# Non-deterministic Matrices and Modular Semantics of Rules

Arnon Avron

**Abstract.** We show by way of example how one can provide in a lot of cases simple modular semantics for rules of inference, so that the semantics of a system is obtained by joining the semantics of its rules in the most straightforward way. Our main tool for this task is the use of finite Nmatrices, which are multi-valued structures in which the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. The method is applied in the area of logics with a formal consistency operator (known as LFIs), allowing us to provide in a modular way effective, finite semantics for thousands of different LFIs.

**Mathematics Subject Classification (2000).** 03B22; 03B50; 03B53.

**Keywords.** Propositional logics, multiple-valued semantics, paraconsistency.

## 1. Introduction

It is well known that every propositional logic satisfying certain minimal conditions can be characterized semantically using a multi-valued matrix ([18]). However, there are many important decidable logics whose characteristic matrices necessarily consist of an infinite number of truth values. In such a case it might be quite difficult to find any of these matrices, or to use one when it is found. Even in case a logic does have a finite characteristic matrix it might be difficult to discover this fact, or to find such a matrix. The deep reason for these difficulties is that in an ordinary multi-valued semantics the rules and axioms of a system should be considered as a whole, and there is no method for separately determining the semantic effects of each rule or axiom alone.

In this paper we show how one can provide in a lot of cases simple modular semantics for rules of inference, so that the semantics of a system is obtained

This research was supported by THE ISRAEL SCIENCE FOUNDATION founded by The Israel Academy of Sciences and Humanities.

by joining the semantics for its rules in the most straightforward way. Our main tool for this task is the use of finite Nmatrices ([6, 4]. Nmatrices are multi-valued structures in which the value assigned by a valuation to a complex formula can be chosen non-deterministically out of a certain nonempty set of options. The use of finite structures of this sort has the benefit of preserving all the advantages of logics with ordinary finite-valued semantics (in particular: decidability and compactness), while it is applicable to a much larger family of logics. The central idea in using Nmatrices for providing semantics for rules is that the main effect of a "normal" rule is to reduce the degree of non-determinism of operations, by forbidding some options (in non-deterministic computations of truth values) which we could have had otherwise. This idea was first applied in [7, 6] for a very special (though extremely important) type of rules (which was called there "canonical rules"). For that type of rules 2-valued Nmatrices suffice. In this paper we show how by employing more than two values we can apply the method for a much larger class of rules. As a case study we have chosen the class of paraconsistent logics knows as LFIs, described in [11, 12] [1]. In what follows we use our method in order to modularly provide effective, finite semantics for thousands of different LFIs.

## 2. Preliminaries

### 2.1. Consequence Relations, Logics, and Pure Rules

**Definition 2.1.**

1. A *Scott consequence relation* (*scr* for short) for a language $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ that satisfies the following conditions:

   | | | |
   |---|---|---|
   | **s-R** | *strong reflexivity*: | if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$. |
   | **M** | *monotonicity*: | if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$. |
   | **C** | *Transitivity* (*cut*): | if $\Gamma \vdash \psi, \Delta$ and $\Gamma', \psi \vdash \Delta'$ then $\Gamma, \Gamma' \vdash \Delta, \Delta'$. |

2. An scr $\vdash$ for $\mathcal{L}$ is *structural* (or substitution-invariant) if for every uniform $\mathcal{L}$-substitution $\sigma$ and every $\Gamma$ and $\Delta$, if $\Gamma \vdash \Delta$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$. $\vdash$ is *finitary* if the following condition holds for all $\Gamma, \Delta \subseteq \mathcal{W}$: if $\Gamma \vdash \Delta$ then there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash \Delta'$. $\vdash$ is *consistent* (or *non-trivial*) if there exist non-empty $\Gamma$ and $\Delta$ s.t. $\Gamma \not\vdash \Delta$. [2]
3. A propositional *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language and $\vdash$ is an scr for $\mathcal{L}$ which is structural and consistent. The logic $\langle \mathcal{L}, \vdash \rangle$ is finitary if $\vdash$ is finitary.

**Definition 2.2.**

1. A *pure rule* in a propositional language $\mathcal{L}$ is any ordered pair $\langle \Gamma, \Delta \rangle$, where $\Gamma$ and $\Delta$ are finite sets of formulas in $\mathcal{L}$ (We shall usually denote such a rule by $\Gamma \vdash \Delta$ rather than by $\langle \Gamma, \Delta \rangle$).

---

[1] The name "LFI" stands for "Logics of Formal Inconsistency". In our opinion it would make more sense to call them "logics of formal consistency", since they are obtained from classical logic by the addition of a new connective $\circ$, with the intended meaning of $\circ\varphi$ being: "$\varphi$ is consistent".

[2] See [7, 6] for the importance of the consistency property.

2. Let $\langle \mathcal{L}, \vdash_1 \rangle$ be a propositional logic, and let $S$ be a set of rules in a propositional language $\mathcal{L}'$. By the extension of $\langle \mathcal{L}, \vdash_1 \rangle$ by $S$ we mean the logic $\langle \mathcal{L}^*, \vdash^* \rangle$, where $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}'$, and $\vdash^*$ is the least *structural* scr $\vdash$ such that $\Gamma \vdash \Delta$ whenever $\Gamma \vdash_1 \Delta$ or $\langle \Gamma, \Delta \rangle \in S$.

*Remark* 2.3. Obviously, the extension of $\langle \mathcal{L}, \vdash_1 \rangle$ by $S$ is well-defined (i.e. a logic) only if $\vdash^*$ is consistent. In all the cases we consider below this will easily be guaranteed by the semantics we provide (and so we shall not even mention it).

*Remark* 2.4. It is easy to see that $\vdash^*$ is the closure under cuts and weakenings of the set of all pairs $\langle \sigma(\Gamma), \sigma(\Delta) \rangle$, where $\sigma$ is a uniform substitution in $\mathcal{L}^*$, and either $\Gamma \vdash_1 \Delta$ or $\langle \Gamma, \Delta \rangle \in S$. This in turn implies that an extension of a finitary logic by a set of pure rules is again finitary.

*Remark* 2.5. Most standard rules used in Gentzen-type systems are equivalent to finite sets of pure rules in the sense of Definition 2.2. For example, the usual $(\supset\Rightarrow)$ rule of classical logic is equivalent (using cuts, weakenings, and the reflexivity axioms $\varphi \vdash \varphi$) to the pure rule $\varphi, \varphi \supset \psi \vdash \psi$, while the classical $(\Rightarrow\supset)$ rule is equivalent to the set $\{\psi \vdash \varphi \supset \psi, \ \ \vdash \varphi, \varphi \supset \psi\}$.

### 2.2. Non-deterministic Matrices

Our main semantical tool in what follows will be the following generalization from [7, 6] of the concept of a matrix: [3]

**Definition 2.6.**

1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:
   (a) $\mathcal{V}$ is a non-empty set of *truth values.*
   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$.
   (c) For every $n$-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding $n$-ary function $\widetilde{\diamond}$ from $\mathcal{V}^n$ to $2^{\mathcal{V}} - \{\emptyset\}$.

   We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{V}$.
2. Let $\mathcal{W}$ be the set of formulas of $\mathcal{L}$. A *(legal) valuation* in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{V}$ that satisfies the following condition for every $n$-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{W}$:
$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$
3. A valuation $v$ in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula $\psi$ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. $v$ is a *model* in $\mathcal{M}$ of a set $\Gamma$ of formulas (notation: $v \models^{\mathcal{M}} \Gamma$) if it satisfies every formula in $\Gamma$.
4. $\vdash_{\mathcal{M}}$, the consequence relation induced by the Nmatrix $\mathcal{M}$, is defined by: $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every $v$ such that $v \models^{\mathcal{M}} \Gamma$, there is $\varphi \in \Delta$ such that $v \models^{\mathcal{M}} \varphi$.

[3] A special two-valued case of this definition was essentially introduced in [9]. Another particular case of the same idea, using a similar name, was used in [13]. It should also be noted that Carnielli's "possible-translations semantics" (see [10]) was originally called "non-deterministic semantics", but later the name was changed to the present one.

5. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is *sound* for an Nmatrix $\mathcal{M}$ (where $\mathcal{L}$ is the language of $\mathcal{M}$) if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{M}}$. $\mathbf{L}$ is *complete* for $\mathcal{M}$ if $\vdash_{\mathbf{L}} \supseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is *characteristic* for $\mathbf{L}$ if $\mathbf{L}$ is both sound and complete for it (i.e.: if $\vdash_{\mathbf{L}} = \vdash_{\mathcal{M}}$). $\mathcal{M}$ is *weakly-characteristic* for $\mathbf{L}$ if for every formula $\varphi$ of $\mathcal{L}$, $\vdash_{\mathbf{L}} \varphi$ iff $\vdash_{\mathcal{M}} \varphi$.

*Remark* 2.7. We shall identify an ordinary (deterministic) matrix with an Nmatrix whose functions in $\mathcal{O}$ always return singletons.

**Theorem 2.8** ([6]). *A logic which has a finite characteristic Nmatrix is finitary and decidable.*

**Definition 2.9.** Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$.

1. A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is a function $F : \mathcal{V}_1 \to \mathcal{V}_2$ such that:
   (a) For every $x \in \mathcal{V}_1$, $x \in \mathcal{D}_1$ iff $F(x) \in \mathcal{D}_2$ (i.e. $D_1 = F^{-1}[D_2]$).
   (b) $F(y) \in \widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n))$ for every $n$-ary connective $\diamond$ of $\mathcal{L}$ and every $x_1, \ldots, x_n, y \in \mathcal{V}_1$ such that $y \in \widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n)$ (in other words: $\widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n) \subseteq F^{-1}[\widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n))]$).
2. A reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$ is called *exact* if it has the following properties:
   (a) $F$ is *onto* $\mathcal{V}_2$.
   (b) For every $n$-ary connective $\diamond$ of $\mathcal{L}$ and every $x_1, \ldots, x_n, y \in \mathcal{V}_1$:
   $$F(y) \in \widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n)) \quad \text{iff} \quad y \in \widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n)$$
   (equivalently: if $\widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n) = F^{-1}[\widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n))]$).
3. $\mathcal{M}_1$ is a *refinement* of $\mathcal{M}_2$ if there exists a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. It is an *exact refinement* of $\mathcal{M}_2$ if this reduction is exact.

**Theorem 2.10.**

1. *If $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.*
2. *If $\mathcal{M}_1$ is an exact refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} = \vdash_{\mathcal{M}_1}$.*

*Proof.* For the first part, assume that $F$ is a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. We show that if $v$ is a legal valuation in $\mathcal{M}_1$ then $v' = F \circ v$ (the composition of $F$ and $v$) is a legal valuation in $\mathcal{M}_2$. Indeed, let $\diamond$ be an $n$-ary connective of $\mathcal{L}$, and let $\varphi_1, \ldots, \varphi_n$ be $n$ formulas of $\mathcal{L}$. We show that $v'(\diamond(\varphi_1, \ldots, \varphi_n)) \in \widetilde{\diamond}_{\mathcal{M}_2}(v'(\varphi_1), \ldots, v'(\varphi_n))$. Let $y = v(\diamond(\varphi_1, \ldots, \varphi_n))$, and $x_i = v(\varphi_i)$ $(i = 1, \ldots, n)$. Then $y \in \widetilde{\diamond}_{\mathcal{M}_1}(x_1, \ldots, x_n)$, and so $F(y) \in \widetilde{\diamond}_{\mathcal{M}_2}(F(x_1), \ldots, F(x_n))$. Since $v'(\diamond(\varphi_1, \ldots, \varphi_n) = F(y)$ and $v'(\varphi_i) = F(x_i)$ $(i = 1, \ldots, n)$, our claim follows.

Now assume that $\Gamma \vdash_{\mathcal{M}_2} \Delta$. We show that $\Gamma \vdash_{\mathcal{M}_1} \Delta$ as well. So let $v$ be a model of $\Gamma$ in $\mathcal{M}_1$. Then $v(\varphi) \in \mathcal{D}_1$ for every $\varphi \in \Gamma$. Hence $F(v(\varphi)) \in \mathcal{D}_2$ for every $\varphi \in \Gamma$. Since $F \circ v$ is a legal valuation in $\mathcal{M}_2$, this means that $F \circ v$ is a model of $\Gamma$ in $\mathcal{M}_2$, and so $F(v(\psi)) = (F \circ v)(\psi) \in \mathcal{D}_2$ for some $\psi \in \Delta$. Since $F$ is a reduction function, this implies that $v(\psi) \in \mathcal{D}_1$ for some $\psi \in \Delta$, as required.

For the second part note that if $F$ is an exact reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$, then every right inverse $G$ of $F$ [4] can easily be shown to be a reduction of $\mathcal{M}_2$ to $\mathcal{M}_1$. Thus by the first part $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$ too, and so $\vdash_{\mathcal{M}_2} = \vdash_{\mathcal{M}_1}$. □

*Remark* 2.11. An important case in which $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ is a refinement of $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ is when $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$, and $\widetilde{\diamond}_{\mathcal{M}_1}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(\vec{x})$ for every $n$-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{V}_1^n$. It is easy to see that the identity function on $\mathcal{V}_1$ is in this case a reduction of $\mathcal{M}_1$ to $\mathcal{M}_2$. A refinement of this sort will be called *simple*.[5]

### 2.3. Positive Classical Logic

**Definition 2.12.** Let $\mathbf{CL}^+ = \langle \mathcal{L}_{\mathrm{cl}}^+, \vdash_{\mathrm{cl}}^+ \rangle$, where $\mathcal{L}_{\mathrm{cl}}^+ = \{\wedge, \vee, \supset\}$, and $\vdash_{\mathrm{cl}}^+$ is the classical consequence relation in the language $\mathcal{L}_{\mathrm{cl}}^+$ (i.e.: $\Gamma \vdash_{\mathrm{cl}}^+ \Delta$ iff every classical two-valued model of $\Gamma$ is a model of at least one formula in $\Delta$).

*Remark* 2.13. For any pure rule in a propositional language containing $\mathcal{L}_{\mathrm{cl}}^+$ it is possible to find an equivalent rule of the form $\vdash \varphi$ (by translating the condition $\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_k$ into $\vdash \varphi_1 \wedge \ldots \wedge \varphi_n \supset \psi_1 \vee \ldots \vee \psi_k$ in case $k > 0$, and to $\vdash \varphi_1 \wedge \ldots \wedge \varphi_n \supset q$, where $q$ is an atomic formula not occurring in $\varphi_1, \ldots, \varphi_n$, in case k=0. Hence it is possible to construct a sound and complete Hilbert-type system (with $MP$ as the sole rule of inference) for any extension of $\mathbf{CL}^+$ by a finite set of pure rules. On the other hand any pure rule is equivalent above $\mathbf{CL}^+$ to a finite set of rules in which none of the formulas has either $\vee, \wedge$ or $\supset$ as its principal connective. For example, a condition of the form $\varphi \wedge \psi, \Gamma \vdash \Delta$ can be replaced by $\varphi, \psi, \Gamma \vdash \Delta$, while $\Gamma \vdash \Delta, \varphi \wedge \psi$ can be replaced by $\{\Gamma \vdash \Delta, \varphi \ , \ \Gamma \vdash \Delta, \psi\}$.

**Definition 2.14.** Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes $\mathcal{L}_{\mathrm{cl}}^+$. We say that $\mathcal{M}$ is *suitable* for $\mathbf{CL}^+$ if the following conditions are satisfied:

- If $a \in \mathcal{D}$ and $b \in \mathcal{D}$ then $a \widetilde{\wedge} b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ then $a \widetilde{\wedge} b \subseteq \mathcal{V} - D$
- If $b \notin \mathcal{D}$ then $a \widetilde{\wedge} b \subseteq \mathcal{V} - D$

- If $a \in \mathcal{D}$ then $a \widetilde{\vee} b \subseteq \mathcal{D}$
- If $b \in \mathcal{D}$ then $a \widetilde{\vee} b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \widetilde{\vee} b \subseteq \mathcal{V} - D$

- If $a \notin \mathcal{D}$ then $a \widetilde{\supset} b \subseteq \mathcal{D}$
- If $b \in \mathcal{D}$ then $a \widetilde{\supset} b \subseteq \mathcal{D}$
- If $a \in \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \widetilde{\supset} b \subseteq \mathcal{V} - D$

---

[4]By this one means a function $G : \mathcal{V}_2 \rightarrow \mathcal{V}_1$ such that $F(G(x)) = x$ for every $x \in \mathcal{V}_2$. Such a function $G$ exists here, since $F$ is onto $\mathcal{V}_2$.

[5]What we call here "a simple refinement" is what was called "a refinement" in [2]. The present definition of "a refinement" is a refinement of the definition given to that concept there.

**Theorem 2.15.** *Suppose $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is suitable for* $\mathbf{CL}^+$. *Let $\mathcal{M}' = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}' \rangle$, where $\mathcal{O}'$ is the subset of $\mathcal{O}$ which corresponds to the connectives of $\mathcal{L}^+_{\text{cl}}$. Then $\vdash^+_{\text{cl}} = \vdash_{\mathcal{M}'}$. Hence $\sigma(\Gamma) \vdash_{\mathcal{M}} \sigma(\Delta)$ whenever $\Gamma \vdash^+_{\text{cl}} \Delta$ and $\sigma$ is a substitution in the language of $\mathcal{M}$.*

*Proof.* Let $\mathcal{M}_{\mathbf{CL}^+}$ be the classical two-valued matrix, where the two truth values are $\mathbf{t}$ and $\mathbf{f}$. Since $\mathcal{M}$ is suitable for $\mathbf{CL}^+$, the function

$$\lambda x \in \mathcal{V}. \begin{cases} \mathbf{t} & x \in \mathcal{D} \\ \mathbf{f} & x \notin \mathcal{D} \end{cases}$$

is a reduction of $\mathcal{M}'$ to $\mathcal{M}_{\mathbf{CL}^+}$. Hence $\vdash^+_{\text{cl}} \subseteq \vdash_{\mathcal{M}'}$. That $\vdash^+_{\text{cl}} = \vdash_{\mathcal{M}'}$ follows from the well-known fact that $\mathbf{CL}^+$ is a maximal nontrivial logic in its language. □

### 2.4. Formal Systems with a Formal Consistency Operation

**2.4.1. The Basic Logic.** Let $\mathcal{L}_{\text{cl}} = \{\wedge, \vee, \supset, \neg\}$. $\mathcal{L}_{\text{cl}}$ is the standard language of the classical propositional logic **CL**. The latter may be characterized as the extension of $\mathbf{CL}^+$ by the rules $\neg\varphi, \varphi \vdash$ and $\vdash \neg\varphi, \varphi$. The two main ideas of da-Costa's school of paraconsistent logics ([14, 11, 12]) are to limit the applicability of the first of these two rules (which amounts to "a single contradiction entails everything") to the case where $\varphi$ is "consistent", and to express the assumption of this consistency of $\varphi$ within the language. The easiest way to implement these ideas is to add to the language of **CL** a new connective $\circ$, with the intended meaning of $\circ\varphi$ being "$\varphi$ is consistent". Then one can explicitly add the assumption of the consistency of $\varphi$ to the problematic (from a paraconsistent point of view) classical rule concerning $\neg$. This leads to the basic system **B** described below.[6]

**Definition 2.16.** Let $\mathcal{L}_{\text{C}} = \{\wedge, \vee, \supset, \neg, \circ\}$.
**Definition 2.17.** The logic **B** is the minimal logic in $\mathcal{L}_{\text{C}}$ which extends $\mathbf{CL}^+$ and satisfies the following two conditions:

- **(t):** $\vdash \neg\varphi, \varphi$
- **(b):** $\circ\varphi, \neg\varphi, \varphi \vdash$

**Lemma 2.18.** *Let LK be the standard Gentzen calculus for classical propositional logic, and let GB be obtained from LK by replacing the $(\neg \Rightarrow)$ rule by:*

$$(\circ, \neg \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\circ\varphi, \neg\varphi, \Gamma \Rightarrow \Delta}$$

*Then for every finite $\Gamma$ and $\Delta$, $\Gamma \vdash_B \Delta$ iff $\Gamma \Rightarrow \Delta$ has a cut-free proof in GB.*

*Proof.* Using cuts, it is straightforward to show that for every finite $\Gamma$ and $\Delta$, $\Gamma \vdash_B \Delta$ iff $\Gamma \Rightarrow \Delta$ has a proof in *GB*. The cut-elimination theorem can then be proved for *GB* by the usual syntactic method of Gentzen (i.e. by using double induction on the complexity of the cut formula and on the height of the cut). □

[6]The logic **B** is called *mbC* in [12]. We prefer to use here a shorter name.

*Remark* 2.19. By Remark 2.13, a Hilbert-type system which is sound and complete for **B** can be obtained by adding the following two axioms to some standard Hilbert-type system for $\mathbf{CL}^+$ having $MP$ as the sole rule of inference:

**(t):** $\neg\varphi \vee \varphi$
**(b):** $(\circ\varphi \wedge \neg\varphi \wedge \varphi) \supset \psi$

The main property of **B** is given in the next theorem from [11, 12]:

**Theorem 2.20.** *Let $\vdash_{cl}$ be the classical consequence relation (in the language $\mathcal{L}_{\rm cl}$). Then $\Gamma \vdash_{cl} \Delta$ iff there exists a subset $\Sigma$ of the set of subformulas of $\Gamma \cup \Delta$ such that $\circ\Sigma, \Gamma \vdash_B \Delta$ (where $\circ\Sigma = \{\circ\psi \mid \psi \in \Sigma\}$).*

*Proof.* Suppose $\Gamma \vdash_{cl} \Delta$. Then there are finite subsets $\Gamma'$ and $\Delta'$ of $\Gamma$ and $\Delta$ (respectively) such that the sequent $\Gamma' \Rightarrow \Delta'$ has a cut-free proof in $LK$. Replace in this proof any application of the classical $(\neg \Rightarrow)$ rule by an application of $(\circ,\neg \Rightarrow)$. The result will be a cut-free proof in $GB$ of a sequent of the form $\circ\Sigma, \Gamma' \Rightarrow \Delta'$, where $\Sigma$ is a subset of the set of subformulas of $\Gamma' \Rightarrow \Delta'$. Hence $\circ\Sigma, \Gamma \vdash_B \Delta$.

For the converse, assume that $\circ\Sigma, \Gamma \vdash_B \Delta$. Then there are finite subsets $\Gamma'$ of $\Gamma$, $\Delta'$ of $\Delta$, and $\Sigma'$ of $\Sigma$, such that $\circ\Sigma', \Gamma' \Rightarrow \Delta'$ has a cut free-proof in $GB$. Replace in that proof every application of $(\circ,\neg \Rightarrow)$ by an application of the classical $(\neg \Rightarrow)$. Since $\circ$ does not occur in $\Gamma' \Rightarrow \Delta'$, the result is a proof in $LK$ of this sequent. It follows that $\Gamma \vdash_{cl} \Delta$. □

**2.4.2. Other Logics with a Formal Consistency Operation.** Rule (**b**) provides the most basic property expected of $\circ$. There are of course many others which might seem plausible to assume. The next two definitions provide a list of rules and systems (not all!) that have been considered in the literature on LFIs.[7]

**Definition 2.21.** Let $RULES$ be the set consisting of the following 10 rules:

**(c):** $\neg\neg\varphi \vdash \varphi$
**(e):** $\varphi \vdash \neg\neg\varphi$
**(k1):** $\vdash \circ\varphi, \varphi$
**(k2):** $\vdash \circ\varphi, \neg\varphi$
**(i1):** $\neg\circ\varphi \vdash \varphi$
**(i2):** $\neg\circ\varphi \vdash \neg\varphi$
**($\mathbf{a}_\neg$):** $\circ\varphi \vdash \circ(\neg\varphi)$
**($\mathbf{a}_\sharp$):** $\circ\varphi, \circ\psi \vdash \circ(\varphi\sharp\psi) \quad (\sharp \in \{\wedge, \vee, \supset\})$

**Definition 2.22.** For $S \subseteq RULES$, $\mathbf{B}[S]$ is the extension of **B** by the rules in $S$.

[7]In [11, 12] what is considered instead of (**i1**) and (**i2**) is actually their combination, the rule (**i**): $\neg\circ\varphi \vdash \varphi \wedge \neg\varphi$. This rule has been split here into two rules as described in Remark 2.13. Conditions (**k1**) and (**k2**) were not considered in [11, 12], but they are natural weaker versions of (**i1**) and (**i2**) (respectively).

## 3. Semantics for the Basic System

The system **B** treats the positive classical connectives exactly as classical logic does. Hence an Nmatrix for $B$ should most naturally be sought among the Nmatrices which are suitable for $\mathbf{CL}^+$. In such Nmatrices the answer to the question whether a sentence of the form $\varphi\sharp\psi$ ($\sharp \in \{\vee, \wedge, \supset\}$) is true or not relative to a given valuation $v$ (i.e. whether $v(\varphi\sharp\psi) \in \mathcal{D}$ or not) is completely determined by the answers to the same question for $\varphi$ and $\psi$. The situation with respect to the unary connectives $\neg$ and $\circ$ is different. The truth/falsity of $\neg\varphi$ or $\circ\varphi$ is not completely determined by the truth/falsity of $\varphi$. More data is needed for this. Now the central idea of the semantics we are about to present is to include all the relevant data concerning a sentence $\varphi$ in the truth value from $\mathcal{V}$ which is assigned to $\varphi$. In our case the relevant data beyond the truth/falsity of $\varphi$ is the truth/falsity of $\neg\varphi$ and $\circ\varphi$. This leads to the use of elements from $\{0,1\}^3$ as our truth values, where the intended intuitive meaning of $v(\varphi) = \langle x, y, z\rangle$ is the following:

- $x = 1$ iff $\varphi$ is "true" (i.e. $v(\varphi) \in \mathcal{D}$ ).
- $y = 1$ iff $\neg\varphi$ is "true" (i.e. $v(\neg\varphi) \in \mathcal{D}$).
- $z = 1$ iff $\circ\varphi$ is "true" (i.e. $v(\circ\varphi) \in \mathcal{D}$).

However, because of the special principles of **B** not all triples can be used. Thus rule (**t**) means that at least one element of the pair $\{\varphi, \neg\varphi\}$ should be true. Hence the truth-values $\langle 0,0,1\rangle$ and $\langle 0,0,0\rangle$ should be rejected. Similarly, rule (**b**) means that $\varphi$, $\neg\varphi$, and $\circ\varphi$ cannot all be true. Hence $\langle 1,1,1\rangle$ should be rejected. We are left with 5 truth-values. Among them those which are designated are those which can be assigned to true formulas, i.e. those whose first component is 1. Then we define the operations in the most liberal way which is coherent with the intended meaning of the truth-values, and with the need to use an Nmatrix suitable for $\mathbf{CL}^+$. The resulting Nmatrix is described in the next definition.

**Definition 3.1.** The Nmatrix $\mathcal{M}_5^B = \langle \mathcal{V}_5, \mathcal{D}_5, \mathcal{O}_5^B\rangle$ is defined as follows:

- $\mathcal{V}_5 = \{t, t_I, I, f_I, f\}$ where:

$$\begin{array}{rcl} t & = & \langle 1,0,1\rangle \\ t_I & = & \langle 1,0,0\rangle \\ I & = & \langle 1,1,0\rangle \\ f & = & \langle 0,1,1\rangle \\ f_I & = & \langle 0,1,0\rangle \end{array}$$

- $\mathcal{D}_5 = \{t, I, t_I\}$ $(= \{\langle x,y,z\rangle \in \mathcal{V}_5 \mid x = 1\})$.
- Let $\mathcal{D} = \mathcal{D}_5$, $\mathcal{F} = \mathcal{V}_5 - \mathcal{D}$. The operations in $\mathcal{O}_5^B$ are defined by:

$$a \tilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D}, \\ \mathcal{F} & \text{if } a, b \in \mathcal{F} \end{cases}$$

$$a \tilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{D} \\ \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \end{cases}$$

$$a \tilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{D} \\ \mathcal{F} & \text{if either } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \end{cases}$$

$$\tilde{\neg} a = \begin{cases} \mathcal{D} & \text{if } a \in \{I, f, f_I\} \\ \mathcal{F} & \text{if } a \in \{t, t_I\} \end{cases}$$

$$\tilde{\circ} a = \begin{cases} \mathcal{D} & \text{if } a \in \{t, f\} \\ \mathcal{F} & \text{if } a \in \{I, t_I, f_I\} \end{cases}$$

**An Explanation.** The rules in **B** related to the positive classical connectives impose no constraints on the truth/falsity of $\neg\varphi$ or $\circ\varphi$. Hence they affect only the first component of truth-values. Thus if the first component of $v(\varphi)$ is 1 (i.e. if $v(\varphi)$ is in $\mathcal{D}_5$) then also the first component of $v(\psi \supset \varphi)$ should be 1, but there are no limitations in this case on the other two components of $v(\psi \supset \varphi)$. Hence $v(\psi \supset \varphi)$ may in this case be any element of $\mathcal{D}_5$. This implies that $a \tilde{\supset} b$ should be $\mathcal{D}_5$ in case $b \in \mathcal{D}_5 = \{t, t_I, I\}$. The other parts of the definitions of $\tilde{\supset}, \tilde{\vee}$, and $\tilde{\wedge}$ are derived similarly. The truth-value of $\neg\varphi$, on the other hand, is dictated by the second component of $v(\varphi)$. If it is 1 then $\neg\varphi$ should be true, implying that $v(\neg\varphi)$ should be an element of $\mathcal{D}_5$. Since **B** imposes no further constraints on $v(\neg\varphi)$ in this case, we get the condition that $\tilde{\neg} a$ should be $\mathcal{D}_5$ in case $a \in \{I, f, f_I\}$. The other parts of the definitions of $\tilde{\neg}$ and $\tilde{\circ}$ are derived similarly (note that in the case of $\circ$ the relevant component is the third).

The five-valued $\mathcal{M}_5^B$ is our basic Nmatrix. In the next section we shall obtain semantics for a lot of extensions of **B** by refining this Nmatrix (where our refinements will be of the special type described in Remark 2.11). However, in many of the systems we discuss (including **B** itself), one needs to include in the truth-value assigned to a formula $\varphi$ only information concerning the truth/falsity of $\varphi$ and $\neg\varphi$. Hence a 3-valued Nmatrix consisting of pairs from $\{0,1\}^2$ (where the pair $\langle 0, 0\rangle$ is rejected because of rule (**t**)) would suffice. The basic 3-valued Nmatrix corresponding to **B** is given in the next Definition.

**Definition 3.2.** The Nmatrix $\mathcal{M}_3^B = \langle \mathcal{V}_3, \mathcal{D}_3, \mathcal{O}_3^B \rangle$ is defined as follows:

- $\mathcal{V}_3 = \{\mathbf{t}, \mathbf{I}, \mathbf{f}\}$ where:
  $\mathbf{t} = \langle 1, 0 \rangle$
  $\mathbf{I} = \langle 1, 1 \rangle$
  $\mathbf{f} = \langle 0, 1 \rangle$
- $\mathcal{D}_3 = \{\mathbf{t}, \mathbf{I}\}$ $(= \{\langle x, y\rangle \in \mathcal{V}_3 \mid x = 1\})$.
- Let this time $\mathcal{D} = \mathcal{D}_3$, $\mathcal{F} = \mathcal{V}_3 - \mathcal{D} = \{f\}$. The operations in $\mathcal{O}_3^B$ corresponding to $\wedge, \vee$ and $\supset$ are defined like in $\mathcal{M}_5^B$. The other two operations are defined as follows:

$$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \{\mathbf{t}\} \\ \mathcal{D} & \text{if } a \in \{\mathbf{I}, \mathbf{f}\} \end{cases}$$

$$\tilde{\circ} a = \begin{cases} \mathcal{V}_3 & \text{if } a \in \{\mathbf{t}, \mathbf{f}\} \\ \mathcal{F} & \text{if } a = \mathbf{I} \end{cases}$$

**Lemma 3.3.** $\vdash_B \subseteq \vdash_{\mathcal{M}_3^B}$

*Proof.* $\mathcal{M}_3^B$ is suitable for $\mathbf{CL}^+$. Hence by Theorem 2.15 it suffices to check that rules ($\mathbf{t}$) and ($\mathbf{b}$) are satisfied by $\vdash_{\mathcal{M}_3^B}$. This is easy. □

**Lemma 3.4.** $\vdash_{\mathcal{M}_3^B} \subseteq \vdash_{\mathcal{M}_5^B}$.

*Proof.* The function $f$ defined by $f(\langle x, y, z\rangle) = \langle x, y\rangle$ is easily seen to be a reduction of $\mathcal{M}_5^B$ to $\mathcal{M}_3^B$. Hence the lemma follows from Theorem 2.10. □

**Lemma 3.5.** $\vdash_{\mathcal{M}_5^B} \subseteq \vdash_B$.

*Proof.* Suppose $\Gamma \nvdash_B \Delta$. We construct a model of $\Gamma$ in $\mathcal{M}_5^B$ which is not a model of any formula in $\Delta$. For this extend $\Gamma$ to a maximal set $\Gamma^*$ of formulas such that $\Gamma^* \nvdash_B \Delta$. $\Gamma^*$ has the following properties:

1. $\varphi \notin \Gamma^*$ iff $\Gamma^*, \varphi \vdash_B \Delta$.
2. $\varphi \vee \psi \in \Gamma^*$ iff either $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$.
3. $\varphi \wedge \psi \in \Gamma^*$ iff both $\varphi \in \Gamma^*$ and $\psi \in \Gamma^*$.
4. $\varphi \supset \psi \in \Gamma^*$ iff either $\varphi \notin \Gamma^*$ or $\psi \in \Gamma^*$.
5. For every sentence $\varphi$ of $\mathcal{L}_{\mathrm{C}}$ either $\varphi \in \Gamma^*$ or $\neg\varphi \in \Gamma^*$.
6. If $\neg\varphi$ and $\varphi$ are both in $\Gamma^*$ then $\circ\varphi \notin \Gamma^*$.

The first property in this list follows from the maximality property of $\Gamma^*$. The last from rule ($\mathbf{b}$). To show the second property, assume first that $\varphi \vee \psi \notin \Gamma^*$. Then $\Gamma^*, \varphi \vee \psi \vdash_B \Delta$. Since also $\varphi \vdash_B \varphi \vee \psi$, we get that $\Gamma^*, \varphi \vdash_B \Delta$, and so $\varphi \notin \Gamma^*$. Similarly, also $\psi \notin \Gamma^*$ in this case. Now assume that neither $\varphi \in \Gamma^*$ nor $\psi \in \Gamma^*$. Then $\Gamma^*, \varphi \vdash_B \Delta$, and $\Gamma^*, \psi \vdash_B \Delta$. Since also $\varphi \vee \psi \vdash_B \varphi, \psi$ (since $\vdash_B$ is an extension of $\vdash^+_{\mathrm{cl}}$), we get that $\Gamma^*, \varphi \vee \psi \vdash_B \Delta$, and so $\varphi \vee \psi \notin \Gamma^*$.

The proofs of the other parts are similar (for the fifth property we use the fact that $\vdash_B$ satisfies rule ($\mathbf{t}$)).

Define now a valuation $v$ by $v(\varphi) = \langle x(\varphi), y(\varphi), z(\varphi)\rangle$, where:

$$x(\varphi) = \begin{cases} 1 & \varphi \in \Gamma^* \\ 0 & \varphi \notin \Gamma^* \end{cases} \qquad y(\varphi) = \begin{cases} 1 & \neg\varphi \in \Gamma^* \\ 0 & \neg\varphi \notin \Gamma^* \end{cases} \qquad z(\varphi) = \begin{cases} 1 & \circ\varphi \in \Gamma^* \\ 0 & \circ\varphi \notin \Gamma^* \end{cases}$$

It is easy to check that the above properties of $\Gamma^*$ imply that $v$ is a legal valuation in $\mathcal{M}_5^B$. Obviously, $v$ is a model of $\Gamma$ which is not a model of any formula in $\Delta$. □

**Theorem 3.6.** *Both $\mathcal{M}_5^B$ and $\mathcal{M}_3^B$ are characteristic Nmatrices for* $\mathbf{B}$.

*Proof.* This is immediate from the last three lemmas. □

**Corollary 3.7.** $\mathbf{B}$ *is decidable.*

*Proof.* This follows from Theorems 3.6 and 2.8. □

## 4. Semantics for the Extensions of B Induced by RULES

One of the main virtues of our semantics is that for many syntactic conditions concerning $\neg$ and $\circ$, it is easy to compute corresponding semantic conditions on *simple* refinements (Remark 2.11) of $\mathcal{M}_5^B$. In the next definition we list the semantic conditions induced by the rules in *RULES* (Definition 2.21).

**Definition 4.1.**

1. The refining conditions induced by the conditions in *RULES* are:
   C(**c**): If $x \in \{f, f_I\}$ then $\tilde{\neg}x \subseteq \{t, t_I\}$.
   C(**e**): $\tilde{\neg}I = \{I\}$
   C(**k1**): $f_I$ should be deleted.
   C(**k2**): $t_I$ should be deleted.
   C(**a**$_\neg$): $\{t, f\}$ is closed under $\tilde{\neg}$ (implying $\tilde{\neg}t = \{f\}$, $\tilde{\neg}f = \{t\}$).
   C(**a**$_\sharp$): $\{t, f\}$ is closed under $\tilde{\sharp}$.
   C(**i1**): $f_I$ should be deleted, and $\tilde{\circ}(f) \subseteq \{t, t_I\}$
   C(**i2**): $t_I$ should be deleted, and $\tilde{\circ}(t) = \{t\}$
2. For $S \subseteq RULES$, let $C(S) = \{\mathrm{C}r \mid r \in S\}$

Here are some examples of how these conditions have been derived:

**Notation.** Let $P_1(\langle a, b, c\rangle) = a$, $P_2(\langle a, b, c\rangle) = b$, $P_3(\langle a, b, c\rangle) = c$.

C(**c**): A refutation of this rule is a valuation $v$ in $\mathcal{M}_5^B$ such that $v(\varphi) \notin \mathcal{D}_5$ (i.e. $P_1(v(\varphi)) = 0$), but $v(\neg\neg\varphi) \in \mathcal{D}_5$ (i.e. $P_2(v(\neg\varphi)) = 1$). This will be impossible iff for every $x \in \mathcal{V}_5$ such that $P_1(x) = 0$ (i.e. for every $x \in \{f, f_I\}$), it is the case that if $y \in \tilde{\neg}x$ then $P_2(y) = 0$ (i.e. $y \in \{t, t_I\}$).

C(**e**): A refutation of this rule is a valuation $v$ in $\mathcal{M}_5^B$ such that $v(\varphi) \in \mathcal{D}_5$ (i.e. $P_1(v(\varphi)) = 1$), but $v(\neg\neg\varphi) \notin \mathcal{D}_5$ (i.e. $P_2(v(\neg\varphi)) = 0$). This will be impossible iff for every $x \in \mathcal{V}_5$ such that $P_1(x) = 1$ (i.e. for every $x$ in $\{t, t_I, I\}$), if $y \in \tilde{\neg}x$ then also $P_2(y) = 1$. For $x \in \{t, t_I\}$ this is already true in $\mathcal{M}_5^B$. For $x = I$ the only element $y$ in $\tilde{\neg}_B x$ which satisfies this condition is $y = I$ (where $\tilde{\neg}_B$ is the interpretation of $\neg$ in $\mathcal{M}_5^B$).

C(**k1**): A refutation of this rule is a valuation $v$ in $\mathcal{M}_5^B$ for which both $v(\varphi)$ and $v(\circ\varphi)$ are not in $\mathcal{D}_5$. This will be impossible iff $f_I$ is not available.

C(**a**$_\sharp$): A refutation of this rule is a valuation $v$ in $\mathcal{M}_5^B$ s.t. $P_3(v(\varphi)) = 1$, $P_3(v(\psi)) = 1$, and $P_3(v(\varphi\sharp\psi)) = 0$ (i.e. $v(\varphi) \in \{t, f\}$, $v(\psi) \in \{t, f\}$, but $v(\varphi\sharp\psi) \notin \{t, f\}$). This will be impossible iff $\{t, f\}$ is closed under $\sharp$.

C(**i2**): A refutation of this rule is a valuation $v$ in $\mathcal{M}_5^B$ s.t. $v(\neg\varphi) \notin \mathcal{D}_5$ (i.e. $P_2(v(\varphi)) = 0$), but $v(\neg\circ\varphi) \in \mathcal{D}_5$ (i.e. $P_2(v(\circ\varphi)) = 1$). This will be impossible iff for every $x \in \mathcal{V}_5$ such that $P_2(x) = 0$, also $P_2(\tilde{\circ}x) = 0$. In other words: iff for every $x \in \{t, t_I\}$, $\tilde{\circ}x \subseteq \{t, t_I\}$. For $x = t_I$ this is incoherent with the value of $\tilde{\circ}(t_I)$ in $\mathcal{M}_5^B$. Hence $t_I$ should be deleted, and so necessarily $\tilde{\circ}(t) = \{t\}$.

**Definition 4.2.** For $S \subseteq RULES$, let $\mathcal{M}_S$ be the weakest simple refinement (Remark 2.11) of $\mathcal{M}_5^B$ in which the conditions in $C(S)$ are all satisfied. In other words: $\mathcal{M}_S = \langle \mathcal{V}_S, \mathcal{D}_S, \mathcal{O}_S \rangle$, where $\mathcal{V}_S$ is the set of values from $\mathcal{V}_5$ which are not deleted by any condition in $S$, $\mathcal{D}_S = \mathcal{D}_5 \cap \mathcal{V}_S$, and for any connective $\diamond \in \mathcal{O}$ and any $\vec{x} \in \mathcal{V}_S^n$ (where $n$ is the arity of $\diamond$), the interpretation in $\mathcal{O}_S$ of $\diamond$ assigns to $\vec{x}$ the set $\widetilde{\diamond}_{\mathcal{M}_S}(\vec{x})$ of all the values in $\widetilde{\diamond}_B(\vec{x})$ which are not forbidden by any condition in $C(S)$ (where $\widetilde{\diamond}_B$ is the interpretation of $\diamond$ in $\mathcal{M}_5^B$).

**An Example.** Let $S = \{(\mathbf{i1}), (\mathbf{a}_\neg)\}$. Then $\mathcal{M}_S = \langle \mathcal{V}_S, \mathcal{D}_S, \mathcal{O}_S \rangle$, where:

- $\mathcal{V}_S = \{t, t_I, I, f\}$
- $\mathcal{D}_S = \{t, I, t_I\}$
- $\widetilde{\vee}, \widetilde{\wedge}$ and $\widetilde{\supset}$ are defined like in the case of $\mathcal{M}_5^B$ (but now $\mathcal{F} = \{f\}$).
- $\widetilde{\neg} t = \widetilde{\neg} t_I = \{f\}$ $\quad \widetilde{\neg} I = \mathcal{D}_S$ $\quad \widetilde{\neg} f = \{t\}$
- $\widetilde{\circ} t = \mathcal{D}_S$ $\quad \widetilde{\circ} t_I = \widetilde{\circ} I = \{f\}$ $\quad \widetilde{\circ} f = \{t, t_I\}$

*Remark* 4.3. It is not difficult to see that for all $S \subseteq RULES$, $\{t, f, I\} \subseteq \mathcal{V}_S$, $\{t, I\} \subseteq \mathcal{D}_S$, and $\widetilde{\diamond}_{\mathcal{M}_S}(\vec{x})$ is never empty (in fact, $\widetilde{\diamond}_{\mathcal{M}_S}(\vec{x}) \cap \{t, f, I\}$ is never empty). Hence $\mathcal{M}_S$ is a well-defined Nmatrix.

*Remark* 4.4. It can easily be checked that in any simple refinement of $\mathcal{M}_5^B$ which satisfies C($\mathbf{a}_\neg$), $\widetilde{\neg}$ behaves on $\{t, f\}$ like classical negation (i.e. $\widetilde{\neg} t = \{f\}, \widetilde{\neg} f = \{t\}$). Similarly, if $\sharp \in \{\vee, \wedge, \supset\}$ then in simple refinements of $\mathcal{M}_5^B$ which satisfy C($\mathbf{a}_\sharp$), $\widetilde{\sharp}$ behaves on $\{t, f\}$ like the classical $\sharp$.

**Theorem 4.5.** *For all $S \subseteq RULES$, $\mathcal{M}_S$ is a characteristic Nmatrix for* $\mathbf{B}[S]$.

*Proof.* It is easy to verify, that for any $r \in RULES$, the satisfaction of C($r$) in some simple refinement of $\mathcal{M}_5^B$ guarantees the validity of $r$ in that refinement. This entails the soundness of $\mathbf{B}[S]$ with respect to $\mathcal{M}_S$. For completeness we repeat the construction done in the proof of Lemma 3.5. It is not difficult to show that the rules of $S$ force the resulting valuation to be a legal valuation in $\mathcal{M}_S$. We do here the case where $S = \{(\mathbf{i1}), (\mathbf{a}_\neg)\}$ as an example. So suppose that $\Gamma \not\vdash_{\mathbf{B}[S]} \Delta$. Construct the set $\Gamma^*$ and the valuation $v$ like in the proof of Lemma 3.5, using $\mathbf{B}[S]$ instead of $\mathbf{B}$. This $v$ is legal for $\mathcal{M}_5^B$, and it is a model of $\Gamma$ which is not a model of any formula in $\Delta$. Now the presence of ($\mathbf{i1}$) implies that $v(\varphi) \neq f_I$ for every $\varphi$ (because there can be no formula $\varphi$ such that both $\varphi \notin \Gamma^*$ and $\circ\varphi \notin \Gamma^*$. Indeed: if $\varphi \notin \Gamma^*$ then because of ($\mathbf{i1}$) $\neg\circ\varphi \notin \Gamma^*$, and so $\circ\varphi \in \Gamma^*$). Hence $v$ is actually a valuation in $\mathcal{V}_S$. It remains to show that it is legal for $\mathcal{M}_S$. Since $v$ is legal for $\mathcal{M}_5^B$, it suffices to show that it respects the conditions imposed by ($\mathbf{i1}$) and ($\mathbf{a}_\neg$):

**C(i1):** Since $f_I$ is not used by $v$, respecting C($\mathbf{i1}$) amounts in the present case to $v(\circ\varphi)$ being in $\{t, t_I\}$ in case $v(\varphi) = f$. But here $v(\varphi) = f$ iff $\varphi \notin \Gamma^*$, and the latter implies (because of ($\mathbf{i1}$)) that $\neg\circ\varphi \notin \Gamma^*$, which means (by definition of $v$) that indeed $v(\circ\varphi) \in \{t, t_I\}$.

**C($\mathbf{a}_\neg$):** Again since $f_I$ is not used by $v$, respecting C($\mathbf{a}_\neg$) amounts in the present case to $v(\neg\varphi) = t$ in case $v(\varphi) = f$. But $v(\varphi) = f$ iff $\varphi \notin \Gamma^*$, $\neg\varphi \in \Gamma^*$, and $\circ\varphi \in \Gamma^*$. Because of ($\mathbf{a}_\neg$) the latter implies that $\circ\neg\varphi \in \Gamma^*$. Since also $\neg\varphi \in \Gamma^*$, necessarily $v(\neg\varphi) = t$. □

## 5. Applications

### 5.1. Decidability

A first important corollary of our semantics is the following:

**Corollary 5.1.** $\mathbf{B}[S]$ *is decidable for any* $S \subseteq RULES$.

*Proof.* Immediate from Theorems 4.5 and 2.8. □

### 5.2. Dependencies between the Conditions

Not all the 1024 systems of the form $\mathbf{B}[S]$ (where $S \subseteq RULES$) are different from each other. Using Theorem 4.5 it is a mechanical matter to check the relations among them, finding what rules in $RULES$ follow from what subsets of the other rules in $RULES$. The next theorem sums up all existing dependencies:

**Theorem 5.2.** *The following is an exhaustive list of all the dependencies among the rules in RULES:*

- $(\mathbf{k1})$ *follows from* $(\mathbf{i1})$.
- $(\mathbf{k2})$ *follows from* $(\mathbf{i2})$.
- $(\mathbf{c})$ *follows from* $\{(\mathbf{a}_\neg), (\mathbf{k1})\}$ *(and from* $\{(\mathbf{a}_\neg), (\mathbf{i1})\}$*).*
- $(\mathbf{a}_\neg)$ *follows from* $\{(\mathbf{c}), (\mathbf{k1}), (\mathbf{k2})\}$ *(and of course also from* $\{(\mathbf{c}), (\mathbf{k1}), (\mathbf{i2})\}$, $\{(\mathbf{c}), (\mathbf{i1}), (\mathbf{k2})\}$*, and* $\{(\mathbf{c}), (\mathbf{i1}), (\mathbf{i2})\}$*).*

*Proof.* The first two items on this list trivially follow from the corresponding semantic conditions. For the third, note that without $f_I$ (i.e.: in the presence of $(\mathbf{k1})$ or $(\mathbf{i1})$), condition $C(\mathbf{c})$ reduces to $\tilde{\neg} f \subseteq \{t, t_I\}$ and this condition immediately follows from $C(\mathbf{a}_\neg)$. Finally, the forth clause is immediate from the fact that if $t$, $f$, and $I$ are the only available truth-values, then $\mathcal{F} = \{f\}$, and both conditions $C(\mathbf{c})$ and $C(\mathbf{a}_\neg)$ reduce to $\tilde{\neg} f = \{t\}$.

A not too difficult examination of the corresponding 10 conditions given in Definition 4.1 (together with the Definition of $\mathcal{M}_5^B$) reveals that the above list is indeed exhaustive. □

**Corollary 5.3.** *Rules* $(\mathbf{c})$ *and* $(\mathbf{a}_\neg)$ *are equivalent in the presence of rules* $(\mathbf{k1})$ *and* $(\mathbf{k2})$*. In particular, they are equivalent in the system* **Bi***, obtained from* **B** *by adding the following schema from* [11, 12]*:*

**(i):** $\neg \circ \varphi \vdash \varphi \wedge \neg \varphi$

### 5.3. Cases Where 3-valued Nmatrices Suffice

In Section 3 We have seen that for the basic system **B** a three-valued reduction of $\mathcal{M}_5^B$ (in which the truth-values include information only on the truth/falsity of a sentence and its negation) suffices. The argument remains almost the same if either $(\mathbf{c})$, $(\mathbf{e})$ or both are added to **B**, since these rules do not involve $\circ$. In the corresponding refinements of $\mathcal{M}_3^B$ we should have $\tilde{\neg} f = \{t\}$ in case $(\mathbf{c})$ is added, and $\tilde{\neg} I = \{I\}$ in case $(\mathbf{e})$ is added.

Another obvious case in which a logic $\mathbf{B}[S]$ ($S \subseteq RULES$) has a characteristic 3-valued Nmatrix is when both $(\mathbf{k1})$ and $(\mathbf{k2})$ are derivable in it (i.e. if either $(\mathbf{k1})$

or (**i1**) is in $S$, and also either (**k2**) or (**i2**) is in $S$). In this case Theorem 4.5 directly provides such an Nmatrix.

**Conjecture.** Except for the cases we have just described, no other system $\mathbf{B}[S]$ ($S \subseteq RULES$) has any characteristic 3-valued Nmatrix.

What we can prove for *all* the systems considered here is the following:

**Theorem 5.4.** $\mathbf{B}[S]$ *($S \subseteq RULES$) does not have a characteristic 2-valued Nmatrix.*

*Proof.* Suppose $\mathcal{M}$ is a 2-valued Nmatrix for which $\mathbf{B}[S]$ is sound and complete. We may assume that the two truth-values of $\mathcal{M}$ are 1 and 0, where 1 is designated and 0 is not. Since condition (**t**) is valid in $\mathcal{M}$, necessarily $\tilde{\neg}0 = \{1\}$. Hence it suffices to consider the following 3 cases:

- Suppose $\tilde{\neg}1 = \{0\}$. Then $\neg\varphi, \varphi \vdash_{\mathcal{M}}$ for all $\varphi$. Since $\mathbf{B}[S]$ is paraconsistent (because by assigning $v(p) = v(\neg p) = I$ we get a model of $\{p, \neg p\}$ in $\mathcal{M}_S$), we get a contradiction.
- Suppose $\tilde{\circ}1 = \{0\}$. Then $\circ\varphi, \varphi \vdash_{\mathcal{M}}$ for all $\varphi$. However, assigning $t$ to both $p$ and $\circ p$ is legal in $\mathcal{M}_S$ (for every $S \subseteq RULES$). Hence $\circ\varphi, \varphi \not\vdash_{\mathbf{B}[S]}$, and we get a contradiction.
- Suppose that 1 is in both $\tilde{\neg}1$ and $\tilde{\circ}1$. Then assigning 1 to all the sentences in $\{p, \neg p, \circ p\}$ is legal in $\mathcal{M}$, contradicting the validity of (**b**) in $\mathcal{M}$.

We got a contradiction in all possible cases. Hence no such Nmatrix exists. □

## 6. Other Plausible Extensions of the Basic System

In addition to the rules considered so far (that were basically taken from [11, 12]), it is of course possible to consider many other rules that might seem plausible. We consider now two natural groups of rules that may also be added to the basic system **B**, and are very easy to handle in our framework.

### 6.1. Rules for Combinations of Negation with the Classical Connectives

(**c**) and (**e**) are just two of the standard classically valid rules concerning negation which are derived from the classical equivalences of $\neg\neg\varphi$ with $\varphi$, $\neg(\varphi \wedge \psi)$ with $\neg\varphi \vee \neg\psi$, $\neg(\varphi \vee \psi)$ with $\neg\varphi \wedge \neg\psi$, and $\neg(\varphi \supset \psi)$ with $\varphi \wedge \neg\psi$. By splitting the last 3 equivalences into simple rules (see Remark 2.13) we get the following list:

**Definition 6.1.** Let $DM$ be the set consisting of the following 9 rules:

**($\neg\wedge 1$):** $\neg\varphi \vdash \neg(\varphi \wedge \psi)$
**($\neg\wedge 2$):** $\neg\psi \vdash \neg(\varphi \wedge \psi)$
**($\neg\wedge 3$):** $\neg(\varphi \wedge \psi) \vdash \neg\varphi, \neg\psi$
**($\neg\vee 1$):** $\neg(\varphi \vee \psi) \vdash \neg\varphi$
**($\neg\vee 2$):** $\neg(\varphi \vee \psi) \vdash \neg\psi$
**($\neg\vee 3$):** $\neg\varphi, \neg\psi \vdash \neg(\varphi \vee \psi)$
**($\neg\supset 1$):** $\neg(\varphi \supset \psi) \vdash \varphi$

**$(\neg\supset 2)$:** $\neg(\varphi \supset \psi) \vdash \neg\psi$
**$(\neg\supset 3)$:** $\varphi, \neg\psi \vdash \neg(\varphi \supset \psi)$

In [1] we have shown how to modularly provide 3-valued non-deterministic semantics for these rules, where the basic logic is $CLuN$ (which is the logic in $\mathcal{L}_{cl}$ obtained from **B** by deleting the schema (**b**)). It is straightforward to adapt those results to the present context, with **B** as the basic logic. All we need to do is to find for the rules in $DM$ equivalent semantic conditions on refinements of $\mathcal{M}_5^B$, using considerations of the type applied for the rules in $RULES$. For this, one should only note that for the rules in $DM$ only the first two components of our truth-values are relevant. We present here as an example the derived semantic conditions which are equivalent to the rules corresponding to the equivalence between $\neg(\varphi \wedge \psi)$ and $\neg\varphi \vee \neg\psi$:

**C$(\neg\wedge 1)$:** If $x \in \mathcal{D}$ then $I \widetilde{\wedge} x = \{I\}$.
**C$(\neg\wedge 2)$:** If $x \in \mathcal{D}$ then $x \widetilde{\wedge} I = \{I\}$.
**C$(\neg\wedge 3)$:** If $x \in \{t, t_I\}$ and $y \in \{t, t_I\}$ then $x \wedge y \subseteq \{t, t_I\}$.

Like in Definition 4.2, We can now define $\mathcal{M}_S$ for every $S \subseteq RULES \cup DM$. It is then easy to prove the following generalization of Theorem 4.5:

**Theorem 6.2.** *For all $S \subseteq RULES \cup DM$, $\mathcal{M}_S$ is a characteristic Nmatrix for* $\mathbf{B}[S]$.

**Corollary 6.3.**

1. $(\neg\wedge 3)$ *is derivable in* $\mathbf{B}[\{(\mathbf{k2}), (\mathbf{a}_\wedge)\}]$.
2. $(\neg\wedge 3)$ *and* $(\mathbf{a}_\wedge)$ *are equivalent in any extension of* $\mathbf{B}[\{(\mathbf{k1}), (\mathbf{k2})\}]$

*Proof.* In the presence of (**k2**) the truth-value $t_I$ is not available. Hence in this case C$(\neg\wedge 3)$ reduces to $t \widetilde{\wedge} t = \{t\}$. This last condition follows from C$(\mathbf{a}_\wedge)$, implying the first part of the corollary. Now in the presence of (**k1**) only $f$ is not in the set $\mathcal{D}$ of designated values, and so in this case C$(\mathbf{a}_\wedge)$ too reduces to $t \widetilde{\wedge} t = \{t\}$. Hence the equivalence in the second part. ☐

*Remark* 6.4. One of the principles behind the construction of da Costa's $C$-systems ([14, 11]) has been that the consistent formulas should be closed under the classical connectives. This has been the reason for including the schemes of the form $(\mathbf{a}_\sharp)$ in the systems. From Corollaries 5.3 and 6.3 it follows that under weak conditions (which are satisfied, e.g., in the presence of axiom (**i**)), the axioms expressing the applications of this principle to $\neg$ and $\wedge$ can be replaced by well-known classical tautologies in which $\circ$ is not mentioned.[8]

*Remark* 6.5. It is interesting to note that the semantics we get for the system $\mathbf{B}[RULES \cup DM]$ itself (or just for the system $\mathbf{B}[DM \cup \{(\mathbf{c}), (\mathbf{e}), (\mathbf{i1}), (\mathbf{i2})\}]$) is

[8] This fact might give some justification why also (**c**) (and not only (**t**)) has been included in the original basic system $C_1$ of da Costa ([14]).

a characteristic 3-valued (ordinary, deterministic) *matrix*. This is the famous 3-valued matrix characteristic for the paraconsistent logic called $LFI1$ in [11, 12], to which $\mathbf{B}[RULES \cup DM]$ is equivalent. [9]

**6.2. Rules Concerning $\circ$**

Finally we turn to rules involving the connective $\circ$ but not $\neg$. We briefly consider two types of rules of this sort.

**Strengthening the closure rules:** The rules of the form $(\mathbf{a}_\sharp)$ express the assumption that if $\varphi$ and $\psi$ are consistent, then so is $\varphi\sharp\psi$. Now it is plausible to consider stronger assumptions. One alternative that is investigated in [11, 12] is that $\varphi\sharp\psi$ should be consistent if either $\varphi$ or $\psi$ is consistent. There is no problem to handle this stronger assumption within our framework by finding corresponding semantic conditions. First, the assumption for $\sharp$ split into the following two rules:

- $(\mathbf{o}^1_\sharp)$ $\circ\,\varphi \vdash \circ(\varphi\sharp\psi)$
- $(\mathbf{o}^2_\sharp)$ $\circ\,\psi \vdash \circ(\varphi\sharp\psi)$

Now the first of them, for example, translates to the condition: if $P_3(x) = 1$ then $P_3(x\widetilde{\sharp}y) = 1$. In other words: If $x \in \{t, f\}$ then $x\widetilde{\sharp}y \in \{t, f\}$. What this implies in specific cases depends on the semantics of $\sharp$. Thus for $\wedge$ we get:

C$(\mathbf{o}^1_\wedge)$**:** $f\widetilde{\wedge}y = \{f\}$ for every $y$, while $t\widetilde{\wedge}y = \{t\}$ for $y \neq t$.

Note that in the presence of **k1**+**k2**, C$(\mathbf{o}^1_\wedge)$ reduces to $t\widetilde{\wedge}y = \{t\}$ for $y \in \{t, I\}$.

It is important to note that by using C$(\mathbf{o}^1_\wedge)$, the rule $(\mathbf{o}^1_\wedge)$ can be added to $RULES$ without essentially affecting the validity of Theorem 4.5. However a new situation arises if we consider $(\mathbf{o}^1_\wedge)$ together with $(\neg\wedge 2)$. C$(\neg\wedge 2)$ implies that $t\widetilde{\wedge}I = \{I\}$, while C$(\mathbf{o}^1_\wedge)$ implies that $t\widetilde{\wedge}I = \{t\}$. This means that we cannot use both $t$ and $I$ in constructing models for theories based on the logic $\mathbf{B}' = \mathbf{B}[\{(\mathbf{o}^1_\wedge), (\neg\wedge 2)\}]$. However, the combination of the corresponding conditions does not decisively rule out any of these two truth-values. Hence the framework we have developed here does not provide a unique characteristic Nmatrix for $\mathbf{B}'$. However, it can be shown that it does provide *two* finite Nmatrices $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\vdash_{\mathbf{B}'} = \vdash_{\mathcal{M}_1} \cap \vdash_{\mathcal{M}_2}$.

**Some common modal rules:** We end with considering the effects of the counterparts for $\circ$ of the three modal axioms of the modal logic $S4$:

(**K**)**:** $\circ\varphi, \circ(\varphi \supset \psi) \vdash \circ\psi$
(**4**)**:** $\circ\varphi \vdash \circ\circ\varphi$
(**T**)**:** $\circ\varphi \vdash \varphi$

In the context of extensions of $\mathbf{B}$ (i.e. refinements of $\mathcal{M}^B_5$) the corresponding semantic conditions can easily be found to be:

C(**K**)**:** If $x \in \{t, f\}$ and $y \in \{I, t_I, f_I\}$ then $x \supset y \subseteq \{I, t_I, f_I\}$.

[9] This logic was originally introduced in [19]. Later it was reintroduced (together with its 3-valued deterministic semantics) in [15, 16], and was called there $J_3$ (see also [17]).

C(**4**): If $x \in \{t, f\}$ then $\widetilde{\circ}x = \{t\}$
C(**T**): The truth-value $f$ should be deleted.

Note that C(**T**) is in direct conflict with C(**k1**), since together they leave no nondesignated element, implying that any formulas is a theorem of the resulting logic (this can of course be verified directly, using a cut). A more interesting observation is that the combined effect of C(**k1**), C(**k2**), and C(**4**) is identical to the combined effect of C(**i1**) and C(**i2**). Hence the axioms (**k1**), (**k2**), and (**4**) are together equivalent to the axiom (**i**) (which is standard in $C$-systems — see [11]).

## 7. Conclusions and Further Research

We have presented an extensive study of the use of Nmatrices in deriving useful semantic for thousands of extensions of one particular basic system: **B**. It should be clear from this case-study that the method has a very large range of applications (far beyond the framework of **B**). However, it is still necessary to formulate it in exact, general terms, and to determine its scope. Another important task is to develop extensions of the framework for cases in which the method used in this paper is too weak. Two such cases (and related questions and tasks) are:

1. The primary constraint on rules to which our method applies seems to be *purity*. A good example of a context in which this constraint is violated, is provided by normal modal logics. As we have seen in the previous section, the usual *axioms* used in these logics pose no real problem. However, the necessitation rule, as it is used in modal logics, is *impure*: if $\vdash$ is supposed to be an extension of the classical consequence relation, then the necessitation rule cannot be translated into $\varphi \vdash \Box\varphi$. Indeed, in classical logic we have that $\Box\varphi \vdash \varphi \supset \Box\varphi$, and that $\vdash \varphi, \varphi \supset \Box\varphi$. Together with $\varphi \vdash \Box\varphi$ these facts entail $\vdash \varphi \supset \Box\varphi$ (using cuts). However, $\varphi \supset \Box\varphi$ is not valid in any interesting modal logic. It seems therefore that extra machinery, like the use of non-deterministic Kripke structures, should be added in order to handle rules of this sort. Steps in this direction have been taken in [1, 2], where hybrid semantics, employing both Nmatrices and Kripke structures, has been provided for many extensions of positive intuitionistic logic [10] (which is another logic which employs impure rules).
2. Two common features of all the rules considered in this paper are that each of them is concerned with at most two different connectives, and also the nesting depth of each formula used in their schematic description is at most two. An example of a rule which lacks both features is rule (**l**) from [11, 12]:

   (**l**): $\neg(\varphi \wedge \neg\varphi) \vdash \circ\varphi$

   Now in [3] it is shown that $\mathbf{B}[\{(\mathbf{c}), (\mathbf{l})\}]$ (which is called there $Cl$) has no finite characteristic Nmatrix. Hence at least one of the two features we have

[10] One of those extensions is da Costa's basic paraconsistent system $C_\omega$ from [14].

mentioned should be essential. Which one? And what can be done in its absence? Concerning the last question, it should be noted that $Cl$ has also been shown in [3] to have an *infinite* characteristic Nmatrix, which is simple enough to yield a decision procedure. Can this fact be generalized?

Another natural (and important) line for further research is to use the semantic ideas presented here for systematically producing tableaux-style proof-systems for the various logics dealt with in this paper. Now general systems of this type have in fact been developed in [5] for every logic which has finite characteristic Nmatrix. However, the central idea of those systems is to use signed formulas, where the signs are (essentially) the truth-values of the characteristic Nmatrix (and so the number of signs equals the number of the truth-values of that Nmatrix). Here it might be more effective to use 6 signs rather than 5, according to the two possible values of the three *components* of each of the five truth-values (or four signs in the cases where 3-valued versions suffice).

Finally, an obvious crucial line of further research is to extend the results and methods of this paper to first-order languages. First steps in this direction have been made in [8].

## References

[1] A. Avron, *Non-deterministic Semantics for Families of Paraconsistent Logics*, To appear in **Paraconsistency with no Frontiers** (J.-Y. Beziau and W. Carnielli, eds.).

[2] A. Avron, *A Non-deterministic View on Nonclassical Negations*, Studia Logica 80, 159-194 (2005).

[3] A. Avron, *Non-deterministic Semantics for Logics with a Consistency Operators*, Forthcoming in the International Journal of Approximate Reasoning.

[4] A. Avron, *Logical Non-determinism as a Tool for Logical Modularity: An Introduction*, In **We Will Show Them: Essays in Honor of Dov Gabbay**, Vol 1 (S. Artemov, H. Barringer, A. S. d'Avila Garcez, L. C. Lamb, and J. Woods, eds.), 105-124, College Publications, 2005.

[5] A. Avron, and B. Konikowska, *Multi-valued Calculi for Logics Based on Non-determinism*, Logic Journal of the IGPL 13, 365-387, (2005).

[6] A. Avron, and I. Lev, *Non-deterministic Multiple-valued Structures*, Journal of Logic and Computation 15, 241-261 (2005).

[7] A. Avron and I. Lev, *Canonical Propositional Gentzen-Type Systems*, in **Proceedings of the 1st International Joint Conference on Automated Reasoning (IJCAR 2001)** (R. Goré, A Leitsch, T. Nipkow, Eds), LNAI 2083, 529-544, Springer Verlag, 2001.

[8] A. Avron, and A. Zamanski, *Quantification in Non-deterministic Multi-valued Structures*, In **Proceedings of the 35th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2005)**, 296-301, IEEE Computer Society Press, 2005.

[9] D. Batens, K. De Clercq, and N. Kurtonina, *Embedding and Interpolation for Some Paralogics. The Propositional Case*, Reports on Mathematical Logic 33 (1999), 29-44.

[10] W. A. Carnielli, *Possible-translations Semantics for Paraconsistent Logics*, in **Frontiers of Paraconsistent Logic** (D. Batens, C. Mortensen, G. Priest, and J. P. Van Bendegem, eds.), 149-163. King's College Publications, Research Studies Press, Baldock, UK, 2000.

[11] W. A. Carnielli and J. Marcos, *A Taxonomy of C-systems*, in **Paraconsistency — the logical way to the inconsistent** (W. A. Carnielli, M. E. Coniglio, I. L. M. D'ottaviano, eds.), Lecture Notes in Pure and Applied Mathematics, 1-94, Marcel Dekker, 2002.

[12] W. A. Carnielli, M. E. Coniglio, and J. Marcos, *Logics of Formal Inconsistency*, to appear in **Handbook of Philosophical Logic** (D. Gabbay and F. Guenthner, eds).

[13] J. M. Crawford and D. W. Etherington, *A Non-deterministic Semantics for Tractable Inference*, in **Proc. of the 15th International Conference on Artificial Intelligence and the 10th Conference on Innovative Applications of Artificial Intelligence**, 286-291, MIT Press, Cambridge, 1998.

[14] N. C. A. da Costa, *On the theory of inconsistent formal systems*, Notre Dame Journal of Formal Logic 15 (1974), 497–510.

[15] I. L. M. D'Ottaviano and N. C. A. da Costa, *Sur un problème de Jaśkowski*, Comptes Rendus de l'Academie de Sciences de Paris (A-B) 270 (1970), 1349–1353.

[16] I. L. M. D'Ottaviano, *The completeness and compactness of a three-valued first-order logic*, Revista Colombiana de Matematicas, vol. XIX (1985), 31–42.

[17] R. L. Epstein, **The semantic foundation of logic**, vol. I: Propositional Logics, ch. IX, Kluwer Academic Publisher, 1990.

[18] J.Łoś and R. Suszko, *Remarks on Sentential Logics*, Indagationes Mathematicae 20 (1958), 177–183.

[19] K. Schütte, **Beweistheorie**, Springer, Berlin, 1960.



Arnon Avron
School of Computer Science
Tel-Aviv University
Ramat Aviv 69978
Israel
e-mail: `aa@math.tau.ac.il`