Multiplicative Conjunction as an Extensional Conjunction

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Abstract

We show that the rule that allows the inference of A from $A \otimes B$ is admissible in many of the basic multiplicative (intensional) systems. By adding this rule to these systems we get, therefore, conservative extensions in which the tensor behaves as classical conjunction. Among the systems obtained in this way the one derived from RMI_m (= multiplicative linear logic together with contraction and its converse) has a particular interest. We show that this system has a simple infinite-valued semantics, relative to which it is *strongly* complete, and a nice cut-free Gentzen-type formulation which employs hypersequents (= finite sequences of ordinary sequents). Moreover: classical logic has a simple, strong translation into this logic. This translation uses definable connectives and preserves the consequence relation of classical logic (not just the set of theorems). Similar results, but with a 3-valued semantics, obtain if instead of RMI_m we use RM_m (the purely multiplicative fragment of RM).

1 Introduction

The purely multiplicative fragment[1] of substructural logics in general, and relevance logics in particular, is universally considered to be the best behaved and best understood fragment, especially from a proof-theoretical point of view (it is called "the multiplicative paradise" in [Gi87]). Relevantists have never considered it as sufficient, though, since its expressive power (so everybody believes) is rather limited. A crucial connective which it seems to lack is conjunction. Sure, it does have the tensor $\otimes$ ("fusion" in relevantists' terminology), which has many intuitive properties of conjunction. But neither A nor B follow from $A \otimes B$ in the

[1] We follow [Do93] and others in using this term, borrowed from [Gi87], for all substructural logics. Relevantists use the term "intensional".

standard substructural logics, contradicting a most fundamental intuition about conjunction. According to this intuition the three classical (or intuitionistic) natural deduction rules should be valid for any "conjunction" operator.

The relevantists solution to this problem is to *enrich* the multiplicative language with a new connective, &, which is intended to represent conjunction. There is a big obstacle for this, though: one cannot add to the standard relevance logics the implications $A\&B \rightarrow A$, $A\&B \rightarrow B$ and $A \rightarrow (B \rightarrow A\&B)$. Such an addition is not conservative even in the case of LL_m. In the case of R_m (the multiplicative fragment of R) the result is simply classical logic. To overcome this difficulty the axiom $A \rightarrow (B \rightarrow A\&B)$ is replaced by the adjunction *rule*: from A and B infer $A\&B$. Alas, by doing this the relevantists destroy the principle according to which their multiplicative logics are constructed: that B should follow from the assumptions $A_1, \ldots, A_n$ iff $A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_n \rightarrow B) \cdots)$ is a valid formula.[2] Moreover: the addition of the so-called extensional connectives leads to systems which are much more complicated (from a proof theoretical point of view) than their purely multiplicative fragments, and even with this addition we do not really get classical conjunction. We do have, of course, a translation of classical logic which is the simplest possible: the classical negation, conjunction and disjunction are interpreted by primitive connectives of the logics. Nevertheless, this interpretation is just a *weak* interpretation: *it does not preserve the consequence relation.* Thus MP for the translation of classical implication is not valid. It is only *admissible* in R (and the related systems) by a famous theorem of Meyer and Dunn ([MD69]. See also [Du86]).[3]

It is worth noting that even this partial success is achieved at the cost of losing decidability and of having no decent proof theory.[4]

In this paper we investigate a different approach. We shall try to take *seriously* $\otimes$ as conjunction. The idea is simple. Instead of adding to the multiplicative language a new connective and force it to be a conjunction by a special rule, we shall remain within this language and force $\otimes$ to be a conjunction by a special rule. The rule we use is just the inference of A from $A \otimes B$.

A first thing that needs to be checked is whether by adding this rule we get completely

[2]It is, in fact, not clear at all what the phrase: "B follows from $A_1, \ldots, A_n$" means for the full system R and its relatives. See [Av92a] for a discussion.

[3]One can add it therefore to R and get a new system, R^{DS}, which has exactly the same theorems as R and in which the interpretation *is* a strong translation. Unfortunately, R^{DS} lacks some of the most desirable properties of R. For example: from a single contradiction one can derive in it every formula.

[4]In Linear Logic the situation is even worse: a translation of classical logic, still only a weak one and less simple, is possible only after introducing also the "exponentials" (Lafont has shown that one can use instead second order propositional language without the exponentials. In any case, the basic multiplicative-additive propositional language is not sufficient). Again, decidability is lost ([LMSS92]. See also [Tr92]).

new logics with new sets of valid formulae. The problem in different terminology is whether the new rule is admissible in the various multiplicative relevance logics or not. In section 3 we show that at least in one important extension of R_m it is indeed admissible: in RMI_m (the system which is obtained from R_m when we add to it the mingle axiom). Moreover, the rule is admissible also in all possible extensions of RMI_m in its language, including RM_m (and also in LL_m, by the way).

In the rest of the paper, we concentrate, therefore, on RMI_m and its extensions. Our main conclusion is that by adding the new rule to them we do change $\otimes$ into conjunction, but without losing any relevant feature of the original systems. Moreover: this approach has the following advantages over the more usual one:

1. Unlike R, $SRMI_m$ ($= RMI_m + \frac{A\otimes B}{A}$) has a simple, useful semantics in the form of the infinite-valued matrix A_ω of [Av84]. $SRMI_m$ is *strongly* complete relative to this matrix (while RMI_m itself is only weakly complete relative to it). This is shown in section 4.

2. Unlike R, $SRMI_m$ is decidable (this follows from (1), since A_ω can be used for a decision procedure).

3. There is a *strong* translation, preserving also the consequence relation, of positive classical logic into $SRMI_m$. Moreover, if we add an appropriate propositional constant $\perp$ to $SRMI_m$ we get a conservative extension, $SRMI_m^\perp$, which has *all* the nice properties of $SRMI_m$ and into which *full* classical logic can strongly be translated ($SRMI_m^\perp$ is investigated in section 5, while the translations of classical logic are described in section 7).

4. Although like in the case of R we lose in $SRMI_m$ the relevant deduction theorem, we have something which is very close: B follows in $SRMI_m$ ($SRMI_m^\perp$) from $A_1, \ldots, A_n$ iff there exists C (which can be chosen to be a theorem of R_m) such that $A_1 \to (A_2 \to \cdots \to (A_n \to B \otimes C)\cdots)$ is a theorem. In addition, $SRMI_m$ and $SRMI_m^\perp$ are paraconsistent, have the variable-sharing property and all the relevant properties.

5. $SRMI_m$ ($SRMI_m^\perp$) has a cut-free Gentzen-type formulation with the subformula property. This formulation employs hypersequents (= finite sets of ordinary sequents). Its rules for all the connectives (including $\otimes$!) are exactly the same as in all other substructural logics. Just like in Dosen's principle ([Do93]), it differs from the formulations of other substructural logics only with respect to structural rules. This shows that $\otimes$ is still the usual multiplicative tensor of all other substructural logics.

The Gentzen-type systems are treated in section 8.

Although there seems to be no real need in $SRMI_m$ for the additives, one can introduce in $SRMI_m$ a purely relevant version of them. This possibility is described in section 6. This section includes also a short discussion of quantifiers (to be expanded in the future). An interesting possibility here is to introduce *multiplicative* versions of the quantifiers instead (or along with) the additives ones of the other substructural logics.

2 Preliminaries

This section summarizes material concerning substructural logics which by now is almost common knowledge. See [AB75], [Du86], [Gi87], [Av88].

2.1 Syntactical Matters

Definition 1. The basic multiplicative language this is the propositional language which has a unary connective $\sim$ and two binary connectives: $\rightarrow, \otimes$.

Notes.

1. The notations $\sim$ and $\rightarrow$ are from relevance logic (Girard used $()^\perp$ and $\perp\!\circ$). $\otimes$ is taken from [Gi87] (relevantists had used $\circ$).

2. In the presence of $\sim$, each of the other two connectives is definable in terms of the other. Thus $A \otimes B = \sim (A \rightarrow \sim B)$.[5]

Definition 2. LL_m, R_m, RMI_m, RM_m and CL_m denote, respectively, the purely multiplicative fragments of LL (Linear Logic – [Gi87], [Tr92]), R (the standard relevance logic of Anderson and Belnap [AB75], [AB92] [Du86]), RMI (see, e.g., [Av90b]), RM (Dunn-McCall R-mingle [AB75] [Du86]) and CL (classical logic).

Hilbert-Type Representations

(I) $\mathbf{LL_m}$

Axioms

[5] Another important multiplicative connective is $+$ (or "par" in [Gi87]), defined by $A + B = \sim (\sim A \otimes \sim B)$. This connective is not important for our present purposes.

(I)	$A \to A$	(Identity)
(T)	$(A \to B) \to ((B \to C) \to (A \to C))$	(Transitivity)
(P)	$(A \to (B \to C)) \to (B \to (A \to C))$	(Permutation)
(R1)	$(A \to (B \to C)) \to (A \otimes B \to C)$	(Residuation)
(R2)	$(A \otimes B \to C) \to (A \to (B \to C))$	
(N1)	$(A \to \sim B) \to (B \to \sim A)$	(Contraposition)
(N2)	$\sim\sim A \to A$	(Double Negation)

Rule of inference.

$$\frac{A \qquad A \to B}{B}$$

(II) $\mathbf{R_m}$: LL_m together with

(C) $A \to A \otimes A$ (Contraction)

(III) $\mathbf{RMI_m}$: R_m together with

(M) $A \otimes A \to A$ (Mingle)

(IV) $\mathbf{RM_m}$: RMI_m together with

(N3) $\sim (A \otimes \sim A \otimes B \otimes \sim B)$ (Mix)[6]

(V) $\mathbf{CL_m}$: R_m together with

(W) $A \otimes B \to A$ (Weakening).

Notes.

1. In RMI_m, RM_m and CL_m Axiom (I) is derivable from the other axioms.

2. The above are not the standard representations of the various logics. In [AB75], for example, the contraction axiom is $(A \to (A \to B)) \to (A \to B)$. Mingle is $A \to (A \to A)$ and weakening is $A \to (B \to A)$. It can easily be seen that these are equivalent to what we present above. Moreover, $A \otimes B$ is *defined* there as $\sim (A \to \sim B)$, and $(R1), (R2)$ become then derivable. The systems are usually called, therefore, $R_{\underset{\to}{\sim}}$, $RMI_{\underset{\to}{\sim}}$ etc. We have chosen here the above representations because they reflect the properties of $\otimes$.

Proposition 1. *(Variable-sharing property): If $A \to B$ is provable in any of the systems above (except classical logic) then A and B share a variable.*

Definition 3. Let L be any of the systems above. The associated (Tarskian) consequence relation $\vdash_L$ is defined in the usual way: $\mathcal{T} \vdash_L A$ iff there exists a sequence $A_1, \ldots, A_n = A$ such that each A_i either belongs to $\mathcal{T}$, or is an instance of an axiom, or follows from two previous ones by MP[7].

[6]This name is taken from [Gi87], since this axiom is equivalent to what was called there "Mix".

[7]See [Av92a], [Av92b] for a discussion of the various other consequence relations that can naturally be associated with substructural logics.

Proposition 2. *(Deduction theorems:)*

(1) [Av92b]: $\mathcal{T}, A \vdash_{LL_m} B$ *iff* $\mathcal{T} \vdash_{LL_m} \overbrace{A \otimes A \otimes \cdots \otimes A}^{m \text{ times}} \to B$ *for some* $m \geq 0$
(2) For $L = R_m$, RMI_m *and* RM_m *(or any other extension of* R_m *by axiom schemes)* $\mathcal{T}, A \vdash_L B$ *iff either* $\mathcal{T} \vdash_L B$ *or* $\mathcal{T} \vdash_L A \to B$.

Proposition 3. $A \leftrightarrow B \vdash_{LL_m} \varphi(A) \leftrightarrow \varphi(B)$ *(where* $\varphi(B)$ *is obtained from the formula* $\varphi(A)$ *by replacing some occurences of* A *by* B*).*

Gentzen-Type Representations

(I) $\boldsymbol{GLL_m}$

Axioms:

$$A \Rightarrow A$$

Logical rules:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2} \qquad \frac{A, \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \otimes B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad \Gamma_2 \Rightarrow \Delta_2, B}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \ A \otimes B}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\sim A, \Gamma \Rightarrow \Delta} \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim A}$$

Structural rules: Exchange:

$$\frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, B, A, \Delta_2}$$

(II) $\boldsymbol{GR_{\mathrm{m}}}$**:** Like GLL_m, with contraction added:

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

(III) $\boldsymbol{GRMI_{\mathrm{m}}}$**:** Like GR_m with expansion (the converse of contraction) added:

$$\frac{A, \Gamma \Rightarrow \Delta}{A, A, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A, A}$$

Alternatively, $GRMI_m$ can be obtained from GR_m by adding to it *mingle* (or *relevant* mix):

$$\frac{A, \Gamma_1 \Rightarrow \Delta_1 \qquad A, \Gamma_2 \Rightarrow \Delta_2}{A, A, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad \Gamma_2 \Rightarrow \Delta_2, A}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A, A}$$

(obviously, mingle is derivable from expansion using cuts, while expansion is directly derivable from mingle in the presence of contraction[8].)

(IV) GRM_m: Like GR_m, with "mix" added:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

(V) GCL_m: Like GR_m, with full weakening added.

Proposition 4. *The cut rule in its multiplicative form:*

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

is admissible in all the systems above (in fact, cuts can systematically be eliminated, so this is equivalent to a cut elimination theorem).

Proposition 5. (1) $\vdash_{GL} \Rightarrow A$ *iff* $\vdash_L A$, *for* $L = LL_m, R_m, RM_m, RMI_m, CL_m$. *More generally,*

$$\vdash_{GL} A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k \quad \text{iff} \quad \vdash_L \sim A_1 + \sim A_2 + \cdots + \sim A_n + B_1 + B_2 + \cdots + B_k \ .$$

(2) $A_n, \ldots, A_n \vdash_L B$ if $\Rightarrow B$ is derivable in GL from $\Rightarrow A_1, \ldots, \Rightarrow A_n$ (using cuts).

Note. Except in CL_m, it is not the case that $A_1, \ldots, A_n \vdash_L B$ iff $\vdash_{GL} A_1, \ldots, A_n \Rightarrow B$. Thus $A \rightarrow (A \rightarrow B), A \vdash_{LL_m} B$, but the corresponding sequent is not derivable in GLL_m.

2.2 Semantical Matters

No *simple* sound and complete semantics for LL_m and R_m is known at the time this paper is written.[9] The situation with respect to the other 3 systems is different, though:

Definition 4 [Av84].

1. The structure $\mathcal{A}_\omega = \langle A_\omega, \sim, \rightarrow, \otimes \rangle$:

[8] When one adds to the language relevant additives then only the version with mingle admits cut-elimination. See [Av91a].

[9] The abstract phase spaces described in [Gi87] for LL_m or the various structures reviewed in [Du86] and [AB92] are not satisfactory from the simplicity point of view, at least in my opinion.

(i) $A_\omega : \{\top, \bot, I_1, I_2, I_3, \ldots\}$

(ii) $\sim \top = \bot$, $\sim \bot = \top$, $\sim I_k = I_k$ $(k = 1, 2, \ldots)$

(iii) $a \to b = \begin{cases} \top & a = \bot \text{ or } b = \top \\ I_k & a = b = I_k \\ \bot & \text{otherwise} \end{cases} \qquad a \otimes b = \begin{cases} \bot & a = \bot \text{ or } b = \bot \\ I_k & a = b = I_k \\ \top & \text{otherwise .} \end{cases}$

2. $\mathcal{A}_n (n \geq 0)$ is the substructure of $\mathcal{A}_\omega$ which consists of $\{\top, \bot, I_1, \ldots, I_n\}$.

3. $\mathcal{T} \Vdash_{A_\omega} \varphi$ iff for every valuation v in $\mathcal{A}_\omega$ which respects the operations, if $v(A) \neq \bot$ for all $A \in \mathcal{T}$ then $v(\varphi) \neq \bot$. $\Vdash_{A_n}$ is defined similarly.

Note. $\mathcal{A}_1$ was first introduced and its set of valid sentences axiomatized in [So52]. It is known, therefore, as Sobociński 3-valued logic. $\Vdash_{A_0}$ is, of course, just classical logic.

Proposition 6. *(Strong soundness of RMI_m relative to $\mathcal{A}_\omega$): If $\mathcal{T} \vdash_{RMI_m} A$ then $\mathcal{T} \Vdash_{A_\omega} A$.*

Proposition 7. *(Weak completeness of RMI_m relative to $\mathcal{A}_\omega$): $\vdash_{RMI_m} A$ iff $\Vdash_{A_\omega} \mathcal{A}$.*

Proposition 8. *If L is any logic in the basic multiplicative language which extends RMI_m then there exist $n \geq 0$ such that for all φ, $\vdash_L \varphi$ iff $\Vdash_{A_n} \varphi$.*

The proofs of propositions 6-8 can be found in [Av84].

Theorem 8 characterizes all the possible extensions of RMI_m from the point of view of *theoremhood*. Two important such extensions are classical logic and RM_m. The first is just $\Vdash_{A_0}$. The other is $\Vdash_{A_1}$:

Proposition 9. *(weak completeness of RM_m) $\Vdash_{A_1} \varphi$ iff $\vdash_{RM_m} \varphi$.*

Proposition 10. *(strong soundness of RM_m) If $\mathcal{T} \vdash_{RM_m} \varphi$ then $\mathcal{T} \Vdash_{A_1} \varphi$.*

Proposition 9 was essentially originally proved in [So52]. An easier proof can be found in [Av84]. Proposition 10 is an easy corollary of the proof of Proposition 9.

3 Turning the Tensor into Conjunction

We start with the following natural question: do the converses of propositions 6 and 10 holdΓ In other words: are RMI_m and RM_m also *strongly* complete relative to (respectively) $\mathcal{A}_\omega$ and $\mathcal{A}_1$Γ The answer, even for finite theories, is negative.

Proposition 11. *$A \otimes B \Vdash_{A_\omega} A$ but in general A does not follow from $A \otimes B$ even in RM_m.*

Proof: Since $v(A \otimes B) = \bot$ whenever $v(A) = \bot$, $A \otimes B \Vdash_{A_\omega} A$. Assume now, for contradiction, that $p \otimes q \vdash_{RM_m} p$ when p and q are atomic. Then either $\vdash_{RM_m} p$ or $\vdash_{RM_m} p \otimes q \rightarrow p$ (proposition 2). The first possibility is obviously false. Letting $v(p) = I_1$, $v(q) = \top$ falsifies the second as well, by proposition 10. ■

The fact that RMI_m and RM_m are only weakly complete relative to their characteristic matrix raises two interesting questions. The first is to find a broader semantics for them, relative to which a strong soundness and completeness theorem obtain. A solution to this problem is described in [Av92a] (see there for full references[10]). The second problem is to find proof-theoretical characterizations of $\Vdash_{A_\omega}$ and $\Vdash_{A_1}$ which *are* strongly complete. This problem is solved below. Before reaching the solution we turn, however, to another interesting problem which is strongly related to proposition 11.

With the obvious exception of classical logic, none of the multiplicative logics reviewed above was accepted as a "complete" logic.[11] The main reason, as we note in the introduction, is that they lack an appropriate conjunction connective (which is usually taken as the simplest connective of classical and intuitionistic logics!). Of course, it can be debated what exactly is meant by "an appropriate conjunction". Nevertheless, there is a certain basic intuition concerning it which is best reflected in the standard natural deduction rules for conjunction:

Definition 5. Let L be a logic with a consequence relation $\vdash_L$. A connective $\wedge$ of L is a *standard conjunction* if the following 3 rules are valid:

$$(i) \quad A \wedge B \vdash_L A \quad (ii) \quad A \wedge B \vdash_L B \quad (iii) \quad A, B \vdash_L A \wedge B \ .$$

Proposition 12. *With the exception of classical logic, none of the systems discussed in I.1 has a definable standard conjunction.*

Proof: It suffices to show that there is no formula $\varphi(p,q)$ (containing only p and q as atomic variables) such that:

$$(i) \quad \varphi(p,q) \vdash_{RM_m} p \quad (ii) \quad \varphi(p,q) \vdash_{RM_m} q \quad (iii) \quad p, q \vdash_{RM_m} \varphi(p,q) \ .$$

Assume otherwise. Then the deduction theorem for RM_m (prop. 2), (i) and (ii) imply:

$$(i') \quad \vdash_{RM_m} \varphi(p,q) \rightarrow p \qquad (ii') \quad \vdash_{RM_m} \varphi(p,q) \rightarrow q \ .$$

[10] For RM_m it had (essentially and implicitly) been solved before by Dunn in [Du70] following the discovery of Meyer [AB75] that Sugihara Matrix is characteristic for RM.

[11] What we call RM_m was designed to be one in [So52], but it became later only a fragment of a richer system: RM.

The same theorem and (iii) imply that either $\vdash_{RM_m} p \to (q \to \varphi(p,q))$ or $\vdash_{RM_m} q \to \varphi(p,q)$ or $\vdash_{RM_m} p \to \varphi(p,q)$ or $\vdash_{RM_m} \varphi(p,q)$. The last three possibilities together with (i') and (ii') contradict the fact that $\vdash_{RM_m} \subseteq \vdash_{CL_m}$. In the first case (i') implies that $\vdash_{RM_m} p \to (q \to p)$, but this formula is not valid in $\mathcal{A}_1$ (take $v(p) = I_1, v(q) = \top$). ■

The solution in the literature on linear and relevance logics to the above problem is invariably to *enrich* the purely multiplicative language by a new connective & for which the conditions in definition 5 are assumed.[12]. To make the resulting extensions more coherent, conditions (i) and (ii) are translated into *implications*: $A\&B \to A$, $A\&B \to B$. It is impossible, though, to translate also the third condition into an implication, because the extension would not be conservative in such a case ($A \to (B \to A)$ becomes a theorem). Hence condition (iii) is forced as a *rule* (called "adjunction"). In addition, also the rule $\frac{C\to A \quad C\to B}{C\to A\&B}$ is added, normally in the form of an implication: $(C \to A)\&(C \to B) \to (C \to A\&B)$. By adding these three implications and the rule to LL_m we get LL_{ma} – the multiplicative – additive fragment of Linear Logic. By adding them to R_m, together with a distribution axiom of & over its De-Morgan dual ($\vee$), we get the full relevance system R. Doing the same additions to RMI_m or RM_m results with the system RM, which is conservative over RM_m, but not over RMI_m. In fact, to RMI_m one cannot even add conservatively the two axioms $A\&B \to A, A\&B \to B$ and the adjunction rule (see [Av84]). Hence in RMI ([Av90b]) a *relevant* conjunction is added, for which $\frac{C\to A \quad C\to B}{C\to A\&B}$ is valid but not (in general) adjunction.

This approach of enriching the language has a very serious drawback. The resulting systems are all notoriously much more complicated than their multiplicative fragments from a proof-theoretical as well as a semantic or a computational point of view.[13] There are also difficulties concerning the consequence relations associated with these logics and the associated deduction theorems (see [Av92a]). All these problems do not exist in classical logic, of course. There, however, we do not have to enrich the language, since the multiplicative fragment is strong enough. In particular: the multiplicative $\otimes$ serves in classical logic as a standard conjunction.

Now in *all* the systems above $\otimes$ has a lot of properties which we expect conjunction to have. Thus condition (iii) of definition 5 obtains, and even in the form of a valid implication: $A \to (B \to A \otimes B)$. The equivalence in all of the systems between $A \otimes B \to C$ and $A \to (B \to C)$ is another extremely important property of usual conjunction connectives. In addition $\otimes$ is associative and commutative in all the systems. In RMI_m (and RM_m) it is even idempotent, while in R – semi-idempotent (idempotency properties of $\otimes$ are the main

[12]& is called "extensional" conjunction in the relevance literature, "additive" in [Gi87].

[13]This applies to a lesser degree to RM and RMI then to LL and R, but is still true also for them.

difference between LL_m, R_m and RMI_m according to the Hilbert-type representations in I.1!). It is no wonder, therefore, that $\otimes$ is usually taken as a sort of conjunction. Still, it is difficult to accept it as such as long as A and B do not follow from $A \otimes B$.

The idea we are about to pursue is that instead of adding a new connective and explicitly demand the validity of adjunction for it – we can turn $\otimes$ into a *standard* conjunction by explicitly *demanding* the validity of $\frac{A\otimes B}{A}$ and $\frac{A\otimes B}{B}$.

Definition 6. Let L be any of the Hilbert-type systems described in I.1. Then SL ("strong L") is the system which is obtained from L by adding $\frac{A\otimes B}{A}$ as a new rule.

Note. Since $\vdash_{LL_m} A \otimes B \rightarrow B \otimes A$, the rule $\frac{A\otimes B}{B}$ is derivable in SLL_m and the other strong systems.

By proposition 11, $\vdash_{SL}$ is strictly stronger than $\vdash_L$ for $L = LL_m, R_m, RMI_m, RM_m$. This leaves open the possibility that SL and L are identical from the narrower point of view of *theoremhood*. With the possible exception of R_m, this indeed is the case:[14]

Theorem 13. *The rule* $\frac{A\otimes B}{A}$ *is admissible in* LL_m*,* RMI_m *and* RM_m*. In other words,* $\vdash_{LL_m} A$ *iff* $\vdash_{SLL_m} A$*, and similarly for* RMI_m *and* RM_m*.*

Proof: The case of LL_m is obvious for proof-theoretical reasons. Since we don't have contraction in GLL_m, any cut-free proof of $\Rightarrow A \otimes B$ should end with an application of $(\Rightarrow \otimes)$ to $\Rightarrow A$ and $\Rightarrow B$. Hence $\Rightarrow A \otimes B$ is provable there iff both $\Rightarrow A$ and $\Rightarrow B$ are.

The cases of RMI_m and RM_m are also easy, but because of semantic reasons. Suppose, e.g., that $\nvdash_{RMI_m} A$. Then there is a valuation v in $\mathcal{A}_\omega$ for which $v(A) = \bot$. But then $v(A \otimes B) = v(A) \otimes v(B) = \bot$, and so $\nvdash_{RMI_m} A \otimes B$, by proposition 7. In the case of RM_m we use $\mathcal{A}_1$ instead of $\mathcal{A}_\omega$. (It is possible here also to give a pure proof-theoretical argument. It relies on the fact that RMI_m and RM_m are closed under "weak weakening": If $\vdash \Gamma \Rightarrow \Delta$ and φ contains only atomic formulas which occur in $\Gamma \cup \Delta$ then $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi$ are also provable (see [Av84, proposition I.6]). We shall present this proof elsewhere). ■

Corollary 14.

(i) $\vdash_L \sim (A \rightarrow B)$ *iff* $\vdash_L A$ *and* $\vdash_L \sim B$ *for* $L = LL_m, RMI_m, RM_m$*.*

(ii) $\vdash_{SRMI_m} A \rightarrow B$ *only if* A *and* B *share a variable (see proposition 1).*

[14] This is another demonstration that a logic is not determined by its theorems, but by its consequence relation!

4 The Strong Semantics of $\mathcal{A}_\omega$

SLL_m seems to be an interesting system, strongly related to LL_m. Unfortunately, I am not aware of an interesting, complete semantics for it. Of course, any semantics for SLL_m is also a semantics for LL_m, and it is already at this stage that no nice semantics is known. For RMI_m and RM_m we *do* have the semantics of $\mathcal{A}_\omega$ and $\mathcal{A}_1$, relative to which they are weakly complete. Our next goal is to show that $SRMI_m$ and SRM_m are *strongly* complete relative to these structures.

Our task is easy when we limit ourselves to *finite* theories.

Theorem 15. *$A_1, \ldots, A_n \Vdash_{A_\omega} B$ iff $A_1, \ldots, A_n \vdash_{SRMI_m} B$.*

Proof: The soundness part is almost immediate from proposition 6. One needs only to check the validity of the extra rule $\frac{A \otimes B}{A}$ in $\mathcal{A}_\omega$. This is easy.

For the converse we need first a lemma.

(*) $A_1, \ldots, A_n \Vdash_{A_\omega} B$ iff $\Vdash_{A_\omega} A_1 \to (A_2 \to \cdots \to (A_n \to (B \otimes (A_1 \otimes A_2 \otimes \cdots \otimes A_n))) \cdots)$.

Proof of (*): The if part is obvious. For the converse, assume $A_1, \ldots, A_n \Vdash_{A_\omega} B$, and let v be any valuation in $\mathcal{A}_\omega$. If $v(A_i) = \perp$ for some i then $v(\cdots \to A_i \to \cdots) = \top$. If $v(A_i) \neq \perp$ for all i then also $v(B) \neq \perp$. Hence $v(B \otimes (A_1 \otimes \cdots \otimes A_n) = \top$, unless $v(B) = v(A_1) = \cdots v(A_n) = I_k$ for some k. In the first case $v(A_1 \to (\cdots \to (A_n \to (B \otimes (A_1 \otimes \cdots \otimes A_n))) \cdots))$ is $\top$, in the second I_k, and in both – different from $\perp$.

Returning to the proof of theorem 15, assume $A_1, \ldots, A_n \Vdash_{A_\omega} B$. By (*) and the weak completeness of RMI_m relative to A_ω, $\vdash_{RMI_m} A_1 \to (A_2 \to \cdots (A_n \to (B \otimes (A_1 \otimes \cdots \otimes A_n))) \cdots)$. From the proof of this in RMI_m one can get a proof of B from $A_1, \ldots, A_n$ in $SRMI_m$ by m applications of MP, followed by a single application of $\frac{\varphi \otimes \psi}{\varphi}$. ■

Note. We have shown, in fact, a stronger result: If $A_1, \ldots, A_n \Vdash_{A_\omega} B$ then there is a proof of this fact which is almost entirely in RMI_m, except for one application of the extra rule of $SRMI_m$ at the very end.

The proof of the last theorem suggests the introduction of the following connective:

Definition 7. $A \supset B =_{Df} A \to (B \otimes A)$.

Theorem 16. *(Deduction theorem for $SRMI_m$) $\mathcal{T}, A \vdash_{SRMI_m} B$ iff $\mathcal{T} \vdash_{SRMI_m} A \supset B$. The same is true for any extension of $SRMI_m$ by axioms.*

Proof: Obviously, $A, A \supset B \vdash_{SRMI_m} B$. Hence the "if part".

For the converse, let L be any extensions of $SRMI_m$ by axioms. We prove by induction on the length of a derivation in L of B from $\mathcal{T} \cup \{A\}$, that $\mathcal{T} \vdash_L A \supset B$. If B is an axiom of L or $B \in \mathcal{T}$ then we use the fact that $\vdash_{LL_m} B \to (A \supset B)$. If $B = A$ we use the fact that $\vdash_{R_m} A \supset A$. In the induction step we have two cases. Suppose, first, that B was inferred from C and $C \to B$. By induction hypothesis, $\mathcal{T} \vdash_L A \supset C$ and $\mathcal{T} \vdash_L A \supset (C \to B)$. But $\vdash_{RMI_m} (A \supset C) \to ((A \supset (C \to B)) \to (A \supset B))$ (use $GRMI_m$ or $\mathcal{A}_\omega$). It follows that $\mathcal{T} \vdash_L A \supset B$ in this case. Assume, finally, that B was inferred from $B \otimes C$. By induction hypothesis, $\mathcal{T} \vdash_L A \supset B \otimes C$. Using $GRMI_m$ or A_ω it is not too difficult to show that $\vdash_{RMI_m} (A \supset B \otimes C) \to ((A \supset B) \otimes (A \supset C))$. Hence again $\mathcal{T} \vdash_L A \supset B$. ■

We turn now to prove the strong completeness theorem in its full generality. In view of theorem 15, this is equivalent to the compactness theorem for $\Vdash_{A_\omega}$. We have found it easier, however, to prove completeness directly, and from this to derive compactness.

Theorem 17.[15] (strong soundness and completeness theorem for $SRMI_m$). *$\mathcal{T} \Vdash_{A_\omega} \varphi$ iff $\mathcal{T} \vdash_{SRMI_m} \varphi$.*

Proof: The soundness part is the same as in theorem 15.

For completeness, assume $\mathcal{T} \nvdash \varphi$ (we shall write just "$\vdash$" instead of "$\vdash_{SRMI_m}$" until the end of this proof). We construct a valuation v which shows that $\mathcal{T} \nVdash_{A_\omega} \varphi$. For this extend first $\mathcal{T}$ to a maximal theory $\mathcal{T}^*$ such that $\mathcal{T}^* \nvdash \varphi$. By the deduction theorem, $\mathcal{T}^* \nvdash A$ iff $\mathcal{T} \vdash A \supset \varphi$.

Fact 1. *$\mathcal{T}^* \vdash \varphi \to A$ for all A.*

Proof: Otherwise $\mathcal{T}^* \vdash (\varphi \to A) \supset \varphi$ for some A. But $(\varphi \to A) \supset \varphi \Vdash_{A_\omega} \varphi$. Hence by theorem 15, $(\varphi \to A) \supset \varphi \vdash \varphi$ and so $\mathcal{T}^* \vdash \varphi$. A contradiction[16].

Fact 2. *$\mathcal{T}^* \nvdash A$ iff $\mathcal{T}^* \vdash A \to \varphi$.*

Proof: The "if" part is obvious. For the converse, suppose $\mathcal{T} \nvdash A$. Then $\mathcal{T}^* \vdash A \supset \varphi$, i.e.: $\mathcal{T}^* \vdash A \to \varphi \otimes A$. But $\mathcal{T}^* \vdash \varphi \to (A \to \varphi)$ by fact 1, and so $\mathcal{T}^* \vdash \varphi \otimes A \to \varphi$. It follows that $\mathcal{T}^* \vdash A \to \varphi$, by transitivity.

Fact 3. *$\mathcal{T}^* \vdash A \to \sim\varphi$, for every A.*

Proof: Immediate from fact 1, by contraposition.

Fact 4. *$\mathcal{T}^* \vdash \sim\varphi$.*

Proof: By fact 3, taking $A = (\varphi \to \varphi)$.

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[15] This theorem was stated without a proof in [Av92a].

[16] A reader who prefers a self-contained proof, not relying on [Av84], can verify directly, using $GRMI_m$, that $\vdash_{RMI_m} ((\varphi \to A) \supset \varphi) \supset \varphi$. It is in fact shorter to show that $\vdash_{RMI_m} ((\varphi \to A) \supset \varphi) \to \varphi \otimes (A \to A)$. Similar remarks apply wherever we invoke theorem 15.

Fact 5. *$\mathcal{T}^* \vdash \sim\varphi \to A$ iff ($\mathcal{T}^* \vdash A$ and $\mathcal{T}^* \nvdash \sim A$) iff $\mathcal{T}^* \not\vdash \sim A$.*

Proof: $\mathcal{T}^* \vdash \sim\varphi \to A$ iff $\mathcal{T}^* \vdash \sim A \to \varphi$ iff $\mathcal{T}^* \not\vdash \sim A$ (by fact 2). In addition, fact 4 implies that if $\mathcal{T}^* \vdash \sim\varphi \to A$ then $\mathcal{T}^* \vdash A$.

Construct now the Lindenbaum Algebra of $\mathcal{T}^*$ by defining $A \equiv B$ iff $\mathcal{T}^* \vdash A \to B$ and $\mathcal{T}^* \vdash B \to A$ (this is obviously a congruence relation. See Proposition 3). Define then the operations on the set of equivalence classes in the obvious way (e.g. $[A] \to [B] = [A \to B]$ etc., where [A] is the equivalence class of A). We shall denote $[\varphi]$ by $\bot$, $[\sim\varphi]$ by $\top$. Obviously, $\top \neq \bot$, since $\mathcal{T}^* \vdash \sim\varphi$ while $\mathcal{T}^* \not\vdash \varphi$. We shall denote the other equivalence classes by $I_1, I_2, \ldots$(the set of these classes is obviously enumerable).

Fact 6. *$[A] = I_n$ for some n iff both $\mathcal{T}^* \vdash A$ and $\mathcal{T}^* \vdash \sim A$.*

Proof: Suppose $[A] = I_n$. Then $[A] \neq \bot = [\varphi]$ and so $\mathcal{T}^* \not\vdash A \to \varphi$, by fact 1. This means, by fact 2, that $\mathcal{T} \vdash A$. Similarly, $[A] \neq \top = [\sim \varphi]$, and so, by fact 3, $\mathcal{T} \not\vdash \sim\varphi \to A$. This entails, by fact 5, that $\mathcal{T}^* \vdash \sim A$.

For the converse, assume both A and $\sim A$ are provable. Then $A \not\equiv \varphi$ since $\mathcal{T}^* \not\vdash \varphi$, and $A \not\equiv \sim\varphi$, since $\mathcal{T}^* \nvdash \sim \sim\varphi$. Hence $[A] \neq \bot$ and $[A] \neq \top$, and so $[A] = I_k$ for some k.

We next show that the Lindenbaum Algebra of $\mathcal{T}^*$ is exactly $\mathcal{A}_\omega$ by showing that the operations are identical to those of $\mathcal{A}_\omega$.

The case of $\sim$: Obviously $\sim\bot = \top$ and $\sim \top = \bot$. Now let $I_n = [A]$. By fact 6, $\mathcal{T}^* \vdash A$ and $\mathcal{T}^* \vdash \sim A$. But both $A \to \sim A \to (A \to \sim A)$ and $A \to \sim A \to (\sim A \to A)$ are theorems of RMI_m. Hence $A \equiv \sim A$ and $\sim I_n = [\sim A] = [A]$.

The case of $\otimes$. Since $\varphi \otimes A \vdash \varphi$, $\mathcal{T}^* \not\vdash \varphi \otimes A$ and so $[\varphi \otimes A] = \bot$ by the first two facts. Hence $\bot \otimes [A] = \bot$ for all A. Similarly, $[A] \otimes \bot = \bot$. If $[A] = [B]$ then $[A] \otimes [B] = [A] \otimes [A] = [A \otimes A] = [A]$, since $A \to A \otimes A$ and $A \otimes A \to A$ are theorems of RMI_m. Finally, suppose $[A] \neq \bot$, $[B] \neq \bot$ and $[A] \neq [B]$. Then $\mathcal{T}^* \vdash A$ and $\mathcal{T}^* \vdash B$. Assume for contradiction that $\top \neq [A] \otimes [B] (= [A \otimes B])$. Then $\mathcal{T}^* \vdash \sim (A \otimes B)$, by facts 3 and 5. But $\vdash_{RMI_m} A \to B \to \sim (A \otimes B) \to (A \to B)$ and $\vdash_{RMI_m} A \to B \to \sim (A \otimes B) \to (B \to A)$. Hence $A \equiv B$, contradicting $[A] \neq [B]$.

The case of $\to$. Since $A \to B$ and $\sim (A \otimes \sim B)$ are equivalent in RMI_m, and the corresponding operations are identical on $\mathcal{A}_\omega$, this case follows from the previous two.

Define now $v(A) = [A]$. This is easily seen to be a valuation (the canonical one). Obviously $v(A) \neq \bot$ for $A \in \mathcal{T}$, while $v(\varphi) = \bot$. Hence $\mathcal{T} \not\Vdash_{A_\omega} \varphi$. ■

Corollary 18. *(Compactness Theorem): $\mathcal{T} \Vdash_{A_\omega} A$ iff there exists a finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Vdash_{A_\omega} A$.*

Corollary 19. *$\mathcal{T} \Vdash_{A_\omega} \varphi$ iff there exists a formula ψ such that $\mathcal{T} \vdash_{RMI_m} \varphi \otimes \psi$.*

Proof: Obviously, if such a formula exists then $\mathcal{T} \vdash_{SRMI_m} \varphi$, and so $\mathcal{T} \Vdash_{A_\omega} \varphi$. Conversely, if $\mathcal{T} \Vdash_{A_\omega} \varphi$ then by the last theorem and its corollary there exist $\Gamma = \{A_1, \ldots, A_n\} \subseteq \mathcal{T}$ such that $\Gamma \vdash_{SRMI_m} \varphi$. But then $\Gamma \vdash_{RMI_m} \varphi \otimes \psi$, where $\psi = A_1 \otimes A_2 \otimes \cdots \otimes A_n$ (see the proof of theorem 17 and the note that follows it). ■

Corollary 20. *$SRMI_m$ is paraconsistent: $\sim p, p \not\vdash_{SRMI_m} q$.*

Proof: Take in $\mathcal{A}_\omega$ $v(p) = I_1$, $v(q) = \perp$. ■

We next provide (like in [Av84]) a full characterization of all the extensions of $SRMI_m$.

Theorem 21. *Let $\mathcal{L}$ be a nontrivial extension of $SRMI_m$ in its language which is obtained by adding axiom-schemes to $SRMI_m$. Then there exists $0 \leq n < \omega$ such that $\vdash_{\mathcal{L}} = \Vdash_{A_n}$.*

Proof: Let $Th(\mathcal{L})$ be the set of theorems of $\mathcal{L}$. $Th(\mathcal{L})$ is closed under substitutions and so (by proposition 8) there exists $0 \leq n < \omega$ such that $\varphi \in Th(\mathcal{L})$ iff $\Vdash_{A_n} \varphi$. We claim that in fact for every theory $\mathcal{T}$ and formula φ $\mathcal{T} \Vdash_{A_n} \varphi$ iff $\mathcal{T} \vdash_{\mathcal{L}} \varphi$. For the "if" part, assume that $\mathcal{T} \vdash_{\mathcal{L}} \varphi$. Then there exist $A_1, \ldots, A_\ell \in \mathcal{T}$ s.t. $A_1, \ldots, A_\ell \vdash_{\mathcal{L}} \varphi$. Hence, by theorem 16, $A_1 \supset A_2 \supset \cdots \supset A_\ell \supset \varphi \in Th(\mathcal{L})$, and so this formula is valid in $\mathcal{A}_n$. This immediately implies that $A_1, \ldots, A_\ell \Vdash_{A_n} \varphi$ and so $\mathcal{T} \Vdash_{A_n} \varphi$.

For the converse, assume $\mathcal{T} \not\vdash_{\mathcal{L}} \varphi$. A construction like in the proof of theorem 17 provides a structure of the form $\mathcal{A}_k$ $(0 \leq k \leq \omega)$ and a valuation v in $\mathcal{A}_k$ such that $v(A) \neq \perp$ for $A \in \mathcal{T}$ while $v(\varphi) = \perp$. Now, since $\mathcal{L}$ is closed under substitutions while every valuation in a Lindenbaum Algebra corresponds to some substitution, all sentences in $Th(\mathcal{L})$ are valid in $\mathcal{A}_k$. It follows that $k \leq n$ (the $\mathcal{A}_i$'s have the property that if $\Vdash_{A_i} \psi$ then $\Vdash_{A_j} \psi$ for $j \leq i$, but there are counterexamples in case $j > i$. See [Av84, p. 340]). Hence v is a valuation also in $\mathcal{A}_n$, and so $\mathcal{T} \not\Vdash_{A_n} \varphi$. ■

Theorem 22. *Let $RMI_m^{(k)}$ be a weakly complete axiomatization of the set of sentences which are valid in $\mathcal{A}_k$ (for example – that given in [Av84]). Then $SRMI_m^{(k)}$ is a strong axiomatization of $\Vdash_{A_k}$.*

Proof: This follows easily from the proof of the previous theorem.

Corollary 23. *SRM_m is a strong axiomatization of $\mathcal{A}_1$.*

A final important note: although $\otimes$ behaves as an extensional connective in $SRMI_m$, its De-Morgan dual $+$ ("par" of [Gi87]) does not. In fact, $+$ is an ideal relevant disjunction: it is commutative, associative, order preserving and idempotent (in R_m, e.g., it lacks the last property). The disjunctive syllogism is valid for it, and $v(A+B) = \top$ iff either $v(A) = \top$ or

$v(B) = \top$. On the other hand $v(A + B) = \bot$ in case A and B have irrelevant truth values (i.e. in case $v(A) = I_i$, $v(B) = I_j$ for $i \neq j$). Hence $A \to (A + B)$ is not valid.

5 Adding Propositional Constants

We shall see below that the basic multiplicative language of $SRMI_m$ suffices for a strong translation of classical positive logic, but not of classical negation. The semantics of $\mathcal{A}_\omega$ provides, however, a very obvious interpretation of absurdity (and "truth"). The corresponding propositional constants obey the laws of Girard's additive constants (but we shall argue below that at least in the present context, if not in general, they really belong to the multiplicative fragment).

In the next definition and later we use the notations which are employed in [Tr92] for the propositional constants, rather than those used in [Gi87][17].

Definition 8. (1) the *extended multiplicative language* is the basic multiplicative language enriched by the propositional constant $\bot$

(2)
$$\top = \bot \to \bot$$

(3) The semantics of $\mathcal{A}_\omega$ is extended by demanding $v(\bot) = \bot$ for every valuation v. $\Vdash_{A_\omega}$ is extended accordingly.
(4) Let L be a logic in the basic multiplicative language. Then $L^\bot$ is L together with the axiom scheme $\bot \to A$. Similarly, if GL is a Gentzen type system in the basic multiplicative language then $GL^\bot$ is GL (extended to the extended multiplicative language) together with the axioms: $\bot, \Gamma \Rightarrow \Delta$.

Theorem 24. *The strong completeness theorem of $SRMI_m$ relative to $\mathcal{A}_\omega$ and its corollaries are true also for $SRMI_m^\bot$.*

Proof: The proof is almost identical to that of theorem 17. All we need to add is a demonstration that $[\bot]$ is indeed the bottom element of the structure defined there. (In other words: that $[\bot] = [\varphi]$.) This, however, is trivial, since from fact 1 in the proof of theorem 17 it follows that $\mathcal{T}^* \vdash \varphi \to \bot$, while $\bot \to \varphi$ is an explicit axiom. ■

Theorem 25. *$RMI_m^\bot$ is weakly (but not strongly!) complete relative to $\mathcal{A}_\omega$.*

Proof: If φ is the language of RMI_m then $\Vdash_{A_\omega} \varphi$ iff $\vdash_{RMI_m^\bot} \varphi$ by proposition 7. Otherwise, we proceed exactly as in the proof of theorem 17. Assume $\nvdash_{RMI_m^\bot} \varphi$, We show that $\nVdash_{A_\omega} \varphi$.

[17] In [Gi87] the roles of $\bot$ and 0 were interchanged.

Let $\mathcal{T}^*$ be a maximal theory such that $\mathcal{T}^* \not\vdash_{RMI_m^\perp} \varphi$. Since we are using here $RMI_m^\perp$, the relevant deduction theorem (proposition (2)) holds. Hence $A \notin \mathcal{T}^*$ iff $\mathcal{T}^* \vdash A \to \varphi$. This corresponds to fact 2 from the proof of theorem 17. The crucial step now is to show that also fact 1 from that proof is again true. For this, we need a lemma:

Lemma. $\vdash_{LL_m^\perp} A \to (\varphi \to \varphi)$ *whenever* φ *contains* $\perp$.

Proof of the Lemma: By induction on the length of φ. If $\varphi = \perp$ then $A \to (\varphi \to \varphi)$ is equivalent to $\perp \to (A \to \perp)$, which is an instance of an axiom. If $\varphi = \sim \psi$ then the claim follows from the induction hypothesis and the fact that $\vdash_{LL_m} (\psi \to \psi) \to (\sim \psi \to \sim \psi)$. Suppose now that $\varphi = \psi_1 \otimes \psi_2$. Then either ψ_1 or ψ_2 contains $\perp$. Assume, e.g., the latter. Then by induction hypothesis $\vdash_{LL_m^\perp} A \to (\psi_2 \to \psi_2)$. But $\vdash_{LL_m} (\psi_2 \to \psi_2) \to (\psi_1 \otimes \psi_2 \to \psi_1 \otimes \psi_2)$. Hence $\vdash_{LL_m^\perp} A \to (\varphi \to \varphi)$. [18]

Returning now to the proof of the theorem, assume that there is A such that $\mathcal{T}^* \not\vdash \varphi \to A$. Then $\mathcal{T}^* \vdash (\varphi \to A) \to \varphi$. But $((A \to A) \to (\varphi \to \varphi)) \to [((\varphi \to A) \to \varphi) \to \varphi]$ is valid in A_ω, and so is provable in RMI_m. Since φ contains $\perp$, this and the lemma imply that $\mathcal{T}^* \vdash \varphi$. A contradiction. It follows that $\mathcal{T}^* \vdash \varphi \to A$ for all A. This last fact implies that $\perp \equiv \varphi$ and $[\perp] = [\varphi]$. From this point the proof proceeds exactly as that of theorem 17. ■

Note. It is easy to see that by the same method we can prove that if φ contains $\perp$ then $\mathcal{T} \Vdash_{A_\omega} \varphi$ iff $\mathcal{T} \vdash_{RMI_m^\perp} \varphi$. The full power of $SRMI_m^\perp$ is needed, however, when φ is in the basic language.

Corollary 26. $\vdash_{SRMI_m^\perp} A$ *iff* $\vdash_{RMI_m^\perp} A$.

Theorem 27. *The characterizations of the extensions of* RMI_m *and* $SRMI_m$ *given by theorems 8, 21, 22 are true also for* $RMI_m^\perp$ *and* $SRMI_m^\perp$.

Proof: This follows from the last two theorems and their proofs in the same way this was shown for RMI_m and $SRMI_m$ in [Av84] and above. ■

Corollary 28. $RM_m^\perp$ *is weakly complete for* $\mathcal{A}_1$ *while* $SRM_m^\perp$ *is strongly complete for it.*

The presence of the propositional constants allows for a generalized version of the deduction theorem:

Theorem 29. *Let* L *be any extension of* $SR_m^\perp$ *by axiom schemes. Then* $\mathcal{T}, A \vdash_L B$ *iff* $\mathcal{T} \vdash_L A \to (B \otimes \top)$.

Proof: The "if" part is easy. The "only if" part is proved by induction on the length of a proof of B from $\mathcal{T} \cup \{A\}$. If B is an axiom or $B \in \mathcal{T}$ we use the theoremhood in

[18] The lemma is not valid with the additives: for example $p \to [(q\&(\perp\to\perp) \to q\&(\perp\to\perp)]$ is not a theorem even in RM_3, which is much stronger than LL, R on even RM.

LL_m of $B \to (A \to . \ B \otimes \top)$. If $B = A$ we use the fact that $\vdash_{LL_m^\perp} A \to . \ A \otimes \top$. If B is inferred from C and $C \to B$ then by induction hypothesis $A \to . \ C \otimes \top$ and $A \to . \ (C \to B) \otimes \top$ are theorems of $\mathcal{T}$. It is easy to show now that $\vdash_{LL_m^\perp} (A \to . \ C \otimes \top) \to (A \to . \ (C \to B) \otimes \top) \to (A \to A \to . \ B \otimes \top)$. Hence $A \to . \ C \otimes \top, A \to (C \to B) \otimes \top \vdash_{R_m^\perp} A \to . \ B \otimes \top$. Finally, if B is inferred from $B \otimes C$ then by induction hypothesis $A \to (B \otimes C) \otimes \top$ is a theorem of $\mathcal{T}$. But $\vdash_{LL_m^\perp} (B \otimes C) \otimes \top \to B \otimes \top$. Hence $\mathcal{T} \vdash_L A \to . \ B \otimes \top$. ■

Note. Theorem 29 suggests an alternative definition of $\supset$ in the presence of $\top$: $A \supset B =_{Df} A \to B \otimes \top$ (note that this is *not* identical to the one given above in the semantics of $\mathcal{A}_\omega$).

What about adding to the basic multiplicative language analogues of Girard's multiplicative constants 0 and 1Γ[19] Well, it is shown in [Av84] that one cannot conservatively add such constants even to RMI_m: the purely multiplicative fragment of the resulting system is $RMI_m^1 = RM_m$. This shows that those "multiplicative" constants are in fact in direct conflict with the basic multiplicative connectives (at least in the framework of RMI_m and its extensions, except for RM_m and CL_m). One can add them conservatively, though, to RM_m. The interpretation of both 1 and 0 in such an extension is I.

We turn now to discuss the nature of the language of $SRMI_m^\perp$. In particular: is it still a "multiplicative" languageΓ Or better: does the addition of $\perp$ to the basic multiplicative language significantly change its character so it cannot be taken as "multiplicative" any longerΓ To answer these questions we should find out first what does the term "multiplicative" actually *mean*, and what is so special about it.

For convenience we shall mainly concentrate in what follows on constants which represent the two types of "truth": $\top$ and 1. The other two propositional constants which are used in the various substructural logics are defined from them using negation. $\top$ is characterized by the axiom $A \to \top$ (in Hilbert-type formulations) or $\Gamma \Rightarrow \Delta, \top$ (in Gentzen-type formulations). 1 is characterized by the axioms 1 and $1 \to (A \to A)$ (Hilbert) or by the axiom $\Rightarrow 1$ and the rule $\frac{\Gamma \Rightarrow \Delta}{1, \Gamma \Rightarrow \Delta}$ (Gentzen). In [Gi87] 1 is "multiplicative" and $\top$ – "additive", but one can find there no definition of these terms,[20] so this fact alone does not answer our questions (at least not the second one).

Girard's distinction between multiplicatives and additives was predated by the relevantists' distinction between intensional connectives and extensional connectives. The idea of this distinction is that the (classical) truth-value of a sentence which is constructed from

[19] t and f in the notations of relevance logics.

[20] The intuition behind the classification in [Gi87] has its origin in the semantics of coherence spaces. This semantics is not of much help for other substructural logics, and is problematic even in Linear Logic when it comes to the propositional constants. Thus some of them become identical in this semantics "for stupid reasons" ([Gi87]).

some components using an extensional connective depends only on the truth-values of the components – and nothing else. (For example: & is extensional since $A\&B$ is true iff both A and B are true). An intensional connective is simply one which is not extensional (for example $\rightarrow$, but also ! or $\Box$). This distinction does not really distinguish between $\top$ and 1: both should be taken according to it as "extensional" (=additiveΓ)

A much clearer distinction can be made within a Gentzen-type framework. Here the distinction primarily applies, however, to *rules* and the way they treat their non-active formulae. A multiplicative rule is a context-free rule: the "non-active" formulae of the premises are transferred into the conclusion, and there should be no connection between the nonactive formulas of the various premises or any side condition on them.[21] A good example is provided by the *cut* rule. Here the only "active" formula is the cut formula, and the multiplicative form of this rule is:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \; .$$

Additive rules in contrast, are context-dependent: the premises and the conclusion should have the same (multiset of) non-active formulas. For example, the additive form of cut is:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

It is plausible to define a *system* to be multiplicative in case all its rules are multiplicative. The main virtue of a multiplicative system is that if a certain sequent follows in it from a finite set of other sequents, then we can add to each side of any of the assumptions any multiset of formulae we like, and all these multisets will be transferred to the (corresponding side of the) conclusion. This seems to me to be the characteristic feature of the various multiplicative logics mentioned above, and what makes them so convenient to deal with. Now, the addition of the various propositional constants to these systems preserves this crucial property (while the addition of the standard binary additive connectives does not). For me this is a sufficient justification for the claim that we are still dealing with a multiplicative *system* when we add the so-called additive propositional constants. As for the question what is a multiplicative *connective*: I prefer to leave it open.

Here are some other facts that show that at least at the context of RMI_m it is more appropriate to take $\top$, rather than 1, as multiplicative:

1. All the constants become *definable* if we allow in the basic multiplicative language of RMI_m *infinite* conjunctions. 1 is equivalent (already in LL_m) to the infinite *additive*

[21] From the point of view of an implementation multiplicative rules are ideal, because of their context-free nature. Additive rules, in contrast, are very problematic.

conjunction of all formulae of the form $A \to A$, while $\top$ is equivalent in RMI_m to the infinite *multiplicative* conjunction of these formulae!

2. In the standard relevance logics of Anderson and Belnap we have the deduction theorem: $\mathcal{T}, A \vdash B$ iff $\mathcal{T} \vdash A\&1 \to B$ (see [MDL74], [Du86]). A similar, (though more complicated) theorem holds in the multiplicative-additive fragment of Linear Logic (see [Av92b]).[22] Theorem 29 is a complementary deduction theorem for $\otimes$ and $\top$.

3. We have noted above that RMI_m collapses to RM_m (which does not even enjoy the variable-sharing property) if we add 1 to it. The addition of & has the same effect on RMI_m (see [Av84]) so again we see here an intimate connection between 1 and &. On the other hand, $\bot$ and $\top$ can conservatively be added to purely multiplicative RMI_m (and $SRMI_m$).

6 Adding Additives and Quantifiers

6.1 Adding Additives

It is at first sight unclear whether there is really a point in trying to add analogues of the "additive" connectives to $SRMI_m$. In relevance logic the motivation for introducing the so-called "extensional connectives" is to have analogues of the *classical*, purely truth-functional conjunction and disjunction. In $SRMI_m$ $\otimes$ is a proper substitution of classical conjunction, and we shall see below that we have a faithful translation also of classical disjunction. On the other hand, the additives are *computationally* important in Linear Logic ([Gi87], [Ab93], [Al94]). Now this paper does not deal with computational aspects of the logics we investigate, but the Gentzen-type systems of section 8 strongly suggest that such exist. The additives might be useful when these aspects are worked out. Moreover: while the language of $SRMI_m$ contains an obvious *relevant* disjunction, it does not have a relevant conjunction (such that $A \wedge B$ can be true only if A and B are mutually "relevant"). Somewhat paradoxically, an appropriate version of the additive conjunction turns out to be an excellent candidate.

The standard way of adding additive conjunction $\wedge$ in substructural logics is to add the axioms:

(C1) $A \wedge B \to A$

[22] $A\&1 \to B$ (or $A \wedge t \to B$) is called "enthymematic implication" (of B by A) by Anderson and Belnap. See [AB61]. It was used by Meyer ([Me73]) for translating intuitionistic logic into relevance logics. Meyer also used $\bot$ (or F) for translating intuitionistic negation as $A\&1 \to \bot$. The axioms for $\bot$ are then not $\bot \to A$ or $\bot \,\&\, 1 \to A$ but rather $\bot \,\&\, 1 \to A^*$, where A^* is a translation of an intuitionistic formula. See also [Du86, p. 141].

(C2) $A \wedge B \to B$

(C3) $(A \to B) \wedge (A \to C) \to A \to B \wedge C$

and the adjunction rule: from A and B infer $A \wedge B$. (The additive disjunction $\vee$ can either be defined as $A \vee B =_{Df} \neg(\neg A \wedge \neg B)$ or be introduced as an independent connective, with similar axioms.[23]) These axioms and rules are equivalent in Gentzen-type systems to the rules:

$$(\wedge \Rightarrow) \quad \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \quad (\Rightarrow \wedge)$$

As was noted already in the previous section, these axioms and rules cannot be added conservatively to RMI_m (and so to $SRMI_m$). Such an addition leads to RM_m (to which these axioms and rules can conservatively be added, of course). Instead, a *relevant* conjunction was conservatively added to RMI_m in [Av90a,b]. The difference is that the adjunction rule is replaced by *relevant adjunction*: from A, B and $R(A, B)$ infer $A \wedge B$ where $R(A, B) = (A \to A)+(B \to B)$ is a sentence which is true iff A and B are mutually relevant. Alternatively, one can replace this rule together with axiom C3 by the rule $\frac{A \to B \qquad A \to C}{A \to B \wedge C}$. In a Gentzen-type formulation this is equivalent to the condition: $\Gamma \cup \Delta \neq \emptyset$ in the rule $(\Rightarrow \wedge)$ [24]

When we start with $SRMI_m$ (or $SRMI_m^{\perp}$) the rule $\frac{A \to B \qquad A \to C}{A \to B \wedge C}$ can be replaced by the following axiom:

$$C3^* \qquad (A \to B) \otimes (A \to C) \supset (A \to B \wedge C)$$

The system $SRMI$ (SRM) is the system which is obtained from $SRMI_m^{\perp}$ ($SRM_m^{\perp}$) by adding to it C1, C2 and C3*

Theorem 16*. *The deduction theorem of Theorem 16 is valid for $SRMI$ and SRM.*

Proof: Immediate from theorem 16.

We next turn to the semantics of $SRMI$ and SRM. The main observation here is that $\mathcal{A}_\omega$ becomes a lattice if we define $a \leq b \Leftrightarrow a \to b \neq \perp$ (the order is simply $\perp \leq I_n \leq \top$ for all n). The axioms for $\wedge$ and $\vee$ obviously mean (as always) that they should correspond to the operations of *lub* and *glb* in this lattice. Defining valuations on $\mathcal{A}_\omega$ accordingly, it is a straightforward matter to extend the soundness and strong completeness results to $SRMI$ and its extensions (including SRM).

[23]In the usual relevance logics like R, but not in linear logic, also the distribution axiom of $\wedge$ over $\vee$ is added. This is certainly not a necessary property of the additives.

[24]In order to get a cut-elimination theorem it is necessary also to use the mingle rule rather than the expansion rule. The resulting system is called $RMI_{\min}$ in [Av90a,b]. The full system RMI is obtained by adding also a weak form of distribution in the Hilbert-type formulation, or using *hypersequents* (see below) in the Gentzen-type version.

Theorem 30. *The soundness and strong completeness theorem of $SRMI_m$ (SRM_m) relative to $\mathcal{A}_\omega$ ($\mathcal{A}_1$) and its corollaries (compactness and paraconsistency) are valid for $SRMI$.*

Corollary. *(Variable-sharing property): If $\vdash_{SRMI} A \to B$ then A and B share a variable.*

Theorem 31. *The characterization of the extensions of $SRMI_m$ given by theorems 21,22 is valid for $SRMI$.*

6.2 Adding Quantifiers

This subsection is only sketchy (a full treatment will be given elsewhere). Its main purpose is to indicate how, in principle, quantifiers can be introduced in the present framework.

The standard method of introducing quantifiers in substructural logics is to use for them exactly the same rules as in classical logic. This makes $\forall$ a sort of infinite additive conjunction and $\exists$ – a sort of infinite additive disjunction. In the case of $SRMI$ we should, like in the case of $\wedge$ and $\vee$, add the condition that the multiset of side formulae should not be empty in ($\Rightarrow \forall$) and ($\exists \Rightarrow$). The rules for $\forall$, e.g., will therefore be:

$$(\forall \Rightarrow) \quad \frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi(y)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \quad (\Rightarrow \forall)$$

(Provided in ($\Rightarrow \forall$) $\Gamma \cup \Delta \neq \emptyset$ and y does not occur free in $\Gamma \cup \Delta$).

In a Hilbert-type version we can, similarly, adopt exactly the axioms and rules of Kleene in [Kl52]:

$$\forall x A \to A(t/x) \qquad\qquad \frac{C \to A(y)}{C \to \forall x A(x)}$$

$$\frac{A(y) \to C}{\exists x A \to C} \qquad\qquad A(t/x) \to \exists x A$$

(where y is not free in C in the rules).

The obvious semantics of $\forall$, e.g., is that $v(\forall x A(x)) = \sup_{d \in D}\{A(d/x)\}$ where D is the domain of discourse (we use here the fact that $\mathcal{A}_\omega$ is a complete lattice). One should note that the generalization rule (from A infer $\forall x A$) is not valid in $\mathcal{A}_\omega$ (but it is valid in $\mathcal{A}_1$). The substitution rule (from A infer $A(t/x)$), on the other hand, *is* valid, and should be adopted as an official rule in both the Hilbert-type and Gentzen-type formulations (it is admissible in both, but not derivable). Alternatively, one can forbid valuations in which $v(p(a_1, \ldots, a_n)) = I_i$, $v(p(b_1, \ldots, b_n)) = I_j$ and $i \neq j$ (where p is atomic). With this limitation on the semantics the generalization rule becomes valid.

It was observed in [Do93] that only additive versions of the quantifiers can be found in existing systems of substructural logics, and that a multiplicative universal quantifier would presumably be something like an infinite tensor. In $SRMI$ it is, in fact, particularly natural to apply this idea. A multiplicative universal quantifier Π can semantically be defined by the conditions:

(i) $v(\Pi x A) = \bot$ iff $v(A(d/x)) = \bot$ for some $d \in D$

(ii) $v(\Pi x A) = I_j$ iff $v(A(d(x)) = I_j$ for all $d \in D$

(iii) $v(\Pi x A) = \top$ in all other cases.

The following 3 rules are valid for Π:

$$\frac{A}{\Pi x A} \qquad \frac{C \to A}{C \to \Pi x A} \qquad \frac{\Pi x A}{A(t/x)} \; .$$

The third rule cannot be replaced here by an implication $\Pi x A \to A(t/x)$ (exactly as $A \otimes B \to A$ is not valid). It can, however, be replaced by the axiom $\Pi x A \supset A(t/x)$. In the presence of the first rule the second can be replaced by the valid axiom $\Pi x(C \to A(x)) \to (C \to \Pi x A)$ (provided x is not free in C). Other important valid formulae are:

$$\Pi x A \leftrightarrow A \quad \text{where} \quad x \quad \text{is not free in} \quad A$$

$$\Pi x(A \otimes B) \leftrightarrow \Pi x A \otimes \Pi x B \; .$$

7 Translations of Classical Logic

We start by showing that the purely positive fragment of classical propositional logic (= the $\{\supset, \wedge, \vee, \equiv\}$-fragment) can strongly be translated into $SRMI_m$.

Theorem 32. *Define a translation I of the positive classical language into the basic multiplicative language as follows:*

$$I(p) = p \quad \text{when} \quad p \quad \textit{is atomic}$$

$$\begin{aligned} I(A \supset B) &= I(A) \to I(B) \otimes I(A) \quad (= I(A) \supset I(B)) \\ I(A \wedge B) &= I(A) \otimes I(B) \\ I(A \vee B) &= I((A \supset B) \supset B) \\ I(A \equiv B) &= I(A \supset B) \otimes I(B \supset A) \; . \end{aligned}$$

Then A follows from $B_1, \ldots, B_n$ in classical positive logic iff

$$I(B_1), \ldots, I(B_n) \vdash_{SRMI_m} I(A) \; .$$

Proof: It is well known that $A \vee B$ is equivalent in classical logic to $(A \supset B) \supset B$ and $A \equiv B$ is equivalent to $(A \supset B) \wedge (B \supset A)$. Hence it is enough to show the claim for the $\{\supset, \wedge\}$ fragment.

To show that if $I(B_1), \ldots, I(B_n) \vdash_{SRMI_m} I(A)$ then $B_1, \ldots, B_n \vdash_{CL} A$ it is enough to note that on $\{\bot, \top\}$ the operations $\otimes$ and $\supset$ behave exactly as the conjunction and implication of the classical two-valued logic ($\{\bot, \top\}$ is, in fact, a submatrix of $\mathcal{A}_\omega$).

For the converse it suffices to take some standard axiomatization of classical logic and show that the translation of all the axioms and rules of inference are valid in A_ω. Now the deduction theorem for $\supset$ and its converse MP are valid by theorem 16, while the standard natural-deduction rules for conjunction are obviously also valid. All we need to check, therefore, is the validity of Pierce law $((A \supset B) \supset A) \supset A$ or, equivalently, that $(A \supset B) \supset A \Vdash_{A_\omega} A$. But it is obvious that if $v(A) = \bot$ then $v(A \supset B) = \top$ and $v(A \supset B) \supset A) = \bot$. Hence $(A \supset B) \supset A \Vdash_{A_\omega} A$. ■

Corollary 33. *The interpretation I of the previous theorem is a weak translation of positive classical logic into RMI_m (i.e. $\vdash_{CL} A$ iff $\vdash_{RMI_m} I(A)$).*[25]

Proof: Immediate from theorem 32 and Theorem 13. ■

What about negationΓ We start with a negative result:

Proposition 34. *There is no translation (of the type given in theorem 32) of full classical logic into $SRMI_m$ or even SRM_m.*

Proof: Suppose such a translation I exists. Since $\neg p \wedge p \vdash_{CL} \neg q \wedge q$ (p, q-atomic), we should have that $I(\neg p \wedge p) \vdash_{SRM_m} I(\neg q \wedge q)$. Define a valuation v by $v(p) = I_1$. Then $v(I(\neg p \wedge p)) = I_1$, since all the operations of the basic multiplicative language have the property that if $v(p) = I_1$ then $v(\varphi) = I_1$ whenever φ is a formula containing only the atomic variable p. Since $I(\neg p \wedge p) \vdash_{SRMI_m} I(\neg q \wedge q)$, $v(I(\neg q \wedge q)) \neq \bot$ for every such v, no matter what $v(q)$ is. This means, in turn, that $I(\neg q \wedge q)$ is valid in $\mathcal{A}_1$ and so $\vdash_{RM_m} I(\neg q \wedge q)$. Hence $\neg q \wedge q$ should be provable in classical logic. This, of course, is not the case. ■

It follows that in order to get a translation of full classical logic, we should enrich the basic multiplicative language. Now an easy (and quite common) method of getting full classical logic from its positive fragment is to add a propositional constant $\bot$ together with the axiom $\bot \supset A$, and then define $\neg A = A \supset \bot$ ($\neg\neg A \supset A$ is then provable using Pierce law). Luckily, the semantics of $\mathcal{A}_\omega$ provides, as we have seen, a very obvious interpretation for such a propositional constant.

[25]This result was also stated without a proof in [Av92a].

Theorem 35. *Add to the definition of I in theorem 32 the clauses:*

$$I(\bot) = \bot \qquad I(\neg A) = I(A \to \bot) \quad (= I(A) \to \bot)\ .$$

Then I is a strong translation of classical logic into $SRMI_m^\bot$ and a weak translation of it into $RMI_m^\bot$ (i.e.: $\vdash_{CL} A$ iff $\vdash_{RMI_m^\bot} I(A)$).

Proof: The second part follows from the first by Corollary 26. The proof of the first part is almost identical to that of theorem 32. One should only note that (i) $\vdash_{RMI_m^\bot} \bot \supset B$ (since $A \to B \vdash_{RMI_m^\bot} A \supset B$) (ii) It is well known that by adding $\bot \supset A$ as an axiom to a complete axiomatization of positive classical logic (and defining $\neg A = A \supset \bot$) we get a complete axiomatization of classical logic (see [Ch56]). (iii) $I(A \supset \bot)$ is equivalent to $I(\neg A)$, since $\varphi \to \bot$ is equivalent in $LL_m^\bot$ to $\varphi \to \varphi \otimes \bot$. ■

The translation given in the last theorem is not the only one possible in the language of $SRMI^\bot$. Thus theorem 29 suggests an interpretation of classical logic in SR_m, which is like the previous one with respect to $\wedge$ and $\neg$, but in which the interpretation of $A \supset B$ is $A \to B \otimes \top$. Since $((A \to \bot) \to \bot) \to A \otimes \top$ is a theorem of $R_m^\bot$, the translation of the double-negation axiom is valid. This fact and theorem 29 easily entail that this is a strong, faithful translation of classical logic into $SR_m^\bot$ (and $SRMI_m^\bot$). It is interesting to note that this interpretation of $A \supset B$ is equivalent (in $LL^\bot$) to $A \to \sim \neg B$.

Problem. Does this provide a *weak* translation of classical logic into $R_m^\bot\Gamma$

If we introduce also the additive connectives we get even more possibilities. The simplest one translates classical conjunction by $\otimes$ (i.e. multiplicative conjunction), classical disjunction by the *additive* disjunction $\vee$ and classical absurdity by $\bot$. It is straightforward to check that this indeed is a strong translation (since $v(A \otimes B) \neq \bot$ iff $v(A) \neq \bot$ and $v(B) \neq \bot$ while $v(A \vee B) \neq \bot$ iff either $v(A) \neq \bot$ or $v(B) \neq \bot$). This translation provides a third translation of classical implication: $\neg A \vee B$, which is equivalent to $(A \to B) \vee B$[26].

8 A Gentzen-Type System

Our goal in this section is to present a Gentzen-type system for the $\Vdash_{A_\omega}$ relation (in the languages without the additive connectives). This might seem strange, since we already have a cut-free system, $GRMI_m(GRMI_m^\bot)$, which can be used to characterize $\Vdash_{A_\omega}$: $A_1, \ldots, A_n \Vdash_{A_\omega} B$ iff $\vdash_{GRMI_m} A_1, \ldots, A_n \Rightarrow B \otimes (A_1 \otimes A_2 \otimes \cdots \otimes A_n)$ (see the proof of theorem 15), or even if $\vdash_{GRMI_m^\bot} A_1, \ldots, A_n \Rightarrow B \otimes \top$ (theorem 29). Nevertheless, we believe

[26]This implication is used also in [Av90a,b] for RMI, but it is weaker there than classical implication, though stronger than the intuitionistic one.

that a *good* logic should have a *good* proof system, with the subformula property. In other words: only subformulas of $A_1, \ldots, A_n$ and B should be involved in a demonstration that $A_1, \ldots, A_n \Vdash_{A_\omega} B$ (while if we use $GRMI_m$ as above then $B \otimes (A_1 \otimes A_2 \otimes \cdots \otimes A_n)$ is also involved) . Another motivation is the following: the characterization using $GRMI_m$ implicitly relies on the rule $\frac{A \otimes B}{A}$. In order to show beyond any reasonable doubt that we are dealing with purely multiplicative systems, we want to have a system which differs from those of the other multiplicative systems only in its *structural* rules (which should all be "multiplicative" by the characterization given in section 5).

Our main tool in achieving this goal is the use of *hypersequents.*[27]

Definition 9. A Hypersequent is a syntactic structure of the form $\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2 | \cdots | \Gamma_n \Rightarrow \Delta_n$, where $\Gamma_i \Rightarrow \Delta_i$ $(i = 1, \ldots, n)$ is an ordinary sequent.

Definition 10.

1. A sequent $\Gamma \Rightarrow \Delta$ is true relative to a valuation v in $\mathcal{A}_\omega$ if either $v(A) = \bot$ for some $A \in \Gamma$, or $v(B) = \top$ for some $B \in \Delta$, or there exists k such that $v(p) = I_k$ for some atomic p, and $v(A) = I_k$ for all $A \in \Gamma \cup \Delta$. (If Γ and Δ are empty this means that v should not be a classical valuation. Otherwise $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$ is true iff $v(\sim (A_1 \otimes A_2 \otimes \cdots \otimes A_n \otimes \sim B_1 \otimes \cdots \otimes \sim B_k)) \neq \bot$).

2. A hypersequent is true relative to a valuation v in $\mathcal{A}_\omega$ if at least one of its components is true relative to v.

3. A hypersequent is valid in $\mathcal{A}_\omega$ if it is true relative to every valuation.

Definition 11. (1) The system $\mathbf{G_{A_\omega}}$
We use G, H as variables for hypersequents, S-for sequents

Axioms

$$A \Rightarrow A$$

External Structural Rules

$$\frac{G}{G|H} \qquad \frac{G|S|S|H}{G|S|H} \qquad \frac{G|S_1|S_2|H}{G|S_2|S_1|H}$$

(External weakening, contraction and permutation, respectively).

[27]Hypersequents were first introduced by Pottinger in [Po83], and independently in [Av87]. Related structures were used before by Mints (see [Mi92]).

Internal Structural Rules

$$\frac{G|\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta|H}{G|\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta|H} \qquad \frac{G|\Gamma \Rightarrow \Delta_1, A, B, \Delta_2|H}{G|\Gamma \Rightarrow \Delta_1, B, A, \Delta_2|H}$$

$$\frac{G|\Gamma, A \Rightarrow \Delta|H}{G|\Gamma, A, A \Rightarrow \Delta|H} \qquad \frac{G|\Gamma \Rightarrow \Delta, A|H}{G|\Gamma \Rightarrow \Delta, A, A|H}$$

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H}{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'|H}$$

$$\frac{G_1|\Gamma_1 \Rightarrow \Delta_1, A|H_1 \qquad G_2|A, \Gamma_2 \Rightarrow \Delta_2|H_2}{G_1|G_2|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H_1|H_2}$$

(Internal permutation, expansion, strong splitting and cut, respectively).

Logical Rules

$$\frac{G|\Gamma \Rightarrow \Delta, A|H}{G|\sim A, \Gamma \Rightarrow \Delta|H} \qquad \frac{G|A, \Gamma \Rightarrow \Delta|H}{G|\Gamma \Rightarrow \Delta, \sim A|H}$$

$$\frac{G_1|\Gamma_1 \Rightarrow \Delta_1, A|H_1 \qquad G_2|B, \Gamma_2 \Rightarrow \Delta_2|H_2}{G_1|G_2|A \to B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H_1|H_2} \qquad \frac{G|\Gamma, A \Rightarrow B, \Delta|H}{G|\Gamma \Rightarrow A \to B, \Delta|H}$$

$$\frac{G|\Gamma, A, B \Rightarrow \Delta|H}{G|\Gamma, A \otimes B \Rightarrow \Delta|H} \qquad \frac{G_1|\Gamma_1 \Rightarrow \Delta_1, A|H_1 \qquad G_2|\Gamma_2 \Rightarrow \Delta_2, B|H_2}{G_1|G_2|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A \otimes B|H_1|H_2}$$

(2) The system $\mathbf{G}^{\perp}_{\mathbf{A}_\omega}$: Add to G_{A_ω} the axioms

$$\perp, \Gamma \Rightarrow \Delta \ .$$

Note. The internal contraction rule is derivable using strong splitting and external contraction. Instead of internal expansion we could have taken the obvious internal relevant mix (this, in fact, is preferable. See below).

Example 1. The following is a proof in G_{A_ω} of a *sequent*, which uses proper hypersequents:

$$\cfrac{A{\to}B \Rightarrow A{\to}B \qquad \cfrac{\cfrac{A \Rightarrow A}{A, A \Rightarrow A} \qquad B \Rightarrow B}{A, A, A{\to}B \Rightarrow B}}{(A{\to}B){\to}A,\ A{\to}B,\ A{\to}B,\ A \Rightarrow B}$$

$$\cfrac{\cfrac{\cfrac{(A{\to}B){\to}A,\ A{\to}B,\ A{\to}B,\ A \Rightarrow B}{(A{\to}B){\to}A,\quad A{\to}B,\quad A \Rightarrow B}}{(A{\to}B){\to}A,\quad A{\to}B \Rightarrow A{\to}B}}{(A{\to}B){\to}A \Rightarrow (A{\to}B){\to}(A{\to}B)} \qquad \cfrac{\cfrac{A \Rightarrow A}{\Rightarrow A \mid A \Rightarrow B}}{\Rightarrow A \mid \ \Rightarrow A{\to}B}$$

$$\frac{(A{\to}B){\to}A \Rightarrow (A{\to}B){\to}(A{\to}B) \qquad \Rightarrow A \mid \ \Rightarrow A{\to}B}{(A{\to}B){\to}A \Rightarrow A \otimes ((A{\to}B){\to}(A{\to}B)) \mid \ \Rightarrow A{\to}B}$$

$$\cfrac{\cfrac{(A{\to}B){\to}A \Rightarrow A \otimes ((A{\to}B){\to}(A{\to}B)) \mid \ \Rightarrow A{\to}B \qquad A \Rightarrow A}{(A{\to}B){\to}A \Rightarrow A \otimes ((A{\to}B){\to}(A{\to}B)) \mid (A{\to}B){\to}A \Rightarrow A} \qquad \cfrac{A{\to}B \Rightarrow A{\to}B}{\Rightarrow (A{\to}B){\to}(A{\to}B)}}{\cfrac{(A{\to}B){\to}A \Rightarrow A \otimes ((A{\to}B){\to}(A{\to}B)) \mid (A{\to}B){\to}A \Rightarrow A \otimes ((A{\to}B){\to}(A{\to}B))}{(A{\to}B){\to}A \Rightarrow A \otimes ((A{\to}B){\to}(A{\to}B))}}$$

Theorem 36. *(Soundness theorem): If* $\vdash_{G^{\perp}_{A_\omega}} G$ *then* G *is valid in* $\mathcal{A}_\omega$.

Proof: Most of the cases are straightforward, and are left to the reader. We show the case of strong splitting as an illustration. Assume that $G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H$ is true relative to a valuation v in $\mathcal{A}_\omega$. We show that $G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'|H$ is true as well. If one of the sequents in G or H is true then this obviously is the case. Otherwise $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ is true. If this is because $v(A) = \perp$ for some $A \in \Gamma_1 \cup \Gamma_2$ or because $v(A) = \top$ for some $A \in \Delta_1 \cup \Delta_2$ then either $\Gamma_1 \Rightarrow \Delta_1$ or $\Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'$ is true, depending where this A is. There remains the case where $v(p) = I_k$ for some p, and $v(A) = I_k$ for all $A \in \Gamma_1 \cup \Gamma_2 \cup \Delta_1 \cup \Delta_2$. In this case $\Gamma_1 \Rightarrow \Delta_1$ is true, since $v(A) = I_k$ for all $A \in \Gamma_1 \cup \Delta_1$ (this includes the case in which $\Gamma_1 = \Delta_1 = \emptyset$!). ■

Corollary 37. *(Conservation theorem): If* $\vdash_{G_{A_\omega}} \Gamma \Rightarrow \Delta$ *then* $\vdash_{GRMI_m} \Gamma \Rightarrow \Delta$ *(i.e. if a sequent is provable then it has a proof which employs only sequents). A similar result holds for* $G^{\perp}_{A_\omega}$.

Proof: If $\vdash_{G_{A_\omega}} \Gamma \Rightarrow \Delta$ then by the soundness theorem it is valid in $\mathcal{A}_\omega$, and so provable in $GRMI_m$ by proposition 7.

Note. The proof of the last proposition is semantic. Indeed, it is not clear how to translate directly proofs which use hypersequents to those which do not. Thus in example 1 above all the components of all the *hypersequents* are single-conclusioned, and so all the internal structural rules which are used in it are left-hand-side rules. It is not difficult to see, however, that every proof in $GRMI_m$ must use contraction and its converse on the right-hand side. In the proof we present below the *only* internal structural rules which are used are right-hand-side rules. It is unclear, therefore, what the connection can be.

Example 2.

$$\dfrac{\dfrac{\dfrac{A\Rightarrow A \qquad \dfrac{\dfrac{\dfrac{A\Rightarrow A \qquad \dfrac{B\Rightarrow B}{B\Rightarrow B, B}}{A, A\to B\Rightarrow B, B}}{A\to B\Rightarrow A\to B,\ B}}{\Rightarrow (A\to B)\to(A\to B), B}}{\dfrac{A\Rightarrow A\otimes((A\to B)\to(A\to B)), B}{\Rightarrow A\otimes((A\to B)\to(A\to B)), A\to B}} \qquad A\Rightarrow A}{(A\to B)\to A\Rightarrow A\otimes((A\to B)\to(A\to B)), A} \qquad \dfrac{(A\to B)\Rightarrow A\to B}{\Rightarrow (A\to B)\to(A\to B)}}{\dfrac{(A\to B)\to A\Rightarrow A\otimes((A\to B)\to(A\to B)), A\otimes((A\to B)\to(A\to B))}{(A\to B)\to A\Rightarrow A\otimes((A\to B)\to(A\to B))}}$$

Theorem 38. *The following are equivalent:*

(i) $\Gamma \Vdash_{A_\omega} A$

(ii) *There exist* $\Gamma_1, \ldots, \Gamma_n \subseteq \Gamma$ *such that* $\Gamma_1 \Rightarrow A | \cdots | \Gamma_n \Rightarrow A$ *has a proof in* G_{A_ω} $(G^{\perp}_{A_\omega})$ *without cuts and without external contractions (but perhaps with internal contractions).*

(iii) *There exists a hypersequent of the form:* $\Gamma \Rightarrow A | \Rightarrow A | \cdots | \Rightarrow A$ *which is provable in* G_{A_ω} $(G^{\perp}_{A_\omega})$ *without cuts and without external contractions (but perhaps with internal contractions).*

(iv) $\Gamma \Rightarrow A | \Rightarrow A$ *is provable in* $G_{A_\omega}(G^{\perp}_{A_\omega})$ *without cuts.*

Proof: Obviously (iii) entails (iv) and (ii). (iv) and (ii), in turn, implies (i), by the soundness theorem. It remains to show that (i) entails (iii). Let $\Gamma = B_1, \ldots, B_k$, and assume $\Gamma \Vdash_{A_\omega} A$. Then $B_1 \otimes \cdots \otimes B_n \Vdash_{A_\omega} A$, and so (theorem 16) there exists C such that $B_1 \otimes B_2 \otimes \cdots \otimes B_n \to A \otimes C$ is valid in $\mathcal{A}_\omega$. Hence $B_1, \ldots, B_n \Rightarrow A \otimes C$ is provable without cuts in $GRMI_m(GRMI^{\perp}_n)$, by Propositions 5 and 7. We'll end the proof by showing that if a sequent of the form $\Gamma \Rightarrow \Delta, \varphi \otimes \psi, \varphi \otimes \psi, \ldots, \varphi \otimes \psi$ is provable in $GRMI_m$ $(GRMI^{\perp}_m)$ then there exists a proof in G_{A_ω} without cuts or external contractions of a hypersequent of the form $\Gamma \Rightarrow \Delta, \varphi | \Rightarrow \varphi | \Rightarrow \varphi | \cdots | \Rightarrow \varphi$. The proof of this is by induction on the length of the cut-free proof of $\Gamma \Rightarrow \Delta, \varphi \otimes \psi, \ldots, \varphi \otimes \psi$. Most of the cases are very easy. Here we'll do the two which are less easy than the others.

Case (i)

Suppose $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2, D \otimes E$, and $\Gamma \Rightarrow \Delta, \overbrace{\varphi \otimes \psi, \ldots, \varphi \otimes \psi}^{n \text{ times}}$ is inferred from $\Gamma_1 \Rightarrow \Delta_1, D, \overbrace{\varphi \otimes \psi, \ldots, \varphi \otimes \psi}^{k \text{ times}}$ and $\Gamma_2 \Rightarrow \Delta_2, E, \overbrace{\varphi \otimes \psi, \ldots, \varphi \otimes \psi}^{\ell \text{ times}}$ where $n = k + \ell$. Then either k or ℓ is greater than 0. Assume, e.g., that $k > 0$. By induction hypothesis there is a proof as desired of a sequent of the form $\Gamma_1 \Rightarrow \Delta_1, D, \varphi | \Rightarrow \varphi | \cdots | \Rightarrow \varphi$. Now if $\ell = 0$ then $\Gamma \Rightarrow \Delta, \varphi | \Rightarrow \varphi | \cdots | \Rightarrow \varphi$ can immediately be inferred. Otherwise we apply the induction hypothesis again to get an appropriate proof of $\Gamma_2 \Rightarrow \Delta_2, E, \varphi | \Rightarrow \varphi | \cdots | \Rightarrow \varphi$. An application of $\Rightarrow \otimes$, followed by an internal contraction gives $\Gamma \Rightarrow \Delta, \varphi | \Rightarrow \varphi | \cdots | \Rightarrow \varphi$.

Case (ii)

$\Gamma \Rightarrow \Delta, \overbrace{\varphi \otimes \psi, \ldots, \varphi \otimes \psi}^{n \text{ times}}$ is inferred from $\Gamma_1 \Rightarrow \Delta_1, \varphi, \overbrace{\varphi \otimes \psi, \ldots, \varphi \otimes \psi}^{k \text{ times}}$ and $\Gamma_2 \Rightarrow \Delta_2, \psi, \overbrace{\varphi \otimes \psi, \ldots, \varphi \otimes \psi}^{\ell \text{ times}}$, where $\Gamma = \Gamma_1, \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$ and $n = k + \ell + 1$. If $k = 0$ we can infer $\Gamma \Rightarrow \Delta, \varphi | \Rightarrow \varphi$ from $\Gamma_1 \Rightarrow \Delta_1, \varphi$ by strong splitting (note that a cut-free proof of a sequent in $GRMI_m$ is also a proof without cuts and external contractions of the same sequent in G_{A_ω}). Otherwise we have by induction hypothesis an appropriate proof of a sequent of the form $\Gamma_1 \Rightarrow \Delta_1, \varphi | \Rightarrow \varphi | \cdots | \Rightarrow \varphi$. By applying strong splitting to this hypersequent we get a proof of the type we seek of $\Gamma \Rightarrow \Delta, \varphi | \Rightarrow \varphi | \cdots | \Rightarrow \varphi$. ■

Corollary 39. $\Gamma \Vdash_{A_\omega} A$ *iff there exists a cut-free proof in* G_{A_ω} *of* $\Gamma \Rightarrow A | \Rightarrow A$ *in which all external contractions, if any, are applied at the end.*

The last theorem entails that we don't need cuts for demonstrating that $\Gamma \Vdash_{A_\omega} A$. Does the system G_{A_ω} as a whole admit cut-eliminationΓ The answer is positive, although the proof is much more complicated.

Theorem 40. G_{A_ω} *admits cut-elimination.*

Proof: As usual (see [Av87]), the main problem is posed here by the rule of external contraction. To overcome this problem, we need the "history" technique of [Av87]. Since details are similar to those in [Av87] (and rather complicated), we omit them. ■

It is possible to give a completeness theorem for G_{A_ω} which applies to arbitrary hypersequents. First – a definition.

Definition 12. Let $\mathcal{T}$ be a set of formulas, H – a hypersequent. We say that $\mathcal{T} \vdash_{G_{A_\omega}} H$ iff there exist sets $\Theta_i, \Gamma_i, \Delta_i$ $(i = 1, \ldots, n)$ such that $\Theta_1, \Gamma_1 \Rightarrow \Delta_1 | \cdots | \Theta_n, \Gamma_n \Rightarrow \Delta_n$ is provable in G_{A_ω}, and for all $1 \leq i \leq n$, $\Theta_i \subset \mathcal{T}$ and $\Gamma_i \Rightarrow \Delta_i$ is a component of H.

Theorem 41. $\mathcal{T} \vdash_{G_{A_\omega}} H$ *iff every model of* $\mathcal{T}$ *in* $\mathcal{A}_\omega$ *is also a model of* H.

Proof: The only if part is a straightforward generalization of the soundness theorem.

For the converse, assume $\mathcal{T} \not\vdash_{G_{A_\omega}} H$. We construct a model of $\mathcal{T}$ which is not a model of H. For this extend $\mathcal{T}$ to a maximal theory $\mathcal{T}^*$ such that $\mathcal{T}^* \not\vdash_{G_{A_\omega}} H$. Obviously, $B \notin \mathcal{T}^*$ iff there exist a theorem of G_{A_ω} of the form:

$$(*) \qquad B, \Theta_1, \Gamma_1 \Rightarrow \Delta_1 | \cdots | B, \Theta_k, \Gamma_k \Rightarrow \Delta_k | \Theta_{k+1}, \Gamma_{k+1} \Rightarrow \Delta_{k+1} | \cdots | \Theta_{k+\ell}, \Gamma_{k+\ell} \Rightarrow \Delta_{k+\ell}$$

where $k > 0$ and for each $1 \leq i \leq k + \ell$ $\Theta_i \subseteq \mathcal{T}^*$ and $\Gamma_i \Rightarrow \Delta_i$ is a component of H. In addition, $\mathcal{T}^*$ has the following properties:

(i) If $\vdash_{RMI_m} B$ then $B \in \mathcal{T}^*$

Proof: $\Rightarrow B$ is provable already in $GRMI_m$. If $B \notin \mathcal{T}^*$ then cuts of $\Rightarrow B$ with the sequent in (*) yield $\Theta_1, \Gamma_1 \Rightarrow \Delta_1 | \cdots | \Theta_{k+\ell}, \Gamma_{k+\ell} \Rightarrow \Delta_{k+\ell}$. This means that $\mathcal{T}^* \vdash H$. A contradiction. Hence $B \in \mathcal{T}^*$.

(ii) If $A \in \mathcal{T}^*$ and $A \to B \in \mathcal{T}^*$ then $B \in \mathcal{T}^*$.

Proof: Otherwise cuts of (*) and $A, A \to B \Rightarrow B$ would yield a demonstration that $\mathcal{T}^* \vdash H$.

(iii) If $B \otimes A \in \mathcal{T}^*$ then $B \in \mathcal{T}^*$.

Proof: Otherwise cuts of (*) with the provable hypersequent $B \otimes A \Rightarrow B| \Rightarrow B$, followed by external contractions, would give $B \otimes A \Rightarrow B|\Theta_1, \Gamma_1 \Rightarrow \Delta_1|\cdots|\Theta_{k+\ell}, \Gamma_{k+\ell} \Rightarrow \Delta_{k+\ell}$. More cuts on B of this and (*), followed again by external contractions would give:

$$B \otimes A, \Theta_1, \Gamma_1 \Rightarrow \Delta_1|\cdots|B \otimes A, \Theta_k, \Gamma_k \Rightarrow \Delta_k|\Theta_1, \Gamma_1 \Rightarrow \Delta_1|\cdots|\Theta_{k+\ell}, \Gamma_{k+\ell} \Rightarrow \Delta_{k+\ell}\ .$$

This will mean that $\mathcal{T}^* \vdash H$. A contradiction.

(iv) For all B, either $B \in \mathcal{T}^*$ or for all A: $B \to A \in \mathcal{T}^*$.

Proof: $\Rightarrow B| \Rightarrow B \to A$ is a theorem of G_{A_ω} (since from $B \Rightarrow B$ one can obtain $\Rightarrow B|B \Rightarrow A$ by strong splitting.) Assume that neither component is in $\mathcal{T}^*$. Then cuts on B of $\Rightarrow B| \Rightarrow B \to A$ with (*), followed by cuts on $B \to A$ of the result with the version of (*) in which $B \to A$ plays the role of B, result in a demonstration that $\mathcal{T}^* \vdash H$.

Examining now the proof of theorem 17 (strong completeness of $SRMI_m$) we see that every theory $\mathcal{T}$ with the properties (i)-(iv) determines a valuation v in $\mathcal{A}_\omega$ such that $v(A) \neq \perp$ iff $A \in \mathcal{T}^*$. Moreover, $v(A) \notin \{\top, \perp\}$ iff both A and $\sim A$ are in $\mathcal{T}^*$. It remains to show that v is not a model of any component of H. Let $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$ be such a component. Assume first that $n+k > 0$. Let $\varphi = \sim (A_1 \otimes A_2 \otimes \cdots \otimes A_n \otimes \sim B_1 \otimes \cdots \otimes \sim B_k)$. Then $\vdash_{GRMI_m} \varphi, A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$. Hence $\varphi \notin \mathcal{T}^*$ (otherwise $\mathcal{T}^* \vdash H$). It follows that $v(\varphi) = \perp$. But φ is true in v if $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$ is true in v. Hence $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$ is false in v. Assume, finally, that $n = k = 0$, so the empty sequent is a component of H. Since $\vdash_{GRMI_m} A, \sim A \Rightarrow$, this means that for all A, either $A \notin \mathcal{T}^*$ or $\sim A \notin \mathcal{T}^*$ (otherwise we take $\Theta_1 = \{A, \sim A\}$, $\Gamma_1 = \Delta_1 = \emptyset$, and get that $\mathcal{T}^* \vdash H$). It follows that $v(A) \in \{\top, \perp\}$ for all A, and so the empty sequent is false. ∎

Corollary 42. *$\vdash_{G_{A_\omega}} H$ iff H is valid in $\mathcal{A}_\omega$.*

Corollary 43. *G_{A_ω} is decidable.*

Proof: Obviously, in order to check the validity of a hypersequent H in $\mathcal{A}_\omega$, it is sufficient to check this in $\mathcal{A}_k$, where k is the number of atomic formulae occurring in H. Hence this corollary follows from the previous one. ∎

Our next result provides another easy translation of classical logic into the logic of $\mathcal{A}_\omega$.

Theorem 44. *$\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n$ is valid in classical logic iff one can prove in G_{A_ω} the sequent $\Gamma_1 \Rightarrow \Delta_1|\Gamma_2 \Rightarrow \Delta_2|\cdots|\Gamma_n \Rightarrow \Delta_n| \Rightarrow$ (in particular, $\Gamma \Rightarrow \Delta$ is a tautology iff $\vdash_{G_{A_\omega}} \Gamma \Rightarrow \Delta| \Rightarrow$).*

Proof: Suppose $\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n$ is a tautology. Let v be a valuation in A_ω. If $v(p) = I_k$ for some k and p then the empty sequent is true in v. Otherwise, v is a classical valuation in $\{\top, \bot\}$, and since $\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n$ is a tautology either $v(A) = \bot$ for some A in some Γ_i, or $v(A) = \top$ for some A in some Δ_i. In either case $\Gamma_i \Rightarrow \Delta_i$ is true in v for some i.

For the converse, assume $\Gamma_1 \Rightarrow \Delta_1 | \cdots | \Gamma_n \Rightarrow \Delta_n | \Rightarrow$ is valid in $\mathcal{A}_\omega$. Let v be any valuation in $\{\top, \bot\}$. Then the empty sequent is false in v and so $\Gamma_i \Rightarrow \Delta_i$ is true in v for some i. It follows that either $v(A) = \bot$ for some $A \in \Gamma_i$ or $v(A) = \top$ for some $A \in \Delta_i$. In either case $\Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta_1, \ldots, \Delta_n$ is true in v. ■

We end this section with suitable versions of its theorems for the three-valued extensions of $(S)RMI_m$.

Theorem 45. *Let $G_{A_1}(G^{\bot}_{A_1})$ be the system obtained from G_{A_ω} $(G^{\bot}_{A_\omega})$ by adding to it the mingle rule:*

$$\frac{G_1|\Gamma_1 \Rightarrow \Delta_1|H_1 \qquad G_2|\Gamma_2 \Rightarrow \Delta_2|H_2}{G_1|G_2|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|H_1|H_2} .$$

Then all the obvious analogues of 36-44 are true for $G_{A_1}(G^{\bot}_{A_1})$ if we replace $\mathcal{A}_\omega$ by $\mathcal{A}_1$ and $(S)RMI^{(\bot)}_m$ by $(S)RM^{(\bot)}_m$ $(= (S)RMI^1_m)$.

Proof: A straightforward adaption of the proofs above.

Note. In [Av91b] an extension of G_{A_1} is presented which includes also rules for the additive connectives. We get there a cut-free Gentzen-type formulation of RM_3 (the unique 3-valued extension of RM) [28]

References

[AB61] Anderson A.R. and Belnap N.D., *Enthymemes*, Journal of Philosophy, 58 (1961) 713-723.

[AB75] Anderson A.R. and Belnap N.D., **Entailment** vol. 1, Princeton University Press, Princeton, N.J., 1975.

[AB92] Anderson A.R., Belnap N.D. and Dunn J.M., **Entailment** vol. 2, Princeton University Press, Princeton, N.J., 1992.

[28] In [Av91b] there is a small mistake. An extra condition is imposed on strong splitting: that $\Gamma_1 \Rightarrow \Delta_1$ should not be empty. This condition is not necessary and prevents full cut-elimination in the presence of empty components.

[Ab93] Abramski S., *Computational interpretations of linear logic*, Theoretical Computer Science, 111 (1993) 3-57.

[Al94] Alexiev V., *Applications of linear logic to computation: an overview*, Bulletin of the Interest Group in Pure and Applied Logics (IGPL), 2 (1994) 77-107.

[Av84] Avron A., *Relevant entailment - semantics and formal systems*, Journal of Symbolic Logic, 49 (1984) 334-342.

[Av87] Avron A., *A constructive analysis of RM*, Journal of Symbolic Logic, 52 (1987) 939-951.

[Av88] Avron A., *The semantics and proof theory of linear logic*, Journal of Theoretical Computer Science, 57 (1988) 161-184.

[Av90a] Avron A., *Relevance and paraconsistency - a new approach*, Journal of Symbolic Logic, 55 (1990) 707-732.

[Av90b] Avron A., *Relevance and paraconsistency - a new approach. Part II: The formal systems*, Notre Dame Journal of Formal Logic, 31 (1990) 169-202.

[Av91a] Avron A., *Relevance and paraconsistency - a new approach. Part 3: Gentzen-type systems*, Notre-Dame Journal of Formal Logic, 32 (1991) 147-160.

[Av91b] Avron A., *Natural 3-valued logics – characterization and proof theory*, Journal of Symbolic Logic, 56 (1991) 276-294.

[Av92a] Avron A., *Whither relevance logic?* Journal of Philosophical Logic, 21 (1992) 243-281.

[Av92b] Avron A., *Axiomatic systems, deduction and implication*, Journal of Logic and Computation, 2 (1992) 51-98.

[Ch56] Church A., **Introduction to Mathematical Logic**, Princeton University Press, Princeton, N.J., 1956.

[Do93] Došen K. *A historical introduction to substructural logics*, in: **Substructural Logics**, Ed. by Schroeder-Heister P. and Došen K, Oxford University Press (1993), pp. 1-30.

[Du70] Dunn J.M., *Algebraic completeness results for R-mingle and its extensions*, Journal of Symbolic Logic, 35 (1970) 1-13.

[Du86] Dunn J.M., *Relevant logic and entailment*, in: **Handbook of Philosophical Logic**, Vol III, ed. by D. Gabbay and F. Guenthner, Reidel: Dordrecht, Holland; Boston, U.S.A. (1986), pp. 117-224.

[Gi87] Girard J.Y., *Linear logic*, Theoretical Computer Science, 50 (1987) 1-101.

[Kl52] Kleene S.C., **Introduction to metamathematics**, New York (D. van Nostrad Co.), (1952).

[LMSS92] Lincoln P.D., Mitchell J, Scedrov A. and Shankar N., *Decision problems for propositional linear logic*, Annals of Pure and Applied Logic, 56 (1992) 239-311.

[MD69] Meyer R.K. and Dunn, J.M., *E, R. and* γ, Journal of Symbolic Logic, 34 (1969) 460-474.

[MDL74] Meyer R.K., Dunn, J.M. and Leblanc, H., *Completeness of relevant quantification theories*, Notre Dame Journal of Formal Logic, 15 (1974) 97-121.

[Me73] Meyer R.K., *Intuitionism, truth and negation* in H. Leblanc (ed.), **Truth, Syntax and Modality**, North-Holland, 1973, pp. 168-198.

[Mi72] Mints G., *Cut-elimination theorem in relevant logics*, Journal of Soviet Mathematics, 6 (1976) 422-428.

[Mi92] Mints G., **A Short Introduction to Modal Logic**, CSLI Lecture Notes, No. 30, 1992.
smallskip

[Po83] Pottinger G. *Uniform, Cut-free formulations of T, S4 and S5*, (abstract), J.S.L., vol. 48 (1983), p. 900.

[So52] Sobociński B., *Axiomatization of partial system of three-valued calculus of propositions*, The Journal of Computing Systems, 11(1) (1952) 23-55.

[Tr92] Troelstra A.S., **Lectures on Linear Logic**, CSLI Lecture Notes No. 29, Center for the Study of Language and Information, Stanford University (1992).