 6.

Theorem 4.6 *There is a compositional bijection between generalized $\mathcal{H}$-proofs (where $\mathcal{H} =$ $\mathbf{K}, \mathbf{K4}$, etc.) of $\langle\varphi_1, l_1\rangle, \ldots, \langle\varphi_n, l_n\rangle \vdash_{\mathcal{H}} \langle\psi, l\rangle$ and terms t such that $\Gamma_X, \gamma_V(\Delta), \gamma_T(\Xi) \vdash_{\Sigma_{2j}(\mathcal{H})}$ $t : (J\ \varepsilon_X(\psi))$, where $\Delta = \{\varphi_i \mid l_i = \text{valid}\}$, $\Xi = \{\varphi_i \mid l_i = \text{true}\}$, and $J = \begin{cases} T & \text{if } l = \text{true} \\ V & \text{otherwise.} \end{cases}$*

Corollary 4.7 *Suppose $\{\varphi_1, \ldots, \varphi_n, \psi\} \subseteq \Phi_X$.*

1. *There is a compositional bijection between proofs in $\mathcal{H}'$ (where $\mathcal{H} = \mathbf{K}, \mathbf{K4}$, etc.) of $\varphi_1 \ldots \varphi_n \vdash_{\mathcal{H}} \psi$ and terms t such that $\Gamma_X, \gamma_V(\{\varphi_1, \ldots, \varphi_n\}) \vdash_{\Sigma_{2j}(\mathcal{H})} t : (V\ \varepsilon_X(\psi))$.*
2. *There is a compositional bijection between proofs in $\mathcal{H}$ (where $\mathcal{H} = \mathbf{K}, \mathbf{K4}$, etc.) of $\varphi_1 \ldots \varphi_n \vdash_{\mathcal{H}} \psi$ and terms t such that $\Gamma_X, \gamma_T(\{\varphi_1, \ldots, \varphi_n\}) \vdash_{\Sigma_{2j}(\mathcal{H})} t : (J\ \psi)$, where J is V if $n = 0$, T otherwise.*[3]

[3] J can, in fact be V even in case $n \neq 0$, provided no φ_i is used in the proof.

• **Judgments** $T, V : o \to \mathbf{Type}$,
• **Axioms and Rules**

$A_1 : \prod_{\varphi,\psi:o} (V\ \varepsilon_X(A_{1_{\varphi,\psi}}))$, with $\mathrm{FV}(\varphi,\psi) \subseteq X$. Similarly for $A_{2_{\varphi,\psi,\vartheta}}, A_{3_{\varphi,\psi}}, K_{\varphi,\psi}$.

$MP_{T,T} : \prod_{\varphi,\psi:o} (T\ (\supset \varphi\psi)) \to (T\ \varphi) \to (T\ \psi)$, $MP_{V,V} : \prod_{\varphi,\psi:o} (V\ (\supset \varphi\psi)) \to (V\ \varphi) \to (V\ \psi)$,

$MP_{T,V} : \prod_{\varphi,\psi:o} (T\ (\supset \varphi\psi)) \to (V\ \varphi) \to (T\ \psi)$, $MP_{V,T} : \prod_{\varphi,\psi:o} (V\ (\supset \varphi\psi)) \to (T\ \varphi) \to (T\ \psi)$,

$NEC : \prod_{\varphi:o} (V\ \varphi) \to (V\ (\Box\varphi))$

L: $\prod_{\varphi:o}(V\ \varepsilon_X(L_\varphi))$, *with* $\mathrm{FV}(\varphi) \subseteq X$. *Similarly for* $T_\varphi, 4_\varphi, 5_\varphi$.

Figure 6: $\Sigma_{2j}(\mathbf{K})$ and its extensions for **KL**, ...

The last corollary is nice, but it is obvious that generalized $\mathcal{H}$-proofs define, in fact, something which is stronger than both $\mathcal{H}$ and $\mathcal{H}'$. What naturally corresponds to them is a sort of a *triple* consequence relation, so that $\Delta; \Xi \vdash_{\mathcal{H}} \varphi$ iff there is a generalized $\mathcal{H}$-proof in which the root is labelled by φ, while every leaf is either labelled by an axiom, or by an element of Δ and *valid*, or by an element of Ξ and *true*. This is the case, it should be emphasized, for any Hilbert-type system of the kind we treat here. In the case of modal logics, however, this triple consequence relation has a clear semantic interpretation (and has already been used, e.g., in [5], where it is denoted like this: $\Delta \models_{\mathcal{H}} \Xi \longrightarrow \varphi$):

$$\Delta; \Xi \vdash_{\mathcal{H}} \varphi \iff \forall \mathcal{M} \in \mathbf{M}. \forall s \in \mathcal{M}. (\models_{\mathcal{M}} \Delta \wedge s \models_{\mathcal{M}} \Xi) \Rightarrow s \models_{\mathcal{M}} \varphi$$

It is clear that what we have constructed is a representation of this triple consequence relation. It is easy, in fact, to show the following generalization of the previous corollary:

Theorem 4.8 *There is a compositional bijection between generalized $\mathcal{H}$-proofs of $\Delta; \Xi \vdash_{\mathcal{H}} \varphi$ and canonical terms t such that*

$$\Gamma_X, \gamma_V(\Delta), \gamma_T(\Xi) \vdash_{\Sigma_{2j}(\mathcal{H})} t : (J\ \varepsilon_X(\varphi))$$

where J is either T or V (depending on whether Ξ is empty or not), and $\Delta \cup \Xi \cup \{\varphi\} \subseteq \Phi_X$.

Remark. In our representation the MP rule has been represented by four constants, each with a different type. In general, a rule of derivation R with n premises will be represented by 2^n constants (while a rule of proof will need just one). We can, in fact, represent any such rule by just two ($R_{V,\dots,V}$ and $R_{T,\dots,T}$), provided we introduce the following extra global constant:

$$C : \prod_{\psi:o} (V\ \psi) \to (T\ \psi)$$

Using this constant we can *define*, e.g., $MP_{T,V}$ and $MP_{V,T}$ as follows:

$$MP_{T,V} \stackrel{\text{def}}{=} \lambda\varphi, \psi : o. \lambda t : (T(\supset \varphi\psi)). \lambda s : (V\varphi). (MP_{T,T}\ t\ (C\ s))$$
$$MP_{V,T} \stackrel{\text{def}}{=} \lambda\varphi, \psi : o. \lambda t : (T(\supset \varphi\psi)). \lambda s : (V\varphi). (MP_{T,T}\ (C\ t)\ s)$$

• **Judgements** $Ta, V : o \to \mathbf{Type}$,	
• **Rules**	$\supset\text{-I} : \prod_{\varphi,\psi:o} ((Ta\ \varphi) \to (Ta\ \psi)) \to (Ta(\supset\ \varphi\ \psi))$,
$\Box_{Ta}\text{-I} : \prod_{\varphi:o} (Ta\ \varphi) \to (V\ \Box\varphi)$,	$\supset\text{-E}_{Ta,Ta} : \prod_{\varphi,\psi:o} (Ta(\supset\varphi\ \psi)) \to (Ta\ \varphi) \to (Ta\ \psi)$,
$\Box_V\text{-I} : \prod_{\varphi:o} (V\ \varphi) \to (V\ \Box\varphi)$,	$\supset\text{-E}_{V,Ta} : \prod_{\varphi,\psi:o} (V(\supset\varphi\ \psi)) \to (Ta\ \varphi) \to (V\ \psi)$,
$\supset\text{-E}_{Ta,V} : \prod_{\varphi,\psi:o} (Ta(\supset\varphi\ \psi)) \to (V\ \varphi) \to (V\ \psi)$,	$\supset\text{-E}_{V,V} : \prod_{\varphi,\psi:o} (V(\supset\varphi\ \psi)) \to (V\ \varphi) \to (V\ \psi)$,
$\supset_\Box\text{-E} : \prod_{\varphi,\psi:o} (Ta\ \Box(\supset\ \varphi\ \psi)) \to (Ta\ \Box\varphi) \to (Ta\ \Box\psi)$	

$\Box\text{-E} : \prod_{\varphi:o} (Ta\ \Box\varphi) \to (Ta\ \varphi)$	$\Box_\Box\text{-I} : \prod_{\varphi:o} (Ta\ \Box\varphi) \to (Ta\ \Box\Box\varphi)$
$\Box_\Diamond\text{-I} : \prod_{\varphi:o} (Ta\ \Diamond\varphi) \to (Ta\ \Box\Diamond\varphi)$	$\Box_\supset\text{-E} : \prod_{\varphi:o} (Ta\ \Box(\supset(\Box\varphi)\ \varphi)) \to (Ta\ \Box\varphi)$

Figure 7: $\Sigma_{2j}(\mathbf{NK}')$ and its extensions for **NK4'**,...

Similar treatment can be given to any rule of derivation. This approach has the advantage that we can require J (in Corollary 4.7 and Theorem 4.8) to be simply T, which is rather intuitive. The disadvantage is that we lose the bijection between proofs and terms: there is some amount of freedom concerning where to apply C, and so more than one term corresponds to a given proof. This can be remedied, e.g., by requiring that in canonical terms C will be applied as late as possible.

5 Encodings of Natural Deduction-style systems

Thoroughout this section, we shall encode only the "minimal" fragment of the modal logics. It should be straightforward to extend the signatures to the full systems.

5.1 Systems for validity

We use an extension of the two-judgements technique seen above. In Fig.7 we give the signature $\Sigma_{2j}(\mathbf{NK}')$ and its extension for systems **NK4'**, **NKT'**, ...

Given $\Delta \subseteq \Phi$ with $\mathrm{FV}(\Delta) \subseteq X$, we define the LF context $\gamma_{Ta}(\Delta)$ as follows:

$$\gamma_{Ta}(\Delta) \stackrel{\text{def}}{=} \begin{cases} \langle\rangle & \text{if } \Delta \equiv \emptyset \\ \gamma_{Ta}(\Delta'), v_\varphi : (Ta\ \varepsilon_X(\varphi)) & \text{if } \Delta \equiv \Delta', \varphi \text{ and } v_\varphi \text{ fresh for } \gamma_{Ta}(\Delta') \end{cases}$$

Theorem 5.1 *For $X \subset \Phi_a$, $\Delta \subseteq \Phi_X$, $\varphi \in \Phi_X$:*

- *There exists a compositional bijection between proofs π, such that $(X, \Delta) \models_{\mathbf{NK}'} \pi : \varphi$, and canonical terms p, such that $\Gamma_X, \gamma_{Ta}(\Delta) \vdash_{\Sigma_{2j}(\mathbf{NK})'} p : (Ta\ \varepsilon_X(\varphi))$.*
- *There exists a compositional bijection between proofs π, such that $(X, \Delta) \Vdash_{\mathbf{NK}'} \pi : \varphi$, and canonical terms p, such that $\Gamma_X, \gamma_{Ta}(\Delta) \vdash_{\Sigma_{2j}(\mathbf{NK})'} p : (V\ \varepsilon_X(\varphi))$.*

Special system for NS4. We can get an alternative Natural Deduction-style system **NS4'** for **NKT4'**, closer in spirit to Prawitz' system for S4 [18], by replacing $\supset_\Box$-E and $\Box_\Box$-I by the rule

$$\supset_\Box\text{-I}\ \frac{\Gamma, \Box\varphi \Vdash \psi}{\Gamma \Vdash \Box\varphi \supset \psi}$$

The resulting system is $\mathbf{NS4'} \stackrel{\text{def}}{=} \mathbf{NC} + \supset_\Box\text{-I} + \Box\text{-E}$. In this system, $\supset_\Box$ and $\Box_\Box$-I are derivable on the level of $\Vdash$, not $\vdash$.

The encoding of system **NS4'** is straightforward, and we get a compositional bijection. This is an improvement of the encoding used in [2, Section 4.2].

One can get an analogue of Prawitz' second system for S4 by using the rule

$$\supset_{EM}\text{-I}\ \frac{\Gamma, \varphi \Vdash \psi}{\Gamma \Vdash \Box\varphi \supset \psi}\ \varphi \text{ is essentially-modal}$$

The side condition can be handled, like in [2], by introducing a special judgement, $EM : o \rightarrow \mathbf{Type}$, which corresponds to the property of being "essentially modal".

5.2 Systems for truth

We present two general solutions for handling the necessitation rule in the classical systems presented in Section 1.4: the first is based on *world parameters*, the second makes use of a *"closed assumption"-judgement.* These solutions extend the corresponding ones introduced for the Hilbert-style case. In Section 5.2.5 we sketch also yet another general solution which makes use of three judgements on formulæ. Strictly speaking, this is an encoding of novel multiple CR systems for the truth CR of Modal Logics.

For the special system **NS4** introduced by Prawitz [18], we consider two more encodings. These adopt an auxiliary judgement on proofs for enforcing Prawitz's conditions ("boxed assumptions" and "boxed-fringe", respectively). Also in this section, we restrict ourselves to the "minimal" fragment of modal logic.

5.2.1 World parameters

In Fig.8 we give the signature $\Sigma_w(\mathbf{NK})$ and its extensions for the other systems ($\mathbf{NK4}, \ldots$). The encoding function $\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}$ is defined on the structure of proofs of **NK**: given a proof $\pi : \Delta \vdash_{\mathbf{NK}} \varphi$, $\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\pi)$ is the proof term corresponding to π.

$$\begin{array}{l}
\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\varphi) \stackrel{\text{def}}{=} v_\varphi\ , if\ \varphi \in \Delta \\
\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\Box'\text{-I}_\varphi(\pi')) \stackrel{\text{def}}{=} \Box\text{-I}\ \varepsilon_X(\varphi)\ (\lambda\omega' : U.\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\emptyset,\omega'}(\pi'))\ \omega \\
\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\supset\text{-I}_{\varphi\psi}(\pi')) \stackrel{\text{def}}{=} \supset\text{-I}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \omega(\lambda v_\varphi : (T\omega\varepsilon_X(\varphi)).\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,(\Delta,\varphi),\omega}(\pi')) \\
\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\supset\text{-E}_{\varphi\psi}(\pi', \pi'')) \stackrel{\text{def}}{=} \supset\text{-E}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \omega\ \varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\pi')\ \varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\pi'') \\
\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\supset_\Box\text{-E}_{\varphi\psi}(\pi', \pi'')) \stackrel{\text{def}}{=} \supset_\Box\text{-E}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \omega\ \varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\pi')\ \varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\pi'')
\end{array}$$

Theorem 5.2 *The function $\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}$ is a compositional bijection between proofs π, such that $(X, \Delta) \models_{\mathbf{NK}} \pi : \varphi$, and canonical terms t, such that $\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{NK})} t : (T\ \omega\ \varepsilon_X(\varphi))$.*

• **Syntactic Categories** $U : \mathbf{Type}$, • **Judgements** $T : U \to o \to \mathbf{Type}$, • **Axioms and Rules** $\supset$-I : $\prod_{\varphi,\psi:o,\omega:U} ((T\ \omega\ \varphi) \to (T\ \omega\ \psi)) \to (T\omega(\supset \varphi\psi))$, $\supset$-E : $\prod_{\varphi,\psi:o,\omega:U} (T\ \omega(\supset \varphi\psi)) \to (T\ \omega\ \varphi) \to (T\ \omega\ \psi)$, $\supset_\Box$-E : $\prod_{\varphi,\psi:o,\omega:U} (T\ \omega\ \Box(\supset \varphi\psi)) \to (T\ \omega\ \Box\varphi) \to (T\omega\Box\psi)$, $\Box'$-I : $\prod_{\varphi:o} \left(\prod_{\omega:U}(T\ \omega\ \varphi)\right) \to \prod_{\omega:U}(T\ \omega\ (\Box\varphi))$	$\Box$-E : $\prod_{\varphi:o,\omega:U} (T\ \omega\ \Box\varphi) \to (T\ \omega\ \varphi)$ $\Box_\Box$-I : $\prod_{\varphi:o,\omega:U} (T\ \omega\ \Box\varphi) \to (T\ \omega\ \Box\Box\varphi)$ $\Box_\Diamond$-I : $\prod_{\varphi:o,\omega:U} (T\ \omega\ \Diamond\varphi) \to (T\ \omega\ \Box\Diamond\varphi)$ $\Box_\supset$-E : $\prod_{\varphi:o,\omega:U} (T\ \omega\ \Box(\supset (\Box\varphi)\ \varphi))$ $\to (T\ \omega\ \Box\varphi)$

Figure 8: $\Sigma_w(\mathbf{NK})$ and its extensions for $\mathbf{NK4}$,...

5.2.2 "Closed Assumptions"-judgement

In Fig.9 we give the signature $\Sigma_{Cl}(\mathbf{NK})$ and its extensions for the other truth systems (**NK4**, **NKT**, ...). Notice that there is a rule for establishing the "closed assumption"-judgement corresponding to each proof constructor, i.e. for each rule in **NK**.

The existence and definition of the encoding function relies upon two technical lemmata:

Lemma 5.3 *$\forall p$ canonical form, if $\Gamma_X, \Delta \vdash_{\Sigma_{Cl}(\mathbf{NK})} p : (T\ t)$ then $\exists c.\Gamma_X, \Delta, \Xi_p(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} c : (Cl\ t\ p)$, where $\Xi_p(\Delta) \stackrel{\text{def}}{=} \{c{:}(Cl\ t\ x) \mid x \in \mathrm{FV}(p) \wedge (x{:}(T\ t)) \in \Delta\}$.*

Lemma 5.3 defines naturally a function from canonical proof forms $p : (T\ t)$ to canonical forms of type $(Cl\ t\ p)$, in the same environment expanded with the "closed assumptions" for the free variables of p. Let us denote such function by α.

Lemma 5.4 *$\forall c$ canonical form, if $\Gamma_X, \Delta, \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} c{:}(Cl\ t\ p)$ then $\Gamma_X, \Delta', \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} c{:}(Cl\ t\ p)$, where Ξ contains all and only the Cl assertions, and $\Delta' = \{x{:}(T\ t) \mid (Cl\ t\ x) \in \Im(\Xi)\}$.*

We can now define the encoding function $\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}$, which relies on the α abovementioned.

$$
\begin{array}{|l|}
\hline
\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\varphi) \stackrel{\text{def}}{=} v_\varphi \qquad\qquad\qquad\qquad , if\ \varphi \in \Delta \\
\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\supset\text{-I}_{\varphi,\psi}(\pi)) \stackrel{\text{def}}{=} \supset\text{-I}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ (\lambda v_\varphi{:}(T\ \varepsilon_X(\varphi)).\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,(\Delta,\varphi)}(\pi)) \\
\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\supset\text{-E}_{\varphi,\psi}(\pi',\pi'')) \stackrel{\text{def}}{=} \supset\text{-E}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\pi')\ \varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\pi'') \\
\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\supset_\Box\text{-E}_{\varphi,\psi}(\pi',\pi'')) \stackrel{\text{def}}{=} \supset_\Box\text{-E}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\pi')\ \varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\pi'') \\
\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\Box'\text{-I}_{\varphi}(\pi)) \stackrel{\text{def}}{=} \Box'\text{-I}\ \varepsilon_X(\varphi)\ \varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\emptyset}(\pi)\ \alpha\left(\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\emptyset}(\pi)\right) \\
\hline
\end{array}
$$

Theorem 5.5 *The function $\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}$ is a compositional bijection between proofs π, such that $(X, \Delta) \models_{\mathbf{NK}} \pi{:}\varphi$, and canonical terms t, such that $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} t{:}(T\ \varepsilon_X(\varphi))$.*

5.2.3 "Boxed Assumptions"-judgement

In Fig.10 we give the signature $\Sigma_\Box(\mathbf{NS4})$, which adopts a special technique for implementing Prawitz' system **NS4** [18].

$$\bullet\ \textbf{Judgements}\qquad T : o \to \textbf{Type},\qquad Cl : \prod_{\varphi:o}(T\ \varphi) \to \textbf{Type},$$

$$\bullet\ \textbf{Axioms and Rules}\qquad \supset_\Box\text{-E} : \prod_{\varphi,\psi:o}(T\ \Box(\supset \varphi\psi)) \to (T\ \Box\varphi) \to (T\ \Box\psi),$$

$$\supset\text{-I} : \prod_{\varphi,\psi:o}((T\ \varphi) \to (T\ \psi)) \to (T(\supset \varphi\psi)),\qquad \supset\text{-E} : \prod_{\varphi,\psi:o}(T\ (\supset \varphi\psi)) \to (T\ \varphi) \to (T\ \psi),$$

$$\Box'\text{-I} : \prod_{\varphi:o}\prod_{d:(T\ \varphi)}(Cl\ \varphi\ d) \to (T\ (\Box\varphi)),\quad Cl_{\Box'\text{-I}} : \prod_{\varphi:o}\prod_{d_1:(T\ \varphi)}\prod_{c_1:(Cl\ \varphi\ d_1)}(Cl\ (\Box\varphi)\ (\Box'\text{-I}\ \varphi\ d_1\ c_1))$$

$$Cl_{\supset\text{-I}} : \prod_{\varphi,\psi:o}\prod_{d:(T\ \varphi)\to(T\ \psi)}\left(\prod_{x:(T\ \varphi)}(Cl\ \varphi\ x) \to (Cl\ \psi\ dx)\right) \to (Cl\ \psi\ (\supset\text{-I}\ \varphi\psi\ d)),$$

$$Cl_{\supset\text{-E}} : \prod_{\varphi,\psi:o}\prod_{d_1:(T(\supset\ \varphi\ \psi))}\prod_{d_2:(T\ \varphi)}(Cl\ \varphi\ d_2) \to (Cl\ (\supset\ \varphi\ \psi)\ d_1) \to (Cl\ \psi\ (\supset\text{-E}\ \varphi\ \psi\ d_1\ d_2)),$$

$$Cl_{\supset_\Box\text{-E}} : \prod_{\varphi,\psi:o}\prod_{d_1:(T\ \Box(\supset\varphi\psi))}\prod_{d_2:(T\ \Box\varphi)}(Cl\ \Box(\supset \varphi\psi)\ d_1) \to (Cl\ \Box\varphi\ d_2) \to (Cl\ \Box\psi\ (\supset_\Box\text{-E}\ \varphi\ \psi d_1\ d_2))$$

$$\Box\text{-E} : \prod_{\varphi:o}(T\ \Box\varphi) \to (T\ \varphi),\quad Cl_{\Box\text{-E}} : \prod_{\varphi:o}\prod_{d:(T\ \Box\varphi)}(Cl\ \Box\varphi\ d) \to (Cl\ \varphi\ (\Box\text{-E}\ \varphi\ d));\quad \ldots,\quad \ldots;$$

Figure 9: $\Sigma_{Cl}(\mathbf{NK})$ and its extensions for **NK4**, ...

Given $\Delta \subseteq \Phi$ with $\mathrm{FV}(\Delta) \subseteq X$, we define the LF context $\gamma_\Box(\Delta)$ as follows:

$$\gamma_\Box(\Delta) \stackrel{\text{def}}{=} \begin{cases} \langle\rangle & \text{if } \Delta \equiv \emptyset \\ \gamma_\Box(\Delta'), v_\varphi : (T\varepsilon_X(\varphi)) & \text{if } \Delta \equiv \Delta', \varphi,\ \varphi \text{ is not boxed and } v_\varphi \text{ fresh for } \gamma_\Box(\Delta') \\ \gamma_\Box(\Delta'), v_\varphi : (T\varepsilon_X(\varphi)), & \text{if } \Delta \equiv \Delta', \varphi,\ \varphi \text{ is boxed and } v_\varphi, vb_\varphi \text{ fresh for } \gamma_\Box(\Delta') \\ \quad vb_\varphi : (Bx\varepsilon_X(\varphi)v_\varphi) & \end{cases}$$

The long proof of adequacy relies upon some very technical lemmata. We report here only those needed for defining the encoding function; the others are in Section A.3.8. For sake of simplicity, we adopt the following definition: for p term and Γ context, we define

$$C(p,\Gamma) \stackrel{\text{def}}{=} \text{for all } v_\psi \in \mathrm{FV}(p),\ \text{if } (v_\psi{:}(T\ \varepsilon_X(\psi))) \in \Gamma \text{ then } (vb_\psi{:}(Bx\ \varepsilon_X(\psi)\ v_\psi)) \in \Gamma$$

Lemma 5.6 *Given a canonical term p such that $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p : (T\ t)$, if $C(p, \gamma_\Box(\Delta))$ holds then there is a canonical term b such that $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b : (Bx\ t\ p)$.*

A consequence of this lemma is the existence of a function β_Δ which maps proof terms p whose free variables are "boxed," to proofs b of $(Bx\ \varphi\ p)$; this "reifies" the fact that p represents a proof which depends only on boxed assumptions. This function is inductively defined as follows.

$$\begin{array}{l}
\beta_\Delta(v_\varphi) \stackrel{\text{def}}{=} vb_\varphi \qquad\qquad ,\text{if } \varphi \in \Delta \text{ and } \varphi \text{ boxed} \\
\beta_\Delta(\supset\text{-I}\ t\ t'(\lambda v_1{:}(T\ t).p)) \stackrel{\text{def}}{=} (Bx_{\supset\text{-I}}\ t\ t'\ (\lambda v_1{:}(T\ t).p)(\lambda v_1{:}(T\ t)\lambda vb_1{:}(Bx\ t\ v_1).\beta_{\Delta,\delta_X(t)}(p))) \\
\beta_\Delta(\supset\text{-E}\ t\ t'\ p_1\ p_2) \stackrel{\text{def}}{=} (Bx_{\supset\text{-E}}\ t\ t'\ p_1\ p_2\ \beta_\Delta(p_1)\ \beta_\Delta(p_2)) \\
\beta_\Delta(\Box\text{-I}\ t\ p\ b) \stackrel{\text{def}}{=} (Bx_{\Box\text{-I}}\ t\ p\ b) \\
\beta_\Delta(\Box\text{-E}\ t\ p) \stackrel{\text{def}}{=} (Bx_{\Box\text{-E}}\ t\ p\ \beta_\Delta(p)) \\
\beta_\Delta(\supset_\Box\text{-I}\ t\ t'\ p_2) \stackrel{\text{def}}{=} (Bx_{\supset_\Box\text{-I}}\ t\ t'\ p_2(\lambda v_1{:}(T\ t)\lambda v_2{:}(Bx\ t\ v_1).\beta_{\Delta,\delta_X(A)}(p))) \\
\text{where } p \stackrel{\text{def}}{=} \lambda v_1{:}(T\ t)\lambda v_2{:}(Bx\ t\ v_1).p
\end{array}$$

• Judgements $\quad T : o \to \mathbf{Type}, \qquad Bx : \prod_{\varphi:o} (T\varphi) \to \mathbf{Type},$

• Axioms and Rules $\quad \supset_{\Box}\text{-I} : \prod_{\varphi,\psi:o} \left(\prod_{d:(T\ \Box\varphi)} (Bx\ \Box\varphi\ d) \to (T\ \psi) \right) \to (T(\supset\ \Box\varphi\ \psi)),$

$\supset\text{-I} : \prod_{\varphi,\psi:o} ((T\ \varphi) \to (T\ \psi)) \to (T(\supset\ \varphi\ \psi)), \qquad \supset\text{-E} : \prod_{\varphi,\psi:o} (T(\supset\ \varphi\ \psi)) \to (T\ \varphi) \to (T\ \psi),$

$\Box\text{-I} : \prod_{\varphi:o} \prod_{d:(T\ \varphi)} (Bx\ \varphi\ d) \to (T\ (\Box\varphi)), \qquad \Box\text{-E} : \prod_{\varphi:o} (T(\Box\varphi)) \to (T\ \varphi),$

$$Bx_{\supset_{\Box}\text{-I}} : \prod_{\varphi,\psi:o} \prod_{d:(\prod_{a:(T\ \Box\varphi)} (Bx\ \Box\varphi\ a) \to (T\ \psi))} \left(\prod_{a:(T\Box\varphi)} \prod_{b:(Bx\ \Box\varphi\ a)} (Bx\ \psi\ (d\ a\ b)) \right) \to (Bx(\supset\ \Box\varphi\ \psi)(\supset_{\Box}\text{-I}\ \varphi\ \psi\ d)),$$

$$Bx_{\supset\text{-I}} : \prod_{\varphi,\psi:o} \prod_{d:(T\ \varphi)\to(T\ \psi)} \left(\prod_{a:(T\ \varphi)} (Bx\ \varphi\ a) \to (Bx\ \psi\ (da)) \right) \to (Bx(\supset\ \varphi\ \psi)(\supset\text{-I}\ \varphi\ \psi\ d)),$$

$$Bx_{\supset\text{-E}} : \prod_{\varphi,\psi:o} \prod_{d_1:(T(\supset\ \varphi\ \psi))} \prod_{d_2:(T\ \varphi)} (Bx(\supset\ \varphi\ \psi)\ d_1) \to (Bx\ \varphi\ d_2) \to (Bx\ \psi\ (\supset\text{-E}\ \varphi\ \psi\ d_1\ d_2)),$$

$$Bx_{\Box\text{-I}} : \prod_{\varphi:o} \prod_{d:(T\varphi)} \prod_{b:(Bx\varphi\ d)} (Bx\Box\varphi(\Box\text{-I}\varphi\ d\ b)),\ Bx_{\Box\text{-E}} : \prod_{\varphi:o} \prod_{d:(T\ \Box\varphi)} (Bx\ \Box\varphi\ d) \to (Bx\ \varphi\ (\Box\text{-E}\varphi\ d))$$

Figure 10: $\Sigma_{\Box}(\mathbf{NS4})$.

Lemma 5.7 *$\forall X, \Delta, \varphi$, if $(X, \Delta) \models_{\mathbf{NS4}} \pi : \varphi$ then there exists a canonical form p such that $\Gamma_X, \gamma_{\Box}(\Delta) \vdash_{\Sigma_{\Box}(\mathbf{NS4})} p : (T\ \varepsilon_X(\varphi))$.*

A consequence of this lemma is the existence of the function $\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}$, which maps proofs of **NS4** to canonical proof terms. This function is inductively defined as follows.

$$\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\varphi) \stackrel{\text{def}}{=} v_{\varphi} \qquad , if\ \varphi \in \Delta$$

$$\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\supset\text{-I}_{\varphi\psi}(\pi')) \stackrel{\text{def}}{=} \begin{cases} (\supset_{\Box}\text{-I}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi) \\ (\lambda v_{\varphi}{:}(T\varepsilon_X(\varphi))\lambda vb_{\varphi}{:}(Bx\varepsilon_X(\varphi)v_{\varphi}).\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta,\varphi}(\pi'))) & \text{if } \varphi \text{ boxed} \\ (\supset\text{-I}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ (\lambda v_{\varphi}{:}(T\varepsilon_X(\varphi)).\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta,\varphi}(\pi'))) & \text{if } \varphi \neg \text{ boxed} \end{cases}$$

$$\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\supset\text{-E}_{\varphi,\psi}(\pi',\pi'')) \stackrel{\text{def}}{=} (\supset\text{-E}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\pi'')\ \varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\pi'))$$

$$\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\Box\text{-I}_{\varphi}(\pi')) \stackrel{\text{def}}{=} (\Box\text{-I}\ \varepsilon_X(\varphi)\ \varepsilon_{X,\Delta}(\pi')\ \beta_{\Delta}(\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\pi')))$$

$$\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\Box\text{-E}_{\varphi}(\pi')) \stackrel{\text{def}}{=} (\Box\text{-E}\ \varepsilon_X(\varphi)\ \varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}(\pi'))$$

Theorem 5.8 *The function $\varepsilon^{\Sigma_{\Box}(\mathbf{NS4})}_{X,\Delta}$ is a compositional bijection between proofs π, such that $(X, \Delta) \models_{\mathbf{NS4}} \pi : \varphi$, and canonical terms t, such that $\Gamma_X, \gamma_{\Box}(\Delta) \vdash_{\Sigma_{\Box}(\mathbf{NS4})} t : (T\ \varepsilon_X(\varphi))$.*

In this signature, besides a rule for establishing the "boxed assumption"-judgement corresponding to each rule in **NS4**, there is also an extra rule, namely $\supset_{\Box}$-I. This subtle rule is necessary in order to discharge "boxed assumption"-judgements: see the following example.

Example 5.1 *We show the derivation of axiom $4 : \Box\varphi \supset \Box\Box\varphi$, both in* **NS4** *and in LF*

$$\bullet\ \textbf{Judgements} \qquad T : o \to \textbf{Type}, \qquad BF : \prod_{\varphi:o} (T\ \varphi) \to \textbf{Type},$$

$$\bullet\ \textbf{Axioms and Rules} \qquad \supset_{\Box}\text{-I} : \prod_{\varphi,\psi:o} \left(\prod_{d:(T\ \Box\varphi)} (BF\ \Box\varphi\ d) \to (T\ \psi) \right) \to (T(\supset\ \Box\varphi\ \psi)),$$

$$\supset\text{-I} : \prod_{\varphi,\psi:o} ((T\ \varphi) \to (T\ \psi)) \to (T(\supset\ \varphi\ \psi)), \qquad \supset\text{-E} : \prod_{\varphi,\psi:o} (T\ (\supset\ \varphi\ \psi)) \to (T\ \varphi) \to (T\ \psi),$$

$$\Box\text{-I} : \prod_{\varphi:o} \prod_{d:(T\ \varphi)} (BF\ \varphi\ d) \to (T\ (\Box\varphi)), \qquad \Box\text{-E} : \prod_{\varphi:o} (T(\Box\varphi)) \to (T\ \varphi),$$

$$BF_{\supset_{\Box}\text{-I}} : \prod_{\varphi,\psi:o} \prod_{d:\prod_{a:(T\ \Box\varphi)}(BF\ \Box\varphi\ a)\to(T\ \psi)} \left(\prod_{a:(T\Box\varphi)} \prod_{b:(BF\ \Box\varphi\ a)} (BF\ \psi\ (d\ a\ b)) \right) \to (BF\ (\supset\ \Box\varphi\ \psi)(\supset_{\Box}\text{-I}\ \varphi\ \psi\ d)),$$

$$BF_{\supset\text{-I}} : \prod_{\varphi,\psi:o} \prod_{d:(T\ \varphi)\to(T\ \psi)} \left(\prod_{a:(T\ \varphi)} (BF\ \varphi\ a) \to (BF\ \psi\ (d\ a)) \right) \to (BF\ (\supset\ \varphi\ \psi)\ (\supset\text{-I}\ \varphi\ \psi\ d)),$$

$$BF_{\supset\text{-E}} : \prod_{\varphi,\psi:o} \prod_{d_1:(T\ (\supset\ \varphi\ \psi))} \prod_{d_2:(T\ \varphi)} (BF(\supset\ \varphi\ \psi)\ d_1) \to (BF\ \varphi\ d_2) \to (BF\ \psi\ (\supset\text{-E}\ \varphi\ \psi\ d_1\ d_2)),$$

$$BF'_{\supset\text{-E}} : \prod_{\varphi,\psi:o} \prod_{d_1:(T\ (\supset\ \varphi\ \Box\psi))} \prod_{d_2:(T\ \varphi)} (BF\ \Box\psi\ (\supset\text{-E}\ \varphi\ \Box\psi\ d_1\ d_2)),$$

$$BF_{\Box\text{-I}} : \prod_{\varphi:o} \prod_{d:(T\varphi)} \prod_{b:(BF\varphi\ d)} (BF\ \Box\varphi\ (\Box\text{-I}\varphi\ d\ b)), \qquad BF_{\Box\text{-E}} : \prod_{\varphi:o} \prod_{d:(T\ \Box\varphi)} (BF\ \varphi\ (\Box\text{-E}\ \varphi\ d))$$

Figure 11: $\Sigma_{Fr}(\textbf{NS4})$.

(for typographical reasons, we omit the function ε_X).

$$\dfrac{\dfrac{\Box\varphi \vdash \Box\varphi}{\Box\varphi \vdash \Box\Box\varphi}\Box\text{-I}}{\vdash \Box\varphi \supset \Box\Box\varphi}\supset\text{-I} \qquad \dfrac{\dfrac{\dfrac{\vdash \Gamma_X, \overbrace{v_{\Box\varphi} : (T(\Box\varphi)), vb_{\Box\varphi} : (Bx\ (\Box\varphi) v_{\Box\varphi})}^{\gamma_{\Box}(\Box\varphi)}}{\Gamma_X, \gamma(\Box\varphi) \vdash \underbrace{(\Box\text{-I}\ (\Box\varphi)\ v_{\Box\varphi}\ vb_{\Box\varphi})}_{d_1} : (T(\Box\Box\varphi))} app(\Box\text{-I})}{\begin{array}{c}\Gamma_X \vdash \underbrace{(\lambda v_{\Box\varphi} : (T(\Box\varphi))\lambda vb_{\Box\varphi} : (Bx\ (\Box\varphi)\ v_{\Box\varphi}).d_1)}_{d_2} : \\ (\prod_{v_{\Box\varphi}:(T\ (\Box\varphi))} \prod_{vb_{\Box\varphi}:(Bx\ (\Box\varphi)\ v_{\Box\varphi})} (T(\Box\Box\varphi)))\end{array}} 2 \times abs}{\Gamma_X \vdash (\supset_{\Box}\text{-I}\ \varphi\ (\Box\Box\varphi) d_2) : (T\ (\Box\varphi \supset \Box\Box\varphi))} app(\supset_{\Box}\text{-I})$$

5.2.4 "Boxed Fringe"-judgement

For the sake of completeness we sketch here how to encode Prawitz's *third version* of system **NS4** [18]. The signature $\Sigma_{Fr}(\textbf{NS4})$ appears in Fig.11.

The judgement $BF : \prod_{\varphi:o}(T\ \varphi) \to \textbf{Type}$ holds only on proofs with a fringe of boxed formulæ (in the minimal fragment of modal logic, boxed formulæ are all the essentially modal formulæ). In the system there are rules for establishing the "boxed fringe" judgement corresponding to each rule in **NS4**. Additional rules for BF can be induced by elimination rules whenever the inferred formula is boxed (and hence belongs to the fringe). This is the case, e.g., of $\supset$-E.

$\Sigma_{2j}(\mathbf{NK}') +$ • **Judgements**	$Ta, V, T\ : o \to \mathbf{Type}$,
• **Axioms and Rules**	$C : \prod_{\varphi:o} (V\ \varphi) \to (T\ \varphi)$
$\supset_T$-I : $\prod_{\varphi,\psi:o} ((T\ \varphi) \to (T\ \psi)) \to (T(\supset \varphi\psi))$,	$\supset_T$-E : $\prod_{\varphi,\psi:o} (T\ (\supset \varphi\psi)) \to (T\ \varphi) \to (T\ \psi)$,
... similarly for negation and *ff*.	

$\Box$-E : $\prod_{\varphi:o} (Ta\ \Box\varphi) \to (Ta\ \varphi)$	$\Box_{\Box}$-I : $\prod_{\varphi:o} (Ta\ \Box\varphi) \to (Ta\ \Box\Box\varphi)$
$\Box_{\Diamond}$-I : $\prod_{\varphi:o} (Ta\ \Diamond\varphi) \to (Ta\ \Box\Diamond\varphi)$	$\Box_{\supset}$-E : $\prod_{\varphi:o} (Ta\ \Box(\supset (\Box\varphi)\ \varphi)) \to (Ta\ \Box\varphi)$

Figure 12: $\Sigma_{3j}(\mathbf{NK}'')$ and its extensions for **NK4”**,....

5.2.5 Three-judgements

We can introduce ND-style systems for “truth” based on the multiple CR ND-style system **NK**’ for validity. We need only to add a third consequence relation, namely $\Vdash$, with exactly the same rules as $\vdash$, and in addition the rule EMBED’. The whole system is called **NK**”:

$$\mathbf{NK}'' \stackrel{\text{def}}{=} \mathbf{NK}' + \begin{array}{|cccc|} \supset_T\text{-I}\ \dfrac{\Gamma, \varphi \Vdash \psi}{\Gamma \Vdash \varphi \supset \psi} & \supset_T\text{-E}\ \dfrac{\Gamma \Vdash \varphi \supset \psi \quad \Gamma \Vdash \varphi}{\Gamma \Vdash \psi} & & \text{RAA}_T\ \dfrac{\Gamma, \neg\varphi \Vdash ff}{\Gamma \Vdash \varphi} \\ \Gamma, \varphi \Vdash \varphi \quad ff_T\text{-I}\ \dfrac{\Gamma \Vdash \varphi \quad \Gamma \Vdash \neg\varphi}{\Gamma \Vdash ff} & ff_T\text{-E}\ \dfrac{\Gamma \Vdash ff}{\Gamma \Vdash \varphi} & & \text{EMBED'}\ \dfrac{\Vdash \varphi}{\Vdash \varphi} \end{array}$$

Soundness of **NK**” is obvious; completeness follows from the fact that $\varphi_1, \ldots, \varphi_n \models \varphi$ iff $\varphi_1 \supset \ldots \supset \varphi_n \supset \varphi$ is valid.

In order to encode this system we add a judgement $T : o \to \mathbf{Type}$, whose constructors are like those of Ta plus a constant C which represents the EMBED’ rule (Fig.12). We can prove then

Theorem 5.9 *There is a compositional bijection between proofs* $\pi : \Delta \vdash_{\mathbf{NK}''} \varphi$ *with* $\mathrm{FV}(\pi) \subseteq X$ *and canonical terms* t *such that* $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{3j}(\mathbf{NK}'')} t : (T\ \varepsilon_X(\varphi))$.

Again, similarly to the case of two-judgement system for **K** (see Section 4.2.3), the resulting system is more powerful than this result points out, since it can deal with both truth and tautology notions, at the same time. Let define the *triple semantic consequence relation* for K as follows:

$$\Delta; \Xi \models \varphi \iff \forall \mathcal{M} \in \mathbf{M}. \forall s \in \mathcal{M}. (\models_{\mathcal{M}} \Delta \wedge s \models_{\mathcal{M}} \Xi) \Rightarrow s \models_{\mathcal{M}} \varphi$$

This semantic consequence relation combines tautuologies and truth CR, just as is done by **NK”** in a syntactical manner:

Theorem 5.10 *For* $X \subset \Phi_a$, $\Delta, \Xi \subseteq \Phi_X$, $\varphi \in \Phi_X$, *the following are equivalent:*

1. $\exists t$ *canonical term such that* $\Gamma_X, \gamma_{Ta}(\Delta), \gamma_T(\Xi) \vdash_{\Sigma_{\Box}(\mathbf{NK})} t : (T\ \varepsilon_X(\varphi))$;
2. $\Delta; \Xi \models \varphi$.

6 Cross soundness

As we have seen, different techniques can be used for encoding the same system; for instance, **K** can be encoded by using either "world parameters" ($\Sigma_w(\mathbf{K})$) or "no assumption"-judgements ($\Sigma_{Na}(\mathbf{K})$), or "two-judgments" ($\Sigma_{2j}(\mathbf{K})$). Morally, these techniques are closely related: for instance, (the encoding of) a proof has no assumptions (in $\Sigma_{Cl}(\mathbf{NK})$) iff it can be carried out from no assumptions (in $\Sigma_w(\mathbf{NK})$).

Theorem 6.1 (Cross-soundness for K) *For $X \subset \Phi_a$, $\Delta \subseteq \Phi_X$, $\varphi \in \Phi_X$, the following are equivalent:*

1. $\exists t.\Gamma_X, \gamma_\omega(\emptyset) \vdash_{\Sigma_w(\mathbf{K})} t : (T\ \omega\ \varphi)$
2. $\exists t', n.\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} n : (Na\ \varphi\ t')$
3. $\exists v.\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{2j}(\mathbf{K})} v : (V\ \varphi)$

Theorem 6.2 (Cross-soundness for NK) *For $X \subset \Phi_a$, $\Delta \subseteq \Phi_X$, $\varphi \in \Phi_X$, the following are equivalent:*

1. $\exists t.\Gamma_X, \gamma_\omega(\emptyset) \vdash_{\Sigma_w(\mathbf{NK})} t : (T\ \omega\ \varphi)$
2. $\exists t', n.\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} n : (Cl\ \varphi\ t')$
3. $\exists v.\Gamma_X, \gamma_{Ta}(\Delta) \vdash_{\Sigma_{3j}(\mathbf{NK})} v : (V\ \varphi)$

These results can be seen as "internal proofs" of adequacy of the encodings. Similar connections can be formulated with respect to other techniques appearing in this paper. These metatheoretic results could be proved formally within some Logical Framework, e.g. Coq.

7 Final Remarks

Applications. Modalities are a common feature of most program logics [6, 8, 19], hence, the techniques we have presented here can be fruitfully employed in developing proof assistants for program logics. The "world parameter" technique was used for encoding a ND-style system for Dynamic Logic [9]. Applications of the other techniques presented in this papers deserve further investigations.

Related Work. A purely semantical approach to the implementation of Modal Logics, alternative to ours, has been studied in [3]. There, the Kripke semantics is built-in the calculus: worlds are reified, and a first order proposition R over worlds is introduced in order to represent the accessibility relation. Introduction of modalities is then reduced to a quantification over accessible worlds; different axiomatizations of R are used to represent the various logics. Although such systems may be easy to implement and use, they force the user to deal directly with specific semantic notions.

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A Proofs

A.1 Proof of Theorems of Section 3

A.1.1 Proof of Theorem 3.1

The encoding function ε_X is clearly injective. It is easy to show by induction on the structure of formulæ that ε_X yields a canonical form of the appropriate type. Surjectivity is established by defining a decoding map δ_X that is left-inverse to ε_X. The decoding δ_X is defined by induction on the structure of the canonical forms as follows:

$\delta_X(\varphi) \stackrel{\text{def}}{=} \varphi \quad , if\ \varphi \in dom(\Gamma_X)$	$\delta_X(\neg\varphi) \stackrel{\text{def}}{=} \neg\delta_X(\varphi)$
$\delta_X(\Box\varphi) \stackrel{\text{def}}{=} \Box\delta_X(\varphi)$	$\delta_X(\supset \varphi\psi) \stackrel{\text{def}}{=} \delta_X(\varphi) \supset \delta_X(\psi)$

Such δ_X is total, for [7, Lemma 2.4.4] and inspection of $\Sigma(\Phi)$ and Γ_X.

The compositionality property is established by a straightforward induction on the structure of modal formulæ (omitted). $\Box$

A.2 Proofs of Theorems of Section 4

A.2.1 Proof of Theorem 4.1

It is straightforward to verify by induction on the structure of proofs that, given the hypothesis of the theorem, $\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(\pi)$ is a canonical term of type $(V\ \varepsilon_X(\varphi))$ in $\Sigma(\mathbf{K}')$ and $\Gamma_X, \gamma_V(\Delta)$. It is a routine matter to show by induction on proofs that $\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}$ is injective. To establish surjectivity we exhibit a left-inverse $\delta^{\Sigma(\mathbf{K}')}_{X,\Delta}$ defined by induction on the structure of the canonical forms as follows:

$\delta^{\Sigma(\mathbf{K}')}_{X,\Delta}(v_\varphi) \stackrel{\text{def}}{=} \varphi, \qquad if\ v_\varphi \in dom(\gamma_V(\Delta)).$
$\delta^{\Sigma(\mathbf{K}')}_{X,\Delta}(A_1\ t'\ t'') \stackrel{\text{def}}{=} A_{1\,\delta_X(t'),\delta_X(t'')}, \qquad analogously\ for\ A_2, A_3, K.$
$\delta^{\Sigma(\mathbf{K}')}_{X,\Delta}(NEC\ t\ p) \stackrel{\text{def}}{=} \mathrm{N\small{EC}}_{\delta_X(t)}\left(\delta^{\Sigma(\mathbf{K}')}_{X,\Delta}(p)\right)$
$\delta^{\Sigma(\mathbf{K}')}_{X,\Delta}(MP\ t\ t'\ p\ p') \stackrel{\text{def}}{=} \mathrm{MP}_{\delta_X(t),\delta_X(t')}\left(\delta^{\Sigma(\mathbf{K}')}_{X,\Delta}(p), \delta^{\Sigma(\mathbf{K}')}_{X,\Delta}(p')\right)$

This function is clearly total and well-defined. It remains to show that $\delta^{\Sigma(\mathbf{K}')}_{X,\Delta}\left(\varepsilon^{\Sigma(\mathbf{K}')}_{X,\Delta}(\pi)\right) = \pi$ and compositionality of the encoding; this is established by induction on the proofs. $\Box$

A.2.2 Proof of Theorem 4.2

We verify by induction on the structure of proofs that $\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi)$ is a canonical term of type $(T\ \omega\ \varepsilon_X(\varphi))$ in $\Sigma_w(\mathbf{K})$ and $\Gamma_X, \gamma_\omega(\Delta)$.

Base Step. We have two cases. If π is instance of an axiom, say $\pi = A_{1\,\psi,\vartheta}$, then it is straightforward to prove that $\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} \varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(A_{1\,\psi,\vartheta}){:}(T\ \omega\ \varepsilon_X(\psi \supset (\vartheta \supset \psi)))$. The cases of A_2, A_3, K are similar.

Otherwise, $\varphi \in \Delta$ is an assumption. Since $\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\varphi) = v_\varphi \in \gamma_\omega(\Delta)$, immediately $\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} p{:}(T\ \omega\ \varepsilon_X(\varphi))$.

Inductive Step. By cases on the last rule applied.

If $\pi \equiv \mathrm{MP}_{\psi,\varphi}(\pi', \pi'')$, then π', π'' are respectively valid proofs of $\psi \supset \varphi$, ψ w.r.t. (X, Δ). By IH, $\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} \varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi') : (T\ \omega\ \varepsilon_X(\psi))$ and
$\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} \varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi'') : (T\ \omega\ \varepsilon_X(\psi \supset \varphi))$. Therefore, we have immediately,

$$\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} \left(MP\ \varepsilon_X(\psi)\ \varepsilon_X(\varphi)\ \omega\ \varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi')\ \varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi'')\right) {:}(T\ \omega\ \varepsilon_X(\varphi)).$$

Otherwise, $\pi \equiv \mathrm{N\small{EC}}_\varphi(\pi')$; then π' is a valid proof of φ w.r.t. $(X, \emptyset)$. By IH, $\Gamma_X, \gamma_\omega(\emptyset) \vdash_{\Sigma_w(\mathbf{K})}$ $\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\emptyset,\omega}(\pi') : (T\ \omega\ \varepsilon_X(\varphi))$.

By abstracting on ω we have $\Gamma_X \vdash_{\Sigma_w(\mathbf{K})} \left(\lambda\omega'{:}U.\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\emptyset,\omega'}(\pi')\right) : \prod_{\omega':U}(T\ \omega'\ \varepsilon_X(\varphi))$. Therefore, we have immediately

$$\Gamma_X, \gamma_\omega(\emptyset) \vdash_{\Sigma_w(\mathbf{K})} \left(\mathit{NEC}\ \varepsilon_X(\varphi)(\lambda\omega'{:}U.\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\emptyset,\omega'}(\pi'))\right) \omega{:}(T\ \omega\ \Box\varepsilon_X(\varphi)).$$

By the above steps, it is easy to show that $\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}$ is injective. Surjectivity is established by exhibiting a left-inverse $\delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}$, defined by induction on the structure of the canonical forms as follows:

$\delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(v_\varphi) \stackrel{\text{def}}{=} \varphi,$	$\mathit{if}\ v_\varphi \in \mathit{dom}(\gamma_\omega(\Delta))$
$\delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(A_1\ t\ t'\ \omega) \stackrel{\text{def}}{=} A_{1\delta_X(t),\delta_X(t')},$	$\mathit{similarly\ for}\ A_2, A_3, K$
$\delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(MP\ t\ t'\ \omega\ p\ p') \stackrel{\text{def}}{=} \mathrm{MP}_{\delta_X(t),\delta_X(t')}\left(\delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(p), \delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(p')\right)$	
$\delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\mathit{NEC}\ t\ (\lambda\omega'{:}U.p)\ \omega) \stackrel{\text{def}}{=} \mathrm{N{\scriptstyle EC}}_{\delta_X(t)}\left(\delta^{\Sigma_w(\mathbf{K})}_{X,\emptyset,\omega}(p)\right)$	

The decoding map $\delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}$ is total and well-defined by the definition of canonical forms and inspection of the signature $\Sigma_w(\mathbf{K})$. By the lemma of characterization, a canonical form p of type $(T\ \omega\ t)$ must have the shape $(\zeta M_1 \ldots M_k)$, where k is the arity of ζ. By inspection of $\Sigma_w(\mathbf{K})$ and $\Gamma_X, \gamma_\omega(\Delta)$ we see that the only choices are $\zeta \in \{v_\varphi, A_1, A_2, A_3, K, MP, \mathit{NEC}\}$.
Base Step. We have two cases. If $p \equiv v_\varphi : (T\ \omega\ \varepsilon_X(\varphi))$ then, taken $\pi = \varphi$ we have a valid proof of φ w.r.t. (X, Δ). Otherwise, $p \in \{A_1, A_2, A_3, K\}$, say $t \equiv A_1\ t'\ t''\ \omega : (T\ \omega(\supset t'(\supset t''t'))$. Then we consider $\pi = A_{1\delta_X(t'),\delta_X(t'')}$. Similarly in the case p is A_2, A_3, K.
Inductive Step. We have two cases. If $p \equiv (MP\ t'\ t''\ \omega\ p'\ p''){:}(T\ \omega\ t'')$, since p is well-typed, we have that $\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} p'{:}(T\ \omega\ t')$ and $\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} p''{:}(T\ \omega\ (\supset t'\ t''))$. By IH there are two proofs such that $(X, \Delta) \models_{\mathbf{K}} \delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(p'){:}\delta_X(t')$ and $(X, \Delta) \models_{\mathbf{K}} \delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(p''){:}\delta_X(\supset t'\ t'')$. Therefore by applying MP we obtain $(X, \Delta) \models_{\mathbf{K}} \pi{:}\delta_X(t'')$.

Otherwise, $p \equiv (\mathit{NEC}\ t'\ (\lambda\omega' : U.p')\omega) : (T\ \omega\ (\Box t'))$. Since p is well-typed, we have that $\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{K})} (\lambda\omega'{:}U.p'){:} \prod_{\omega':U}(T\omega' t')$. Notice that each canonical term p of type $(T\ w\ t)$ has exactly one free variable of type U, namely w. This can be proved by induction on the structure of p (look at the previous steps). Hence, $(\lambda\omega'{:}U.p')$ has no free variable of type U. We can drop therefore the hypotheses $\gamma_\omega(\Delta)$, since if they appear free in p there should be two free variables of type U in p' — a contradiction. Hence, $\Gamma_X \vdash_{\Sigma_w(\mathbf{K})} (\lambda\omega'{:}U.p'){:} \prod_{\omega':U}(T\ \omega'\ t')$, that is $\Gamma_X, \omega'{:}U \vdash_{\Sigma_w(\mathbf{K})} p'{:}(T\ w'\ t')$. By IH there is a valid proof $(X, \emptyset) \models_{\mathbf{K}} \delta^{\Sigma_w(\mathbf{K})}_{X,\emptyset,\omega'}(p'){:}\delta_X(t')$. Hence by applying NEC we obtain $(X, \Delta) \models_{\Sigma_w(\mathbf{K})} \pi{:}\delta_X(\Box t')$.

It remains to show that $\delta^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}\left(\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}(\pi)\right) = \pi$, and that $\varepsilon^{\Sigma_w(\mathbf{K})}_{X,\Delta,\omega}$ is compositional. This is proved by induction on the structure of π, following the steps above. □

A.2.3 Proof of Lemma 4.3

By lemma of characterization, a canonical form p of type $(T\ t)$ must have the form $\zeta M_1 \ldots M_k$, where k is the arity of ζ. By inspection of $\Sigma_{Na}(\mathbf{K})$ and Γ_X we see that the only choices for ζ are $\zeta \in \{A_1, A_2, A_3, K, MP, \mathit{NEC}\}$.
Base Step: p is an instance of an axiom scheme; say $p \equiv (A_1\ t\ t')$; we take $n = (Na_{A_1}\ t\ t')$. The cases of schemata A_2, A_3, K are similar.
Inductive Step. We have two cases.

If $p \equiv (MP\ t\ t'\ p'\ p'')$, since p is well-typed we have that $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} p'{:}(T\ (\supset\ t\ t'))$ and $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} p''{:}(T\ t)$. By IH there are n', n'' such that $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} n'{:}(Na\ (\supset$

$t\ t')\ p')$ and $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} n''{:}(Na\ t\ p'')$. Then, $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} (Na_{MP}\ t\ t'\ p''\ p'\ n''\ n') : (Na\ \ t\ (MP\ t\ t'\ p'\ p''))$.

Otherwise, $p \equiv (NEC\, t\, p'\ n)$; since p is well-typed we have that $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} p'{:}(T\ t)$ and $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} n{:}(Na\ t\ p')$. Then $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} (Na_{NEC}\ t\ p'\ n){:}(Na\ \Box t\ (NEC\, t\ p'\ n))$. □

A.2.4 Proof of Theorem 4.4

It is straightforward to verify by induction on the structure of proofs that $\varepsilon_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(\pi)$ is a canonical term of type $(T\ \varepsilon_X(\varphi))$ in $\Sigma_{Na}(\mathbf{K})$ and $\Gamma_X, \gamma_T(\Delta)$.
Base Step. We have two cases. If φ is an axiom instance, say $\pi \equiv A1_{\psi,\vartheta}$, then we take $p = \varepsilon_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(A1_{\psi,\vartheta})$, it is straightforward to prove that $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} p{:}(T\ \varepsilon_X(\psi \supset (\vartheta \supset \psi)))$. Similarly in the cases A_2, A_3, K.

Otherwise, φ is an assumption, say $\pi = \varphi$ con $\varphi \in \Delta$; then we take $p = v_\varphi = \varepsilon_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(\varphi)$. It is straightforward to prove that $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} p{:}(T\ \varepsilon_X(\varphi))$.
Inductive Step. By cases on the last rule applied.

If $\pi \equiv \mathrm{MP}_{\psi,\varphi}(\pi', \pi'')$, then π', π'' are respectively valid proofs of $\psi \supset \varphi, \psi$ w.r.t. (X, Δ). By IH there are two canonical terms such that $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} \varepsilon_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(\pi') : (T\varepsilon_X(\psi \supset \varphi))$ and $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} \varepsilon_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(\pi''){:}(T\omega\varepsilon_X(\psi))$. Therefore, we have immediately, $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} MP\ \varepsilon_X(\psi)\ \varepsilon_X(\varphi)\ \varepsilon_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(\pi'')\ \varepsilon_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(\pi'){:}(T\ \varepsilon_X(\varphi))$.

Otherwise, $\pi \equiv \mathrm{NEC}_\varphi(\pi')$; then, we have that π' is a valid proof of φ w.r.t. $(X, \emptyset)$. So by IH, $\Gamma_X, \gamma_T(\emptyset) \vdash_{\Sigma_{Na}(\mathbf{K})} \varepsilon_{X,\emptyset}^{\Sigma_{Na}(\mathbf{K})}(\pi') : (T\ \varepsilon_X(\varphi))$. Now, by Lemma 4.3 we obtain that there exists a term n such that $\Gamma_X \vdash_{\Sigma_{Na}(\mathbf{K})} n : \left(Na\ \varepsilon_X(\varphi)\ \varepsilon_{X,\emptyset}^{\Sigma_{Na}(\mathbf{K})}(\pi')\right)$. Then we have $\Gamma_X, \gamma(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} \left(NEC\ \varepsilon_X(\varphi)\ \varepsilon_{X,\emptyset}^{\Sigma_{Na}(\mathbf{K})}(\pi')\ n\right) : (T\ \Box\varepsilon_X(\varphi))$.

By above, $\varepsilon_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}$ is injective. Surjectivity is established by exhibiting a left-inverse $\delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}$, defined by induction on canonical forms as follows:

$\delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(v_\varphi) \stackrel{\text{def}}{=} \varphi,$	$if\ v_\varphi \in dom(\gamma_T(\Delta))$.
$\delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(A_1\ t'\ t'') \stackrel{\text{def}}{=} A1_{\delta_X(t'),\delta_X(t'')},$	similarly for A_2, A_3, K.
$\delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(NEC\ t\ p\ n) \stackrel{\text{def}}{=} \mathrm{NEC}_{\delta_X(t)}(\beta(n))$	
$\delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(MP\ t\ t'\ p\ p') \stackrel{\text{def}}{=} \mathrm{MP}_{\delta_X(t),\delta_X(t')}\left(\delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(p), \delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(p')\right)$	

The decoding map $\delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}$ is total and well-defined follows from the definition of canonical forms and inspection of the signature $\Sigma_{Na}(\mathbf{K})$. By lemma of characterization, a canonical form p of type $(T\ t)$ must have the form $\zeta M_1 \ldots M_k$, where k is the arity of ζ. By inspection of $\Sigma_{Na}(\mathbf{K})$ and $\Gamma_X, \gamma_T(\Delta)$ we see that the only choices are $\zeta \in \{v_\varphi, A_1, A_2, A_3, K, MP, NEC\}$.
Base Step. We have two cases. If φ is an assumption, say $p \equiv v_\varphi{:}(T\ \omega\ \varepsilon_X(\varphi))$, then, taken $\pi = \varphi$ we have a valid proof of φ w.r.t. (X, Δ).

Otherwise, $p \in \{A_1, A_2, A_3, K\}$; say $p \equiv (A_1\ t'\ t''){:}(T(\supset t'(\supset t''t'))$. Then we take $\pi = A1_{\delta_X(t'),\delta_X(t'')}$. Similarly in the other cases.
Inductive Step. We have two cases.

If $p \equiv (MP\ t'\ t''\ p'\ p''){:}(T\ \omega\ t'')$, since p is well-typed, $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} p'{:}(T\ (\supset t'\ t''))$ and $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} p''{:}(T\ t')$. By IH there are two proofs such that $(X, \Delta) \models_{\mathbf{K}} \delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(p') : \delta_X(\supset t'\ t'')$ and $(X, \Delta) \models_{\mathbf{K}} \delta_{X,\Delta}^{\Sigma_{Na}(\mathbf{K})}(p'') : \delta_X(t')$. By applying MP we obtain $(X, \Delta) \models_{\mathbf{K}} \pi{:}\delta_X(t'')$.

Otherwise, $p \equiv (NEC\ t'\ p'\ n) : (T\ (\Box t'))$; then, since p is well-typed, $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} p' {:} (T\ t')$ and $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} n{:}(Na\ t'\ p')$. Here we need a technical lemma (Lemma A.1) which relates Na and derivations from the empty set of assumptions; it appears below. So, for this lemma, there is π' such that $(X, \emptyset) \models_{\mathbf{K}} \pi' : \delta_X(t')$. By applying NEC to π' we obtain $(X, \Delta) \models_{\mathbf{K}} \pi{:}\delta_X(\Box t')$.

It remains to show that $\delta^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}\left(\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}(\pi)\right) = \pi$, and that $\varepsilon^{\Sigma_{Na}(\mathbf{K})}_{X,\Delta}$ is compositional. This is proved by induction on the structure of proofs. □

Lemma A.1 *$\forall n$ canonical:* $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} n : (Na\ t\ p) \Rightarrow \exists \pi.(X, \emptyset) \models_{\mathbf{K}} \pi : \delta_X(t)$.

Proof. By lemma of characterization, a canonical form p of type $(Na\ t\ p)$ must have the form $\zeta M_1 \ldots M_k$, where k is the arity of ζ. By inspection of $\Sigma_{Na}(\mathbf{K})$ and $\Gamma_X, \gamma_T(\Delta)$ we see that the only choices are $\zeta \in \{Na_{A_1}, Na_{A_2}, Na_{A_3}, Na_K, Na_{MP}, Na_{NEC}\}$.
Base Step: n is one of Na_{A_1}, Na_{A_2}, Na_K, say $n = (Na_{A_1}\ t\ t')$. Then, $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} n{:}(Na\ (\supset t\ (\supset\ t'\ t))\ (A_1\ t\ t'))$; hence we take $\pi = A_{1\,\delta_X(t),\delta_X(t)}$. The cases of other schemata are similar.
Inductive Step. We have two cases.

If $n \equiv (Na_{MP}\ t\ t'\ p\ p'\ n'\ n'')$, then since n is well-typed we have that $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} n' : (Na\ t\ p)$ and $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} n''{:}(Na\ (\supset\ t\ t')\ p')$. By IH $(X, \emptyset) \models_{\mathbf{K}} \pi'{:}\delta_X(t)$ and $(X, \emptyset) \models_{\mathbf{K}} \pi''{:}\delta_X(\supset\ t\ t')$. Then we take $\pi = \mathrm{MP}_{\delta_X(t),\delta_X(t')}(\pi', \pi'')$ with $(X, \emptyset) \models_{\mathbf{K}} \pi{:}\delta_X(t')$.

Otherwise, $n \equiv (Na_{NEC}\ t\ p\ n')$; since n is well-typed we have that $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Na}(\mathbf{K})} n'{:}(Na\ t\ p)$. By IH, $(X, \emptyset) \models_{\mathbf{K}} \pi'{:}\delta_X(t)$; then we take $\pi = \mathrm{NEC}_{\delta_X(t)}(\pi')$ with $(X, \emptyset) \models_{\mathbf{K}} \pi{:}\delta_X(\Box t)$. □

A.2.5 Proof of Theorem 4.6

Similar to that of Theorem 4.1. □

A.3 Proofs of Theorems of Section 5

A.3.1 Proof of Theorem 5.1

The proof follows the standard methodology of [7]. We exhibit the encoding function, and its inverse, for the $\vdash$ CR (the case of validity CR is similar). These functions are defined by induction on the proofs in $\mathbf{NK}'$ and on the terms of $\Sigma_{2j}(\mathbf{NK}')$ respectively.

$$
\begin{array}{|l|}
\hline
\alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\varphi) \stackrel{\text{def}}{=} v_\varphi \qquad\qquad ,if\ \varphi \in \Delta \\
\alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\supset\text{-I}_{\varphi,\psi}(\pi)) \stackrel{\text{def}}{=} \supset\text{-I}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,(\Delta,\varphi)}(\pi) \\
\alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\supset\text{-E}_{\varphi,\psi}(\pi', \pi'')) \stackrel{\text{def}}{=} \supset\text{-E}_{Ta,Ta}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\pi')\ \alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\pi'') \\
\alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\supset_\Box\text{-E}_{\varphi,\psi}(\pi', \pi'')) \stackrel{\text{def}}{=} \supset_\Box\text{-E}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ \alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\pi')\ \alpha^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\pi'') \\
\hline
\end{array}
$$

$$
\begin{array}{|l|}
\hline
\beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(v_\varphi) \stackrel{\text{def}}{=} \varphi \qquad\qquad ,if\ v_\varphi \in dom(\gamma_{Ta}(\Delta)) \\
\beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\supset\text{-I}\ t\ t'\ p) \stackrel{\text{def}}{=} \supset\text{-I}_{\delta_X(t),\delta_X(t')}(\beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,(\Delta,\delta_X(t))}(p)) \\
\beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\supset\text{-E}_{Ta,Ta}\ t\ t'\ p\ p') \stackrel{\text{def}}{=} \supset\text{-E}_{\delta_X(t),\delta_X(t')}(\beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(p), \beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(p')) \\
\beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(\supset_\Box\text{-E}\ t\ t'\ p\ p') \stackrel{\text{def}}{=} \supset_\Box\text{-E}_{\delta_X(t),\delta_X(t')}(\beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(p), \beta^{\Sigma_{2j}(\mathbf{NK}')}_{X,\Delta}(p')) \\
\hline
\end{array}
$$
□

A.3.2 Proof of Theorem 5.2

Very similar to Theorem 4.2. We have only to take care of the $\supset$-I rule, which envolves a dischargement, as a new case of inductive steps.

If $\pi \equiv \supset\text{-I}_{\varphi,\psi}(\pi')$, then $(X,(\Delta,\varphi)) \models_{\mathbf{NK}} \pi':\psi$. By IH, we have $\Gamma_X, \gamma_\omega(\Delta,\varphi) \vdash_{\Sigma_w(\mathbf{NK})} \varepsilon^{\Sigma_w(\mathbf{NK})}_{X,(\Delta,\varphi),\omega}(\pi') : (T\ \omega\ \varepsilon_X(\psi))$. By abstracting on v_φ, we obtain

$$\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{NK})} (\lambda v_\varphi{:}(T\ \omega\ \varepsilon_X(\varphi)).\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,(\Delta,\varphi),\omega}(\pi')) : \prod_{v_\varphi:(T\ \omega\ \varepsilon_X(\varphi))} (T\ \omega\ \varepsilon_X(\psi)).$$

By applying the constant $\supset$-I, we obtain

$$\Gamma_X, \gamma_\omega(\Delta) \vdash_{\Sigma_w(\mathbf{NK})} \supset\text{-I}\, \varepsilon_X(\varphi)\, \varepsilon_X(\psi)\, \omega\, (\lambda v_\varphi : (T\, \omega\, \varepsilon_X(\varphi)).\varepsilon^{\Sigma_w(\mathbf{NK})}_{X,(\Delta,\varphi),\omega}(\pi')) : (T\, \omega\, \varepsilon_X(\supset\ \varphi\, \psi)).$$

The rest of the proof follows closely that of Theorem 4.2. We show just the left-inverse:

$$\begin{array}{|l|}\hline
\delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(v_\varphi) \stackrel{\text{def}}{=} \varphi \qquad\qquad\qquad\qquad ,if\ v_\varphi \in dom(\gamma_\omega(\Delta)) \\
\delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\supset\text{-I}\ t\ t'\ \omega(\lambda p : (T\ \omega\ t).p')) \stackrel{\text{def}}{=} \supset\text{-I}_{\delta_X(t),\delta_X(t')}\left(\delta^{\Sigma_w(\mathbf{NK})}_{X,(\Delta,\delta_X(t)),\omega}(p')\right) \\
\delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\supset_\Box\text{-E}\ t\ t'\ \omega\ p\ p') \stackrel{\text{def}}{=} \supset_\Box\text{-E}_{\delta_X(t),\delta_X(t')}\left(\delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(p), \delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(p')\right) \\
\delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\supset\text{-E}\ t\ t'\ \omega\ p\ p') \stackrel{\text{def}}{=} \supset\text{-E}_{\delta_X(t),\delta_X(t')}\left(\delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(p), \delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(p')\right) \\
\delta^{\Sigma_w(\mathbf{NK})}_{X,\Delta,\omega}(\Box'\text{-I}\ t\ (\lambda\omega'{:}U.p)\ \omega) \stackrel{\text{def}}{=} \Box'\text{-I}_{\delta_X(t)}\left(\delta^{\Sigma_w(\mathbf{NK})}_{X,\emptyset,\omega}(p)\right) \\ \hline
\end{array} \Box$$

A.3.3 Proof of Lemma 5.3

By lemma of characterization, a canonical form p of type $(T\ t)$ must be $\zeta M_1 \ldots M_k$, where k is the arity of ζ. By inspection of $\Sigma_{Cl}(\mathbf{NK})$ and $\Gamma_X, \gamma_T(\Delta)$, we see that the only choices for ζ are $\zeta \in \{v_\varphi, \supset\text{-I}, \supset\text{-E}, \Box'\text{-I}, \supset_\Box\text{-E} \ldots\}$.

Base Step. If p is an assumption of type $(p{:}(T\ t)) \in \Delta$ then we have $p \in \mathrm{FV}(p)$ and hence $c{:}(Cl\ t\ p) \in \Xi_p(\Delta)$.

Inductive Step. By cases on the last rule applied. We will see only some significant cases, the other being similar.

•$p \equiv (\supset\text{-E}\ t\ t'\ p'\ p'')$: since p is well-typed we have that $\Gamma_X, \Delta \vdash_{\Sigma_{Cl}(\mathbf{NK})} p'{:}(T\ (\supset\ t\ t'))$ and $\Gamma_X, \Delta \vdash_{\Sigma_{Cl}(\mathbf{NK})} p''{:}(T\ t)$. By IH, $\Gamma_X, \Delta, \Xi_{p'}(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} c'{:}(Cl\ (\supset\ t\ t')\ p')$ and $\Gamma_X, \Delta, \Xi_{p''}(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} c''{:}(Cl\ t\ p'')$. Since $\Xi_p(\Delta) \supseteq \Xi_{p'}(\Delta), \Xi_{p''}(\Delta)$, then $\Gamma_X, \Delta, \Xi_p(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} (Cl_{\supset\text{-E}}\ t\ t'\ p'\ p''\ c''\ c') : (Cl\ t'\ (\supset\text{-E}t\ t'\ p'\ p''))$.

•$p \equiv (\supset\text{-I}\ t\ t'\ p')$: since p is well-typed we have $\Gamma_X, \Delta \vdash_{\Sigma_{Cl}(\mathbf{NK})} p'{:}(T\ t) \to (T\ t')$. Since p' is a canonical form, it must be $p' = \lambda x : (T\ t).p''$, where $\Gamma_X, \Delta, x{:}(T\ t) \vdash_{\Sigma_{Cl}(\mathbf{NK})} p'' : (T\ t')$. By IH, $\Gamma_X, \Delta, x{:}(T\ t), \Xi_{p''}(\Delta, x{:}(T\ t)) \vdash_{\Sigma_{Cl}(\mathbf{NK})} c''{:}(Cl\ t'\ p'')$. Now we have that $\Xi_{p''}(\Delta, x{:}(T\ t)) \subseteq \Xi_{p''}(\Delta), c'{:}(Cl\ t\ x)$, then by abstracting on c' and x we obtain

$$\Gamma_X, \Delta, \Xi_{p''}(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} (\lambda x{:}(T\ t).\lambda c'{:}(Cl\ t\ x).c'') : \prod_{x:(T\ t)} (Cl\ t\ x) \to (Cl\ t'\ p'').$$

Moreover we have that $\Xi_p(\Delta) = \Xi_{p''}(\Delta)$ because $\mathrm{FV}(p) = \mathrm{FV}(p'') \setminus \{x\}$ and $x{:}(T\ t) \notin \Delta$ (otherwise $\Delta, x{:}(T\ t)$ would be not a valid context). Then, defining $t_1 \stackrel{\text{def}}{=} (\lambda x{:}(T\ t)\lambda c'{:}(Cl\ t\ x).c'')$, we have $\Gamma_X, \Delta, \Xi_p(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} t_1 : \prod_{x:(T\ t)}(Cl\ t\ x) \to (Cl\ t'\ p'')$. We apply now $Cl_{\supset\text{-I}}$ obtaining $\Gamma_X, \Delta, \Xi_p(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} (Cl_{\supset\text{-I}}\ t\ t'\ p'\ t_1) : (Cl\ t'\ (\supset\text{-I}\ t\ t'\ p'))$.

•$p \equiv (\Box'\text{-I}\ t\ p'\ c')$: since p is well-typed we have $\Gamma_X, \Delta \vdash_{\Sigma_{Cl}(\mathbf{NK})} p' : (T\ t)$ and $\Gamma_X, \Delta \vdash_{\Sigma_{Cl}(\mathbf{NK})} c' : (Cl\ t\ p')$. Then we apply $\Box'$-I obtaining $\Gamma_X, \Delta(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} (Cl_{\Box'\text{-I}}\ t\ p'\ c') : (Cl\ (\Box t)(\Box'\text{-I}\ t\ p'\ c'))$, and therefore

$$\Gamma_X, \Delta(\Delta), \Xi_p(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} (Cl_{\Box'\text{-I}}\ t\ p'\ c'):(Cl\ (\Box t)\ (\Box'\text{-I}\ t\ p'\ c')).\Box$$

A.3.4 Proof of Lemma 5.4

By lemma of characterization, c of type $(Cl\ t\ p)$ is of the form $\zeta M_1 \ldots M_k$, where k is the arity of ζ. By inspection of $\Sigma_{Cl}(\mathbf{NK})$ and $\Gamma_X, \gamma_T(\Delta)$ we see that the only choices are $\zeta \in dom(\Xi) \cup \{Cl_{\supset\text{-I}}, Cl_{\supset\text{-E}}, Cl_{\Box'\text{-I}}, Cl_{\supset_\Box\text{-E}} \cdots\}$.
Base Step: $c{:}(Cl\ t'\ p)) \in \Xi$; then, the claim is trivial.
Inductive Step: by cases on the top constructor. We see only some significant cases, the other being similar.

•$c \equiv (Cl_{\supset\text{-I}}\ t\ t'\ p\ t'') : (Cl\ t'\ (\supset\text{-I}\ t\ t'\ p))$: since c is well-typed we have that $\Gamma_X, \Delta, \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} p{:}(T\ t) \to (T\ t')$ and $\Gamma_X, \Delta, \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} t'' : \prod_{x:t}(Cl\ t\ x) \to (Cl\ t'\ (p\ x))$. Since t'' is a canonical form then it must be $t'' = \lambda x{:}(T\ t)\lambda c'{:}(Cl\ t\ x).t'''$. Then by some introductions we obtain $\Gamma_X, \Delta, x{:}(T\ t), \Xi, c'{:}(Cl\ t\ x) \vdash_{\Sigma_{Cl}(\mathbf{NK})} t'''{:}(Cl\ t'\ (p\ x))$. By the IH we know $\Gamma_X, \Delta'', \Xi, c'{:}(Cl\ t\ x) \vdash_{\Sigma_{Cl}(\mathbf{NK})} t'''{:}(Cl\ t'\ (p\ x))$, where $\Delta'' \stackrel{\text{def}}{=} \{p{:}(T\ t') | (Cl\ t'\ p) \in \Im(\Xi, c'{:}(Cl\ t\ x))\} = \Delta' \cup \{x{:}(T\ t)\}$. Then, by abstracting on x, c' we find that $\Gamma_X, \Delta', \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} t'' : \prod_{x:(T\ t)}(Cl\ t\ x) \to (Cl\ t'\ (p\ x))$. Finally, by applying $Cl_{\supset\text{-I}}$ we obtain $\Gamma_X, \Delta', \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} (Cl_{\supset\text{-I}}\ t\ t'\ p\ t''){:}(Cl\ t'\ (\supset\text{-I}\ t\ t'\ p))$.

•$c \equiv (Cl_{\Box'\text{-I}}\ t\ p\ c') : (Cl\ \Box t\ (\Box'\text{-I}\ t\ p\ c'))$: since c is well-typed we have that $\Gamma_X, \Delta, \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} p{:}(T\ t)$ and $\Gamma_X, \Delta, \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} c'{:}(Cl\ t\ p)$. By IH we have that $\Gamma_X, \Delta', \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} c'{:}(Cl\ t\ p)$. Hence, $\Gamma_X, \Delta', \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} p{:}(T\ t)$. Then we apply $Cl_{\Box'\text{-I}}$ obtaining $\Gamma_X, \Delta', \Xi \vdash_{\Sigma_{Cl}(\mathbf{NK})} (Cl_{\Box'\text{-I}}\ t\ p\ c') : (Cl\ \Box t\ (\Box'\text{-I}\ t\ p\ c'))$.

•$c \equiv (Cl_{\supset_\Box\text{-E}}\ t\ t'\ p\ p'\ c'\ c'')$: an immediate application of IH on c', c''. □

A.3.5 Proof of Theorem 5.5

It is straightforward to verify by induction on the structure of proofs that, given the hypothesis of the theorem, $\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\pi)$ is a canonical term of type $(T\ \varepsilon_X(\varphi))$ in $\Sigma_{Cl}(\mathbf{NK})$ and $\Gamma_X, \gamma_T(\Delta)$.
Base Step: φ is an assumption, i.e. $\pi = \varphi \in \Delta$. Then immediately $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} v_\varphi{:}(T\ \varepsilon_X(\varphi))$.
Inductive Step. By cases on the last rule applied. We see only some significant cases, the other being similar.

•$\pi \equiv \supset\text{-I}_{\varphi,\psi}(\pi')$: then $(X, (\Delta, \varphi)) \models_{\mathbf{NK}} \pi'{:}\psi$. By IH we have that $\Gamma_X, \gamma_T(\Delta, \varphi) \vdash_{\Sigma_{Cl}(\mathbf{NK})} t'{:}(T\ \varepsilon_X(\psi))$. Let $t'' \stackrel{\text{def}}{=} \lambda v_\varphi{:}(T\ \varepsilon_X(\varphi)).t'$; then $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} t'' : \prod_{v_\varphi:(T\ \varepsilon_X(\varphi))}(T\ \varepsilon_X(\psi))$. By applying $\supset$-I we obtain $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} (\supset\text{-I}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ t''){:}(T\ (\supset\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)))$.

•$\pi \equiv \supset\text{-E}_{\varphi,\psi}(\pi', \pi'')$: then $(X, \Delta) \models_{\mathbf{NK}} \pi' : \varphi \supset \psi$ and $(X, \Delta) \models_{\mathbf{NK}} \pi'' : \varphi$. By IH, $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} t' : (T\ \varepsilon_X(\varphi \supset \psi))$ and $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} t'' : (T\ \varepsilon_X(\varphi))$. Therefore by applying $\supset$-E we obtain $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} (\supset\text{-E}\ \varepsilon_X(\varphi)\ \varepsilon_X(\psi)\ t'\ t''){:}(T\ \varepsilon_X(\psi))$.

•$\pi \equiv \Box'\text{-I}_\varphi(\pi')$: then $(X, \emptyset) \models_{\mathbf{NK}} \pi' : \varphi$. By IH, $\Gamma_X \vdash_{\Sigma_{Cl}(\mathbf{NK})} t_1 : (T\ \varepsilon_X(\varphi))$, and hence by Lemma 5.3 there is a term c_{t_1} such that $\Gamma_X \vdash_{\Sigma_{Cl}(\mathbf{NK})} c_{t_1}{:}(Cl\ \varepsilon_X(\varphi)\ t_1)$. Therefore, by applying $\Box'$-I, we obtain $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} (\Box'\text{-I}\ \varepsilon_X(\varphi)\ t_1\ c_{t_1}) : (T\ \Box\varepsilon_X(\varphi))$.

By the above steps, it is easy to see that $\varepsilon^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}$ is injective. Surjectivity is estabilished by exhibiting a left-inverse $\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}$, defined by induction on the structure of the canonical forms as follows:

$\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(v_\varphi) \stackrel{\text{def}}{=} \varphi$	$,if\ v_\varphi \in dom(\gamma_T(\Delta))$
$\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\Box'\text{-I}\ t\ p\ c) \stackrel{\text{def}}{=} \Box'\text{-I}_{\delta_X(t)}(p)$	
$\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\supset\text{-I}\ t\ t'\ (\lambda p{:}(T\ t).p')) \stackrel{\text{def}}{=} \supset\text{-I}_{\delta_X(t),\delta_X(t')}(\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,(\Delta,\delta_X(t))}(p'))$	
$\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\supset\text{-E}\ t\ t'\ p\ p') \stackrel{\text{def}}{=} \supset\text{-E}_{\delta_X(t),\delta_X(t')}(\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(p), \delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(p'))$	
$\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(\supset_\Box\text{-E}\ t\ t'\ p\ p') \stackrel{\text{def}}{=} \supset_\Box\text{-E}_{\delta_X(t),\delta_X(t')}(\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(p), \delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}(p'))$	

The $\delta^{\Sigma_{Cl}(\mathbf{NK})}_{X,\Delta}$ is total and well-defined for the definition of canonical forms and inspection of the signature $\Sigma_{Cl}(\mathbf{NK})$. The application of $\Box'$-I is sound, for the presence of $c : (Cl\ \delta_X(t)\ p)$ and the fact that no Cl assumptions are made by the encoding of the context $(\gamma_T(\Delta))$.

By lemma of characterization, a canonical form p of type $(T\ t)$ must be $\zeta M_1 \dots M_k$, where k is the arity of ζ. By inspection of $\Sigma_{Cl}(\mathbf{NK}), \Gamma_X, \gamma_T(\Delta)$ we see that the only choices for ζ are $\zeta \in \{v_\varphi, \Box'\text{-I}, \supset\text{-I}, \supset\text{-E}, \supset_\Box\text{-E} \dots\}$.

Base Step: $p = v_\varphi \in \gamma_T(\Delta)$, then we take $\pi = \varphi$.

Inductive Step: we see only some significant cases.

•$p \equiv \Box'\text{-I}\ t\ p'\ c$: since p is well-typed we have that $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} p' : (T\ t)$ and $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} c : (Cl\ t\ p')$. By Lemma 5.4, there is a term c such that $\Gamma_X, \emptyset \vdash_{\Sigma_{Cl}(\mathbf{NK})} c : (Cl\ t\ p')$; since c is well-typed, $\Gamma_X, \emptyset \vdash_{\Sigma_{Cl}(\mathbf{NK})} p'{:}(T\ t)$. By the IH, we obtain that there exists π' such that $(X, \emptyset) \models_{\mathbf{NK}} \pi' : \delta_X(t)$ and hence we conclude $\pi = \Box'\text{-I}_{\delta_X(t)}(\pi')$.

•$p \equiv \supset\text{-I}\ t\ t'\ p'$: since p is well-typed, $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} p' : (T\ t) \to (T\ t')$, then $\Gamma_X, \gamma_T(\Delta), a : (T\ t) \vdash_{\Sigma_{Cl}(\mathbf{NK})} p'a : (T\ t')$. By IH there exists π' such that $(X, (\Delta, \delta_X(t))) \models_{\mathbf{NK}} \pi'{:}\delta_X(t')$, and hence we conclude taking $\pi = \supset\text{-I}_{\delta_X(t),\delta_X(t')}(\pi')$.

•$p \equiv \supset\text{-E}\ t\ t'\ p'\ p''$: since p is well-typed, $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} p' : (T\ (\supset\ t\ t'))$ and $\Gamma_X, \gamma_T(\Delta) \vdash_{\Sigma_{Cl}(\mathbf{NK})} p''{:}(T\ t)$. By IH there exist π', π'' such that $(X, \Delta) \models_{\mathbf{NK}} \pi' : \delta_X(\supset\ t\,t')$ and $(X, \Delta) \models_{\mathbf{NK}} \pi'' : \delta_X(t)$. We conclude taking $\pi = \supset\text{-E}_{\delta_X(t),\delta_X(t')}(\pi', \pi'')$. □

A.3.6 Proof of Lemma 5.6

By the lemma of characterization, a canonical form p of type $(T\ t)$ is $\zeta M_1 \dots M_k$, where k is the arity of ζ, which is $\zeta \in \{v_\varphi, \supset\text{-I}, \supset\text{-E}, \Box\text{-I}, \Box\text{-E}, \supset_\Box\text{-I} \dots\}$.

Base Step: $p \equiv v_{\delta_X(t)} : (T\ t)$. By definition of $\gamma_\Box$, $C(p, \gamma_\Box(\Delta))$ holds, then there is the assumption $(vb_{\delta_X(t)}{:}(Bx\ t\ v_{\delta_X(t)})) \in \gamma_\Box(\Delta)$. Hence, $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} vb_{\delta_X(t)} : (Bx\ t\ v_{\delta_X(t)})$.

Inductive Step: we see only some significant cases.

•$p \equiv (\Box\text{-I}\ t\ p_1\ b) : (T\ \Box t)$. Since p is well-typed, we have $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ t)$ and $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b : (Bx\ t\ p_1)$. Hence $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (Bx_{\Box\text{-}\mathbf{I}}\ t\ p_1\ b) : (Bx\ \Box t\ (\Box\text{-I}\ t\ p_1\ b))$.

•$p \equiv (\supset\text{-I}\ t\ t'\ (\lambda v_{\delta_X(t)} : (Tt).p_1)) : (T(\supset\ t\,t'))$: since p is well-typed, $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (\lambda v_{\delta_X(t)}{:}(T\ t).p_1) : (\Pi_{v_{\delta_X(t)}:(T\ t)}(T\ t'))$, that is $\Gamma_X, \gamma_\Box(\Delta), v_{\delta_X(t)} : (T\ t) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ t')$. Moreover, chosen a fresh variable $vb_{\delta_X(t)}$, $C\left(p_1, (\gamma(\Delta), v_{\delta_X(t)}{:}(T\ t), vb_{\delta_X(t)}{:}(Bx\ t\ v_{\delta_X(t)}))\right)$ holds. Then, by IH there is b_1 such that $\Gamma_X, \gamma_\Box(\Delta), v_{\delta_X(t)}{:}(T\ t), vb_{\delta_X(t)}{:}(Bx\ t\ v_{\delta_X(t)}) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1{:}(Bx\ t'\ p_1)$, and hence

$$\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} t'' : \prod_{v_{\delta_X(t)}:(T\ t)} \prod_{vb_{\delta_X(t)}:(Bx\ t\ v_{\delta_X(t)})} (Bx\ t'\ p_1),$$

where $t'' \stackrel{\text{def}}{=} \lambda v_{\delta_X(t)}{:}(T\ t)\lambda vb_{\delta_X(t)}{:}(Bx\ t\ v_{\delta_X(t)}).b_1$. Then, finally

$\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (Bx_{\supset\text{-}\mathbf{I}}\ t\ t'\ (\lambda v_{\delta_X(t)}{:}(T\ t).p_1)t'') : (Bx\ (\supset\ t\,t')\ (\supset\text{-I}\ t\ t'\ (\lambda v_{\delta_X(t)}{:}(T\ t).p_1)))$.

$\bullet p \equiv (\supset\text{-E}\ t\ t' p_1\ p_2) : (T\ t')$: since p is well-typed, $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ (\supset\ t\ t'))$ and $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_2 : (T\ t)$. Since $FV(p_1), FV(p_2) \subseteq FV(p)$, then both $C(p_1, \gamma_\Box(\Delta))$ and $C(p_2, \gamma_\Box(\Delta))$ hold. Then, by IH there exist b_1, b_2 such that $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1 : (Bx\ (\supset\ t\ t')\ p_1)$ and $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_2 : (Bx\ t\ p_2)$. Therefore $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (Bx_{\supset\text{-E}}\ t\ t'\ p_1\ p_2\ b_1\ b_2) : (Bx\ t'\ (\supset\text{-E}\ t\ t'\ p_1\ p_2))$.

$\bullet p \equiv (\supset_\Box\text{-I}\ t\ t'\ (\lambda v_{\delta_X(\Box t)}{:}(T\ \Box t)\lambda vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}).p_1) : (T\ (\supset\ \Box t\ t')))$: since p is well-typed, $\Gamma_X, \gamma_\Box(\Delta), v_{\delta_X(\Box t)}{:}(T\ \Box t), vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ t')$. Moreover $C\left(p1, (\gamma_\Box(\Delta), v_{\delta_X(\Box t)}{:}(T\ \Box t), vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}))\right)$ holds. Then by IH there exists b_1 such that $\Gamma_X, \gamma_\Box(\Delta), v_{\delta_X(\Box t)}{:}(T\ \Box\ t), vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1 : (Bx\ t'\ p_1)$. By abstracting we obtain

$$\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} t'' : \prod_{v_{\delta_X(\Box t)}:(T\ \Box t)} \prod_{vb_{\delta_X(\Box t)}:(Bx\ \Box t\ v_{\delta_X(\Box t)})} (Bx\ t'\ p_1).$$

where $t'' \stackrel{\text{def}}{=} \lambda v_{\delta_X(\Box t)}{:}(T\ \Box t)\lambda vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}).b_1$. Then

$$\begin{aligned}\Gamma_X, \gamma_\Box(\Delta) \quad & \vdash_{\Sigma_\Box(\mathbf{NS4})} (Bx_{\supset_\Box\text{-I}}\ t\ t'(\lambda v_{\delta_X(\Box t)}{:}(T\ \Box t).\lambda vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}).p_1)(t'')) : \\ & (Bx\ (\supset\ \Box t\ t')\ (\supset_\Box\text{-I}\ t\ t'\ (\lambda v_{\delta_X(\Box t)}{:}(T\ \Box t)\lambda vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}).p_1))).\end{aligned}$$

□

A.3.7 Proof of Lemma 5.7

By induction on the structure of π.
Base Step: φ is an assumption, i.e. $\varphi \in \Delta$. Then we take $p = v_\varphi \in dom(\gamma_\Box(\Delta))$.
Inductive Step: by cases on the last rule applied. We see only some significant cases, the other being similar.

$\bullet \pi \equiv \supset\text{-I}_{\psi\theta}(\pi')$: then $(X, (\Delta, \psi)) \models_{\mathbf{NS4}} \pi' : \theta$ and hence by IH there exists a canonical term p such that $\Gamma_X, \gamma_\Box(\Delta, \psi) \vdash_{\Sigma_\Box(\mathbf{NS4})} p : (T\ \varepsilon_X(\theta))$. Now there are two cases, depending on whether ψ is boxed or not.

- if ψ is boxed, then $\gamma_\Box(\{\psi\}) = v_\psi{:}(T\ \varepsilon_X(\psi)), vb_\psi{:}(Bx\ \varepsilon_X(\psi)\ v_\psi)$. Then

  $$\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} t : \prod_{v_\psi:(T\ \varepsilon_X(\psi))} \prod_{vb_\psi(Bx\ \varepsilon_X(\psi)\ v_\psi)} (T\ \varepsilon_X(\theta))$$

  where $t \stackrel{\text{def}}{=} \lambda v_\psi{:}(T\ \varepsilon_X(\psi)).\lambda vb_\psi{:}(Bx\ \varepsilon_X(\psi)\ v_\psi).p$. Hence
  $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (\supset_\Box\text{-I}\ \varepsilon_X(\psi)\ \varepsilon_X(\theta)\ t){:}(T\ (\supset \varepsilon_X(\varphi)\ \varepsilon_X(\theta)))$

- otherwise, ψ is not boxed; then $\gamma_\Box(\{\psi\}) = v_\psi{:}(T\ \varepsilon_X(\psi))$. Then $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (\lambda v_\psi{:}(T\ \varepsilon_X(\psi)).p) : (\Pi v_\psi{:}(T\ \varepsilon_X(\psi))).(T\ \varepsilon_X(\theta))$. Hence immediately
  $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (\supset\text{-I}\ \varepsilon_X(\psi)\varepsilon_X(\theta)(\lambda v_\psi{:}(T\varepsilon_X(\psi)).p)) : (T(\supset\ \varepsilon_X(\psi)\ \varepsilon_X(\theta)))$.

$\bullet \pi \equiv \Box\text{-I}_\psi(\pi')$; then $(X, \Box\Delta) \models_{\mathbf{NS4}} \pi' : \psi$. By IH there exists a canonical term p_1 such that $\Gamma_X, \gamma_\Box(\Box\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ \varepsilon_X(\psi))$. Since $C(p_1, \gamma_\Box(\Box\Delta))$ always holds, by Lemma 5.6 there exists a canonical term b_1 such that $\Gamma_X, \gamma_\Box(\Box\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1 : (Bx\ \varepsilon_X(\psi)\ p_1)$. Hence $\Gamma_X, \gamma_\Box(\Box\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (\Box\text{-I}\ \varepsilon_X(\psi)\ p_1\ b_1) : (T\ \Box\varepsilon_X(\psi))$.

$\bullet \pi \equiv \supset\text{-E}_{\psi,\varphi}(\pi', \pi'')$: then $(X, \Delta) \models_{\mathbf{NS4}} \pi' : \psi$ and $(X, \Delta) \models_{\mathbf{NS4}} \pi'' : \psi \supset \varphi$. Therefore by IH there exist two canonical terms p_1, p_2 such that $\Gamma_X, \gamma_\Box(\Box\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ \varepsilon_X(\psi))$ and $\Gamma_X, \gamma_\Box(\Box\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_2 : (T\ \varepsilon_X(\psi \supset \varphi))$. Then, $\Gamma_X, \gamma_\Box(\Box\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} (\supset\text{-E}\ \varepsilon_X(\psi)\ \varepsilon_X(\varphi)\ p_2\ p_1) : (T\varepsilon_X(\varphi))$ □

A.3.8 Proof of Theorem 5.8

The result follows immediately from Lemma 5.7 and the following two technical Lemma A.2, A.3. For sake of simplicity, we adopt the following definitions: for p term and Γ context, we define

$$C'(p,\Gamma) \stackrel{\text{def}}{=} \text{for all } c \in FV(p), \text{ for all } (c{:}(T\ t)) \in \Gamma, \text{ there exists } (b{:}(Bx\ t\ c)) \in \Gamma$$
$$\alpha_p(\Gamma) \stackrel{\text{def}}{=} \{\varphi \mid (v_\varphi{:}(T\ \varepsilon_X(\varphi))) \in \Gamma \text{ and } v_\varphi \in FV(p)\}.$$

Intuitively, the set $\alpha_p(\Gamma)$ contains the "active assumptions" in the context Γ for p.

Lemma A.2 *If there is a canonical term b such that $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b : (Bx\ \varphi\ p)$ then $C'(p, \gamma_\Delta(\Delta))$ holds.*

Proof. By lemma of characterization, a canonical form d of type $(Bx\ \varphi\ p)$ must be $\zeta M_1 \ldots M_k$, where k is the arity of ζ. By inspection of $\Sigma_\Box(\mathbf{NS4})$ and $\Gamma_X, \gamma_\Box(\Delta)$ we see that the only choises are $\zeta \in \{vb_\varphi, Bx_{\supset\text{-I}}, Bx_{\supset\text{-E}}, Bx_{\Box\text{-I}}, Bx_{\Box\text{-E}}, \ldots\}$.
Base Step: $b = vb_\varphi \in \gamma_\Box(\Delta)$; then, immediately, $v_\varphi \in \gamma_\Box(\Delta)$ and hence $C'(p, \gamma_\Box(\Delta))$ holds.
Inductive Step: we see some significant cases, the other being similar.

•$b \equiv (Bx_{\supset\text{-I}}\ t\ t'(\lambda v_{\delta_X(t)}{:}(T\ t).p_1)(\lambda v_{\delta_X(t)} : (T\ t)\lambda vb_{\delta_X(t)}{:}(Bx\ t\ v_{\delta_X(t)}).b_1))$: then $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b : (Bx\ (\supset\ t\ t')\ p)$. Since b is well-typed we have that

$$\Gamma_X, \gamma_\Box(\Delta), v_{\delta_X(t)} : (T\ t), vb_{\delta_X(t)} : (Bx\ t\ v_{\delta_X(t)}) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1 : (Bx\ t'\ p_1).$$

By IH $C'(p_1, (\gamma_\Box(\Delta), v_{\delta_X(t)} : (T\ t), vb_{\delta_X(t)} : (Bx\ t\ v_{\delta_X(t)})))$. Since $p \equiv \supset\text{-I}\ t\ t'(\lambda v_{\delta_X(t)} : (T\ t).p_1)$, we have $FV(p) = FV(p_1) \setminus \{v_{\delta_X(t)}\}$, therefore $C'(p, \gamma(\Delta))$ holds.

•$b \equiv (Bx_{\supset\text{-E}}\ t\ t'\ p_1\ p_2\ b_1\ b_2)$: then $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b : (Bx\ t'\ p)$ where $p \stackrel{\text{def}}{=} (\supset\text{-E}\ t\ t'\ p_1\ p_2)$. Since b is well-typed we have that $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1 : (Bx\ (\supset\ t\ t')\ p_1)$ and $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_2 : (Bx\ t\ p_2)$. By IH, $C'(p1, \gamma_\Box(\Delta))$ and $C'(p2, \gamma_\Box(\Delta))$ hold; then $C'(p, \gamma_\Box(\Delta))$ holds, since $FV(p_1) \cup FV(p_2) = FV(p)$.

•$b \equiv (Bx_{\supset_\Box\text{-I}}\ t\ t'\ p_1\ p_2)$, where $p_1 \stackrel{\text{def}}{=} \lambda v_{\delta_X(\Box t)}{:}(T\ \Box t).\lambda vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}).p'_1$ and $p_2 \stackrel{\text{def}}{=} \lambda v_{\delta_X(\Box t)} : (T\ \Box t).\lambda vb_{\delta_X(\Box t)}{:}(Bx\ \Box t\ v_{\delta_X(\Box t)}).b_1$. Then $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b : (Bx\ (\supset\ \Box t\ t')\ p)$ where $p \equiv (\supset_\Box\text{-I}\ t\ t'\ p_1)$. Since b is well-typed we have that

$$\Gamma_X, \gamma_\Box(\Delta), v_{\delta_X(\Box t)}{:}(T\ \Box t), vb_{\delta_X(\Box t)} : (Bx\ \Box t\ v_{\delta_X(t)}) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1{:}(Bx\ t'\ p'_1).$$

By inductive hypothesis, $C'(p'_1, (\gamma_\Box(\Delta), v_{\delta_X(\Box t)}{:}(T\ \Box t), vb_{\delta_X(\Box t)} : (Bx\ \Box t\ v_{\delta_X(\Box t)})))$ holds, and then $C'(p_1, \gamma_\Box(\Delta))$ holds too. Therefore $C'(p, \gamma_\Box(\Delta))$ holds.

•$b \equiv (Bx_{\Box\text{-I}}\ t\ p_1\ b_1)$: then $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b : (Bx\ (\Box t)\ p)$ with $p \equiv (\Box\text{-I}\ t\ p_1\ b_1)$. Since b is well-typed we have $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1 : (Bx\ t\ p_1)$, and hence by IH $C'(p_1, \gamma(\Delta))$ holds. Therefore $C'(p, \gamma(\Delta))$ holds as well, because the free variables in b_1 are typed in $\gamma(\Delta)$ only by the Bx judgement. $\Box$

Lemma A.3 *Given a canonical term p such that $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p : (T\ t)$, there exists a proof π such that $(X, \alpha_p(\gamma_\Box(\Delta))) \models_{\mathbf{NS4}} \pi : \delta_X(t)$.*

Proof. By lemma of characterization, a canonical form p of type $(T\ t)$ must be $\zeta M_1 \ldots M_k$, where k is the arity of ζ. By inspection of $\Sigma_\Box(\mathbf{NS4})$ e $\Gamma_X, \gamma_\Box(\Delta)$ we see that the only choices for ζ are $\zeta \in \{v_\varphi, \supset\text{-I}, \supset\text{-E}, \Box\text{-I}, \Box\text{-E}, \supset_\Box\text{-I} \ldots\}$. We proceed by induction.
Base Step: $p = v_{\delta_X(t)}$. Then, $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p : (T\ t)$ and moreover we have that $\alpha_p(\gamma_\Box(\Delta)) = \{\delta_X(t)\}$. Taken $\pi = \delta_X(t)$, we obtain $(X, \delta_X(t)) \models_{\mathbf{NS4}} \pi : \delta_X(t)$.

Inductive Step. We see only some significant cases.

$\bullet p \equiv (\supset\text{-E } t\ t'\ p_1\ p_2)$: then $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p : (T\ t')$. Since p is well-typed we have that $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 {:} (T(\supset t\ t'))$ and $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_2 : (T\ t)$. By IH there exist π', π'' such that $(X, \alpha_{p_1}(\gamma_\Box(\Delta))) \models_{\mathbf{NS4}} \pi' : \delta_X(\supset t\ t')$ and $(X, \alpha_{p_2}(\gamma_\Box(\Delta))) \models_{\mathbf{NS4}} \pi'' : \delta_X(t)$. Then taken $\pi = \supset\text{-E}_{\delta_X(t), \delta_X(t')}(\pi', \pi'')$ we obtain that $(X, \alpha_p(\gamma_\Box(\Delta))) \models_{\mathbf{NS4}} \pi : \delta_X(t')$.

$\bullet p \equiv (\supset\text{-I } t\ t'(\lambda v_{\delta_X(t)} : (T\ t).p_1))$: then, $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p : (T(\supset t\ t'))$. Since p is well-typed we have that $\Gamma_X, \gamma_\Box(\Delta), v_{\delta_X(t)} {:} (T\ t) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ t')$
(i.e. $\Gamma_X, \gamma_\Box(\Delta, \delta_X(t)) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ t')$ because $\gamma_\Box(\Delta, \delta_X(t)) \supseteq \gamma_\Box(\Delta), v_{\delta_X(t)} : (T\ t)$). By IH there exists π' such that $(X, \alpha_{p_1}(\gamma_\Box(\Delta, \delta_X(t)))) \models_{\mathbf{NS4}} \pi' {:} \delta_X(t')$. Moreover, $\alpha_{p_1}(\gamma_\Box(\Delta, \delta_X(t))) \subseteq \alpha_p(\gamma(\Delta)), \delta_X(t)$ since $\mathrm{FV}(p_1) \subseteq \mathrm{FV}(p) \cup \{v_{\delta_X(t)}\}$. Then $(X, \alpha_p(\gamma_\Box(\Delta)), \delta_X(t)) \models_{\mathbf{NS4}} \pi' {:} \delta_X(t')$; taken $\pi = \supset\text{-I}_{\delta_X(t), \delta_X(t')}(\pi')$, we finally obtain $(X, \alpha_p(\gamma_\Box(\Delta))) \models_{\mathbf{NS4}} \pi : \delta_X(\supset t\ t')$.

$\bullet p \equiv (\supset_\Box\text{-I } t\ t'\ (\lambda v_{\delta_X(t)} {:} (T\ t) \lambda vb_{\delta_X(t)} {:} (Bx\ t\ v_{\delta_X(t)}).p_1))$: then $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p : (T\ (\supset t\ t'))$. Since p is well-typed we have

$$\Gamma_X, \gamma_\Box(\Delta), v_{\delta_X(t)} {:} (T\ t), vb_{\delta_X(t)} {:} (Bx\ t\ v_{\delta_X(t)}) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ t').$$

Moreover, since $\delta_X(t)$ is boxed, it is $\Gamma_X, \gamma_\Box(\Delta, \delta_X(t)) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ t')$. By IH there exists π' such that $(X, \alpha_{p_1}(\gamma_\Box(\Delta, \delta_X(t)))) \models_{\mathbf{NS4}} \pi' {:} \delta_X(t')$. Now, $\alpha_{p_1}(\gamma_\Box(\Delta, \delta_X(t))) \subseteq \alpha_p(\gamma_\Box(\Delta)) \cup \{\delta_X(A)\}$, since $\mathrm{FV}(p_1) \subseteq \mathrm{FV}(p) \cup \{v_{\delta_X(t)}, vb_{\delta_X(t)}\}$. Then $(X, \alpha_p(\gamma_\Box(\Delta)), \delta_X(t)) \models_{\mathbf{NS4}} \pi' : \delta_X(t')$. Taken $\pi = \supset_\Box\text{-I}_{\delta_X(t), \delta_X(t')}(\pi')$ we obtain that $(X, \alpha_p(\gamma_\Box(\Delta))) \models_{\mathbf{NS4}} \pi : \delta_X(\supset t\ t')$.

$\bullet p \equiv (\Box\text{-I } t\ p_1\ b_1)$: then $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p : (T\ (\Box t))$. Since p is well-typed we have $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} b_1 : (Bx\ t\ p_1)$ and $\Gamma_X, \gamma_\Box(\Delta) \vdash_{\Sigma_\Box(\mathbf{NS4})} p_1 : (T\ t)$. By IH there exists π' such that $(X, \alpha_{p_1}(\gamma(\Delta))) \models_{\mathbf{NS4}} \pi' {:} \delta_X(t)$. By the lemma A.2, $C'(p_1, \gamma_\Box(\Delta))$ holds. Now, for each $\psi \in \alpha_{p_1}(\gamma_\Box(\Delta))$, by definition of α there is an assumption $(v_\psi {:} (T\ \varepsilon_X(\psi))) \in \gamma_\Box(\Delta)$ such that $v_\psi \in \mathrm{FV}(p_1)$. Since $C'(p_1, \gamma_\Box(\Delta))$ holds, we have that there is an assumption $(vb_\psi : (Bx\ \varepsilon_X(\psi)\ v_\psi)) \in \gamma_\Box(\Delta)$, but by definition of $\gamma_\Box$, this means that ψ is boxed. Then $\alpha_{p1}(\gamma_\Box(\Delta))$ contains only boxed formulæ. Then we can take $\pi = \Box\text{-I}_{\delta_X(t)}(\pi')$ obtaining $(X, \alpha_p(\gamma_\Box(\Delta))) \models_{\mathbf{NS4}} \pi : \Box\delta_X(t)$. $\Box$

From Lemma A.3 follows the definition of the decoding function $\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}$:

$$\begin{array}{l}
\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}(v_\varphi) \stackrel{\text{def}}{=} \varphi \qquad\qquad , if\ v_\varphi \in dom(\gamma_\Box(\Delta)). \\
\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}(\supset\text{-E } t\ t'\ p_1\ p_2) \stackrel{\text{def}}{=} \supset\text{-E}_{\delta_X(t), \delta_X(t')}(\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}(p_1), \delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}(p_2)) \\
\delta_{X,\Delta}(\supset\text{-I } t\ t'(\lambda v_{\delta_X(t)} {:} (T\ t).p_1)) \stackrel{\text{def}}{=} \supset\text{-I}_{\delta_X(t), \delta_X(t')}(\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta,\delta_X(t)}(p_1)) \\
\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}(\supset_\Box\text{-I } t\ t'\ (\lambda v_{\delta_X(t)} {:} (T\ t) \lambda vb_{\delta_X(t)} {:} (Bx\ t\ v_{\delta_X(t)}).p_1)) \stackrel{\text{def}}{=} \supset_\Box\text{-I}_{\delta_X(t), \delta_X(t')}(\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta,\delta_X(t)}(p_1)) \\
\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}(\Box\text{-I } t\ p_1\ b_1) \stackrel{\text{def}}{=} \Box\text{-I}_{\delta_X(t)}(\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}(p_1)) \\
\delta_{X,\Delta}(\Box\text{-E } t\ p_1) \stackrel{\text{def}}{=} \Box\text{-E}_{\delta_X(t)}(\delta^{\Sigma_\Box(\mathbf{NS4})}_{X,\Delta}(p_1))
\end{array}$$

A.3.9 Proof of Theorem 6.1

Actually, only some hints:

- $1 \Rightarrow 2$: by induction on t.
- $2 \Rightarrow 3$: by induction on n.
- $3 \Rightarrow 1$: by induction on v. Alternatively, replace everywhere $(V\ \psi)$ by $\prod_{w:U}(T\ w\ \psi)$, and $(T\ \psi)$ by $(T\ w\ \psi)$.

- 3 $\Rightarrow$ 2: it is possible to express V in terms of Na, by means of Σ-types: $(V\ \varphi) = \sum_{x:(T\ \varphi)}(Cl\ \varphi\ x)$. Hence, proof of $(V\ \varphi)$ is, a proof of $(T\ \varphi)$ together with the proof that it does not depend on any assumptions. This is not possible in LF but in some higher-order logical framework, such as CIC. □

A.3.10 Proof of Theorem 6.2

Similar to Theorem 6.1.