Multiplicative Conjunction and an Algebraic Meaning of Contraction and Weakening

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Abstract

We show that the elimination rule for the multiplicative (or intensional) conjunction $\wedge$ is admissible in many important multiplicative substructural logics. These include LL_m (the multiplicative fragment of Linear Logic) and RMI_m (the system obtained from LL_m by adding the contraction axiom and its converse, the mingle axiom.) An exception is R_m (the intensional fragment of the relevance logic R, which is LL_m together with the contraction axiom). Let SLL_m and SR_m be, respectively, the systems which are obtained from LL_m and R_m by adding this rule as a new rule of inference. The set of theorems of SR_m is a proper extension of that of R_m, but a proper subset of the set of theorems of RMI_m. Hence it still has the variable-sharing property. SR_m has also the interesting property that classical logic has a strong translation into it. We next introduce general algebraic structures, called strong multiplicative structures, and prove strong soundness and completeness of SLL_m relative to them. We show that in the framework of these structures, the addition of the weakening axiom to SLL_m corresponds to the condition that there will be exactly one designated element, while the addition of the contraction axiom corresponds to the condition that there will be exactly one nondesignated element (in the first case we get the system BCK_m, in the second - the system SR_m). Various other systems in which multiplicative conjunction functions as a true conjunction are studied, together with their algebraic counterparts.

1 Introduction

It is well known that multiplicative linear logic (LL_m) is obtained from classical logic by deleting the structural rules of contraction and weakening (and using multiplicative versions of the rules for the connectives[1]). By adding contraction to LL_m we get R_m (or $R_{\rightarrow}$) – the

[1] This roughly means that the applicability of rules does not depend on the side formulae. A more exact definition is given in the next section.

multiplicative fragment of the standard relevance logic R. By adding weakening to LL_m we get BCK_m – the multiplicative fragment of BCK (or affine) logic.

The main purpose of this paper is to show that each of these two rules of inference has a clear semantic interpretation. Weakening corresponds to the assumption that there is exactly one designated truth-value (and so everything which is not "strictly true" is "false"). Contraction corresponds to the opposite assumption – that there is exactly one nondesignated truth-value (and so everything which is not "strictly false" is "true").

For reasons that are explained at the beginning of section 3, it seems more appropriate to make the comparison between the effects of weakening and contraction not in the framework of LL_m itself, but relative to a stronger version SLL_m. SLL_m is obtained from LL_m by adding to it the $\otimes$-elimination rule: from $A \otimes B$ infer A.[2] Luckily, this does not change the set of valid formulae of LL_m, since the new rule is ***admissible*** there, but it does change the ***consequence relation***[3].

Adding $\otimes$-elimination as a rule of inference has the obvious aspect of turning the multiplicative (or intensional) "conjunction" into a ***real*** conjunction. With it all the standard classical/intuitionistic natural deduction rules for conjunction become valid for $\otimes$. Moreover: it makes the situation symmetrical with respect to $\otimes$ and the "extensional" (or "additive") conjunction &. In LL and R the elimination rules for & are valid as ***entailments***: $A \& B \to A$, $A \& B \to B$. The introduction rule, in contrast, is valid only as an ***inference*** (the "adjunction" rule). In SLL_m (and SR_m – the extension of R_m by $\otimes$-elimination) the opposite is true for $\otimes$. $A \to B \to A \otimes B$ is a valid entailment, but deducing A (and B) from $A \otimes B$ is valid only as an inference.

Exploring the possibility of using $\otimes$, rather then &, as the counterpart of classical conjunction is the second goal of this paper (this is mainly relevant to relevance logics, since in BCK_m all the natural deduction rules for $\otimes$ are already valid as entailments). The main result here is that there is a ***strong*** translation (i.e., one that preserves the consequence relation) of classical logic into SR_m, in which classical conjunction translates into $\otimes$. In R, in contrast, only a ***weak*** translation is available (in which & serves as the translation of classical conjunction). At this point it is important to note that we show below that although SR_m

[2] since $\vdash_{LL_m} A \otimes B \to B \otimes A$, the dual rule $A \otimes B / B$ also becomes derivable.

[3] There are, in any case, several consequence relations which naturally correspond to LL and have been used in the literature. See [Av92]. Girard himself emphasized in [Gi87] that he is not defining *any* consequence relation, only theoremhood of formulae.

is a proper extension of R_m, it is still a purely relevant logic: it has the variable-sharing property, and its set of theorems is properly included in that of RMI_m ($= R_m +$ the mingle axiom).[4]

The structure of the rest of the paper is as follows: in Section 2 we review familiar material concerning the most important multiplicative systems: LL_m, BCK_m, R_m, RMI_m and CL_m. In Section 3 we investigate the strong versions of these systems, obtained by the addition of $\otimes$-elimination. We prove the results mentioned above, as well as: the admissibility of the new rule in all systems except R_m, deduction theorems, etc. In Section 4 we introduce and investigate the general algebraic structures which correspond to SLL_m. Two important subclasses of these structures, which represent two opposite extreme cases are T-structures and F-structures. In Section 5 we relate the algebraic structures of Section 4 to the systems of Section 3, proving appropriate soundness and completeness theorems, with some examples of applications.

2 Preliminaries

This section summarizes material concerning substructural logics which by now is almost common knowledge. See [AB75], [Du86], [Gi87], [Av88], [Do93].

Definition 1. The basic multiplicative language: this is the propositional language which has a unary connective $\sim$ and two binary connectives: $\rightarrow, \otimes$.

Notes.

1. The name "multiplicative" is from [Gi87]; it is the most common nowadays (see, e.g. [Do93]). In the relevance logic literature the name "intensional" had been used before.

2. The notation $\sim$ and $\rightarrow$ are from relevance logic (Girard used $()^\perp$ and $\multimap$). $\otimes$ is taken from [Gi87] (relevantists had used $\circ$).

3. In the presence of $\sim$, each of the other two connectives is definable in terms of the other. Thus $A \otimes B = \sim (A \rightarrow \sim B)$. Another important multiplicative connective is $+$ (or "par" in [Gi87]), defined by $A + B = \sim (\sim A \otimes \sim B)$. This connective is not important for our present purposes.

[4] Unlike RM, RMI_m is known to be a purely relevant system. See [AB75, pp. 148-9] and [Av84].

Definition 2. LL_m, R_m, RMI_m, BCK_m and CL_m denote, respectively, the purely multiplicative fragments of LL (Linear Logic – [Gi87], [Tr92]), R (the standard relevance logic of Anderson and Belnap [AB75], [AB92] [Du86]), RMI (see, e.g., [Av90]), BCK (see [Do93] for references. This logic is also called Affine logic) and CL (classical logic).

Hilbert-Type Representations

(I) $\boldsymbol{LL_m}$

Axioms

(I)	$A \to A$	(Identity)
(B)	$(A \to B) \to ((B \to C) \to (A \to C))$	(Transitivity)
(C)	$(A \to (B \to C)) \to (B \to (A \to C))$	(Permutation)
(R1)	$(A \to (B \to C)) \to (A \otimes B \to C)$	
		(Residuation)
(R2)	$(A \otimes B \to C) \to (A \to (B \to C))$	
(N1)	$(A \to \sim B) \to (B \to \sim A)$	(Contraposition)
(N2)	$\sim\sim A \to A$	(Double Negation)

Rule of inference.

$$\frac{A \qquad\qquad A \to B}{B}$$

(II) $\boldsymbol{R_m}$: LL_m together with

(W) $A \to A \otimes A$ (Contraction)

(III) $\boldsymbol{RMI_m}$: R_m together with

(M) $A \otimes A \to A$ (Mingle)

(IV) $\boldsymbol{BCK_m}$: LL_m together with

(K) $A \otimes B \to A$ (Weakening)

(V) $\boldsymbol{CL_m}$: LL_m together with contraction and weakening.

Proposition 1. *(Variable-sharing property): If $A \to B$ is provable in LL_m, R_m or RMI_m then A and B share a variable.*

Definition 3. Let L be any of the systems above. The associated (Tarskian) consequence relation $\vdash_L$ is defined in the usual way: $\mathcal{T} \vdash_L A$ iff there exists a sequence $A_1, \ldots, A_n = A$

such that each A_i either belongs to $\mathcal{T}$, or is an instance of an axiom, or follows from two previous ones by MP[5].

Proposition 2. *$A \leftrightarrow B \vdash_{LL_m} \varphi(A) \leftrightarrow \varphi(B)$ (where $\varphi(A)$ is a formula of which A is a subformula, and $\varphi(B)$ is obtained from $\varphi(A)$ by replacing some occurences of A with B).*

Gentzen-Type Representations.

(I) $\boldsymbol{GLL_m}$

Axioms

$$A \Rightarrow A$$

Logical Rules:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2} \qquad \frac{A, \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \otimes B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad \Gamma_2 \Rightarrow \Delta_2, B}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2,\ A \otimes B}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\sim A, \Gamma \Rightarrow \Delta} \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim A}$$

Structural rules. Permutation.

(II) $\boldsymbol{GR_m}$: Like GLL_m, with contraction added.

(III) $\boldsymbol{GRMI_m}$: Like GR_m, with expansion (the converse of contraction) added:

$$\frac{A, \Gamma \Rightarrow \Delta}{A, A, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A, A}$$

Alternatively, $GRMI_m$ can be obtained from GR_m by adding to it ***mingle*** (or ***relevant*** mix):[6]

$$\frac{A, \Gamma_1 \Rightarrow \Delta_1 \qquad A, \Gamma_2 \Rightarrow \Delta_2}{A, A, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad \Gamma_2 \Rightarrow \Delta_2, A}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A, A}$$

(IV) $\boldsymbol{GBCK_m}$: Like GLL_m, with weakening added.

(V) $\boldsymbol{GCL_m}$: Like GR_m, with weakening added.

Proposition 3. *The cut elimination theorem in its multiplicative form:*

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

obtains for all the systems above.

[5] This is called the "external consequence relation" in [Av88]. See also [Av92] for a discussion of the various consequence relations that can naturally be associated with substructural logics.

[6] When one adds to the language relevant additives then only the version with mingle admits cut-elimination. See [Av91].

Proposition 4. *Let L be one of the systems above.*

1. $\vdash_{GL} \Rightarrow A$ *iff* $\vdash_L A$.

2. $A_1, \ldots, A_n \vdash_L B$ *iff* $\Rightarrow B$ *is derivable in GL from* $\Rightarrow A_1, \ldots, \Rightarrow A_n$ *(using cuts).*

3. $\mathcal{T} \vdash_L B$ *iff there exists a (possibly empty) multiset* Γ*, all elements of which belong to* $\mathcal{T}$*, such that* $\vdash_{GL} \Gamma \Rightarrow B$.

Note. Unless $L = CL_m$, it is not the case that $A_1, \ldots, A_n \vdash_L B$ iff $\vdash_{GL} A_1, \ldots, A_n \Rightarrow B$. Thus $A \vdash_{LL_m} A \otimes A$, but the corresponding sequent is not derivable in GLL_m.

Definition 4.

1. The ***extended multiplicative language*** is the basic multiplicative language enriched by the propositional constant $\top$. We shall denote $\sim \top$ by $-$.[7]

2. Let L be a logic in the basic multiplicative language. L^T will denote the logic which is obtained from L by extending it to the extended multiplicative language and adding as axioms $A \to \top$ (Hilbert-type formulations) or $\Gamma \Rightarrow \Delta, \top$ (Gentzen-type formulations).

Proposition 5. *Let L be one of the systems above. Then Propositions 3 and 4 obtain also for* L^T.

Proposition 6. *For L as above, L^T is a strongly conservative extension of L (In other words: If $\mathcal{T}$ and A are in the basic multiplicative language, then $\mathcal{T} \vdash_{L^T} A$ iff $\mathcal{T} \vdash_L A$).*

Definition 5.

1. The ***full multiplicative language*** is the extended multiplicative language, enriched by the propositional constant 1.

2. Let L be one of the basic logics above. L^b will denote the logic which is obtained from L^T by extending its axioms and rules to the full multiplicative language and by adding to it the axioms 1 and $1 \to (A \to A)$ (Hilbert-type formulations) or the axiom $\Rightarrow 1$ and the rule: from $\Gamma \Rightarrow \Delta$ infer $1, \Gamma \Rightarrow \Delta$ (Gentzen-type formulations).

Proposition 7. *Proposition 4 obtains for L^b (L as above).*

[7] We follow here [Tr92]. Girard used 0 instead ([Gi87]).

Proposition 8. *The cut-elimination theorem obtains for GLL^b_m, GR^b_m, $GBCK^b_m$ and GCL^b_m, but not for $GRMI^b_m$.*

Proposition 9. *For $L = LL_m$, R_m, BCK_m and CL_m, L^b is a strongly conservative extension of L and L^T. RMI^b_m, in contrast, is not a conservative extension of RMI_m.*

Proposition 9 is an easy corollary of Propositions 7 and 8 in the case of LL^b_m, R^b_m, BCK^b_m and CL^b_m. As for $GRMI^b_m$, one can easily derive in it (using cuts) the "mix" rule: from $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. This fact entails both propositions in the case of RMI^b_m.[8]

Note. In [Gi87] the constant $\top$ belongs to the "additives", not to the official multiplicative fragment of Linear Logic. In [Av97] we argue in some length why considering it as a multiplicative constant is more reasonable.[9] In what follows we shall encounter some other indications that it is very natural to include $\top$ (and $-$) in the multiplicative fragment. This inclusion will prove to be very useful in what follows.

3 Multiplicative Conjunction as a Real Conjunction

Both the contraction axiom (W) and the weakening axiom (K) above are not valid in LL_m. There is, however, no real symmetry in the relations between LL_m and these axioms. While $A \to A \otimes A$ is not valid in LL_m, the corresponding ***rule*** (from A infer $A \otimes A$) ***is*** valid: $A \vdash_{LL_m} A \otimes A$. On the other hand not only is the weakening axiom $A \otimes B \to A$ invalid in LL_m, but so is the corresponding rule: from $A \otimes B$ infer A (it fails also in R_m and RML_m, in fact). To make an honest comparison between the effects of the two axioms we should start with a system in which ***both*** rules are valid, while both axioms are not. This leads us to consider a stronger version of (some of) the systems described in Section 2.

Definition 6.

1. The rule of $\otimes$-elimination is the following rule:

$$(\otimes - E) \qquad \frac{A \otimes B}{A} \; .$$

[8]Cut-elimination can be restored in $GRMI^b_m$ if we add "mix" as an extra rule. The system we get is equivalent to RM_m – the multiplicative fragment of the system RM of Dunn-McCall.

[9]The main point is that the characteristic property of a multiplicative logic is that if $\Gamma \Rightarrow \Delta$ can be derived in it from $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ then $\Gamma'_1, \ldots, \Gamma'_n, \Gamma \Rightarrow \Delta, \Delta'_1, \ldots, \Delta'_n$ can also be derived (with practically the same proof) from $\Gamma'_1, \Gamma_1 \Rightarrow \Delta_1, \Delta'_1, \ldots, \Gamma'_n, \Gamma_n \Rightarrow \Delta_n, \Delta'_n$. The addition of $\top$ preserves this property (while that of the additive *connectives* does not).

2. Let L be a logic in the basic multiplicative language or some extension of it. SL is the system which is obtained from L by adding $(\otimes - E)$ as a rule of inference.

Theorem 10.

1. *$SL = L$ for $L = BCK_m$ (or any extension, like CL_m).*

2. *$(\otimes - E)$ is admissible, but not derivable in LL_m and RMI_m (in other words: If $\vdash_{LL_m} A \otimes B$ then $\vdash_{LL_m} A$, but $A \otimes B \nvdash_{LL_m} A$. Similarly for RMI_m).*

3. *$(\otimes - E)$ is not admissible in R_m. Hence SR_m is a proper extension of R_m.*

Proof:

1. Trivial.

2. The case of LL_m is obvious for proof-theoretical reasons. Since we do not have contraction in GLL_m, any cut-free proof of $\Rightarrow A \otimes B$ should end with an application of $(\Rightarrow \otimes)$ to $\Rightarrow A$ and $\Rightarrow B$. Hence $\Rightarrow A \otimes B$ is provable there iff both $\Rightarrow A$ and $\Rightarrow B$ are.

For RMI_m, prove by induction on length of proofs, that if $\Gamma \Rightarrow \Delta, A \otimes B, \ldots, A \otimes B$ is provable in $GRMI_m$ and every atomic formula which occurs in $\Gamma \cup \Delta$ occurs also in A, then $\Gamma \Rightarrow \Delta, A$ is also provable. This, in turn, relies on the fact that RMI_m is closed under "weak weakening": If $\vdash \Gamma \Rightarrow \Delta$ and φ contains only atomic formulas which occur in $\Gamma \cup \Delta$ then $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi$ are also provable (see [Av84, Proposition I.6]). For example, in the main case, where $\Gamma \Rightarrow \Delta, A \otimes B, \ldots, A \otimes B$ is obtained by $\Rightarrow \otimes$, we can apply the induction hypothesis to the premise with A, and get a provable sequent of the form $\Gamma' \Rightarrow \Delta', A$, where $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$. Weak weakening and the assumption about $\Gamma \cup \Delta$ allows us to conclude that $\Gamma \Rightarrow \Delta, A$ is also provable.[10]

To show that $A \otimes B \nvdash_{RMI_m} A$ use part (3) of Proposition 4 and the cut-elimination theorem or Proposition 1.

3. We first need a lemma:

Lemma 1. $A \to C, (A \to B) \to C \vdash_{R_m} C \otimes (\sim B \to \sim B \otimes \sim B)$.

Proof of the Lemma: It is not difficult to check that the corresponding sequent is provable in GR_m. Alternatively, one can reason as follows: Let φ be $\sim B \to \sim B \otimes \sim B$. Then both

[10] A shorter proof, using a semantic argument, can be found in [Av97].

φ and $\sim B \to \varphi$ are theorems of R_m. From the assumption $(A \to B) \to C$ it follows in R_m that $\sim C \to A \otimes \sim B$. Thus, the assumption $A \to C$ and the validity of $\sim B \to \varphi$ together imply $\sim C \to C \otimes \varphi$. But $C \to C \otimes \varphi$ is a theorem of R_m, because φ is. This and $\sim C \to C \otimes \varphi$ yield $C \otimes \varphi$.

Proof of part (3): It can easily be seen that if we take $C = ((A \to B) \to A) \to A \otimes ((A \to B) \to A)$ (the source of this C will become clear after the proof of Theorem 14 below) then $\vdash_{R_m} A \to C$ and $\vdash_{R_m} (A \to B) \to C$. Hence $\vdash_{R_m} C \otimes (\sim B \to \sim B \otimes \sim B)$ by the lemma. It remains to show that $0_{R_m} C$. Assume otherwise. Since C is in the $\{\to, \otimes\}$ language, C should be provable from the $\{\to, \otimes\}$ fragment of R_m, by Meyer's well-axiomatization results (see [AB75]). This fragment, however, is contained in intuitionistic logic (interpreting $\otimes$ as conjunction). We conclude therefore that $((A \to B) \to A) \to [A \wedge ((A \to B) \to A)]$ is valid intuitionistically. This, in turn, immediately implies that Pierce's law, $((A \to B) \to A) \to A$, is valid intuitionistically. This, of course, is false. $\square$

It follows from the last theorem that only SR_m (among the systems we consider) is a completely new system. In the rest of this section we study the effects of $\otimes - E$ mainly in the framework of SR_m and its extensions by axioms. We start with the following "upper bound" on SR_m:

Proposition 11. *SR_m is a proper subsystem of $SRMI_m$.*

Proof: That if $\vdash_{SR_m} A$ then $\vdash_{RMI_m} A$ follows from the fact that RMI_m is closed under $(\otimes - E)$ (see Theorem 10). To refute the converse, use the four truth-values $\{0, 1, 2, 3\}$ with $\sim a = 3 - a$, $a \otimes b = \min(3, ab)$ and $a \to b = \max(3 - 3a + ab, 0)$. It is straightforward to check that if $\vdash_{SR_m} A$ then $v(A) \neq 0$ for all v, while $v(p \to (p \to p)) = 0$ in case $v(p) = 2$. $\square$

Corollary. $0_{SR_m} A \otimes B \to A$.

Note. Since RMI_m has the variable-sharing property ([AB75, pp. 148-9], [Av84]), so does SR_m. SR_m is, therefore, still a relevance logic.

The next proposition shows that the use of the extra new rule can be reduced to one single application at the end of a proof.

Proposition 12. *Let L be LL_m or an extension of LL_m by axioms. Then $\mathcal{T} \vdash_{SL} B$ iff there exist a sentence φ such that $\mathcal{T} \vdash_L B \otimes \varphi$.*

Proof: The "if" direction is trivial. We prove the "only if" part by induction on the length of the proof of B from $\mathcal{T}$ in SL. If B is an axiom of L or $B \in \mathcal{T}$ then take $\varphi = B$. If B follows from $B \otimes C$ by $(\otimes - E)$ then by induction hypothesis there is ψ such that $\mathcal{T} \vdash_L (B \otimes C) \otimes \psi$. Take then $\varphi = C \otimes \psi$. Finally, if B follows from C and $C \to B$ by MP then by induction hypothesis there are φ_1 and φ_2 such that $\mathcal{T} \vdash_L C \otimes \varphi_1$ and $\mathcal{T} \vdash_L (C \to B) \otimes \varphi_2$. Take $\varphi = \varphi_1 \otimes \varphi_2$. □

We next show that SR_m allows a strong translation of positive classical logic in which $\otimes$ takes the role of classical conjunction. First, however, we introduce an appropriate translation of classical implication.

Definition 7. $A \supset B = \quad A \to A \otimes B$.

Theorem 13. *The deduction theorem for $\supset$ obtains in SR_m (or any extension of it by axioms[11]): $\mathcal{T} \vdash_{SR_m} A \supset B$ iff $\mathcal{T}, A \vdash_{SR_m} B$.*

Proof: The "only if" part is again trivial. For the "if" part assume that $\mathcal{T}, A \vdash_{SR_m} B$. We show that $\mathcal{T} \vdash_{SR_m} A \supset B$ by induction on the length of the proof of B from $\mathcal{T} \cup \{\mathcal{A}\}$. If $B \in \mathcal{T}$ or B is an axiom then we use the fact that $\vdash_{R_m} B \to (A \supset B)$, while if $B = A$ we use the fact that $\vdash_{R_m} A \supset A$ (note that this is exactly the contraction axiom (W)!) For the cases where B is obtained by one of the two inference rules we need some lemmas.

Lemma 2. *Let $A_1, \ldots, A_n$ be sentences in some extension of the $\{\to, \otimes\}$-language. Then there exists an instance φ of a theorem of R_m such that $\vdash_{R_m} A_i \to \varphi$ for $i = 1, \ldots, n$.*

Proof: Take $\varphi = (A_1 \to A_1 \otimes A_1) \otimes (A_2 \to A_2 \otimes A_2) \otimes \cdots \otimes (A_n \to A_n \otimes A_n))$ (note the crucial role of the contraction axiom here!).

Lemma 3. *Let Γ_i, Δ_i $(i = 1, \ldots, n)$ be multisets of formulas in some extension of the basic multiplicative language. Then there exists a theorem φ of R_m such that $\vdash_{GR_m} \Gamma_i \Rightarrow \Delta_i, \varphi$, for $i = 1, \ldots, n$.*

Proof: Immediate from Lemma 2, since $\vdash_{GR_m}, A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k, \varphi$ iff $\vdash_{R_m} A_1 \otimes A_2 \otimes \cdots \otimes A_n \otimes \sim B_1 \otimes \cdots \otimes \sim B_k \to \varphi$.

Note. In contrast to Lemma 2, where we need only the availability of $\to$ and $\otimes$, in Lemma 3 we need also $\sim$ (and so the whole basic multiplicative language).

Lemma 4. $A \supset C$, $A \supset (C \to B) \vdash_{SR_m} A \supset B$.

Proof: By Lemma 3 there exists φ such that both $\Rightarrow \varphi$ and $A \Rightarrow B, \varphi$ are theorems of GR_m. We show now that $A \supset C$, $A \supset (C \to B) \vdash_{GR_m} (A \supset B) \otimes \varphi$:

[11] This result was first shown for $SRMI_m$ in [Av97]. The proof here for SR_m is, however, more complicated.

$$
\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{A \Rightarrow A \qquad \dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{C \Rightarrow C \qquad \dfrac{A \Rightarrow \varphi, B \qquad \dfrac{\dfrac{B \Rightarrow B \qquad A \Rightarrow A}{B, A \Rightarrow A \otimes B}}{B \Rightarrow A \supset B}}{A, B \Rightarrow (A \supset B) \otimes \varphi, B}}{C, A, C \to B \Rightarrow (A \supset B) \otimes \varphi, B}}{A, C, A, C \to B \Rightarrow (A \supset B) \otimes \varphi, A \otimes B}}{A \otimes C, A \otimes (C \to B) \Rightarrow (A \supset B) \otimes \varphi, A \otimes B}}{A, A \otimes C, A \supset (C \to B) \Rightarrow (A \supset B) \otimes \varphi, A \otimes B}}{A, A, A \supset C, A \supset (C \to B) \Rightarrow (A \supset B) \otimes \varphi, A \otimes B}}{\dfrac{A, A \supset C, A \supset (C \to B) \Rightarrow (A \supset B) \otimes \varphi, A \otimes B}{A \supset C, A \supset (C \to B) \Rightarrow (A \supset B) \otimes \varphi, A \supset B \qquad \Rightarrow \varphi}}}{\dfrac{A \supset C, A \supset (C \to B) \Rightarrow (A \supset B) \otimes \varphi, (A \supset B) \otimes \varphi}{A \supset C, A \supset (C \to B) \Rightarrow (A \supset B) \otimes \varphi}}
$$

Lemma 5. $A \supset B \otimes C \vdash_{SR_m} A \supset B$.

Proof: By Lemma 3 there exists φ such that both $\Rightarrow \varphi$ and $A, C \Rightarrow \varphi, B$ are provable in GR_m. We show now that $A \supset B \otimes C \vdash_{GR_m} (A \supset B) \otimes \varphi$:

$$
\dfrac{\dfrac{\Rightarrow \varphi \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{\dfrac{A, C \Rightarrow \varphi, B \qquad \dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, B \Rightarrow A \otimes B}}{B \Rightarrow A \supset B}}{A, B, C \Rightarrow (A \supset B) \otimes \varphi, \quad B}}{A \otimes B \otimes C \Rightarrow (A \supset B) \otimes \varphi, \quad B}}{A \supset B \otimes C, \quad A \Rightarrow (A \supset B) \otimes \varphi, \quad B} \qquad A \Rightarrow A}{A \supset B \otimes C, A, A \Rightarrow (A \supset B) \otimes \varphi, \quad A \otimes B}}{A \supset B \otimes C, A \Rightarrow (A \supset B) \otimes \varphi, \quad A \otimes B}}{A \supset B \otimes C \Rightarrow (A \supset B) \otimes \varphi, \quad A \supset B}}{A \supset B \otimes C \Rightarrow (A \supset B) \otimes \varphi, \quad (A \supset B) \otimes \varphi}}{A \supset B \otimes C \Rightarrow (A \supset B) \otimes \varphi}
$$

End of Proof of Theorem 13: Immediate now from Lemmas 4 and 5. $\square$

We present now the translation of positive classical logic:

Theorem 14. *Define a translation I of the positive classical language into the basic multiplicative language as follows:*

$$I(p) = p \quad \text{when} \quad p \quad \textit{is atomic}$$

$$I(A \supset B) = I(A) \to I(B) \otimes I(A) \ \ (= I(A) \supset I(B))$$

$$I(A \wedge B) = I(A) \otimes I(B)$$

$$I(A \vee B) = I((A \supset B) \supset B)$$

$$I(A \equiv B) = I(A \supset B) \otimes I(B \supset A)\ .$$

Then A follows from $B_1, \ldots, B_n$ in classical logic iff

$$I(B_1), \ldots, I(B_n) \vdash_{SR_m} I(A)\ .$$

Proof: It is well known that $A \vee B$ is equivalent in classical logic to $(A \supset B) \supset B$ and $A \equiv B$ is equivalent to $(A \supset B) \wedge (B \supset A)$. Hence it is enough to show the claim for the $\{\supset, \wedge\}$ fragment.

Suppose first that $I(B_1), \ldots, I(B_n) \vdash_{SR_m} I(A)$. Then $I(B_1), \ldots, I(B_n) \vdash_{Cl_m} I(A)$ as well, since $\boldsymbol{Cl_m}$ is an extension of SR_m. But in $\boldsymbol{Cl_m}$ $I(\varphi)$ and φ are obviously equivalent. Hence $B_1, \ldots, B_m \vdash_{Cl_m} A$.

For the converse it suffices to take some standard axiomatization of classical logic and show that the translation of all the axioms and rules of inference are valid in SR_m. Now the deduction theorem for $\supset$ and its converse MP are valid by Theorem 13, while the standard natural-deduction rules for conjunction are obviously also valid. All we need to check, therefore, is the validity of the Pierce law $((A \supset B) \supset A) \supset A$ or, equivalently, (by the deduction theorem), that $(A \supset B) \supset A \vdash_{SR_m} A$. Let φ be a sentence such that $\Rightarrow \varphi$, $\Rightarrow B \otimes A, \varphi$ and $B \Rightarrow \varphi, A$ are all provable in GR_m (such φ exists by Lemma 3 from the proof of Theorem 13). We end by showing that $(A \supset B) \supset A \Rightarrow A \otimes \varphi$ is provable in GR_m:

$$
\dfrac{
  \dfrac{
    \dfrac{A \Rightarrow A \qquad \Rightarrow A \otimes B, \varphi}{A \Rightarrow A \otimes B, A \otimes \varphi}
  }{\Rightarrow A \otimes \varphi, A \supset B}
  \qquad
  \dfrac{
    \dfrac{A \Rightarrow A \qquad \dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow \varphi, A}{A, B \Rightarrow A \otimes \varphi, A}}{A \otimes B \Rightarrow A \otimes \varphi, A}}{A, \quad A \supset B \Rightarrow A \otimes \varphi, A}
  }{A \otimes (A \supset B) \Rightarrow A \otimes \varphi, A}
}{
  \dfrac{
    \dfrac{(A \supset B) \supset A \Rightarrow A \otimes \varphi, A \otimes \varphi, A \qquad \Rightarrow \varphi}{(A \supset B) \supset A \Rightarrow A \otimes \varphi, A \otimes \varphi, A \otimes \varphi}
  }{(A \supset B) \supset A \Rightarrow A \otimes \varphi}
}
\qquad \square
$$

Note. Again, Theorem 14 was first shown (with an easier proof) for $SRMI_m$ (see [Av97]). Since $SRMI_m$ and RMI_m have the same set of theorems, the theorem implies that the above translation is a ***weak*** translation of classical logic into RMI_m ($\vdash_{Cl_m} A \Leftrightarrow \vdash_{RMI_n} I(A)$). In contrast, I is not even a weak translation of CL_m into R_m. Thus the translation of Pierce's law is not valid in R_m (because the implication-conjunction fragment of R_m is a subsystem of intuitionistic logic, and in that logic A and $I(A)$ are equivalent if A is in the implication-conjunction fragment). Note, in this connection, that the example above of a theorem of SR_m which is not provable in R_m (which was given in the proof of Theorem 10) is just $((A \supset B) \to A) \supset A$.

What about translating classical negationΓ In [Av97] it is shown that this is impossible even in $SRMI_m$.

In order to get a translation of full classical logic we should, therefore, extend the basic multiplicative language. Now an easy (and quite common) method of getting full classical logic from its positive fragment is to add propositional constant $-$, together with the axiom $- \supset A$ (one defines then $\neg A$ as $A \supset -$). A natural extension of SR_m in which this can be done is, of course, SR_m^T (where $-$ denotes, recall, $\sim \top$). In this system $- \supset A$ is indeed a theorem, and $A \supset -$ is equivalent to $A \to -$.[12] First, however, we should show that the addition of $\top$ and $-$ is conservative also in the framework of the strong systems.

Proposition 15. *SLL_m^T is strongly conservative extension of SLL_m. Similarly, L^T is a strongly conservative extension of L whenever L is an extension of R_m (this includes SR_m and R_m itself).*

[12]These two facts are true already in LL_m^T!

Proof: By Proposition 12, if $\mathcal{T} \vdash_{SLL_m^T} A$ then there exists a sentence φ (which might contain $\top$) such $\mathcal{T} \vdash_{LL_m^T} A \otimes \varphi$. Hence, by Proposition 4, there exists a multiset Γ_0, containing only sentences from $\mathcal{T}$, such that $\vdash_{GLL_m^T} \Gamma_0 \Rightarrow A \otimes \varphi$. It remains to show that if Γ, Δ and A do not contain $\top$, and $\vdash_{GLL_m^T} \Gamma \Rightarrow \Delta, A \otimes \varphi$ then there exists B in the basic multiplicative language such that $\vdash_{GLL_m} \Gamma \Rightarrow \Delta, A \otimes B$. The proof is by induction on the length of the proof of $\Gamma \Rightarrow \Delta, A \otimes \varphi$. The only interesting case is when $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$ and $\Gamma \Rightarrow \Delta,\ A \otimes \varphi$ is inferred from $\Gamma_1 \Rightarrow \Delta_1, A$ and $\Gamma_2 \Rightarrow \Delta_2, \varphi$, In this case we take $B = A \to A$ in case $\Gamma_2 = \Delta_2 = \emptyset$ and $B = C_1 \otimes C_2 \otimes \cdots \otimes C_n \otimes \sim D_1 \otimes \cdots \otimes \sim D_k$ in case $n + k > 0$ and $\Gamma_2, \Delta_2 = [C_1, \ldots, C_n, D_1, \ldots, D_k]$. In both cases $\vdash_{CLL_m} \Gamma_2 \Rightarrow \Delta_2, B$ and so $\vdash_{GLL_m} \Gamma \Rightarrow \Delta, A \otimes B$.

Suppose, next, that L is an extension of R_m. Let $A_1 \to \top, \ldots, A_n \to \top$ be all the new axioms of L^T which are used in the proof in L^T of A from $\mathcal{T}$. By Lemma 2 there exists a theorem φ of R_m such that $\vdash_{R_m} A_i \to \varphi$ for $i = 1, \ldots, n$. Replace $\top$ by φ everywhere in the proof. The result is proof of A from $\mathcal{T}$ in L. □

Notes.

1. Proposition 15 trivially holds also for BCK_m and its extensions, since in BCK_m any theorem can take the role of $\top$.

2. We needed a different argument in the case of R_m, since the argument given for SLL_m fails in the presence of contraction.

3. It is well known that the propositional constant 1 is equivalent, in a certain sense, to the ***additive*** conjunction of all the identity axioms ($A \to A$). The proofs of the last proposition and of Lemma 2 show, on the other hand, that in R_m (and its extensions) the constant $\top$ is equivalent to the infinite ***multiplicative*** conjunction of all the contraction axioms ($A \to A \otimes A$). This is another indication, I believe, that at least in the framework of R_m $\top$ and $-$ belong to the multiplicative fragment (and even more so than 1!).

Theorem 16. *Add to the definition of I in Theorem 14 the clause:*

$$I(\neg A) = I(A) \to - \ (= I(A) \supset -)\ .$$

Then I is a strong translation of classical logic into SR_m^T.

Proof: Similar to that of theorem 14, using the comments before Proposition 15. □

Note. Proposition 15 naturally raises the question: what about SL^b, where L is one of the strong logics discussed hereΓ Is it also a conservative extension of SL^T (and so of SL itself)Γ The answer for SLL_m^b, BCK^b and SR_m^b is in fact positive, but for SR_m^b we shall be able to show it only using semantic methods (see Section 5, Theorem 37). The answer for $SRMI_m^b$ is negative, like in the case of RMI_m^b, and for the same reasons (see Proposition 9). For SLL_m^b and BCK_m^b the proof is very similar to that given above for SLL_m^T and BCK_m^T. We record it below for future use:

Proposition 15*. *SLL_m^b and BCK_m^b are conservative extensions of SLL_m^T and BCK_m^T (and so of SLL_m and BCK_m), respectively.*

We end this section with a note which provides another perspective on the role of $(\otimes - E)$ in the context of R_m.

Proposition 17. *In the context of R_m $(\otimes - E)$ is equivalent to the rule: from A and $\sim (A \to B)$ infer $\sim B$.*

Proof: In LL_m, $\sim (A \to B)$ is equivalent to $A \otimes \sim B$. Hence the new rule is derivable from $(\otimes - E)$. For the converse, assume the new rule. We derive A from $A \otimes B$ as follows:

1.	$A \otimes B \to \sim ((A \otimes B) \to \sim A)$	(a theorem of R_m)
2.	$A \otimes B$	(assumption)
3.	$\sim (A \otimes B \to \sim A)$	(1,2,MP)
4.	$\sim\sim A$	(2,3, the new rule)
5.	A	(from 4)

□

It follows that SR_m can be formalized using its negation-implication axioms (the axioms of $R_{\underset{\to}{\sim}}$) and the following two rules:

$$\frac{A \qquad A \to B}{B} \qquad\qquad \frac{A \qquad \sim (A \to B)}{\sim B}$$

4 Corresponding Algebraic Structures

In this section we introduce the algebraic structures which correspond to the logics we investigated above.

4.1 Multiplicative structures and strong multiplicative structures

Definition 8. A *multiplicative structure* (m.s.) is a structure $\overline{S} = \langle S, \leq, \top, -, \sim, \otimes, D\rangle$ in which:

1. $\langle S, \leq\rangle$ is a bounded poset, with $\top$ and $-$ as the greatest and least elements, respectively.

2. $\sim$ is an involution on $\langle S, \leq\rangle$ i.e.:

   (i) $\sim\sim a = a$

   (ii) $a \leq b \Rightarrow \sim b \leq \sim a$.

3. $\otimes$ is an associative and commutative binary operation on S.

4. D is a nonempty, proper subset of S which is upward closed ($a \leq b, a \in D \Rightarrow b \in D$).

5. $a \leq b$ iff $a \rightarrow b \in D$, where $a \rightarrow b = \sim (a \otimes \sim b)$.

Notes.

1. "Multiplicative structures" are, more or less, the structures which were given in [Av88] the somewhat unattractive name: "basic relevant disjunction structures with truth subset". The only difference is the demand here of the existence of $\top$ and $-$ (in [Av88] there is one more condition: that $\otimes$ should be order-preserving. This condition is derivable, however, from the others, as we show below.)

2. Intuitively, S is the set of truth-values, D – the subset of designated truth-values, $\otimes$ and $\sim$ correspond to connectives of the basic multiplicative language, $\top$ and $-$ represent, respectively, absolutely true and absolutely false propositions and $\leq$ corresponds to the internal entailment relation (represented by the connective $\rightarrow$).

Proposition 18. *In every m.s.* $\overline{S}$*:*

(i) $- \notin D$, $\top \in D$.

(ii) $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$ *(i.e.* $\rightarrow$ *is a residual operation).*

(iii) $a \otimes b \leq c \Leftrightarrow a \otimes \sim c \leq \sim b$.

(iv) $a \leq b \Rightarrow a \otimes c \leq b \otimes c$ *(i.e.* $\otimes$ *is order preserving).*

(v) $a \otimes - = - \otimes a = -$.

(vi) $a \in D \Rightarrow b \leq a \otimes b$.

(vii) $b \leq \top \otimes b$ *(in particular* $\top \otimes \top = \top$)

(viii) $a \in D$, $b \in D \Rightarrow a \otimes b \in D$.

Proof:

(i) Obvious from the properties of D, $\top$ and $-$.

(ii) By condition (5) of Definition 8, $a \leq b \to c$ iff $\sim (a \otimes \sim (b \to c)) \in D \overset{(2)(i)}{\Longleftrightarrow} \sim (a \otimes (b \otimes \sim c)) \in D \Longleftrightarrow \sim ((a \otimes b) \otimes \sim c) \in D \overset{(5)}{\Leftrightarrow} a \otimes b \leq c$.

(iii) $b \to c = \sim c \to \sim b$ by (2)(i) and the commutativity of $\otimes$. Hence (iii) follows from (ii).

(iv) Since $b \otimes c \leq b \otimes c$, $b \leq c \to b \otimes c$ by (ii). This and $a \leq b$ together imply that $a \leq c \to b \otimes c$. Hence $a \otimes c \leq b \otimes c$, by (ii) again.

(v) Since $- \leq a \to -$, $- \otimes a \leq -$ by (ii). Hence $- \otimes a = -$.

(vi) Like in (iv), $a \leq b \to a \otimes b$ for all a, b. It follows that if $a \in D$ then $b \to a \otimes b \in D$, and so $b \leq a \otimes b$ (by (5)).

(vii) Immediate from (i) and (vi).

(viii) Immediate from (vi), and the fact that D is upward closed. □

A particularly important class of multiplicative structures is given in the next definition.

Definition 9. A ***multiplicative monoid*** (m.m.) is an m.s. which has an identity element 1 for $\otimes$ (i.e. $a \otimes 1 = a$). In other words: m.m. is an m.s. $\overline{S}$ in which $\langle S, \otimes, 1 \rangle$ is a commutative monoid.

Proposition 19. *If* $\overline{S}$ *is an m.m. then* $D = \{a \in S | a \geq 1\}$*. Conversely, if* S *is an m.s. in which* D *has a least element then this element is an identity element for* $\otimes$ *(and so* $\overline{S}$ *is an m.m.).*

Proof: Assume $\overline{S}$ is an m.m. Since $a \otimes 1 \leq a$, $a \leq 1 \rightarrow a$ by Proposition 18. It follows that if $a \in D$ then also $1 \rightarrow a \in D$ and so $a \geq 1$. On the other hand $1 \leq 1 \Rightarrow \sim (1 \otimes \sim 1) \in D \Rightarrow \sim\sim 1 \in D$. Hence $1 \in D$ and so $a \geq 1 \Rightarrow a \in D$. Hence $D = \{a \in S | a \geq 1\}$.

For the converse, assume S is an m.s. and that e is the least element of D. Since $b \leq b$, $b \rightarrow b \in D$ for every $b \in S$ and so $e \leq b \rightarrow b$. It follows that $e \otimes b \leq b$ (Proposition 18). On the other hand, the fact that $e \in D$ implies that $b \leq e \otimes b$ for all b (part (vi) of Proposition 18). Hence $e \otimes b = b$. □

Proposition 19 allows us to give an alternative definition of a multiplicative monoid:

Proposition 20. *Multiplicative monoids can be characterized as structures* $\overline{S} = \langle S, \leq, \sim \otimes, 1\rangle$ *such that:*

1. *$\langle S, \leq\rangle$ is a non-trivial bounded poset.*

2. *$\sim$ is an involution on $\langle S, \leq\rangle$.*

3. *$\langle S, \otimes, 1\rangle$ is a commutative monoid.*

4. *$a \leq b \Leftrightarrow a \otimes \sim b \leq \sim 1$.*

Proof: By Proposition 18(iii), every m.m. satisfies properties (1)–(4). For the converse, suppose $\overline{S} = \langle S, \leq, \sim, \otimes, 1\rangle$ satisfies (1)–(4). Since $\overline{S}$ is not trivial, ~ 1 cannot be the greatest element of S (because of (4)), and so 1 is not the least element of S. Define $D = \{a \in S | a \geq 1\}$. It is easy to see that this D has all the needed properties, so we get an m.m. By proposition 19, this is the only possible choice for D. □

An important subclass of the class of multiplicative monoids, which was shown in [Av88] to be equivalent to the class of Girard's "phase spaces" is the following.

Definition 10. A *Girard structure* is an m.m in which the underlying poset is a complete lattice.

Theorem 21 [Av88]. *Every m.m $\overline{S}$ can be embedded in a Girard's structure $\overline{S}^*$ so that existing infima and suprema of subsets in S are preserved, and $\top, -$ and 1 remain, respectively, the greatest, least and identity elements of $\overline{S}^*$.*

Proof (outline): Define $C : P(S) \rightarrow P(S)$ by $C(X) = X^{+\perp}$, where

$$X^{+} = \{y \in S | \forall x \in X, \quad x \leq y\} \quad X^{\perp} = \{y \in S | \forall x \in X,\ y \leq x\}\ .$$

Let $S^* = \{X \in P(S) | X = C(X)\}$. Define:

$$XY = \{a \otimes b | a \in X, b \in Y\}$$
$$X \otimes^* Y = C(XY)$$
$$\sim^* X = \{\sim y | y \in X^+\}$$
$$a^* = \{a\}^{+\perp} = \{x \in S | x \leq a\} \ .$$

Then $\langle S^*, \subseteq, \otimes^*, \sim^*, 1^* \rangle$ is a Girard structure and $\lambda x \in S.\ x^*$ is an embedding of S in it of the type required. □

All of the various types of structures considered above provide an adequate semantics for LL_m. In order to have an adequate semantics for SLL_m we need to consider special subclasses of them, which we call the "strong versions" of these structures.

Definition 11. A strong multiplicative structure (*sms*) is a multiplicative structure which satisfies the following condition:

$$(sd) \qquad a \otimes b \in D \Leftrightarrow a \in D \quad \text{and} \quad b \in D$$

strong multiplicative monoids (*smm*) and strong Girard structures are defined similarly.

Notes.

1. The extra condition (sd) makes the set of designated elements something close to the standard definition of a *filter*. Recall, however, that here the order relation $\leq$ and the conjunction operator $\otimes$ are not connected to each other (or, rather, the connection is much more complicated than in lattices).

2. By part (viii) of Proposition 18, Condition (sd) equivalent to: $a \otimes b \in D \Rightarrow a \in D$ and $b \in D$ (or even just $a \otimes b \in D \Rightarrow a \in D$). Similarly, *smm* can be defined as an *mm* in which $a \otimes b \geq 1$ implies that $a \geq 1$ and $b \geq 1$, or as a structure which satisfies the four conditions in Proposition 20 as well as the last condition.

Proposition 22. *In an sms, $a \otimes \top = \top$ iff $a \in D$.*

Proof: If $a \otimes \top = \top$ then $a \otimes \top \in D$ and so $a \in D$ by the definition of an *sms*. The converse is true, in fact, in every m.s., since by Proposition 18(vi) if $a \in D$ then $\top \leq a \otimes \top$ and so $\top = a \otimes \top$.

Theorem 23. *Every smm $\overline{S}$ can be embedded into a strong Girard's structure $\overline{S}^*$ so that existing infima and suprema of subsets of S, as well as the identity of $\top, -$ and 1, are preserved.*

Proof: Like that of Theorem 21. The only thing that should be added is that $\overline{S}^*$ (as constructed in the proof of Theorem 21) is strong, i.e., that if $X \otimes^* Y \geq 1^*$ then $X \supseteq 1^*$. For this note first that since $Z = C(Z)$ for every $Z \in S^*$, $1^* \subseteq Z$ iff $1 \in Z$. What we should show, therefore, is that if $1 \notin X$ then $1 \notin X \otimes^* Y$. So assume $1 \notin X$. Since $X = C(X) = X^{+\perp}$, this means that there exist $z \in X^+$ such that $z \not\geq 1$. It follows that also $z \otimes \top \not\geq 1$, by the defining property of an *smm*. It is easy however to see that if $z \in X^+$ then $z \otimes \top \in (XY)^+$. Hence $1 \notin (XY)^{+\perp} = X \otimes^* Y$. □

4.2 T-structures and F-structures

In this subsection we introduce and investigate two especially important types of (strong) multiplicative structures, representing two extreme possibilities concerning the subset D. From Proposition 18(i) it follows that $\{\top\} \subseteq D \subseteq S - \{-\}$. The two most extreme cases are, therefore, when $D = \{\top\}$ (only one designated value) and when $D = S - \{-\}$ (only one nondesignated value). This observation naturally leads to the following definition.

Definition 12.

1. A T-structure (T-monoid, Girard's T-structure) is a multiplicative structure (multiplicative monoid, Girard's structure) in which $D = \{\top\}$.

2. An F-structure (F-monoid, Girard's F-structure) is a multiplicative structure (multiplicative monoid, Girard's structure) in which $D = S - \{-\}$.

Proposition 24. *Both T-structures and F-structures are strong multiplicative structures.*

Proof: For F-structures we should show that if $a \otimes b \neq -$ then $a \neq -$ and $b \neq -$. This is obvious from Proposition 18(v).

For T-structures we first prove a lemma which is important for its own sake:

Lemma 6. *In T-structures $a \otimes b \leq a$ for all a, b.*

Proof of the lemma: Since $a \leq a$, $\sim (a \otimes \sim a) = \top$ in T-structures and so $a \otimes \sim a = -$. Hence $a \otimes \sim a \otimes b = -$ (by Proposition 18(v)), and so $\sim ((a \otimes b) \otimes \sim a) = \top$. This entails that $a \otimes b \leq a$.

End of the proof of Proposition 24: Let $\overline{S}$ be a T-structure and suppose that $a \otimes b \in D$. This means that $a \otimes b = \top$, and so also $a = \top$, by Lemma 6. Hence $a \in D$. □

Our next proposition provides alternative characterizations of T-structures and F-structures, which resemble that given to multiplicative monoids in Proposition 20.

Proposition 25.

1. *T-structures can be characterized as structures $\overline{S} = \langle S, \leq, \sim \otimes, \top \rangle$ such that*
   - *(a) $\langle S, \leq \rangle$ is a non-trivial bounded poset and $\top$ is its greatest element.*
   - *(b) $\sim$ is an involution on $\langle S, \leq \rangle$.*
   - *(c) $\otimes$ is a commutative and associative operation on S.*
   - *(d) $a \leq b \Leftrightarrow a \otimes \sim b = \sim \top\ (= -)$.*
2. *F-structures can be characterized as structures $\overline{S} = \langle S, \leq, \sim, \otimes, \top \rangle$ such that:*
   - *(a) $\langle S, \leq \rangle$ is a nontrivial bounded poset and $\top$ is its greatest element.*
   - *(b) $\sim$ is an involution of $\langle S, \leq \rangle$.*
   - *(c) $\otimes$ is a commutative and associative operation on S.*
   - *(d) $a \leq b \Leftrightarrow a \otimes \sim b < \top$.*

Proof:

1. Obviously, every T-structure satisfies these conditions. Conversely, if $\langle S, \leq, \sim, \otimes, \top \rangle$ satisfies (a)-(d) then by defining $- = \sim \top$ and $D = \{\top\}$ we get a T-structure.
2. Again, every F-structure obviously satisfies the conditions. Conversely, if $\langle S, \leq, \sim, \otimes, \top \rangle$ satisfies (a)–(d) then by defining $- = \sim \top$ and $D = S - \{-\}$ we get an F-structure. □

Proposition 26. *Every T-structure is a T-monoid in which $1 = \top$. Conversely, if $\overline{S}$ is an m.m. in which $1 = \top$ then $\overline{S}$ is a T-structure.*

Proof: That $a \otimes \top = a$ in every T-structure follows from Lemma 6 (from the proof of the last proposition) and Proposition 18(vii). The converse is a corollary of Proposition 19, since $\{a \in S | a \geq \top\} = \{\top\}$. □

Because of the last proposition, we shall use the terms "T-structure" and "T-monoid" synonymously.

Unlike T-structures, F-structures are not necessarily F-monoids. We shall see examples below that will be very important for giving strong semantics to RMI_m. Still, every F-structure can be turned into an F-monoid by adding only two more elements.

Proposition 27. *Every F-structure $\overline{S}$ can be embedded into an F-monoid which has exactly two new elements.*

Proof: Let $\overline{S} = \langle S, \leq, \sim, \otimes, \top\rangle$ and let $1, f$ be two new entities not in S. Define $\overline{S}^* = \langle S^*, \leq^*, \sim^*, \otimes^*, \top\rangle$ as follows:

$$S^* = S \cup \{1, f\}$$

$$- \leq^* 1 \leq^* a \leq^* f \leq^* \top \quad \text{for all} \quad a \in S - \{\top, -\}$$

$$\sim^* a = \begin{cases} f & a = 1 \\ 1 & a = f \\ \sim a & a \in S \end{cases}$$

$$a \otimes^* b = a \otimes b \quad \textit{if} \quad a, b \in S$$

$$a \otimes^* 1 = 1 \otimes^* a = a$$

$$a \otimes^* f = f \otimes^* a = \begin{cases} - & a = - \\ f & a = 1 \\ \top & \text{otherwise}\ . \end{cases}$$

It is straightforward to check that $\overline{S}^*$ is indeed an F-monoid as required. We only note that to prove associativity when f is involved, we use Proposition 22, which in the present case implies (by Proposition 24) that $a \neq - \Rightarrow a \otimes \top = \top \otimes a = \top$. □

The equations used in the proof of the last theorem are in fact the only possibility:

Proposition 28. *Let $\overline{S}$ be an F-monoid. Then*

1. $1 \leq a \Leftrightarrow a \neq -$.

2. $a \leq \sim 1 \Leftrightarrow a \neq \top$.

3. $\sim 1 \otimes a = a \otimes \sim 1 = \begin{cases} - & a = - \\ \sim 1 & a = 1 \\ \top & \textit{otherwise}\ . \end{cases}$

4. *Unless* $1 = \top$, *we have that* $- < 1 \leq a \leq \sim 1 < \top$ *for all* $a \notin \{\top, -\}$.

Proof:

1. This follows from Proposition 19.

2. Immediate from 1.

3. The cases $a = -$ and $a = 1$ are obvious (see Proposition 18(v)). Assume next that $a \notin \{-, 1\}$. Then a $\not\leq 1$ by part (1), and so $a \otimes \sim 1 = \top$, by Proposition 25 (part (2)).

4. This follows from parts (1) and (2). □

Note. The exception in part (4) corresponds to the classical two-valued Boolean algebra, which is the only F-structure in which $1 = \top$ (i.e., the only structure which is both a T-structure and an F-structure).

Theorem 29. *Every T-monoid can be embedded in a Girard's T-structure and every F-monoid can be embedded in a Girard's F-structure, so that in both cases existing infima and suprema as well as the identity of $\top, -$ and 1 are preserved.*

Proof: We only need to check that the construction in the proof of the Theorem 23 provides a T-monoid (F-monoid) if we start from a T-monoid (F-monoid). Now for T-monoids this follows from Proposition 26 and the fact that the equation $1 = \top$ is preserved by the construction (since the identities of both 1 and $\top$ are preserved). For an F-structure it suffices to observe that if $X = C(X)$ then either $X = \{-\}$ or $1 \in X$ (since 1 is a lower bound for every $a \in S$ such that $a \neq -$). It follows that $1^* \subseteq X$ for all $X \in S^*$ such that $X \neq -^*$. Hence $\overline{S}^*$ is an F-monoid, by Proposition 19. □

4.3 Examples

Basic Examples

1. As noted above, there is just one multiplicative structure which is both a T-structure and an F-structure: the two valued Boolean algebra.

2. There are exactly two multiplicative structures which have three elements. Both are strong Girard's structures. In fact one of them is a T-structure, the other, an F-structure. The T-structure is Lukasiewicz $\mathcal{L}_3$. The F-structure is Sobociński 3-valued

logic [So52], denoted by M_3 in [AB75] and A_1 in [Av84].[13] Both structures consist of the three elements $-$, I and $\top$ with $- < I < \top$ and $\sim I = I$. In both $-\otimes X = X \otimes - = -$ and $\top \otimes \top = \top$. In $\mathcal{L}_3$ $\top \otimes I = I \otimes \top = I$ and $I \otimes I = -$. In A_1 $\top \otimes I = I \otimes \top = \top$ and $I \otimes I = I$.

Generalizations

1. Not only the two-valued, but of course every Boolean algebra is a T-structure.

2. The example of $\mathcal{L}_3$ can be generalized in at least two different ways.

   (i) Lukasiewicz's n-valued matrices. These are usually defined as follows: let

$$L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\right\} \quad (n \geq 2)$$
$$\sim a = 1 - a$$
$$a \otimes b = \begin{cases} 0 & a+b \leq 1 \\ a+b-1 & a+b > 1 \end{cases} \qquad a \to b = \begin{cases} 1 & a \leq b \\ 1-a+b & a > b \end{cases}$$

   Then $\mathcal{L}_n = \langle L_n, \leq, \sim, \otimes, 0, 1\rangle$ is a T-structure (in fact, a Girard's T-structure) for all $n \geq 2$.

   (ii) Let $\mathcal{L}^*_n$ be defined like $\mathcal{L}_n$, except that $\otimes$ is defined this time as follows:

$$a \otimes b = \begin{cases} 0 & a+b \leq 1 \\ \min(a,b) & a+b > 1 \end{cases}$$

   $\mathcal{L}^*_n$ is also a Girard's T-structure for all $n \geq 2$.

   Both examples can be extended to an infinite Girard's T-structure by taking the set of truth-values to be the whole interval [0,1] (and the same definitions of $\leq, \sim$ and $\otimes$).

3. The example of M_3 can be generalized in at least three interesting ways:

(i) The structures $\overline{A}_n = \langle A_n, \leq, \sim, \otimes, \top, -\rangle$ $(n \geq 0)$ are defined as follows:

$$A_n = \{\top, -, I_1, I_2, \ldots, I_n\}$$
$$a \leq b \Leftrightarrow a = - \quad \text{or} \quad b = \top \quad \text{or} \quad a = b$$
$$\sim - = \top, \ \sim \top = -, \ \sim I_k = I_k$$
$$a \otimes b = \begin{cases} - & a = - \lor b = - \\ I_k & a = b = I_k \\ \top & \text{otherwise} \end{cases} \qquad a \to b = \begin{cases} \top & a = - \lor b = \top \\ I_k & a = b = I_k \\ - & \text{otherwise}\ . \end{cases}$$

[13] The different name reflects a different generalization. As M_3 this structure is the second in the sequence $\{M_m | 2 \leq m \leq w\}$ of Sugihara Matrices. As A_1 it is the second in the sequence $\{A_n | 0 \leq n \leq w\}$ of [Av84] (which is also described below). $M_2 = A_0$ is the two-valued Boolean algebra.

$\overline{A}_n$ are all F-structures, which are based on complete lattices. They are not F-monoids, though, for $n \geq 2$. (Note that $\overline{A}_1$ is exactly M_3, while A_0 is the two-valued Boolean algebra.)

Again we can easily generalize to infinite matrices. For example $\overline{A}_\omega$ is defined exactly like A_n, only $A_\omega = \{\top, -, I_1, I_2, I_3, \cdots\}$.[14] More generally, if c is any cardinal, define $\overline{A}_c$ exactly as $\overline{A}_\omega$, only $A_c = \{\top, -\} \cup \{I_\alpha | \alpha < c\}$.

(ii) $\overline{F}^*_n$, Girard's F-structures, analogous to Ł^*_n above, can be defined by taking the same set of truth-values as in Ł^*_n, the same definitions of $\leq$ and $\sim$, but the following definition of $\otimes$ (and $\rightarrow$):

$$a \otimes b = \begin{cases} 0 & a = 0 \vee b = 0 \\ 1 & a + b > 1 \\ \max(a, b) & \text{otherwise} \end{cases} \qquad a \rightarrow b = \begin{cases} 0 & a > b \\ 1 & a = 0 \vee b = 1 \\ \min(1 - a, b) & 0 < a \leq b < 1 \end{cases}$$

Like Ł^*_n, the various $\overline{F}^*_n$ are all substructures of the infinite F-structure in which the set of truth-values is the interval [0,1] (with the same definitions of $\leq, \sim$ and $\otimes$). This F-structure is not an F-monoid (although it is based on a complete lattice). However, every subset of it which contains 0 and 1 and is closed under $\lambda a.1 - a$ is also an F-structure, and many of these structures are F-monoids or even Girard's F-structures. Again $\overline{F}^*_2$ is the two-valued Boolean algebra, while $\overline{F}^*_3 = M_3 = A_3$.

(iii) It is possible to define finite F-monoids which are a kind of analogue of Lukasiewicz's finite-valued matrices. Define $\overline{F}_n$ as follows. $\overline{F}_n$ has the same set of truth-values and the same definitions of $\sim$ and $\leq$ as Ł_n (and Ł^*_n, and $\overline{F}^*_n$...). For $\otimes$ (and $\rightarrow$), however, we use the following definition:

$$a \otimes b = \begin{cases} 0 & a = 0 \vee b = 0 \\ 1 & a + b > 1 \\ a + b - \frac{1}{n \perp 1} & \text{otherwise} \end{cases} \qquad a \rightarrow b = \begin{cases} 1 & a = 0 \vee b = 1 \\ 0 & a > b \\ b - a + \frac{1}{n \perp 1} & 0 < a \leq b < 1 \end{cases}$$

Another possible description of $\overline{F}_n$ (equivalent up to isomorphism) can be given as follows. We take the truth values to be $-, 1, 2, \ldots, n-2, \top$, with $- < 1 < 2 < \cdots < n - 2 < \top$, $\sim - = \top$, $\sim \top = -$ and otherwise $\sim a = n - 2 - a$. In addition:

$$a \otimes b = \begin{cases} - & a = - \vee b = - \\ a + b - 1 & a + b \leq n - 1 \;\; (a, b \in N) \\ \top & \text{otherwise} \end{cases}$$

$$a \rightarrow b = \begin{cases} - & a > b \\ \top & a = - \vee b = \top \\ b - a + 1 & \text{otherwise .} \end{cases}$$

[14] The matrices $\overline{A}_n$, for $0 \leq n \leq \omega$, were first introduced in [Av84]. The corresponding logic is further investigated in [Av97].

Again, $\overline{F}_2 = \overline{F}^*_2 = A_0$ and $\overline{F}_3 = \overline{F}^*_3 = A_3 = M_3$ (F_4, by the way, is isomorphic to the matrix used in the proof of Proposition 11).

We note that unlike the other examples, there does not seem to be an obvious way of generalizing $\overline{F}_n$ to an infinite matrix (unless we use nonstandard natural numbers...).

Constructions. T-lattices (i.e. T-structures which are based on lattices rather than just on posets) have an equational characterization, since the condition $a \leq b \Leftrightarrow a \otimes \sim b = -$ can be replaced by the two conditions:

$$\text{(i)} \quad a \otimes (a \to b) \leq b \qquad\qquad \text{(ii)} \quad (a \wedge b) \otimes \sim b = - \ .$$

It follows that the class of T-lattices is closed under the various standard operations (like products) under which all algebraic classes having this property are closed.[15] F-structures, in contrast, are not closed under product (the two-valued Boolean algebra is the only structure which belongs to both classes, and so it is impossible for both classes to be closed under product). Now Proposition 27 provides one simple method of obtaining a new F-structure from an old one by adding two new elements. In fact, $\overline{F}^*_{2n}$ is obtained from $\overline{F}^*_2$ by $n-1$ applications of this method, while $\overline{F}^*_{2n+1}$ is obtained from $\overline{F}^*_3$ $(=\overline{A}_1)$ by $n-1$ such applications. The next definition and proposition provides another method, which is a sort of a substitute for the product operation.

Definition 13. Let, for each $\alpha \in I$, $\overline{S}_\alpha = \langle S_\alpha, \leq_\alpha, \sim_\alpha, \otimes_\alpha, \top, - \rangle$ be an F-structure. Assume that $S_\alpha \cap S_\beta = \{\top, -\}$ for $\alpha \neq \beta$. The ***composition*** of $\{S_\alpha | \alpha \in I\}$ is the structure $\overline{S} = \langle S, \leq, \sim, \otimes, \top, - \rangle$ defined as follows:

$$S = \bigcup_{\alpha \in I} S_\alpha$$
$$\sim a = \sim_\alpha a \quad \text{if} \quad a \in S_\alpha$$
$$a \leq b \Leftrightarrow \exists \alpha (a \in S_\alpha \wedge b \in S_\alpha \wedge a \leq_\alpha b)$$
$$a \otimes b = \begin{cases} a \otimes_\alpha b & a \in S_\alpha \wedge b \in S_\alpha \\ \top & \text{otherwise} \ . \end{cases}$$

Proposition 30. *The composition of $\{\overline{S}_\alpha | \alpha \in I\}$ is also an F-structure (under the conditions of Definition 13). If all the $\overline{S}_\alpha$ are F-lattices then so is their composition.*

Proof: Left to the reader.

[15] In the case of product it is easy to see that T-structures in general are also closed under it.

An example. $\overline{A}_n$, for $1 \leq n \leq \omega$ is the composition of n copies of Sobociński's F-structure $M_3(= A_3)$, since for each k, $\{-, I_k, \top\}$ forms such a copy.

Note. The composition of F-monoids is not an F-monoid. However, we can define m-composition of F-monoids similarly. Only now we have to assume that $S_\alpha \cap S_\beta = \{\top, -, 1, \sim 1\}$ (all other definitions remain the same). It is easy to see that m-composition of F-monoids is again an F-monoid (which is a lattice if all the original F-monoids are).

The composition method can also be used to get an *sms* from a pair of an F-structure and a T-structure.

Definition 14. Let $\overline{S}_F$ and $\overline{S}_T$ be an F-structure and a T-structure, respectively. Assume that $S_F \cap S_T = \{\top, -\}$. Define $\overline{S}_F * \overline{S}_T$ as the following structure $\langle S, \leq, \sim, \otimes, \top, -, D >$

$$S = S_F \cup S_T$$

$$\sim a = \begin{cases} \sim_F a & a \in S_F \\ \sim_T a & a \in S_T \end{cases}$$

$$a \leq b \Leftrightarrow (a \in S_F \wedge b \in S_F \wedge a \leq_F b) \vee (a \in S_T \wedge b \in S_T \wedge a \leq_T b)$$

$$a \otimes b = \begin{cases} a \otimes_F b & a \in S_F \wedge b \in S_F \\ a \otimes_T b & a \in S_T \wedge b \in S_T \\ a & a \in S_T \wedge b \notin S_T \\ b & b \in S_T \wedge a \notin S_T \end{cases}$$

$$D = \{a \in S_F | a \neq -\}$$

Proposition 31. *The composition of an F-structure (F-monoid) and a T-structure is an sms (smm). If both of the original structures are non-trivial (i.e., both have at least three elements), then this sms is neither a T-structure nor an F-structure.*

Proof: Again, left for the reader.

Example: 4-valued strong multiplicative structures. An exhaustive enumeration reveals that there are exactly six such non-isomorphic strong multiplicative structures. Three of them are T-structures. These are $\mathcal{L}_4$, $\mathcal{L}_4^*$ and the four-valued Boolean algebra (which is the product of two copies of the two-valued Boolean algebra). Two of the other structures are F-structures. These are $\overline{F}_4$ and $\overline{F}_4^*$. Finally, the last one is an *smm* which is neither a T-structure nor an F-structure. This *smm* is obtained by composing $\overline{A}_1$ and $\mathcal{L}_3$ according to Definition 14.

Structures which are not based on lattices. All the examples we have given so far are of T-lattices or F-lattices. We end this section with examples of a T-structure and an F-monoid which are not based on lattices.

(i) $TSIX$ consists of the six elements $\{-, \sim a, \sim b, a, b, \top\}$. $\sim$ is defined in the obvious way. We have $- \leq x \leq \top$ for all x and $\sim x \leq y$ when $x, y \in \{a, b\}$. (Hence $a \wedge b$ does not exist.) The operation $\otimes$ is defined according to the following matrix (together, of course, with the conditions that $- \otimes x = x \otimes - = -$ and $x \otimes \top = \top \otimes x = x$):

x	a	b	$\sim a$	$\sim b$
a	$\sim b$	$\sim a$	$-$	$\sim b$
b	$\sim a$	b	$\sim a$	$-$
$\sim a$	$-$	$\sim a$	$-$	$-$
$\sim b$	$\sim b$	$-$	$-$	$-$

It is easy to check that $TSIX$ is indeed a T-structure.

(ii) $FEIGHT$ consist of the eight elements $\{-, \top, 1, \sim 1, \sim a, \sim b, a, b\}$. $\sim$ is again defined in the obvious way, and $- \leq 1 \leq x \leq \sim 1 \leq \top$ for $x \in \{\sim a, \sim b, a, b\}$. $x \otimes y$ is defined to be $\top$ if $x \not\leq \sim y$, $-$ if $x = -$ or $y = -$, x if $y = 1$, y if $x = 1$ and ~ 1 in any other case. Again, it is easy to see that this is an F-monoid which is not based on a lattice.

The example of $FEIGHT$ can easily be generalized as follows: Let $\overline{O} = \langle O, \leq_O, \sim_O \rangle$ be a poset with an involution. Define an F-structure based on $\overline{O}$ as follows. Let $\top, -, 1$ and f be four new different objects, not in O. Let

$$S = O \cup \{\top, -, 1, f\}$$

$$\sim \top = -, \ \sim -, = \top, \ \sim 1 = f, \ \sim f = 1, \ \sim a = \sim_O a \text{ if } a \in O$$

$$- \leq 1 \leq a \leq f \leq \top \quad \text{for every} \quad a \in O$$

$$a \leq b \Leftrightarrow a \leq_O b \quad \text{if} \quad a, b \in O$$

$$a \otimes b = \begin{cases} - & a = - \vee b = - \\ a & b = 1 \\ b & a = 1 \\ \top & a \not\leq \sim b \\ f & \text{otherwise} . \end{cases}$$

It is easy to see that $\langle S, \leq, \sim, \otimes, \top, -, 1 \rangle$ is an F-monoid.

4.4 Idempotent T-structures and F-structures

In this section we investigate and characterize a particularly important type of T-structures and F-structures.

Definition 15. A multiplicative structure is called *idempotent* if $a \otimes a = a$ for all a.

Proposition 32. *A T-structure is idempotent iff it is a Boolean algebra.*

Proof: Obviously, idempotency of $\otimes$ is a necessary condition for a T-structure $\overline{S}$ to be a Boolean algebra. Conversely, suppose $\overline{S}$ is an idempotent T-structure. Since $c \leq a$ and $c \leq b$ imply that $c \otimes c \leq a \otimes b$, this means that if $c \leq a$ and $c \leq b$ then $c \leq a \otimes b$. Hence $a \otimes b$ is the meet of a and b (recall that in T-structures $a \otimes b \leq a$ and $a \otimes b \leq b$). Since $\sim$ is an involution, this entails that $a + b = \sim (\sim a \otimes \sim b)$ is the join of a and b. It can easily be checked now that $\langle S, \sim, \otimes, +, \top, - \rangle$ is a Boolean algebra. For example, the distributive law is proved as follows:

$$\begin{array}{ll} a \otimes b \leq a \otimes b & \\ a \otimes \sim (a \otimes b) \leq \sim b & \text{(by Proposition 18(iii))} \\ a \otimes (\sim (a \otimes b) \otimes \sim (a \otimes c)) \leq \sim b & \\ a \otimes (\sim (a \otimes b) \otimes \sim (a \otimes c)) \leq \sim c & \text{(similarly)} \\ a \otimes (\sim (a \otimes b) \otimes \sim (a \otimes c)) \leq \sim b \otimes \sim c & \text{(see above)} \\ a \otimes \sim (\sim b \otimes \sim c) \leq \sim (\sim (a \otimes b) \otimes \sim (a \otimes c)) & \text{(Proposition 18(iii))} \\ a \otimes (b + c) \leq a \otimes b + a \otimes c \ . & \end{array}$$

□

Lemma 7. *In any F-structure:*

(i) If $b \neq -$ then $a \leq a \otimes b$.

(ii) If $a \neq -$ and $b \neq -$ then $a \otimes b$ is an upper bound of a and b.

(iii) $a \leq a \otimes a$.

Proof:

(i) This follows from Proposition 18(vi).

(ii) Immediate from (i).

(iii) This is obvious in case $\boldsymbol{a} = -$, and follows from part (i) in case $\boldsymbol{a} \neq -$.

Theorem 33. *An F-structure is idempotent iff it is isomorphic to $\overline{A}_c$ for some cardinal c.*

Proof: Every $\overline{A}_c$ is idempotent by definition. Conversely, suppose that $\overline{S}$ is an idempotent F-structure. Let $\boldsymbol{a} \in S - \{-, \top\}$. Then $\boldsymbol{a} \neq -$ and $\sim \boldsymbol{a} \neq -$ and so $\sim \boldsymbol{a} \leq \boldsymbol{a} \otimes \sim \boldsymbol{a}$, by Lemma 7(i). On the other hand the idempotency condition $\boldsymbol{a} \otimes \boldsymbol{a} \leq \boldsymbol{a}$ implies, by Proposition 18(iii), that $\boldsymbol{a} \otimes \sim \boldsymbol{a} \leq \sim \boldsymbol{a}$. It follows that $\sim \boldsymbol{a} = \boldsymbol{a} \otimes \sim \boldsymbol{a}$. By substituting $\sim \boldsymbol{a}$ for $\boldsymbol{a}$ in this identity we get that also $\boldsymbol{a} = \boldsymbol{a} \otimes \sim \boldsymbol{a}$. Hence $\boldsymbol{a} = \sim \boldsymbol{a}$ if $\boldsymbol{a} \neq -, \top$. Since $\sim$ is an involution, this implies that there are no $\boldsymbol{a}, \boldsymbol{b}$ such that $- < \boldsymbol{a} < \boldsymbol{b} < \top$. This and Lemma 7(ii) together imply that if $\boldsymbol{a} \neq -, \boldsymbol{b} \neq -$ and $\boldsymbol{a} \neq \boldsymbol{b}$ then $\boldsymbol{a} \otimes \boldsymbol{b} = \top$. It follows that $\overline{S}$ is isomorphic to $\overline{A}_c$, where $\boldsymbol{c}$ is the cardinality of $S - \{-, \top\}$. □

5 Soundness and Completeness Theorems

In this section we prove the soundness and completeness theorems of the various strong systems relative to the corresponding classes of structures. The main theorems are those concerning SR_m and F-structures in §5.3, since for BCK_m the theorems are not really new, while for SLL_m the proof is just a straightforward adaption of a known proof in the case of LL_m.

5.1 SLL_m and strong multiplicative structures

Definition 16.

(i) An ***interpretation*** for the basic or extended multiplicative languages is a pair $\langle \overline{S}, \boldsymbol{v} \rangle$ where $\overline{S}$ is a strong multiplicative structure and $\boldsymbol{v}$ a valuation in S (for the sentences of the language) which respects the operations (these include $\top$ and $-$ in the case of the extended language). $\langle \overline{S}, \boldsymbol{v} \rangle$ is called an ***m-interpretation*** if $\overline{S}$ is a strong multiplicative monoid. An ***interpretation*** for the ***full*** multiplicative language is an $\boldsymbol{m}$-interpretation for which $\boldsymbol{v}(1) = 1_s$ (the identity of $\overline{S}$).[16]

(ii) A ***model*** of a sentence A of the (basic, extended, full) multiplicative language is an interpretation $\langle \overline{S}, \boldsymbol{v}\rangle$ (of the appropriate type) such that $\boldsymbol{v}(A) \in D$. An $\boldsymbol{m}$-model of A is a model $\langle \overline{S}, \boldsymbol{v}\rangle$ of A in which $\overline{S}$ is an ***smm***, and a ***G-model*** of A is a model $\langle \overline{S}, \boldsymbol{v}\rangle$ of A in which $\overline{S}$ is a strong Girard structure. An ***($m^\perp, G^\perp$) model*** of a theory $\mathcal{T}$ is an ($\boldsymbol{m}^\perp, G^\perp$) model of every element of $\mathcal{T}$.

Theorem 34 (strong soundness and completeness theorem for SLL_m). ***$\mathcal{T} \vdash_{SLL_m} \varphi$ iff every model of $\mathcal{T}$ is a model of φ, iff every m-model of $\mathcal{T}$ is an m-model of A iff every G-model of $\mathcal{T}$ is a G-model of A. The same result obtains for SLL_m^T and SLL_m^b.***

Proof: The "only if" parts (i.e. soundness) are easy and are left for the reader. We do the "if" part (i.e. strong completeness) for the case of SLL_m. The proof for the other cases is the same.

So assume that $\mathcal{T} \nvdash_{SLL_m} \boldsymbol{\varphi}$. By Proposition 15*, $\mathcal{T} \nvdash_{SLL_m^b} \boldsymbol{\varphi}$. Construct $\overline{S}$, the Lindenbaum algebra of $\mathcal{T}$ in the usual way. Define $A \equiv B$ if both $\mathcal{T} \vdash_{SLL_m^b} A \to B$ and $\mathcal{T} \vdash_{SLL_m^b} B \to A$. This is an equivalence relation. Let $[A]$ denote the equivalence class of A. Let S be the set of equivalence classes (in the full language). Define $[A] \le [B]$ iff $\mathcal{T} \vdash_{SLL_m^b} A \to B$, $\sim [A] = [\sim A]$, $[A] \otimes [B] = [A \otimes B]$, and $D = \{[A] \mid \mathcal{T} \vdash_{SLL_m^b} A\}$. It is easy to see that $\overline{S} = \langle S, \le, \sim, \otimes, D\rangle$ is an ***smm***, in which $[\top]$ is the greatest element, $[-]$ – the least element and $[1]$ – the identity element. Let $\boldsymbol{v}$ be the canonical valuation $\boldsymbol{v}(A) = [A]$. This indeed is a valuation, and $\langle \overline{S}, \boldsymbol{v}\rangle$ is clearly an $\boldsymbol{m}$-model of $\mathcal{T}$ which is not a model of $\boldsymbol{\varphi}$. Now by Theorem 23, $\overline{S}$ can be embedded into a strong Girard structure $\overline{S}^*$ with the same $\top, -$ and 1. This $\langle \overline{S}^*, \boldsymbol{v}\rangle$ is a G-model of $\mathcal{T}$ which is not a model of $\boldsymbol{\varphi}$. □

Note. The $(\otimes - E)$ rule is needed to ensure that the Lindenbaum algebra in the last proof is indeed a ***strong*** multiplicative structure. Without it we would have got the corresponding strong soundness and completeness theorem of LL_m (say) relative to multiplicative structures (or monoids or Girard structures) as given in [Av88].

[16]Note that if φ is in the basic or extended multiplicative language then what we take as an interpretation for it depends on the context. When we consider it as a sentence of the full language an interpretation is really an m-interpretation.

5.2 BCK_m and T-structures

Definition 17. A *T-model* (*GT-model*) of a sentence φ (a theory $\mathcal{T}$) is a model $\langle \overline{S}, v\rangle$ of it in which $\overline{S}$ is a T-structure (Girard's T-structure).

Theorem 35 (strong completeness and soundness theorem for BCK_m). *$\mathcal{T} \vdash_{BCK_m} \varphi$ iff every T-model of $\mathcal{T}$ is a model of φ iff every GT-model of $\mathcal{T}$ is a model of φ. The same is true for BCK_m^T and BCK_m^b.*

Proof: Soundness follows from Lemma 6 (in the proof of Proposition 24). The proof of completeness is similar to that of Theorem 34. We only have to show that in BCK_m^b the Lindenbaum algebra of a theory $\mathcal{T}$ is a T-structure. This is obvious from Proposition 26 and the fact that in BCK_m^b $1 \to \top$ and $\top \to 1$ are both theorems, and so $[1] = [\top]$. At the end of the proof we should use Theorem 29 (rather than Theorem 23). □

5.3 SR_m and F-structures

Definition 18. (i) An *F-model* (*mF-model, GF-model*) of a sentence φ (a theory $\mathcal{T}$) in the basic, extended, or full multiplicative language is a model (m-model, G-model) $\langle \overline{S}, v\rangle$ of φ (of $\mathcal{T}$) in which $\overline{S}$ is an F-structure (F-monoid, Girard's F-structure).

Note again that what we take as an F-model of a sentence φ (a theory $\mathcal{T}$) in the extended multiplicative language depends on the context. When we view it as a sentence in the full multiplicative language its F-models are what we take as mF-models when we view it as a sentence in the extended multiplicative language. This should cause no confusion: when we refer to $\vdash_{SR_m}$ we assume that all sentences belong to the basic language, when we refer to $\vdash_{SR_m^T}$ we assume that all sentences belong to the extended language, and when we refer to $\vdash_{SR_m^b}$ we assume that all sentences belong to the full language (even those which do not mention any propositional constant).

Theorem 36 (strong completeness and soundness theorem for SR_m). *$\mathcal{T} \vdash_{SR_m} \varphi$ iff every F-model of $\mathcal{T}$ is an F-model of φ, iff every mF-model of $\mathcal{T}$ is an mF-model of φ, iff every GF-model of $\mathcal{T}$ is a GF-model of φ. The same results are true also for SR_m^T and SR_m^b.*

Proof: The soundness part follows from Lemma 7(iii) (which means that $A \to A \otimes A$ is valid in every F-interpretation) and the validity of the other rules and axioms in any interpretation.

We show now the completeness part in the case of SR_m. The proofs for the other two systems are similar. So assume that $\mathcal{T} \nvdash \varphi$ (we will write just "$\vdash$" for "$\vdash_{SR_m}$" until the end of this proof). We construct an F-model of $\mathcal{T}$ which is not an F-model of φ.

As a first step we extend $\mathcal{T}$ to a maximal theory $\mathcal{T}^*$ such that $\mathcal{T}^* \nvdash \varphi$. The maximality of $\mathcal{T}^*$ entails that $A \notin \mathcal{T}^*$ iff $\mathcal{T}^* \cup \{A\} \vdash \varphi$ iff (by the deduction Theorem 13) $\mathcal{T}^* \vdash A \supset \varphi$. We now show two crucial facts about $\mathcal{T}^*$.

Fact 1. $\mathcal{T}^* \vdash \varphi \to A$ for all A.

Suppose otherwise. Then $\mathcal{T}^* \vdash (\varphi \to A) \supset \varphi$ for some A. Now by Lemma 3 (from the proof of Theorem 13) there exists a sentence ψ such that both $\Rightarrow A, \psi$ and $\varphi \to A \Rightarrow \psi$ are provable in GR_m. From these two sequents one can derive $(\varphi \to A) \supset \varphi \Rightarrow \varphi \otimes \psi$ as follows:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \Rightarrow A,\psi}{\varphi \Rightarrow A, \varphi\otimes\psi}}{\Rightarrow \varphi\to A, \varphi\otimes\psi} \qquad \dfrac{\dfrac{\varphi\Rightarrow\varphi \qquad \varphi\to A\Rightarrow\psi}{\varphi,\varphi\to A \Rightarrow \varphi\otimes\psi}}{\varphi\otimes(\varphi\to A)\Rightarrow\varphi\otimes\psi}}{\dfrac{(\varphi\to A)\supset\varphi\Rightarrow\varphi\otimes\psi,\varphi\otimes\psi}{(\varphi\to A)\supset\varphi\Rightarrow\varphi\otimes\psi}}$$

It follows that $\mathcal{T}^* \vdash \varphi \otimes \psi$ and so $\mathcal{T}^* \vdash \varphi$, by $(\otimes - E)$. This is a contradiction.

Fact 2. $\mathcal{T}^* \nvdash A$ iff $\mathcal{T}^* \vdash A \to \varphi$.

The "if" part here is obvious. For the "only if" part, assume $\mathcal{T}^* \nvdash A$. Then $\mathcal{T}^* \vdash A \supset \varphi$. In other words: $\mathcal{T}^* \vdash A \to \varphi \otimes A$. But $\mathcal{T}^* \vdash \varphi \to (A \to \varphi)$ by Fact 1, and so $\mathcal{T}^* \vdash \varphi \otimes A \to \varphi$. It follows that $\mathcal{T}^* \vdash A \to \varphi$, by transitivity.

Now construct the Lindenbaum algebra $\overline{S}$ of $\mathcal{T}^*$ as in the proof of Theorem 34, and let v be the canonical valuation ($v(A) = [A]$). As in the proof of Theorem 34, $\overline{S}$ is a strong multiplicative structure, only this time $[\varphi]$ is the least element $-$, by Fact 1 (and $[\sim \varphi]$ is the upper bound $\top$). Again, as in the proof of Theorem 34, $\langle \overline{S}, v \rangle$ is a model of $\mathcal{T}^*$ (and so of $\mathcal{T}$) but not of φ. It remains to show that it is in fact an F-model. But facts 1 and 2 together imply that $S - D = \{[A] | \mathcal{T}^* \nvdash A\} = \{[A] | A \equiv \varphi\} = \{[\varphi]\} = \{-\}$.

We have constructed an F-model of $\mathcal{T}$ which is not a model of φ. By Proposition 27 we can construct from this F-model an mF-model with exactly the same properties, and by Theorem 29 we can construct from that mF-model a GF-model with the same properties. These facts (and the soundness parts) entail the three completeness theorems. □

We next present some applications of Theorem 36.

Theorem 37. *SR_m^b is a strongly conservative extension of SR_m and SR_m^T.*

Proof: Assume φ and $\mathcal{T}$ are in the basic multiplicative language and that $\mathcal{T} \vdash_{SR_m^b} \varphi$. Then every GF-model of $\mathcal{T}$ is a model of φ (by the soundness of SR_m^b) and so $\mathcal{T} \vdash_{SR_m} \varphi$, by the completeness of SR_m relative to GF-models. The proof for SR_m^T is identical.

Note. I have found no purely syntactical proof of Theorem 37, although I am sure that one exists.

Proposition 38.

(i) $A \otimes A \to A$ *is not provable in* SR_m.

(ii) $((A \to B) \to A) \supset A$ *is provable in* SR_m.

Proof:

(i) This is obvious from Theorem 36, Lemma 7(iii), Theorem 33 and the examples in §4.3 of F-structures which are not idempotent (i.e., not isomorphic to any $\overline{A}_c$).

(ii) An easy computation (using Proposition 18, parts (v) and (vii)) shows that in every multiplicative structure $(- \to b) \to - = -$ for all b. It follows that if $v((A \to B) \to A) \neq -$ then $v(A) \neq -$. Hence every F-model of $(A \to B) \to A$ is an F-model of A, and so $(A \to B) \to A \vdash_{SR_m} A$ by Theorem 36. This entails that $\vdash_{SR_m} ((A \to B) \to A) \supset A$, by the deduction theorem. □

Notes. It is instructive to compare this easy proof of (ii) to the direct derivation of the same sentence in the proof of Theorem 10. This example shows that the completeness theorem can actually be useful in showing theoremhood in SR_m without presenting direct proofs.

5.4 Extensions of SR_m and BCK_m

A careful examination of the various completeness proofs above, which use the Lindenbaum algebra $\overline{S}$ of a certain theory $\mathcal{T}^*$, reveals the following important fact: if A is a sentence all instances of which are theorems of $\mathcal{T}^*$ then A is ***valid*** in $\overline{S}$, i.e., true under ***all*** valuations, not only the canonical one. This observation easily entails the following generalization of Theorems 35 and 36:

Theorem 39. *Let L be an extension of SR_m (BCK_m) by axiom-schemes, and let $m_F(L)$ ($m_T(L)$) be the class of F-structures (T-structures) in which all axioms of L are valid. Then L is strongly sound and complete relative to the semantics of $m_F(L)$ ($m_T(L)$).*

Proof: Soundness is trivial. For completeness, assume $\mathcal{T} \nvdash_L \varphi$. Let $\mathcal{T}^* = \mathcal{T} \cup \{A \mid \vdash_L A\}$. Then $\mathcal{T}^* \nvdash_{SR_m} \varphi$ (the case of BCK_m is similar). It follows, by the proof of Theorem 36 and the observation above that there is an F-model $(\overline{S}, v)$ of $\mathcal{T}^*$, which is not a model of φ and such that all axioms of L are valid in $\overline{S}$. In other words: $\overline{S} \in m_F(L)$, and $(\overline{S}, v)$ is a model of $\mathcal{T}$ which refutes φ. This is exactly what is needed for strong completeness. □

Note. The above theorem (and its proof) is applicable also in case the language of L is an extension of the basic multiplicative language, provided the extra operations get appropriate interpretations in $m_F(L)$ (or $m_T(L)$).

We now examine important applications of Theorem 39.

5.4.1 Classical logic

We can look here at classical logic from two different points of view. First, it can be obtained from BCK_m by adding $A \to A \otimes A$ as an axiom-scheme. This fact, Theorem 39 and Proposition 32 entail that classical logic is strongly sound and complete relative to the semantics of Boolean algebras. Alternatively, classical logic is obtained from SR_m by adding $A \otimes B \to A$ as an axiom-scheme. Now from Lemma 7 it easily follows that $\overline{A}_0$ (the two-valued B.A.) is the only F-structure in which $A \otimes B \to A$ is valid. Hence Theorem 39 entails that classical logic is strongly complete relative to the semantics of $\overline{A}_0$.

5.4.2 $SRMI_m$

$SRMI_m$ is obtained from SR_m by adding $A \otimes A \to A$ as an axiom-scheme. By Theorem 33, the only F-structures in which this axiom is valid are those of the form $\overline{A}_c$. The same is true for $SRMI_m^T$ (with the obvious interpretation of $\top$). Hence we get:

Theorem 40. *$SRMI_m$ and $SRMI_m^T$ are strongly sound and complete relative to the semantics of the $\overline{A}_c$'s (i.e., idempotent F-structures).*

Assuming that we are dealing only with denumerable languages, we can obviously restrict Theorem 40 to the case in which $0 \leq n \leq \omega$. Since $\overline{A}_n$, for $n < \omega$, is a submatrix of $\overline{A}_\omega$, we can conclude:

Theorem 40*. *$SRMI_m$ and $SRMI_m^T$ are strongly sound and complete relative to $\overline{A}_\omega$.*

For $SRMI_m^b$ the situation is different. We have:

Theorem 41. *$SRMI_m^b$ is strongly sound and complete relative to $\overline{A}_1$.*

Proof: $SRMI_m^b$ is obtained from SR_m by adding the axiom-schemes 1, $1 \to (A \to A)$ and $A \otimes A \to A$. The only F-structures in which all these axioms are ***valid*** are the idempotent F-monoids, and using Theorem 33, we see that the only idempotent F-monoids are $\overline{A}_0$ and $\overline{A}_1$. Since $\overline{A}_0$ is a substructure of $\overline{A}_1$, Theorem 41 follows from Theorem 39. □

Notes.

1. $\overline{A}_\omega$ can be embedded, of course in an F-monoid by Proposition 27, but this F-monoid is not idempotent, and so $SRMI_m$ is not sound relative to it.

2. Theorems 40* and 41 were first proved [Av97]. The weak completeness of RMI_m relative to $\overline{A}_\omega$ was first proved in [Av84]. In these papers it is shown that RMI_m and $SRMI_m$ are strongly decidable and have both cut-free Gentzen-type formulations (the one for $SRMI_m$ uses hypersequents rather than ordinary sequents). These facts and the simple semantics of $SRMI_m$ mean that this logic is a really nice relevance logic. It should be noted that the decidability of SR_m is still open, and no decent Gentzen-type system for it is known at the moment.

5.4.3 SRM_m

RM_m is the system which is obtained from R_m by adding to it the axiom-scheme $\sim (A \to A) \to (B \to B)$, and by adding to GR_m the "mix" rule:[17] from $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. Now $A \otimes A \to A$ is provable in RM_m. Using this fact and Theorem 33, it is easy to see that the above axiom-scheme is again valid only in $\overline{A}_0$ and $\overline{A}_1$ (among the F-structures). Hence, as in the case of $SRMI_m^b$ we get:

Theorem 42. *SRM_m is strongly sound and complete relative to $\overline{A}_1$.* □

Note. Theorem 42 was also first proved in [Av97]. The ***weak*** soundness of completeness of RM_m itself relative to $\overline{A}_1$ was shown already in [So52] (see also [Av84]).

[17] This name is taken from [Gi87]. In the past, I preferred the name "combining" for this rule.

5.4.4 Adding additive connectives

Although the need for additive connectives is not clear, in my opinion, in systems in which $\otimes$ really functions as an extensional conjunction, they can easily be introduced in the framework of T-structures and F-structures. They have a clear interpretation in structures which are based on ***lattices***. Accordingly, we define:

Definition 19.

(i) A ***T-lattice*** is a T-structure in which the underlying poset is a lattice.

(ii) An ***F-lattice*** is an F-structure in which the underlying poset is a lattice, and the following condition is satisfied (where $a \wedge b$ denotes the g.l.b of a and b):

(*) If $a \neq -$ and $b \neq -$ then $a \wedge b \neq -$.

Note. Since $\sim$ is an involution, already the existence of $a \wedge b$ for every a, b means that the underlying poset is a lattice.

Theorem 43. *Any T-lattice can be embedded in a Girard's T-structure, and any F-lattice can be embedded in a Girard's F-structure.*

Proof: For T-lattices this is an immediate corollary of Theorem 29. For F-lattices we note first that the embedding of an F-lattice $\overline{S}$ in an F-monoid $\overline{S}'$ which is described in Proposition 27 preserves the lattice operations (here condition (*) in the definition of an F-lattice is crucial, since if a, b are two elements of $\overline{S}$ such that $a > -$, $b > -$ and $a \wedge b = -$ then in $\overline{S}'$ $a \wedge b = 1 \neq -$). By applying Theorem 29 to $\overline{S}'$ we therefore get a Girard's F-structure as required. □

Now the "additive" (or "extensional") conjunction $\wedge$ is usually characterized in Gentzen-type systems by the following three rules:

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \; .$$

In the framework of BCK_m these three rules are easily seen to be equivalent to the following three axiom-schemes:

$$A \wedge B \to A$$

$$A \wedge B \to B$$

$$((C \to A) \otimes (C \to B)) \to (C \to A \wedge B) \; .$$

Definition 20. $HBCK$ is the system (in the multiplicative-additive language) which is obtained from BCK_m by the addition of the above three axioms.

Theorem 44. *$HBCK$ is strongly sound and complete relative to the semantics of T-lattices, as well as relative to the narrower semantics of Girard's T-structures.*

Proof: This follows from Theorems 39 and 43. □

When we turn our attention to R_m, things become more complicated, since it is well known that in order to translate the above Gentzen-type rules into a Hilbert-type formalism, it is necessary to add a new rule of inference (usually adjunction: from A and B infer $A \wedge B$). However, most of our proofs above are not valid if we have this extra rule. Luckily, in the stronger framework of SR_m we ***can*** translate the adjunction rule into an equivalent ***axiom***: $A \otimes B \supset A \wedge B$.

Definition 21.

1. $SR_{\min}$ is the system which is obtained from SR_m by the addition of the following four axioms:

$$A \wedge B \to A$$
$$A \wedge B \to B$$
$$(C \to A) \wedge (C \to B) \to (C \to A \wedge B)$$
$$A \otimes B \supset A \wedge B \ .$$

2. SR is obtained from $SR_{\min}$ by adding the distributivity axiom $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$ (where $A \vee B = \sim(\sim A \wedge \sim B)$).

Proposition 45. *SR is equivalent to $R + (\otimes - E)$.*

Proof: Left to the reader.

Theorem 46.

(i) $SR_{\min}$ is strongly sound and complete relative to the semantics of F-lattices, as well as relative to the narrower semantics of Girard's F-structures.

(ii) SR is strongly sound and complete relative to the semantics of distributive F-lattices.

Proof:

(i) Soundness of the first three axioms is due to the fact that we are dealing with lattices, while that of the fourth is ensured by condition (*) from the definition of an F-lattice. The completeness parts are again corollaries of Theorems 39 and 43.

(ii) This again follows from Theorem 39. □

A final note: the frameworks of Girard's T-structures and Girard's F-structures seem to provide a natural semantics for the quantifiers (the additive quantifiers, to be precise). We believe that this indeed is the case. There is, however, a difficulty in applying the methods and results above: it seems that in any corresponding Hilbert-type formulation a new rule (like generalization) is needed. We leave this problem to future research.

Another interesting possibility here is generalizing $\otimes$ to ***multiplicative*** universal quantifiers (exactly as $\forall$ is a kind of an infinite additive conjunction). This possibility, as well as the whole subject of quantifiers in the frameworks of T-structures and F-structures, is also left for future research.

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