Classical Gentzen-type Methods in Propositional Many-Valued Logics

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Abstract. A classical Gentzen-type system is one which employs two-sided sequents, together with structural and logical rules of a certain characteristic form. A decent Gentzen-type system should allow for direct proofs, which means that it should admit some useful forms of cut elimination and the subformula property. In this survey we explain the main difficulty in developing classical Gentzen-type systems with these properties for many-valued logics. We then illustrate with numerous examples the various possible ways of overcoming this difficulty, and the strong connection between semantic completeness and cut-elimination in each case. Our examples include practically all 3-valued and 4-valued logics, as well as Gödel finite-valued logics and some well-known infinite-valued logics.

1 Introduction

Many-valued logics are, above all, *Logics*. This means that like any other logic, the main issue with which they deal is: what formulas follow from what sets of premises (under certain assumptions). The "many-valued" part of their name refer to the type of semantics which determines the answers they give to this basic question. However, any logic needs efficient *proof* systems in order to actually be applied (as well as for a deeper understanding of its logical properties), and many-valued logics are no exception. The problem of adapting to many-valued logics the usual methods of automated reasoning (like Gentzen-type systems, tableaux, and resolution) has therefore attracted a lot of attention in recent years. The main idea which is used in most of the works on this subject is to use for any particular n some n-valued counterparts of the structures used in the usual proof systems for classical logic (like sequents with n components, tableaux systems with n signs, etc.). This fact is well reflected in two recent survey papers [Häh99] and [BFS00] (see there for extensive list of references). In this paper we concentrate, in contrast, on proofs systems which use the same syntactic structures as the classical ones (with particular emphasis on Gentzen-type systems, on which all other systems are based). This approach has two advantages. First, the implementation of the proofs systems described here can be based on existing systems, because no new data structure is used. Second, two sided sequents (like those used in classical logic) can directly represent the consequence relation of *any* given logic, and this after all is what logics are all about.

A decent Gentzen-type system for a logic should allow for direct proofs, which means that it should admit some useful forms of cut elimination and the subformula property. In the rest of this survey we explain the main difficulty in developing classical Gentzen-type systems with these properties for many-valued logics, and discuss the possible methods of overcoming this difficulty. We then illustrate with numerous examples these methods. Our examples include practically all 3-valued and 4-valued logics, as well as Gödel finite-valued logics and some well-known infinite-valued logics which can be taken to be semi-finite. One of our main goals is to demonstrate the strong connection which exists in all cases between semantic completeness and the admissibility of the cut rule. The various proofs we present (most of which are new) establish therefore both (strong) completeness and (strong) cut-elimination at the same time.

2 Gentzen-type Systems— A Background

In what follows $\mathcal{L}$ is a propositional language, p, q, r denote atomic formulas, $A, B, C, \psi, \varphi, \phi$ denote arbitrary formulas (of $\mathcal{L}$), and Γ, Δ denote finite sets of formulas of $\mathcal{L}$. A sequent of $\mathcal{L}$ has the form $\Gamma \Rightarrow \Delta$. We use s as a variable for sequents and S as a variable for a finite set of sequents. Following tradition, we write Γ, φ and Γ, Δ for $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \Delta$ (respectively).

Definition 1.

1. [Sco74b,Sco74a] *A (*Scott*)* consequence relation *(*scr *for short) for $\mathcal{L}$ is a binary relation $\vdash$ between (finite) sets of formulas of $\mathcal{L}$ that satisfies the following conditions:*

   R reflexivity*:* $\{\varphi\} \vdash \{\varphi\}$ *for every* φ.
   M monotonicity*:* *if* $\Gamma \vdash \Delta$ *and* $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ *then* $\Gamma' \vdash \Delta'$.
   C cut*:* *if* $\Gamma \vdash \psi, \Delta$ *and* $\Gamma', \psi \vdash \Delta'$ *then* $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

2. *An scr* $\vdash$ *is called* structural *if* $\Gamma \vdash \Delta$ *implies* $\sigma(\Gamma) \vdash \sigma(\Delta)$ *for every uniform substitution* σ *and for every* Γ *and* Δ. $\vdash$ *is* consistent *(or* non-trivial*) if there exist non-empty* Γ *and* Δ *s.t.* $\Gamma \not\vdash \Delta$.
3. *A propositional* logic *is a pair* $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, *where* $\mathcal{L}$ *is a propositional language and* $\vdash_{\mathbf{L}}$ *is a structural consistent scr for* $\mathcal{L}$.

Some notes concerning this Definition are in order:

– The notion of an scr is a natural symmetric generalization of the notion of a *Tarskian* consequence relation (tcr), which is defined similarly, but is a relation between sets of formulas and *formulas* (the monotonicity condition applies in it therefore only to the l.h.s.).

- There are exactly four inconsistent finitary scrs in any given language: the one in which $\Gamma \vdash \Delta$ iff Γ and Δ are non-empty; the one in which $\Gamma \vdash \Delta$ iff Γ is non-empty; the one in which $\Gamma \vdash \Delta$ iff Δ is non-empty; and the one in which $\Gamma \vdash \Delta$ for all Γ and Δ. All of them should be considered trivial, and are excluded from our definition of a *logic*.
- A useful generalization of the concept of an scr is obtained by giving up the monotonicity condition. In what follows we shall call a reaction which satisfies conditions **R** and **C** of Definition 1 a *generalized scr*. It is easy to see that any generalized scr $\vdash$ can be extended to an ordinary scr $\vdash^*$ by letting $\Gamma \vdash^* \Delta$ iff there exist $\Gamma' \subseteq \Gamma, \Delta' \subseteq \Delta$ such that $\Gamma' \vdash \Delta'$. There are further, less natural, generalizations of the concept of an scr in which the relation is taken to be between multisets, or even sequences, of formulas rather than sets thereof. We shall have no use here for such generalizations. In fact, we shall see that a many-valued logic always directly defines an scr, and any other entailment relation associated with it is reducible to that scr (at least in all cases I know).

A Gentzen-type calculus ([Gen69]) over $\mathcal{L}$ is an axiomatic system which manipulates higher-level constructs called *sequents*, rather than the formulae themselves. There are several variants of what constitutes a sequent. Here we shall always take it to be a construct of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite *sets* of formulae of $\mathcal{L}$ and $\Rightarrow$ is a new symbol, not occurring in $\mathcal{L}$. In other variants Γ, Δ may be either multisets or sequences of formulae. There is no real difference, if we assume the following *standard structural rules* (permutation, contraction, and expansion, respectively):

$$\frac{\Gamma_1, \varphi, \psi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \psi, \varphi, \Gamma_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta_1, \varphi, \psi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \psi, \varphi, \Delta_2}$$

$$\frac{\Gamma_1, \varphi, \varphi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \varphi, \Gamma_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta_1, \varphi, \varphi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi, \Delta_2}$$

$$\frac{\Gamma_1, \varphi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \varphi, \varphi, \Gamma_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta_1, \varphi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi, \varphi, \Delta_2}$$

Definition 2. *A Gentzen-type system* **G** *is called* standard *if its set of axioms includes the standard axioms: $\psi \Rightarrow \psi$ (for all ψ), and its set of rules includes (the standard structural rules and) the following* Weakening *and* Cut *rules:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad \varphi, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Definition 3. *A rule of a Gentzen-type system* **G** *is called* pure *(*[Avr91b]*), or* multiplicative *(*[Gir87a]*), if whenever $\Gamma \Rightarrow \Delta$ can be inferred by it from $\Gamma_i \Rightarrow \Delta_i$ $(i = 1, \ldots, n)$ then, for arbitrary $\Gamma'_1, \ldots, \Gamma'_n, \Delta'_1, \ldots, \Delta'_n$, the sequent $\Gamma, \Gamma'_1, \ldots, \Gamma'_n \Rightarrow \Delta, \Delta'_1, \ldots, \Delta'_n$ can also be inferred by it from $\Gamma_i, \Gamma'_i \Rightarrow \Delta_i, \Delta'_i$ $(i = 1, \ldots, n)$*[1]. **G** *is called pure if all its rules are pure.*

[1] In other words: a rule is pure if it is context-free. In practice, this means that there are no side conditions on how it can be applied.

What a Gentzen-type system **G** directly defines is a Tarskian consequence relation between *sequents*: A sequent s follows in **G** from a set S of sequents ($S \vdash_{\mathbf{G}} s$) iff s can be derived from the sequents in S and the axioms of **G** using the rules of **G**. Usually, however, a Gentzen-type system **G** is mainly used as a tool for investigating scrs between the *formulas* of the underlying language. Any non-trivial standard Gentzen-type system **G** naturally defines in fact an scr (which we shall also denote by $\vdash_{\mathbf{G}}$):

Definition 4.

1. *Let* **G** *be a standard Gentzen-type system. We say that* $\Gamma \vdash_{\mathbf{G}} \Delta$ *iff* $\Gamma' \Rightarrow \Delta'$ *is a theorem of* **G** *for some* $\Gamma' \supseteq \Gamma, \Delta' \supseteq \Delta$. $\vdash_{\mathbf{G}}$ *is called the* standard scr defined by **G**.
2. *A Gentzen-type system* **G** *over a language* $\mathcal{L}$ *is* sound and complete *for a logic* $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ *if* $\vdash_{\mathbf{L}} = \vdash_{\mathbf{G}}$. **G** *is* weakly sound and complete *for* **L** *if for all* ψ, $\vdash_{\mathbf{L}} \psi$ *iff* $\Rightarrow \psi$ *is provable in* **G**.

Proposition 1. *If* **G** *is standard then the relation* $\vdash_{\mathbf{G}}$ *is a finitary scr. Moreover: if* Γ *and* Δ *are finite then* $\Gamma \vdash_{\mathbf{G}} \Delta$ *iff* $\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$.

We leave the easy proof for the reader. We note only that the first part of this proposition (but not the second) remains true even if the set of rules of **G** does not include the weakening rule.

Proposition 2. *If a Gentzen-type system* **G** *is both standard and pure then* $\varphi_1, \ldots, \varphi_n \vdash_{\mathbf{G}} \psi_1, \ldots, \psi_m$ *iff the empty sequent* $\Rightarrow$ *follows in* **G** *from*

$$S = \{(\Rightarrow \varphi_1), \ldots, (\Rightarrow \varphi_n), (\psi_1 \Rightarrow), \ldots, (\psi_m \Rightarrow)\}$$

This remains true even if **G** *is not closed under weakening.*

Proof: If some subsequent of $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$ is provable in **G** then $\Rightarrow$ can be derived from it and the sequents in S using cuts. For the converse, assume that P is a proof in **G** of $\Rightarrow$ from S. Obtain P^* from P by adding (for each i and j) φ_i to the left hand side of all the sequents in P which depend on $\Rightarrow \varphi_i$, and ψ_j to the right hand side of all the sequents which depend on $\psi_j \Rightarrow$. These changes turn the premises from S which are used in P into axioms, while the conclusion $\Rightarrow$ of P is turned into a subsequent of $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$. Since all the rules of **G** are pure, all inference steps in P remain valid in P^*. Hence P^* is a valid proof in **G** of its conclusion, and so $\varphi_1, \ldots, \varphi_n \vdash_{\mathbf{G}} \psi_1, \ldots, \psi_m$.

Note: $\vdash_{\mathbf{G}}$ is no doubt the most natural consequence relation associated with a given Gentzen-type system **G**. However, it is not the only possible or useful one. Thus the identification of a formula ψ with the singleton $\Rightarrow \psi$ (implicitly used already in Definition 4) naturally leads to another important *Tarskian* consequence relation frequently associated with **G**, according to which a formula ψ follows from the set $\{\varphi_1, \ldots, \varphi_n\}$ if $\{(\Rightarrow \varphi_1), \ldots, (\Rightarrow \varphi_n)\} \vdash_{\mathbf{G}} \Rightarrow \psi$.

An ideal Gentzen-type system (of which the usual systems for classical logic provide the principal examples) is a pure, standard system in which every logical rule is an introduction rule for one connective. Moreover: it should introduce exactly one occurrence of that connective in its conclusion, no other occurrence of connectives should be mentioned anywhere else in its formulation, and its side formulas should be immediate subformulas of the principal formula. The next definition formulates this idea in exact terms, and provides a method for describing such rules.

Definition 5. ([AL01])

1. *A* canonical rule *of arity n is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C$, where $m \geq 0$, C is either $\diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ or $\Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ for some connective $\diamond$ (of arity n), and for $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a clause such that $\Pi_i, \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.*[2]
2. *An application of a canonical rule of the form:*

$$\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} \;/\; \diamond(p_1, \ldots, p_n) \Rightarrow$$

is any inference step of the form:

$$\frac{\{\Gamma_i, \Pi_i^* \Rightarrow \Delta_i, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \Delta}$$

where Π_i^* and Σ_i^* are obtained from Π_i and Σ_i (respectively) by substituting ψ_j for p_j (for $1 \leq j \leq n$), Γ_i, Δ_i are any finite sets of formulas, $\Gamma = \bigcup_{i=1}^m \Gamma_i$, and $\Delta = \bigcup_{i=1}^m \Delta_i$. An application of a canonical rule for introducing a connective on the right hand side is defined similarly.

Notes:

1. Definition 5 allows the case $m = 0$, when canonical rules reduce to canonical *axioms* of the form $\diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ or $\Rightarrow \diamond(p_1, p_2, \ldots, p_n)$. This amounts to allowing canonical propositional constants.
2. We have used in the last definition the multiplicative form of an application of a rule. In what follows we shall usually prefer the *additive* ([Gir87a]) version, in which all premises of a rule have the same side formulas. Thus an application of a canonical rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} \;/\; \diamond(p_1, \ldots, p_n) \Rightarrow$ has in the additive version the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \Delta}$$

where Π_i^* and Σ_i^* are as above. The two versions are easily seen to be equivalent in the framework of standard systems (since both weakening and contraction are available in them).

[2] By a clause we mean a sequent which consists of atomic formulas only. When propositional clauses are written in this way, resolution and cut amount to the same thing.

Example 1. The two usual introduction rules for classical conjunction can be formulated as the two canonical rules:

$$\{(p_1, p_2 \Rightarrow)\} /\ p_1 \wedge p_2 \Rightarrow \qquad \{(\Rightarrow p_1), (\Rightarrow p_2)\}\ /\Rightarrow p_1 \wedge p_2$$

Applications of these rules have the form:

$$\frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma' \Rightarrow \Delta', \phi}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \psi \wedge \phi}$$

(for the additive versions of these rules see the system GCPL below).

Definition 6. *A Gentzen-type system is called* canonical *if its axioms include the standard axioms, its rules include the standard structural rules, cut and weakening, and all its other rules and axioms are canonical.*

A canonical system is obviously standard and pure. However, in order for $\vdash_{\mathbf{G}}$ to really be an scr, one further condition should be satisfied:

Definition 7. The coherence condition: *Whenever* $S_1 / \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ *and* $S_2 / \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ *are rules of* $\mathbf{G}$*, the set of clauses* $S_1 \cup S_2$ *is classically inconsistent (and so the empty clause can be derived from it using cuts).*

Theorem 1. ([AL01]) *If* $\mathbf{G}$ *is a canonical system then* $\vdash_{\mathbf{G}}$ *is consistent (and so an scr) iff the coherence condition is satisfied. Moreover: any canonical system which satisfies this condition admits cut-elimination.*

Example 2. The two classical rules for conjunction described in Example 1 form a coherent pair of rules, where $S_1 = \{(p_1, p_2 \Rightarrow)\}$, $S_2 = \{(\Rightarrow p_1), (\Rightarrow p_2)\}$. $S_1 \cup S_2$ is here the inconsistent set $\{(p_1, p_2 \Rightarrow), (\Rightarrow p_1), (\Rightarrow p_2)\}$.

3 A Classical Example

The classical examples of canonical Gentzen-type systems are those for Classical Propositional Logic (CPL) and its fragments. We present now a well-known canonical Gentzen-type systems for CPL. We follow [Gen69] in using the *additive* form of the rules. The reason is that in the classical case (as well as in most of the systems we discuss later) the additive rules are *invertible* (i.e.: their premises can be derived in the system from their conclusion using the other rules of the system, including cut). This property is very useful, and so in describing standard systems we shall use the additive form of a rule whenever it is invertible.

THE SYSTEM GCPL

Axioms: $A \Rightarrow A$

Structural Rules: Cut, Weakening (and the standard rules)

Logical Rules:

$$(\neg\Rightarrow)\quad \frac{\Gamma\Rightarrow\Delta, A}{\neg A,\Gamma\Rightarrow\Delta} \qquad \frac{A,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg A}\quad(\Rightarrow\neg)$$

$$(\supset\Rightarrow)\quad \frac{\Gamma\Rightarrow\Delta, A \quad B,\Gamma\Rightarrow\Delta}{A\supset B,\Gamma\Rightarrow\Delta} \qquad \frac{\Gamma, A\Rightarrow\Delta, B}{\Gamma\Rightarrow\Delta, A\supset B}\quad(\Rightarrow\supset)$$

$$(\wedge\Rightarrow)\quad \frac{\Gamma, A, B\Rightarrow\Delta}{\Gamma, A\wedge B\Rightarrow\Delta} \qquad \frac{\Gamma\Rightarrow\Delta, A \qquad \Gamma\Rightarrow\Delta, B}{\Gamma\Rightarrow\Delta, A\wedge B}\quad(\Rightarrow\wedge)$$

$$(\vee\Rightarrow)\quad \frac{\Gamma, A\Rightarrow\Delta \qquad \Gamma, B\Rightarrow\Delta}{\Gamma, A\vee B\Rightarrow\Delta} \qquad \frac{\Gamma\Rightarrow\Delta, A, B}{\Gamma\Rightarrow\Delta, A\vee B}\quad(\Rightarrow\vee)$$

The semantic of CPL is of course the classical, two-valued semantics, with the classical interpretation of the connectives. A valuation v in $\{t, f\}$ which respects the classical operations is a model of sentence φ if $v(\varphi) = t$. The corresponding classical scr is defined by: $\Gamma \vdash_{CPL} \Delta$ if every model of all the sentences in Γ is a model of some sentence in Δ. A sequent $\Gamma \Rightarrow \Delta$ is *classically valid* if $\Gamma \vdash_{CPL} \Delta$. The concepts of truth and consequence can be extended to sequents as follows: A valuation v is a model of a sequent $\Gamma \Rightarrow \Delta$ if $v(\varphi) = f$ for some $\varphi \in \Gamma$, or $v(\varphi) = t$ for some $\varphi \in \Delta$. A sequent s classically follows from a set S of sequents if every model of all the sequents in S is also a model of s. It can easily be seen then that a sequent is classically valid iff it classically follows from $\emptyset$.

It is not difficult to show that $GCPL$ is strongly sound for the classical semantics described above, i.e.: if $S \vdash_{GCPL} s$ than the sequent s classically follows from S (in particular: if s is provable in GCPL then it is classically valid). The completeness of $GCPL$, on the other hand, is one of the two most important theorems concerning this calculus. The other one is Gentzen's celebrated cut-elimination theorem ([Gen69,TS00]). We present and simultaneously prove now strengthened forms of both theorems.

Theorem 2. Strong Completeness and Cut-elimination for GCPL: *A sequent s classically follows from $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ iff s has a proof in GCPL from S in which* all cuts are done on formulas in $\bigcup_{i=1}^n \Gamma_i \cup \bigcup_{i=1}^n \Delta_i$.

Proof: We start by introducing some definitions. Let $F = \bigcup_{i=1}^n \Gamma_i \cup \bigcup_{i=1}^n \Delta_i$. An S-cut is an application of the cut rule in which the cut formula belongs to F. An S-proof of a sequent s is a derivation of s in GCPL in which the sequents of S may be used as extra axioms, and all the cuts are S-cuts. Finally, a sequent $\Gamma^* \Rightarrow \Delta^*$ is called *S-saturated* if:

(1) It has no S-proof.
(2) If $\varphi \in F$ then $\varphi \in \Gamma^* \cup \Delta^*$
(3) If $\varphi \supset \psi \in \Delta^*$ then $\varphi \in \Gamma^*$ and $\psi \in \Delta^*$

If $\varphi \supset \psi \in \Gamma^*$ then either $\varphi \in \Delta^*$ or $\psi \in \Gamma^*$

(4) Similar conditions, corresponding to the other rules of GCPL, obtain for the other connectives.

We shall show now that if s does not have an S-proof then there is a model of S which is not a model of s. This will be done in two stages, reflected in the two lemmas below. The theorem easily follows from these two lemmas.

Lemma 1. If $\Gamma \Rightarrow \Delta$ has no S-proof then it can be extended to an S-saturated sequent $\Gamma^* \Rightarrow \Delta^*$.

Proof of Lemma 1: Let $\Gamma^* \Rightarrow \Delta^*$ be a maximal extension of $\Gamma \Rightarrow \Delta$ which does not have an S-proof, and such that $\Gamma^* \cup \Delta^*$ contains only formulas from F^*, where F^* is the set of subformulas of formulas in $F \cup \Gamma \cup \Delta$ (since F^* is finite, such a maximal extension exists). Obviously, if $\varphi \in F^*$ then $\varphi \notin \Gamma^*$ ($\varphi \notin \Delta^*$) iff $\varphi, \Gamma^* \Rightarrow \Delta^*$ ($\Gamma^* \Rightarrow \Delta^*, \varphi$) has an S-proof. It follows that if $\varphi \in F^*$ but $\varphi \notin \Gamma^* \cup \Delta^*$ then both $\Gamma^* \Rightarrow \Delta^*, \varphi$ and $\varphi, \Gamma^* \Rightarrow \Delta^*$ have S-proofs. φ cannot therefore be an element of F in such a case, since otherwise an S-cut on φ would provide an S-proof of $\Gamma^* \Rightarrow \Delta^*$. Hence $\Gamma^* \Rightarrow \Delta^*$ satisfies the second condition in the definition of an S-saturated sequent. We show now that it satisfies the others as well. So assume, e.g., that $\varphi \supset \psi \in \Delta^*$. In such a case $\Gamma^* \Rightarrow \Delta^*$ can be derived from $\varphi, \Gamma^* \Rightarrow \Delta^*, \psi$ in a single logical inference step. It follows that $\varphi, \Gamma^* \Rightarrow \Delta^*, \psi$ has no S-proof, and so the maximality property of $\Gamma^* \Rightarrow \Delta^*$ implies that $\varphi \in \Gamma^*$ and $\psi \in \Delta^*$. The other conditions are proved similarly.

Lemma 2. If $\Gamma^* \Rightarrow \Delta^*$ is S-saturated then there is a model of S which is not a model of $\Gamma^* \Rightarrow \Delta^*$.

Proof of Lemma 2: Define

$$v(p) = \begin{cases} t & p \in \Gamma^* \\ f & p \notin \Gamma^* \end{cases}$$

We show by induction on the complexity of φ that $v(\varphi) = t$ for every $\varphi \in \Gamma^*$, and $v(\varphi) = f$ for every $\varphi \in \Delta^*$ (and so v is not a model of $\Gamma^* \Rightarrow \Delta^*$). If p is atomic this follows immediately from the definition of v and the fact that if $p \in \Delta^*$ then $p \notin \Gamma^*$ (otherwise $\Gamma^* \Rightarrow \Delta^*$ would have a trivial S-proof). Assume now that $\varphi = \phi \supset \psi$ (other cases are treated similarly). In such a case if $\varphi \in \Delta^*$ then $\phi \in \Gamma^*$ and $\psi \in \Delta^*$. Hence $v(\phi) = t$ and $v(\psi) = f$ by induction hypothesis, and so $v(\varphi) = f$. If, on the other hand, $\varphi \in \Gamma^*$ then either $\phi \in \Delta^*$ or $\psi \in \Gamma^*$. Hence either $v(\phi) = f$ or $v(\psi) = t$, and in either case $v(\varphi) = t$.

It remains to show that v *is* a model of S. Let $\Gamma_i \Rightarrow \Delta_i$ be an element of S. Then $\Gamma_i \cup \Delta_i \subseteq \Gamma^* \cup \Delta^*$ because $\Gamma^* \Rightarrow \Delta^*$ is S-saturated. It cannot be the case that both $\Gamma_i \subseteq \Gamma^*$ and $\Delta_i \subseteq \Delta^*$, because in such a case $\Gamma^* \Rightarrow \Delta^*$ would have a trivial S-proof (using only weakenings). Hence either $\varphi \in \Delta^*$ for some $\varphi \in \Gamma_i$, or $\varphi \in \Gamma^*$ for some $\varphi \in \Delta_i$. In the first case $v(\varphi) = f$ and so v is a model of $\Gamma_i \Rightarrow \Delta_i$; in the second $v(\varphi) = t$, and again v is a model of $\Gamma_i \Rightarrow \Delta_i$.

Theorem 2 has the following immediate corollaries:

Corollary 1. Completeness of GCPL:

1. *A sequent s classically follows from a set S of sequents iff* $S \vdash_{GCPL} s$.
2. $\Gamma \vdash_{CPL} \Delta$ *iff* $\Gamma \vdash_{GCPL} \Delta$.

Corollary 2. Strong Cut-elimination for GCPL ([Gir87b,Avr93]):

1. $\Gamma \Rightarrow \Delta$ *is derivable in* $GCPL$ *from* $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ *iff it has a proof in* $GCPL$ *from* S *in which all cuts are done on formulas in* $\bigcup_{i=1}^{n} \Gamma_i \cup \bigcup_{i=1}^{n} \Delta_i$.
2. $\vdash_{GCPL} \Gamma \Rightarrow \Delta$ *iff it has in* $GCPL$ *a cut-free proof.*

Note: The proof we have just presented demonstrates the semantic approach to the issue of cut-elimination, and it is paradigmatic: all our proofs below for other systems use similar methods and lines of thought (though the details might be more complicated). It should be noted that Gentzen's original method of proof, in contrast, was completely syntactic: it shows how to constructively eliminate cuts by a double induction on the complexity of the cut-formula and on the sum of the lengths (or of the heights) of the proofs of their premises. This approach has its own advantages, but for our present purpose of demonstrating the deep connection between the admissibility of the cut rule and semantic completeness, the semantic approach is superior.

The (strong) cut-elimination theorem for $GCPL$ has many applications, like: decidability, the interpolation theorem, and the subformula property. It is also the basis for the two main proof search methods for CPL (see [Avr93] for more details):

The Tableaux Method: Try to show that $\Gamma \vdash_{CPL} \Delta$ by searching for a cut-free proof of $\Gamma \Rightarrow \Delta$ in $GCPL$. This is done by applying "backwards" the invertible versions of the rules. In particular: one shows that φ is valid by searching for a cut-free proof of $\Rightarrow \varphi$. This gives either such a proof, or an equivalent set of *clauses* (i.e.: sequents consisting of atomic formulas) which can be translated into a conjunctive normal form for φ.

The Resolution Method: Try to show that $\varphi_1, \ldots, \varphi_n \vdash_{CPL} \psi_1, \ldots, \psi_k$ by showing that the empty sequent $\Rightarrow$ can be derived in $GCPL$ from the set:

$$\{(\Rightarrow \varphi_1), , \ldots, (\Rightarrow \varphi_n), (\psi_1 \Rightarrow), \ldots, (\psi_k \Rightarrow)\}$$

(In particular, show that a formula φ is valid by proving that $(\varphi \Rightarrow) \vdash_{GCPL} \Rightarrow)$. For this replace (using tableaux) each $(\Rightarrow \varphi_i)$ and $(\psi_j \Rightarrow)$ by an equivalent set of *clauses*. By the *strong* cut-elimination theorem, if $\Rightarrow$ is derivable from the original set of sequents, then it can be derived from the union of the equivalent sets of clauses using only *cuts*.

4 The Problem with Many-Valued Logics

We start by defining in precise terms what we mean by "a finite-valued logic", and, more generally, "a many-valued logic".

Definition 8.

1. *A* matrix $\mathcal{M}$ *for a language* $\mathcal{L}$ *is a triple* $\langle M, D, O\rangle$ *such that:*
   (a) *M is a nonempty set (of "truth-values").*
   (b) *D is a proper, nonempty subset of M (the "designated values").*[3]
   (c) *O is a set of operations on M, so that for each connective of $\mathcal{L}$ there is a corresponding operation on M.*
2. *Let $\mathcal{M} =< \mathcal{M}, \mathcal{D}, \mathcal{O} >$ be a matrix for $\mathcal{L}$. A function v from the set of formulas of $\mathcal{L}$ into M is called a valuation in $\mathcal{M}$ if it respects the operations in O. A valuation v is an $\mathcal{M}$-model of a formula φ of $\mathcal{L}$ if $v(\varphi) \in D$. v is an $\mathcal{M}$-model of a set T of formulas if it is an $\mathcal{M}$-model of each element of T.*
3. *Let $\mathcal{M}$ be a matrix for $\mathcal{L}$. $\vdash_{\mathcal{M}}$, the consequence relation induced by $\mathcal{M}$, is defined by: $\Gamma \vdash_{\mathcal{M}} \Delta$ iff every model of Γ is a model of some formula in Δ.*
4. *A sequent $\Gamma \Rightarrow \Delta$ is* $\mathcal{M}$-valid *if $\Gamma \vdash_{\mathcal{M}} \Delta$.*
5. *Let $\mathcal{M}$ be a matrix for $\mathcal{L}$. A valuation v is an $\mathcal{M}$-model of a sequent $\Gamma \Rightarrow \Delta$ if $v(\varphi) \notin D$ for some $\varphi \in \Gamma$, or $v(\varphi) \in D$ for some $\varphi \in \Delta$. A sequent s follows in $\mathcal{M}$ from a set S of sequents if every $\mathcal{M}$-model of all the sequents in S is also an $\mathcal{M}$-model of s.*

It is easy to see that $\vdash_{\mathcal{M}}$ is indeed a consequence relation (this is true in fact for any relation which is based on some notion of a "model" in a way similar to that of $\vdash_{\mathcal{M}}$), that $\langle \mathcal{L}, \vdash_{\mathcal{M}}\rangle$ is a logic (see Definition 1), and that again a sequent is $\mathcal{M}$-valid iff it follows in $\mathcal{M}$ from $\emptyset$. Obviously, the various parts of the last Definition are all straightforward generalizations of the corresponding classical notions.

Definition 9.
Let $n \geq 2$ be a natural number. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}}\rangle$ is called n-valued if there exists a matrix $\mathcal{M}$ for $\mathcal{L}$ such that M has exactly n elements, and $\vdash_{\mathcal{M}} = \vdash_{\mathbf{L}}$. $\mathbf{L}$ is called weakly n-valued *if there exists an n-valued matrix $\mathcal{M}$ for $\mathcal{L}$ such that for every sentence φ, $\vdash_{\mathcal{M}} \varphi$ iff $\vdash_{\mathbf{L}} \varphi$.*

Notes:

1. A matrix $\mathcal{M}$ is frequently used for defining entailment relations (which are usually not scrs) other than $\vdash_{\mathcal{M}}$. In all cases we know the definitions of these alternative consequence relations can be reduced to the validity of certain formulas in $\mathcal{M}$ (usually connected to what is taken to be the "official" implication connective of the logic), and so to $\vdash_{\mathcal{M}}$. Thus the "consequence relation" usually associated with Łukasiewicz 3-valued logic can be characterized

[3] In [Urq86] and elsewhere the only condition concerning D is that it should be a subset of M. We exclude here the two extreme cases ($D = M$ and $D = \emptyset$) because the corresponding scrs (as defined below) are not consistent.

as follows: $\varphi_1, \ldots, \varphi_n \vdash \psi$ iff the formula $\varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$ is valid in Łukasiewicz 3-valued matrix. Similar definitions are used for many other finite-valued "logics". Therefore by finding an appropriate Gentzen-type system which is sound and complete for $\vdash_{\mathcal{M}}$ (or at least weakly complete for it) we almost always solve also the problem of Gentzenizing other logics which are based on $\mathcal{M}$.

2. In the literature the term "many-valued logic" usually has a broader sense than the notion of a "finite-valued logic" which we have just defined. In particular: it includes some logics which do not have a finite characteristic matrix. However, the scope of this notion should be restricted somehow, since if we allow arbitrary characteristic matrices then *every* logic (closed under substitution) would become "many-valued" by a famous theorem of Łos and Suszko ([ŁS58,Urq86]). Our next definition demarcates the class of "many-valued" logics which are dealt with in this paper:

Definition 10. *A logic* $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ *is called (weakly)* many-valued *if there exists a matrix* $\mathcal{M}$ *for* L *such that* $\vdash_{\mathcal{M}} = \vdash_{\mathbf{L}}$ *(*$\vdash_{\mathcal{M}} \varphi$ *iff* $\vdash_{\mathbf{L}} \varphi$ *for every sentence* φ*), and for every finite* Γ *and* Δ *(for every sentence* φ*) there is a finite submatrix* $\mathcal{M}^*$ *of* $\mathcal{M}$ *such that* $\Gamma \vdash_{\mathbf{L}} \Delta$ *(*$\vdash_{\mathbf{L}} \varphi$*) iff* $\Gamma \vdash_{\mathcal{M}^*} \Delta$ *(*$\vdash_{\mathcal{M}^*} \varphi$*).*

The following theorem is the main obstacle to providing decent Gentzen-type calculi for many-valued logics:

Theorem 3. ([AL01]) *Let* $\mathbf{G}$ *be a consistent canonical calculus. Then either* $\mathbf{G}$ *defines a logic which is a fragment of classical logic, or it is not sound and complete for any many-valued logic.*

It follows that a Gentzen-type calculus for a given many-valued logic should use at least one of the following:

- Noncanonical rules And/or axioms which are neither standard nor canonical
- Impure rules (i.e.: rules with side conditions on their applications)
- A nonstandard set of structural rules

In the following sections we shall see examples of all these alternatives.

5 Three-Valued Logics

We start with the simplest type of non-classical many-valued logics: the three-valued ones. We assume (w.l.o.g.) that those three values are t, f and I, where t is designated and f is not. The 3-valued logics are accordingly divided into two classes: those in which I is designated (i.e. $D = \{t, I\}$), and those in which it is not (i.e. $D = \{t\}$). The logics inside each class differ only with respect to the expressive power of their languages. Hence a logic the language of which includes a functionally complete sets of 3-valued operations contains all other logics in its class. Essentially there are therefore just two 3-valued logics, and all the rest are just fragments of them. Our main strategy in what follows is therefore to select for each of the two cases an appropriate functionally complete set of connectives, and then to find a sound and complete set of rules for that set. Under one minimal requirement from the language (that it includes the most standard three-valued negation), this will allow us to find an adequate set of rules for *every* connective, and so for every 3-valued logic (having this negation).

A crucial guiding line in choosing in each case an appropriate set of connectives is to use, as far as possible, connectives which are obvious counterparts of the common classical connectives. Moreover: we want our systems to closely resemble GCPL (so that their use and implementation would require almost no new efforts). We also want to employ connectives which are actually used in current applications of 3-valued logics. Now the most famous 3-valued logics (including, e.g., Łukasiewicz' Ł$_3$ ([Łuk67])) and Kleene's K$_3$ ([Kle50])) employ generalizations of the classical connectives $\neg$, $\vee$ and $\wedge$, in which $\vee$, $\wedge$ are interpreted as the *max* and *min* operations (respectively) according to the order $f \leq I \leq t$, while the interpretation of $\neg$ is given by:

$$\neg t = f \qquad \neg f = t \qquad \neg I = I$$

The above three connectives are not expressive enough for defining (in either of the two cases) an implication connective for which the two classical rules (or just both MP and the deduction theorem) are valid. For this we use the following general construction from [AA96]:

Definition 11. *Let $\mathcal{M} = \langle M, D, O \rangle$ be a matrix such that $t \in D$. The* natural implication operation *of $\mathcal{M}$ is defined by:*

$$a \supset b \ = \begin{cases} b & \text{if } a \in D \\ t & \text{if } a \notin D \end{cases}$$

Unlike in the two-valued case, the resulting set of connectives $\{\neg, \vee, \wedge, \supset\}$ is not functionally complete yet (in both cases). In [Avr99] it is proved, however, that by adding the propositional constants f and I (interpreted as the corresponding truth values), one does get a functionally complete set of connectives. In what follows we shall use therefore the set $\{\neg, \vee, \wedge, \supset, f, I\}$ as our basic set of connectives. Note that among the connectives of this set only the propositional constant I is not a counterpart of a standard classical operation and is peculiar to three-valued logics (*any* functionally complete set of 3-valued connectives should contain at least one connective of this sort, of course).

Notes:

1. Unlike the other connectives, the interpretation of $\supset$ depends on the choice of D. Hence we use here in fact two different implications: $\supset_{\{t\}}$ (in the case $D = \{t\}$), and $\supset_{\{t,I\}}$ (in the case $D = \{t, I\}$). The two implications are however definable in terms of each other using negation and the propositional constant f. In fact, $a \supset_{\{t\}} b = \neg(\neg a \supset_{\{t,I\}} f) \vee b$ and similarly $a \supset_{\{t,I\}} b = \neg(\neg a \supset_{\{t\}} f) \vee b$. In what follows we shall usually use just $\supset$ for both, relying on the context for determining which one we have in mind.

2. The connective $\supset_{\{t\}}$ was originally introduced by Słupecki in [Słu36]. It was independently reintroduced in [Mon67,Woj84,Sch86] and [Avr91a] (see also [Bus96]). The language $\{\neg, \vee, \wedge, \supset_{\{t\}}\}$ is equivalent ([Avr91a]) to that used in the logic LPF of the VDM project ([Jon86]), as well as to the language of Łukasiewicz 3-valued logic Ł$_3$ ([Łuk67])).[4]
3. The connective $\supset_{\{t,I\}}$ was first introduced in [DdC70,dC74]. It was independently introduced also in [Avr86]. The language $\{\neg, \vee, \wedge, \supset_{\{t,I\}}\}$ is equivalent to that used in the standard 3-valued paraconsistent logic J_3 ([D'o85,Avr86,Roz89,Eps90]. In [Avr91a], it is called *Pac*), as well as to that used in the semi-relevant system RM_3 ([AB75,AB92,Dun86]. See also [Avr86,Avr91a]).[5]

Definition 12.

1. $\mathcal{M}_3^{\{t\}}$ *is the 3-valued matrix for the language* $\{\neg, \vee, \wedge, \supset, f, I\}$ *in which* $D = \{t\}$ *and the interpretation of* $\supset$ *is* $\supset_{\{t\}}$.
2. $\mathcal{M}_3^{\{t,I\}}$ *is the 3-valued matrix for the language* $\{\neg, \vee, \wedge, \supset, f, I\}$ *in which* $D = \{t, I\}$ *and the interpretation of* $\supset$ *is* $\supset_{\{t,I\}}$.

5.1 The Use of Noncanonical Rules and Axioms

The most important feature of canonical rules is that they introduce exactly one new occurrence of a connective at a time. Most of the Gentzen-type systems for many-valued logics give up this feature by allowing rules which introduce two occurrences of connectives at the same time. To see how to satisfactorily do it in the present case we check first what rules of GCPL remain valid according to their 3-valued interpretations. It can easily be seen that this is the case with the classical rules for $\vee, \wedge$, and $\supset$. This is true for both $\mathcal{M}_3^{\{t\}}$ and $\mathcal{M}_3^{\{t,I\}}$. The situation with negation is different: in both matrices one of the two rules for $\neg$ is valid, while the other is not. In $\mathcal{M}_3^{\{t\}}$ the rule $(\neg \Rightarrow)$ is valid while $(\Rightarrow \neg)$ is not, and the opposite is true in $\mathcal{M}_3^{\{t,I\}}$. It follows that $\neg$ is the connective which needs noncanonical rules. It seems that the best way to handle negation is to replace, first of all, its two classical rules with standard rules for the *combination* of negation with the classical connectives of the language. This is what is done in the following basic system (from [Avr91a]):

THE SYSTEM GBS: This is the systems obtained from $GCPL$ by deleting the two rules for negation and adding instead the following rules and axioms:

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[4] It is in fact the language of all 3-valued operations which are *classically closed* (See [Avr99] for further details and references).

[5] It is the language of all 3-valued operations which are classically closed and *free* ([Avr99]).

$$(\neg\neg\Rightarrow)\quad \frac{A,\Gamma\Rightarrow\Delta}{\neg\neg A,\Gamma\Rightarrow\Delta} \qquad \frac{\Gamma\Rightarrow\Delta,A}{\Gamma\Rightarrow\Delta,\neg\neg A}\quad(\Rightarrow\neg\neg)$$

$$(\neg\supset\Rightarrow)\quad \frac{A,\neg B,\Gamma\Rightarrow\Delta}{\neg(A\supset B),\Gamma\Rightarrow\Delta} \qquad \frac{\Gamma\Rightarrow\Delta,A \qquad \Gamma\Rightarrow,\neg B}{\Gamma\Rightarrow\Delta,\neg(A\supset B)}\quad(\Rightarrow\neg\supset)$$

$$(\neg\vee\Rightarrow)\quad \frac{\Gamma,\neg A,\neg B\Rightarrow\Delta}{\Gamma,\neg(A\vee B)\Rightarrow\Delta} \qquad \frac{\Gamma\Rightarrow\Delta,\neg A \qquad \Gamma\Rightarrow\Delta,\neg B}{\Gamma\Rightarrow\Delta,\neg(A\vee B)}\quad(\Rightarrow\neg\vee)$$

$$(\neg\wedge\Rightarrow)\quad \frac{\Gamma,\neg A\Rightarrow\Delta \qquad \Gamma,\neg B\Rightarrow\Delta}{\Gamma,\neg(A\wedge B)\Rightarrow\Delta} \qquad \frac{\Gamma\Rightarrow\Delta,\neg A,\neg B}{\Gamma\Rightarrow\Delta,\neg(A\wedge B)}\quad(\Rightarrow\neg\wedge)$$

$$(f\Rightarrow)\quad f\Rightarrow \qquad\qquad \Rightarrow\neg f\quad(\Rightarrow\neg f)$$

Theorem 4. Strong Soundness of GBS: *If $S\vdash_{GBS} s$ than the sequent s follows from S in any matrix $\langle M,D,O\rangle$ (for a language which includes that of GBS) that satisfies the following conditions:*

1. *$t\in D$ and $f\notin D$*
2. *the interpretation of the propositional constant f is the truth value f, and the interpretation of $\supset$ is like in Definition 11.*
3. *$a\vee b\in D$ iff $a\in D$ or $b\in D$*
4. *$a\wedge b\in D$ iff $a\in D$ and $b\in D$*
5. *The operation $\neg$ is an extension of the classical negation which satisfies the De Morgan laws as well as the double negation law, i.e.:*

$$\neg t=f,\ \neg f=t,\ \neg\neg a=a,\ \neg(a\vee b)=\neg a\wedge\neg b,\ \neg(a\wedge b)=\neg a\vee\neg b$$

Proof: We show here the validity of the $(\neg\supset\Rightarrow)$ rule, leaving the other cases for the reader. So assume that v is a model of $A,\neg B,\Gamma\Rightarrow\Delta$. We show that it is also a model of $\neg(A\supset B),\Gamma\Rightarrow\Delta$. This is obvious in case $v(\varphi)\notin D$ for some $\varphi\in\Gamma$ or $v(\varphi)\in D$ for some $\varphi\in\Delta$. Otherwise either $v(A)\notin D$, or $v(A)\in D$ and $v(\neg B)\notin D$. In the first case $v(\neg(A\supset B))=f$, in the second $v(\neg(A\supset B))=v(\neg B)$, and so in both $v(\neg(A\supset B))\notin D$.

It immediately follows from the last theorem that GBS is strongly sound for both $\mathcal{M}_3^{\{t\}}$ and $\mathcal{M}_3^{\{t,I\}}$. It is obvious therefore that in order to get corresponding sound and *complete* systems one needs to extend GBS.

THE SYSTEM $GM_3^{\{t\}}$: This is GBS together with the following axioms:

$$\neg A,A\Rightarrow$$

$$I\Rightarrow \qquad\qquad \neg I\Rightarrow$$

THE SYSTEM $GM_3^{\{t,I\}}$: This is GBS together with the following axioms:

$$\Rightarrow \neg A, A$$

$$\Rightarrow I \qquad\qquad \Rightarrow \neg I$$

Theorem 5. Strong Soundness, Completeness and Cut-elimination for $GM_3^{\{t\}}$ and $GM_3^{\{t,I\}}$: *For $D \in \{\{t\},\{t,I\}\}$, a sequent $\Gamma \Rightarrow \Delta$ follows in $\mathcal{M}_3^D$ from $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ iff $\Gamma \Rightarrow \Delta$ has a proof in GM_3^D from S in which all cuts are done on formulas in $\bigcup_{i=1}^n \Gamma_i \cup \bigcup_{i=1}^n \Delta_i$.*

Proof: We give the proof for $GM_3^{\{t,I\}}$ (the case of $GM_3^{\{t\}}$ is dual).

Since it is easy to check that the extra axioms of $GM_3^{\{t,I\}}$ are valid in $\mathcal{M}_3^{\{t,I\}}$, the strong soundness of $GM_3^{\{t,I\}}$ follows from Theorem 4.

The simultaneous proof of strong completeness and the admissibility of Cut closely follows the proof of Theorem 2. We define the notions of an S-cut, an S-proof, and an S-saturated sequent like there (only in the definition of an S-saturated sequent one should replace, of course, the conditions which corresponds to the classical rules of negation by the conditions induced by the negation rules of GBS. For example: the condition that if $\neg(A \supset B) \in \Gamma^*$ then $A \in \Gamma^*$ and $\neg B \in \Gamma^*$). It remains then to prove the obvious counterparts of lemmas 1 and 2 of that proof. The proof here of Lemma 1 is identical to its proof there. To prove Lemma 2 (i.e.: that if $\Gamma^* \Rightarrow \Delta^*$ is S-saturated, then there is a model of S which is not a model of $\Gamma^* \Rightarrow \Delta^*$) we define:

$$v(p) = \begin{cases} t & \neg p \in \Delta^* \\ I & p \notin \Delta^*, \neg p \notin \Delta^* \\ f & p \in \Delta^* \end{cases}$$

Since $\Rightarrow p, \neg p$ is an axiom of $GM_3^{\{t,I\}}$, and $\Gamma^* \Rightarrow \Delta^*$ is S-saturated, it is impossible that both p and $\neg p$ are elements of Δ^*. Hence v is well defined. We prove now that it has the property that if $\varphi \in \Gamma^*$ then $v(\varphi) \in D$ (which is $\{t, I\}$ in the case we are doing) and if $\varphi \in \Delta^*$ then $v(\varphi) \notin D$. From this the fact that v is model of S which is not a model of $\Gamma^* \Rightarrow \Delta^*$ follows exactly as in the proof of Theorem 2. To show this crucial property of v we use induction on the complexity of φ. The case where φ is a literal (i.e.: an atomic formula or a negation of such a formula) other than $f, \neg f, I$, and $\neg I$ is immediate from the definition of v and the fact that if $\varphi \in \Gamma^*$ then $\varphi \notin \Delta^*$ (since $\Gamma^* \Rightarrow \Delta^*$ is S-saturated). The cases where $\varphi \in \{f, \neg f, I, \neg I\}$ also follow from this fact, using the relevant axioms of $GM_3^{\{t,I\}}$ (for example, if $I \in \Gamma^*$ then trivially $v(\varphi) \in D$, because always $v(\varphi) = I \in D$, while it cannot be the case that $I \in \Delta^*$, because otherwise $\Gamma^* \Rightarrow \Delta^*$ would have had a trivial S-proof from the axiom $\Rightarrow I$). The induction step is very much like in the classical case, and basically follows from the fact that all the logical rules of $GM_3^{\{t,I\}}$ are semantically invertible [6]. We do

[6] This is true of course also for $GM_3^{\{t\}}$, and so the proof of the induction step is *identical* in both systems!

the case where $\varphi = \neg(A \supset B)$ as an example. Well, if $\neg(A \supset B) \in \Gamma^*$ then also $A \in \Gamma^*$ and $\neg B \in \Gamma^*$. Hence, by induction hypothesis, both $v(A)$ and $v(\neg B)$ are in D. But if $v(A) \in D$ then $v(\neg(A \supset B)) = v(\neg B)$, and so $v(\neg(A \supset B)) \in D$ as well. Similarly, if $\neg(A \supset B) \in \Delta^*$ then either $A \in \Delta^*$ or $\neg B \in \Delta^*$. Hence either $v(A) \notin D$ or $v(\neg B) \notin D$. If $v(A) \notin D$ then $v(\neg(A \supset B)) = f \notin D$. If not, then $v(\neg(A \supset B)) = v(\neg B)$, and so again $v(\neg(A \supset B)) \notin D$.

Corollary 3. Strong Completeness of $GM_3^{\{t\}}$ and $GM_3^{\{t,I\}}$: *Let D be either $\{t\}$ or $\{t, I\}$.*

1. *A sequent s follows in $\mathcal{M}_3^D$ from a set S of sequents iff $S \vdash_{GM_3^D} s$.*
2. *$\Gamma \vdash_{\mathcal{M}_3^D} \Delta$ iff $\Gamma \vdash_{GM_3^D} \Delta$.*

Corollary 4. Strong Cut-elimination for $GM_3^{\{t\}}$ and $GM_3^{\{t,I\}}$: *Let D be either $\{t\}$ or $\{t, I\}$.*

1. *$\Gamma \Rightarrow \Delta$ is derivable in GM_3^D from $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ iff it has a proof in GM_3^D from S in which all cuts are done on formulas in $\bigcup_{i=1}^n \Gamma_i \cup \bigcup_{i=1}^n \Delta_i$.*
2. *$\vdash_{GM_3^D} \Gamma \Rightarrow \Delta$ iff it has in GM_3^D a cut-free proof.*

Notes:

1. From the proof of Theorem 5 it is clear that we may restrict the various axioms of $GM_3^{\{t\}}$ and $GM_3^{\{t,I\}}$ to the case in which they contain only literals.
2. The second parts of each of the last two corollaries were proved in [Avr91a]. The first parts are new here.

Theorem 5 and its corollaries have the same applications and consequences in the 3-valued case as in the two-valued one. Examples are the interpolation theorem, and an appropriate version of the subformula property (according to which a proof of a sequent contains only subformulas of this sequent, or negations of proper subformulas of it). It also leads to versions of the tableaux and resolution methods which are very similar to those used in the classical case. The differences are as follows:

1. A "clause" in the present context is a sequent which contains only *literals* on both sides. Since all the rules of the systems above are invertible, every sequent can be reduced to an equivalent finite set of clauses (in this sense) by the corresponding tableaux rules.
2. A clause $\Gamma \Rightarrow \Delta$ is valid not only when $\Gamma \cap \Delta \neq \emptyset$, but also when it contains other axioms of the corresponding system (for example: if for some atomic p, $\{p, \neg p\} \subseteq \Gamma$ (in case $D = \{t\}$), or $\{p, \neg p\} \subseteq \Delta$ (in case $D = \{t, I\}$)).
3. For proving $\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_k$ using resolution, the set of clauses obtained from the set $\{(\Rightarrow \varphi_1), \ldots, (\Rightarrow \varphi_n), (\psi_1 \Rightarrow), \ldots, (\psi_k \Rightarrow)\}$ should be extended with all nonstandard axioms of the corresponding system which contain atomic formulas occurring in the original sequent (for example: sequents of the form $p, \neg p \Rightarrow$ (in case $D = \{t\}$) or $\Rightarrow p, \neg p$ (in case $D = \{t, I\}$).

Note: In the tableaux method one should basically consider 4 types of signed formulas: $\mathbf{T}\varphi$, $\mathbf{T}\neg\varphi$, $\mathbf{F}\varphi$, and $\mathbf{F}\neg\varphi$. It might be more convenient to use instead four different signs: $\mathbf{T}$, $\mathbf{T_N}$, $\mathbf{F}$ and $\mathbf{F_N}$. The resulting rules become very similar to the classical ones (but $\mathbf{T}\neg\varphi$, e.g., is reduced to $\mathbf{F_N}\varphi$ rather than to $\mathbf{F}\varphi$). There are then also several ways to close a branch, each closely related to the classical case (except those in which I is involved). For example: in the language without the propositional constants a branch is closed iff for some φ, it contains either $\mathbf{T}\varphi$ and $\mathbf{F}\varphi$, or $\mathbf{T_N}\varphi$ and $\mathbf{F_N}\varphi$, or (in case $D = \{t\}$) $\mathbf{T}\varphi$ and $\mathbf{F_N}\varphi$, or (in case $D = \{t, I\}$) $\mathbf{T_N}\varphi$ and $\mathbf{F}\varphi$.

5.2 Systems for Łukasiewicz Logic Ł$_3$ and for RM_3

Since the sets of connectives used in the complete systems above are functionally complete, we can use the rules of these systems as a basis for finding an adequate set of rules for *every* connective (and choice of D), and so for every 3-valued logic (provided its language includes $\neg$). As an example, we show how to handle Łukasiewicz 3-valued logic. This logic is in fact equivalent ([Avr99]) to the logic of $\mathcal{M}_3^{\{t\}}$ in the language of $\neg, \vee, \wedge, \supset$. However, instead of $\supset$ another connective is taken as primitive (and as the official "implication" connective of the logic): Łukasiewicz 3-valued implication $\rightarrow_L$ (or just $\rightarrow$, when no confusion may arise). Giving a decent Gentzen-type system for Łukasiewicz 3-valued logic amounts therefore to providing an appropriate set of rules for his implication. Now $\varphi \rightarrow \psi$ is equivalent (in the strong sense of always having the same truth-value) to the formula $(\varphi \supset_{\{t\}} \psi) \wedge (\neg\psi \supset_{\{t\}} \neg\varphi)$. This leads to the following four rules for $\rightarrow$:

$$(\rightarrow\Rightarrow) \quad \frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \psi, \neg\varphi, \Delta}{\Gamma, \psi \rightarrow \varphi \Rightarrow \Delta}$$

$$(\Rightarrow\rightarrow) \quad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta \quad \Gamma, \neg\varphi \Rightarrow \neg\psi, \Delta}{\Gamma \Rightarrow \psi \rightarrow \varphi, \Delta}$$

$$(\neg\rightarrow\Rightarrow) \quad \frac{\Gamma, \psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \rightarrow \varphi) \Rightarrow \Delta}$$

$$(\Rightarrow\neg\rightarrow) \quad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma \Rightarrow \neg(\psi \rightarrow \varphi), \Delta}$$

Let us explain, as an example, how the first (and most complicated) rule in this list is obtained. Well, the sequent $\Gamma, \psi \rightarrow \varphi \Rightarrow \Delta$ is equivalent to the sequent $\Gamma, (\psi \supset \varphi) \wedge (\neg\varphi \supset \neg\psi) \Rightarrow \Delta$. Using the invertibility of the rules of $GM_3^{\{t\}}$ (which can easily be established using cuts or the strong completeness of $GM_3^{\{t\}}$) we find that this sequent is equivalent to the following set of sequents:

$$\{(\varphi, \neg\psi, \Gamma \Rightarrow \Delta), \quad (\varphi, \Gamma \Rightarrow \Delta, \neg\varphi), \quad (\neg\psi, \Gamma \Rightarrow \Delta, \psi), \quad (\Gamma \Rightarrow \Delta, \psi, \neg\varphi)\}$$

Now the second sequent in this list, $\varphi, \Gamma \Rightarrow \Delta, \neg\varphi$, can be replaced by the simpler $\varphi, \Gamma \Rightarrow \Delta$, to which it is equivalent in $GM_3^{\{t\}}$ (the first can be derived from the second using weakening, the second can be derived from the first by using a cut with the axiom $\neg\varphi, \varphi \Rightarrow$). Similarly, the third sequent can be replaced by the equivalent $\neg\psi, \Gamma \Rightarrow \Delta$. After these replacements the first sequent in the list becomes superfluous (since it can be derived from either of the two new sequents using weakening), and can be deleted. We are left with the set of premises used in the rule ($\rightarrow\Rightarrow$) above.

Note: It should be emphasized again, that the sequents of the Gentzen-type system we have just presented for Łukasiewicz 3-valued logic do *not* reflect the "consequence relation" which is induced by using $\rightarrow$ as the official *implication*. That relation is not even a Tarskian consequence relation, since it is a relation between *multisets* of formulas and formulas, not between sets of formulas and formulas. Indeed, the contraction rule fails for this relation. However, a sentence ψ follows in it from a multiset of sentences $\varphi_1, \ldots, \varphi_n$ iff the singleton sequent $\Rightarrow \varphi_1 \rightarrow (\varphi_2 \rightarrow (\ldots(\varphi_n \rightarrow \psi)\ldots))$ is provable in the system we have just described.

The counterpart of Łukasiewicz 3-valued logic in the case where $D = \{t, I\}$ is the semi-relevant system RM_3 ([AB75,AB92,Dun86]), which is the strongest logic in the family of the relevant logics. Its language is equivalent in its expressive power to that of $\{\neg, \vee, \wedge, \supset_{\{t,I\}}\}$ (sometimes also f is added), but again instead of $\supset$ another connective is taken as primitive (and as the official "implication" connective of the logic). This time this is Sobociński's implication $\rightarrow_S$ from [Sob52]. Again $\varphi \rightarrow_S \psi$ is strongly equivalent ([Avr86]) to $(\varphi \supset_{\{t,I\}} \psi) \wedge (\neg\psi \supset_{\{t,I\}} \neg\varphi)$, and again this leads to a set of rules for $\rightarrow_S$ which is very similar to that for Łukasiewicz implication. The only difference is that the rule ($\rightarrow\Rightarrow$) above should be replaced by the following dual (all other rules remain the same):

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma, \neg\psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \rightarrow \varphi \Rightarrow \Delta}$$

5.3 A Semi-Canonical System for Sobociński Logic

As we have emphasized above, there can be no canonical system which is sound and complete for a given finite-valued logic (unless it is sound and complete for some two-valued matrix). Nevertheless, in this subsection we present GRM_m, a Gentzen-type system for the important 3-valued logic of Sobociński, which is almost canonical in the sense that all its logical rules are canonical, its axioms are standard, and in addition to these rules and axioms it has only purely structural rules (the only reason we call it "semi-canonical" is that instead of the full weakening rule it has a certain weaker version). One should note, however, that GRM_m is only *weakly* complete (see Definition 4) for the ordinary scr defined by Sobociński's 3-valued matrix, although it is strongly complete for a certain generalized scr (see the third note after Definition 1) which is based on this matrix and is described below.

Sobociński 3-valued matrix was first introduced and weakly axiomatized in [Sob52]. It was later shown ([AB75]) to be equivalent to RM_m, the purely multiplicative (or "intensional") fragment of Dunn-McCall semi-relevant system RM. The idea behind this logic is that the content of certain sentences may be taken as totally insignificant for certain purposes. When this is the case no real truth-value should be attached to such sentences. This situation is represented by assigning to them the "truth-value" I (meaning "Insignificant", or "Irrelevant", or "don't care"), which should not be taken as a real truth-value. Now the the main principles that guide the semantics of the logic are that a complex is significant iff at least one of its components is significant, and that only significants components should be taken into account in computing truth values and in deductions. Thus a conjunction of sentences should be taken as true iff it has some significant conjunct, and all its *significant* conjuncts are true. It should be taken as false iff at least one of its conjuncts is false (such a conjunct is then significant by definition). This leads to the following interpretation for a binary conjunction, which we denote here by $\otimes$: [7]

$$a \otimes b = \begin{cases} f & \text{if } a = f \text{ or } b = f \\ I & \text{if } a = I \text{ and } b = I \\ t & \text{otherwise} \end{cases}$$

The principles described above imply that a 3-valued n-ary connective $\diamond$ ($n \geq 1$) may be allowed in the language iff it satisfies the following condition:

$$\diamond(a_1, \ldots, a_n) = I \quad \text{iff} \quad a_1 = a_2 = \ldots = a_n = I$$

It can be proved that a connective satisfies this condition iff it is definable in the language of $\{\neg, \otimes\}$ (where $\neg$ is the standard 3-valued negation). We take therefore these connectives as the primitive connectives of the logic (instead of $\otimes$ we could have taken the implication $\rightarrow_S$ described at the end of the previous subsection, since it is easy to see that $\otimes$ and $\rightarrow_S$ are definable from each other using $\neg$ exactly like the conjunction and implication of CPL).

The principles described above determine also the logic's concept of validity, which takes into account only valuations which are *relevant* to the formulas or sequents under consideration:

Definition 13.

1. *A valuation v is* relevant *to a formula φ iff it assigns a real truth value (either t or f) to at least one atomic formula which occurs in φ (this happens iff $v(\varphi) \in \{t, f\}$). v is* relevant *to a sequent $\Gamma \Rightarrow \Delta$ iff it is relevant to at least one formula in $\Gamma \cup \Delta$ (iff $v(p) \in \{t, f\}$ for some atomic formula which occurs in $\Gamma \Rightarrow \Delta$).*

[7] This is the notation which was used by Girard in [Gir87a] for his multiplicative conjunction, which is characterized by the multiplicative versions the classical canonical rules for conjunction (Example 1 above, and the system below). Nowadays this is the common notation for multiplicative conjunction in all substructural logics, including relevant logics.

2. *A sequent $\Gamma \Rightarrow \Delta$ is called RM_m-valid iff for every v which is relevant to it we have that $v(\varphi) = f$ for some $\varphi \in \Gamma$ or $v(\varphi) = t$ for some $\varphi \in \Delta$.*

It can easily be seen that a sequent $\Gamma \Rightarrow \Delta$ is RM_m-valid if *every* valuation v *agrees* with it, when we say that v agrees with $\Gamma \Rightarrow \Delta$ if either $v(\varphi) = f$ for some $\varphi \in \Gamma$, or $v(\varphi) = t$ for some $\varphi \in \Delta$, or $v(\varphi) = I$ for all $\varphi \in \Gamma \cup \Delta$ [8]. It follows that a sequent of the form $\Rightarrow \varphi$ is RM_m-valid iff $v(\varphi) \in \{t, I\}$ for every valuation v, i.e.: iff φ is valid in $\mathcal{M}_3^{\{t,I\}}$. Hence a Gentzen-type system which proves exactly the RM_m-valid sequents would be weakly complete for the multiplicative (i.e.: the $\{\neg, \rightarrow_S\}$-) fragment of RM_3. We present now such a Gentzen-type system.

THE SYSTEM GRM_m

Axioms: $$A \Rightarrow A$$

Structural Rules: Cut, and the following *Mingle* [9] rule:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \quad (\Rightarrow \neg)$$

$$(\otimes \Rightarrow) \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \otimes B \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad \Gamma_2 \Rightarrow \Delta_2, B}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, A \otimes B} \quad (\Rightarrow \otimes)$$

Notes:

1. Recall that Γ, Δ etc. are here finite *sets* of formulas, and that we should in principle have written $\{A\}$ rather than just A.
2. As in GCPL, it is easy to see that it suffices to take as axioms only sequents of the form $p \Rightarrow p$ where p is atomic. In the rest of this section we assume that this is the case.

Theorem 6. Soundness of GRM_m: *If $\vdash_{GRM_m} s$ then s is RM_m-valid.*

Proof: From the definition of validity it follows that any axiom of GRM_m is valid, because given a valuation v, the three possibilities ($v(p) = f$, $v(p) = t$, and $v(p) = I$) exactly match the three cases in which v agrees with $p \Rightarrow p$. We next prove for every rule of GRM_m that if a valuation v agrees with its premises then it agrees also with its conclusion. We do here the two more difficult cases, leaving the other cases for the reader.

[8] It is easy to see that by defining $\Gamma \vdash_{Sob} \Delta$ iff this condition obtains for all v we get a generalized scr $\vdash_{Sob}$.

[9] Also called "mix" in the literature on Linear Logic ([Gir87a]).

- *The case of Cut*: Suppose that v agrees with both $\Gamma_1 \Rightarrow \Delta_1, \varphi$ and $\varphi, \Gamma_2 \Rightarrow \Delta_2$. We show that it also agrees with $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. This is obvious if $v(A) = f$ for some $A \in \Gamma_1, \Gamma_2$ or $v(A) = t$ for some $A \in \Delta_1, \Delta_2$. If this is not the case then $v(\varphi)$ cannot be t (since otherwise v would not agree with $\varphi, \Gamma_2 \Rightarrow \Delta_2$), and it is not f either (since otherwise v would not agree with $\Gamma_1 \Rightarrow \Delta_1, \varphi$). It follows that $v(\varphi) = I$, and the only possibility that remains for v to agree with the two premises is that $v(A) = I$ for all $A \in \Gamma_1, \Gamma_2, \Delta_1, \Delta_2$. But in this case it again agrees with $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.
- *The case of* $(\Rightarrow \otimes)$: Suppose that v agrees with $\Gamma_1 \Rightarrow \Delta_1, \varphi$ as well as with $\Gamma_2 \Rightarrow \Delta_2, \psi$. We show that it also agrees with $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \varphi \otimes \psi$. This is obvious if $v(A) = f$ for some $A \in \Gamma_1, \Gamma_2$ or $v(A) = t$ for some $A \in \Delta_1, \Delta_2$. If this is not the case then there are four possibilities: $v(\varphi) = t$ and $v(\psi) = t$, $v(\varphi) = t$ and $v(A) = I$ for all $A \in \Gamma_2 \cup \Delta_2 \cup \{\psi\}$, $v(\psi) = t$ and $v(A) = I$ for all $A \in \Gamma_1 \cup \Delta_1 \cup \{\varphi\}$, and $v(A) = I$ for all $A \in \Gamma_1 \cup \Gamma_2 \cup \Delta_1 \cup \Delta_2 \cup \{\varphi, \psi\}$. In the first three cases $v(\varphi \otimes \psi) = t$. In the forth $v(A) = I$ for every formula A in $\Gamma_1 \cup \Gamma_2 \cup \Delta_1 \cup \Delta_2 \cup \{\varphi \otimes \psi\}$.

Our next goal is to prove the completeness of GRM_m and the cut-elimination theorem for it. As usual in this paper, this will be done simultaneously.

Notations: For the rest of this section "$\vdash$" and "provable" mean "provable in GRM_m without a cut". $A(X)$ denotes the sets of atomic formulas which occurs in X (X may be a formula or a sequent).

The following Lemma shows that an important special case of weakening is admissible in GRM_m:

Lemma 1: If $\vdash \Gamma \Rightarrow \Delta$ and $A(\varphi) \subseteq A(\Gamma \Rightarrow \Delta)$ then $\vdash \varphi, \Gamma \Rightarrow \Delta$ and $\vdash \Gamma \Rightarrow \Delta, \varphi$.

Proof of Lemma 1: By induction on the complexity of φ. The base case (where φ is atomic) is done by induction on the length of the proof of $\Gamma \Rightarrow \Delta$. The base case of this inner induction uses the special form we use for the axioms of $GRMI_m$, while both the induction step of the inner induction and the induction step of the main one are easy consequence of the fact that all rules of GRM_m are pure (multiplicative).

Definition 14. *Let $\Gamma \Rightarrow \Delta$ be a sequent such that $\not\vdash \Gamma \Rightarrow \Delta$. $\Gamma \Rightarrow \Delta$ is called saturated if it also satisfies the following conditions:*

(i) If $\neg\varphi \in \Gamma$ then $\varphi \in \Delta$
(ii) If $\neg\varphi \in \Delta$ then $\varphi \in \Gamma$
(iii) If $\varphi \otimes \psi \in \Gamma$ then $\varphi \in \Gamma$ and $\psi \in \Gamma$
(iv) If $\varphi \otimes \psi \in \Delta$ and $\not\vdash \Gamma \Rightarrow \Delta, \varphi$ then $\varphi \in \Delta$
(v) If $\varphi \otimes \psi \in \Delta$ and $\not\vdash \Gamma \Rightarrow \Delta, \psi$ then $\psi \in \Delta$.

Lemma 2: If $\nvdash \Gamma \Rightarrow \Delta$ then there exists a saturated sequent $\Gamma^* \Rightarrow \Delta^*$ such that $\Gamma \subseteq \Gamma^*$, $\Delta \subseteq \Delta^*$, $\nvdash \Gamma^* \Rightarrow \Delta^*$ and $A(\Gamma^* \Rightarrow \Delta^*) = A(\Gamma \Rightarrow \Delta)$.

Proof of Lemma 2: If $\nvdash \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta$ is not saturated then it is possible to properly extend $\Gamma \Rightarrow \Delta$ by some of its subformulas without making the new sequent provable (this is obvious and standard if one of the conditions (i)–(iii) is violated by $\Gamma \Rightarrow \Delta$, and trivial in the special cases (iv)–(v)). Since $\Gamma \Rightarrow \Delta$ has only finitely many subformulas, this process must stop with a saturated sequent.

Lemma 3: If $\Gamma \Rightarrow \Delta$ is saturated and $\nvdash \Gamma \Rightarrow \Delta$ then $\Gamma \Rightarrow \Delta$ has a countermodel (i.e.: a valuation v which does not agree with it).

Proof of Lemma 3: Assume that $\Gamma \Rightarrow \Delta$ has these properties. Define:

$$I(\Gamma \Rightarrow \Delta) = \{p \in A(\Gamma \Rightarrow \Delta) \mid p \in \Gamma \cap \Delta\}$$

$$v(p) = \begin{cases} I & p \in I(\Gamma \Rightarrow \Delta) \\ t & p \in \Gamma, p \notin \Delta \\ f & p \notin \Gamma \end{cases}$$

We show that this v is a countermodel of $\Gamma \Rightarrow \Delta$. For this we first show by induction on the complexity of φ that if $\varphi \in \Gamma$ then $v(\varphi) \neq f$, and if $\varphi \in \Delta$ then $v(\varphi) \neq t$. This is obvious in case φ is atomic. In case $\varphi = \neg\psi$ the claim follows easily from the induction hypothesis and conditions (i)–(ii) from Definition 14. If $\varphi = \psi_1 \otimes \psi_2$ and $\varphi \in \Gamma$ then the claim follows from the induction hypothesis concerning ψ_1 and ψ_2 and condition (iii) of Definition 14. Finally assume that $\varphi = \psi_1 \otimes \psi_2$ and $\varphi \in \Delta$. Had both $\Gamma \Rightarrow \Delta, \psi_1$ and $\Gamma \Rightarrow \Delta, \psi_2$ been provable, so $\Gamma \Rightarrow \Delta$ would have been (since $\varphi \in \Delta$). Hence one of those sequents is unprovable. Assume, e.g., that $\nvdash \Gamma \Rightarrow \Delta, \psi_1$. Then $\psi_1 \in \Delta$ by condition (iv) of Definition 14. Hence $v(\psi_1) \neq t$ by induction hypothesis. If $v(\psi_1) = f$ then $v(\varphi) = f \neq t$. Assume, therefore, that $v(\psi_1) = I$. Hence $v(p) = I$ for every $p \in A(\psi_1)$, and so $A(\psi_1) \subseteq I(\Gamma \Rightarrow \Delta)$. Now $\vdash A(\psi_1) \Rightarrow A(\psi_1)$ because of the Mingle rule, and so $\vdash A(\psi_1) \Rightarrow A(\psi_1), \psi_1$ by Lemma 1. It is not possible therefore that $\vdash \Gamma \Rightarrow \Delta, \psi_2$, since otherwise we would have that $\vdash \Gamma, A(\psi_1) \Rightarrow \Delta, A(\psi_1), \psi_1 \otimes \psi_2$, and so that $\vdash \Gamma \Rightarrow \Delta$ (since $\psi_1 \otimes \psi_2 = \varphi \in \Delta$, $A(\psi_1) \subseteq \Gamma$, and $A(\psi_1) \subseteq \Delta$). It follows by condition (v) of Definition 1 that $\psi_2 \in \Delta$, and so $v(\psi_2) \neq t$ by induction hypothesis. Now if $v(\psi_2) = f$ then $v(\varphi) = f \neq t$, while if $v(\psi_2) = I$ then $v(\varphi) = I \otimes I = I \neq t$.

To show that v is a countermodel of $\Gamma \Rightarrow \Delta$ it remains now to eliminate the possibility that $v(\varphi) = I$ for all $\varphi \in \Gamma \cup \Delta$. Well, had this been the case we would have that $v(p) = I$ for all $p \in A(\Gamma \Rightarrow \Delta)$, and so that $A(\Gamma \Rightarrow \Delta) = I(\Gamma \Rightarrow \Delta)$. Since $I(\Gamma \Rightarrow \Delta) \subseteq \Gamma$, $I(\Gamma \Rightarrow \Delta) \subseteq \Delta$, and $\vdash I(\Gamma \Rightarrow \Delta) \Rightarrow I(\Gamma \Rightarrow \Delta)$ (using mingle), Lemma 1 would have implied then that $\vdash \Gamma \Rightarrow \Delta$. A contradiction.

Theorem 7. Completeness and Cut-elimination for GRM_m**:** *A sequent $\Gamma \Rightarrow \Delta$ in the language of GRM_m is RM_m-valid iff it has a proof in GRM_m without cuts.*

Proof: Assume that $\not\vdash \Gamma \Rightarrow \Delta$. By lemma 2 there exists an unprovable saturated sequent $\Gamma^* \Rightarrow \Delta^*$ such that $A(\Gamma^* \Rightarrow \Delta^*) = A(\Gamma \Rightarrow \Delta)$, $\Gamma \subseteq \Gamma^*$, and $\Delta \subseteq \Delta^*$. These last 3 properties are easily seen to entail that every model of $\Gamma \Rightarrow \Delta$ is also a model of $\Gamma^* \Rightarrow \Delta^*$.[10] Hence the countermodel of $\Gamma^* \Rightarrow \Delta^*$ given by lemma 3 is also a countermodel of $\Gamma \Rightarrow \Delta$.

Corollary 5. *(***Completeness of** GRM_m*)*

1. *A sequent in the language of GRM_m is RM_m-valid iff it is provable in GRM_m*
2. *A sentence φ in the language of GRM_m is valid in $\mathcal{M}_3^{\{t,I\}}$ iff $\Rightarrow \varphi$ is provable in GRM_m*

Corollary 6. *The system GRM_m admits cut-elimination.*

Note: The last two corollaries were proved (using two unrelated proofs) already in [Avr87] (but were known to relevance logicians much before).

6 Four-Valued Logics

There are basically 3 four-valued logics, differing according to the number of designated truth-values in their matrices (1, 2, or 3). The most important of them is by far the one in which there are exactly two. We devote most of this section to this useful logic.

6.1 Belnap's Four-valued Logic and Its Extensions

The methods used above for three-valued logics can be extended with very slight changes to four-valued logics in which there are exactly two designated elements. Let the truth values of these logics be $t, f, \top$, and $\bot$, where t and f are the classical values. According to Belnap's suggestion in [Bel77b,Bel77a], $\top$ should represent the truth-value of formulas about which there is inconsistent data (such a formula is "both true and false"), while $\bot$ is the truth-value of formulas on which no data at all is available ("neither true nor false"). This intuition is the basis of what is known as Belnap four-valued logic and of various extensions suggested in the literature [11]. Obviously, according to these interpretations $\top$ is the four-valued counterpart of the 3-valued designated I, while $\bot$ is the counterpart of the 3-valued non-designated I. The corresponding four-valued matrix may therefore be taken as a combination of the two 3-valued matrices. We make our choice of connectives and their interpretations accordingly. In particular: the partial order $\leq_t$ we use for defining conjunction and disjunction is simply the union of the partial orders which are used for this purpose in $\mathcal{M}_3^{\{t\}}$ and $\mathcal{M}_3^{\{t,I\}}$ (where I is replaced, respectively, by $\bot$ and $\top$). $\leq_t$ is defined therefore by: $f \leq_t \top, \bot \leq_t t$.

[10] Note that the first one is crucial here, since ordinary monotonicity fails.

[11] Belnap's structure is nowadays known also as the basic (distributive) *bilattice*, and its logic — as the basic logic of (distributive) bilattices (see [Gin87,Gin88,Fit90b,Fit90a,Fit91,Fit94,AA94,AA96,AA98]).

Definition 15. *Let* $L_4 = \{\neg, \vee, \wedge, \supset, f, \bot, \top\}$.

The matrix $\mathcal{M}_4 = \langle M_4, D_4, O_4 \rangle$ *for* L_4 *is defined as follows:*

- $M_4 = \{t, f, \top, \bot\}$
- $D_4 = \{t, \top\}$
- *The operations in* O_4 *are defined by:*
  1. $\neg t = f, \ \ \neg f = t, \ \ \neg\top = \top, \ \ \neg\bot = \bot$
  2. $\supset$ *is defined like in Definition 11*
  3. $a \vee b = sup_{\leq_t}(a, b), \ \ a \wedge b = inf_{\leq_t}(a, b)$
  4. *The propositional constants* $f, \bot$ *and* $\top$ *are interpreted by the corresponding truth values.*

We present now a Gentzen-type system which is strongly sound and complete for $\mathcal{M}_4$, and for which the (strong) cut-elimination theorem obtains. Since L_4 is a functionally complete set of connectives for 4-valued logics ([Avr99]), this again allows us to find a complete, cut-free Gentzen-type system for any 4-valued logic in which $D = \{t, \top\}$.

THE SYSTEM GM_4: This is the systems obtained from GBS by adding to it the following axioms:

$$\bot \Rightarrow \qquad\qquad \neg\bot \Rightarrow$$

$$\Rightarrow \top \qquad\qquad \Rightarrow \neg\top$$

Theorem 8. Strong Soundness, Completeness and Cut-elimination for GM_4**:** *A sequent s follows in* $\mathcal{M}_4$ *from* $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ *iff s has a proof in* GM_4 *from S in which all cuts are done on formulas in* $\bigcup_{i=1}^n \Gamma_i \cup \bigcup_{i=1}^n \Delta_i$.

Proof: The new axioms concerning $\top$ and $\bot$ are obviously valid in $\mathcal{M}_4$. Hence the strong soundness of GM_4 easily follows from Theorem 4.

The simultaneous proof of the strong completeness and of the strong Cut-elimination Theorem closely follows the proofs of Theorems 2 and 5. The main difference is that this time to prove Lemma 2 (that if $\Gamma^* \Rightarrow \Delta^*$ is S-saturated, then there is a model of S which is not a model of $\Gamma^* \Rightarrow \Delta^*$) we define:

$$v(p) = \begin{cases} t & p \in \Gamma^*, \neg p \notin \Gamma^* \quad or \quad p \notin \Delta^*, \neg p \in \Delta^* \\ \bot & p \in \Delta^*, \neg p \in \Delta^* \\ \top & p \in \Gamma^*, \neg p \in \Gamma^* \\ f & otherwise \end{cases}$$

As usual, we next prove that v is well-defined, and that if $\varphi \in \Gamma^*$ then $v(\varphi) \in D_4$, while if $\varphi \in \Delta^*$ then $v(\varphi) \notin D_4$. Details are left for the reader.

Corollary 7. Strong Completeness of GM_4**:**

1. *A sequent s follows in* $\mathcal{M}_4$ *from a set S of sequents iff* $S \vdash_{GM_4} s$.
2. $\Gamma \vdash_{\mathcal{M}_4} \Delta$ *iff* $\Gamma \vdash_{GM_4} \Delta$.

Corollary 8. *(***Strong Cut-elimination for** GM_4**:**

1. *s is derivable in GM_4 from $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ iff it has a proof in GM_4 from S in which all cuts are done on formulas in $\bigcup_{i=1}^{n} \Gamma_i \cup \bigcup_{i=1}^{n} \Delta_i$.*
2. *$\vdash_{GM_4} \Gamma \Rightarrow \Delta$ iff it has in GM_4 a cut-free proof.*

Notes:

1. Again from the proof of Theorem 8 it is clear that one may restrict the axioms of GM_4 to the case in which they contain only literals.
2. The second parts of each of the last two corollaries were claimed in [Avr91a] and proved in [AA94]. The first parts are new here.
3. The results of this section have the same applications here as in the 3-valued case. In fact, both of the resulting tableaux and resolution methods are simpler here than in the 3-valued case, because the set of axioms is simpler in the present system than in its 3-valued counterparts.

6.2 Other Four-valued Logics

Except for the class just dealt with, there are two other possible classes of 4-valued logics: those with $D = \{t\}$, and those with $D = \{t, \top, \bot\}$. These two classes are dual to each other: it is easy to see that a sequent $\Gamma \Rightarrow \Delta$ is valid according to one iff $\neg\Delta \Rightarrow \neg\Gamma$ is valid according to the other (where $\neg$ is the connective defined in the previous subsection, and $\neg\{A_1, \ldots, A_n\} = \{\neg A_1, \ldots, \neg A_n\}$). In principle it suffices therefore to find an appropriate system for one of these cases. This can indeed be done by methods which are similar to those used above. We demonstrate this claim for the case $D = \{t\}$ in subsection 7.2 below, using the four-valued Gödel's logic as a basis. We note, however, that at present the known systems for the two other classes are more complicated and less elegant than those in the case $D = \{t, \top\}$.

7 Infinite-valued Logics and Related Logics

Up to now, we have only considered three- or four-valued logics. In this section we want to show that our methods may be applicable to n-valued logics with n having bigger values, and even to infinite-valued logics.

7.1 Gödel-Dummett Logics

In [Göd86] Gödel introduced a sequence $\{\mathcal{G}_n\}$ $(n \geq 2)$ of n-valued matrices. He used these matrices to show some important properties of intuitionistic logic. An infinite-valued matrix $\mathcal{G}_\omega$ in which all the $\mathcal{G}_n$s can be embedded was later introduced by Dummett in [Dum59]. The logic of $\mathcal{G}_\omega$ was axiomatized in the same paper, and has been known since then as Gödel-Dummett's LC. It is probably the most important intermediate logic, which turns up in several places, such as the provability logic of Heyting's Arithmetics ([Vis82]), and relevance logic ([DM71]). Recently it has again attracted a lot of attention because of its recognition as one of the three most basic fuzzy logics ([Haj98]).

Definition 16. *Let* $L_{LC} = \{\to, \vee, \wedge, f\}$.

1. *The matrix* $\mathcal{G}_\omega = \langle G_\omega, D_{LC}, O_{LC}\rangle$ *for* L_{LC} *is defined as follows:*
   - $G_\omega = N \cup \{t, f\}$ [12]
   - $D_{LC} = \{t\}$
   - *The operations in* O_{LC} *are defined by:*
     (a) $a \to b \;= \begin{cases} t & a \leq b \\ b & a \not\leq b \end{cases}$
     Here $\leq$ *is the usual order on* N *extended by a greatest element* t *and a smallest element* f.
     (b) $a \vee b = max_{\leq}(a, b)$ *and* $a \wedge b = min_{\leq}(a, b)$.
     (c) *The propositional constant* f *is interpreted by the corresponding truth value.*
2. $LC = \langle L_{LC}, \vdash_{\mathcal{G}_\omega}\rangle$
3. *for* $n \geq 2$ *the matrix* $\mathcal{G}_n$ *is defined like* $\mathcal{G}_\omega$, *but its the set of truth values is just* $\{1, ..., n-2\} \cup \{t, f\}$.

Notes:

1. The matrices we have just defined are not those given by Gödel and Dummett, but their duals. We note also that for the application as a fuzzy logic it is more useful ([Haj98]) to use instead of $N \cup \{t, f\}$ the real interval [0,1], with 1 playing the role of t (and 0 that of f). This makes a difference only when we consider inferences from infinite theories.
2. It is not difficult to see that LC is indeed a many-valued logic according to Definition 10, since $\Gamma \vdash_{\mathcal{G}_\omega} \Delta$ iff $\Gamma \vdash_{\mathcal{G}_k} \Delta$, where k is the number of atomic formulas occurring in $\Gamma \cup \Delta$.

A cut-free Gentzen-type formulation for LC was first given by Sonobe in [Son75]. His approach was improved in [AFM99] and [Dyc99]. All those systems have, however, the serious drawback of using the following rule, which introduces an arbitrary number of implications, and has an arbitrary number of premises, all of which contain formulas of essential importance for the inference:

$$\frac{\{\Gamma, \varphi_i \Rightarrow \psi_i, \Delta^i\}}{\Gamma \Rightarrow \Delta}\, R\to$$

Here Δ contains exactly $m > 0$ implicational formulas $\varphi_i \to \psi_i, i = 1, \ldots, m$ (Δ may also contain some other kinds of formulas). Δ^i denotes the multiset consisting of exactly the $m-1$ implicational formulas of Δ other than $\varphi_i \to \psi_i$.

An alternative sound and complete system GLC_{RS} for LC has been given in [AK01]. Like the former systems, GLC_{RS} admits cut-elimination, but it does not have a rule with arbitrary number of premises. The idea behind GLC_{RS} is again to use noncanonical invertible rules which decompose formulas in a sequent into simpler ones, until a set of "clauses" equivalent to the original sequent is reached. However, unlike in the 3-valued and 4-valued logics described above, in

[12] Instead of adding f one may identify it with the number 0.

GLC_{RS} a "clause" is defined as a sequent consisting only of atomic formulas or implications between atomic formulas. Accordingly, instead of having for each connective rules for its combinations with negation, we have rules for all its possible combinations with $\rightarrow$. The four needed rules for $\rightarrow$ itself are:

$$(\Rightarrow\rightarrow(\rightarrow))\ \frac{\Gamma \Rightarrow \Delta, \psi_1 \rightarrow \psi_2, \varphi \rightarrow \psi_2}{\Gamma \Rightarrow \Delta, \varphi \rightarrow (\psi_1 \rightarrow \psi_2)}$$

$$(\rightarrow(\rightarrow)\Rightarrow)\ \frac{\Gamma, \psi_1 \rightarrow \psi_2 \Rightarrow \Delta \qquad \Gamma, \varphi \rightarrow \psi_2 \Rightarrow \Delta}{\Gamma, \varphi \rightarrow (\psi_1 \rightarrow \psi_2) \Rightarrow \Delta}$$

$$(\Rightarrow(\rightarrow)\rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi_2 \rightarrow \psi \qquad \Gamma, \varphi_1 \rightarrow \varphi_2 \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, (\varphi_1 \rightarrow \varphi_2) \rightarrow \psi}$$

$$((\rightarrow)\rightarrow\Rightarrow)\ \frac{\Gamma, \varphi_2 \rightarrow \psi \Rightarrow \Delta, \varphi_1 \rightarrow \varphi_2 \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, (\varphi_1 \rightarrow \varphi_2) \rightarrow \psi \Rightarrow \Delta}$$

The number of rules needed for each binary connective other than $\rightarrow$ is six. The rules for $\vee$, for example, are:

$$(\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\vee\Rightarrow)\ \frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$$

$$(\Rightarrow\vee\rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi_1 \rightarrow \psi \qquad \Gamma \Rightarrow \Delta, \varphi_2 \rightarrow \psi}{\Gamma \Rightarrow \Delta, (\varphi_1 \vee \varphi_2) \rightarrow \psi}$$

$$(\vee\rightarrow\Rightarrow)\ \frac{\Gamma, \varphi_1 \rightarrow \psi, \varphi_2 \rightarrow \psi \Rightarrow \Delta}{\Gamma, (\varphi_1 \vee \varphi_2) \rightarrow \psi \Rightarrow \Delta}$$

$$(\Rightarrow\rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi_1, \varphi \rightarrow \psi_2}{\Gamma \Rightarrow \Delta, \varphi \rightarrow (\psi_1 \vee \psi_2)}$$

$$(\rightarrow\vee\Rightarrow)\ \frac{\Gamma, \varphi \rightarrow \psi_1 \Rightarrow \Delta \qquad \Gamma, \varphi \rightarrow \psi_2 \Rightarrow \Delta}{\Gamma, \varphi \rightarrow (\psi_1 \vee \psi_2) \Rightarrow \Delta}$$

Like in the previous cases, these rules are not sufficient for completeness. We need to have also means for determining which "clauses" (or "basic sequents") are valid. The straightforward way of doing so is again to add appropriate non-standard axioms. Unfortunately, the needed axioms are here much more complicated than those employed in the 3-valued and 4-valued cases described above:

Definition 17. *Let $\Gamma \Rightarrow \Delta$ be a clause of GLC_{RS}.*

- *We say that $(p \leq q) \in (\Gamma \Rightarrow \Delta)$ iff $(p \rightarrow q) \in \Gamma$.*
- *We say that $(t \leq q) \in (\Gamma \Rightarrow \Delta)$ iff $q \in \Gamma$.*
- *We say that $(p < q) \in (\Gamma \Rightarrow \Delta)$ iff $(q \rightarrow p) \in \Delta$.*
- *We say that $(q < t) \in (\Gamma \Rightarrow \Delta)$ iff $q \in \Delta$.*
- *Let p, q be either atomic formulas or t* [13]*. We say that $(p \lhd q) \in (\Gamma \Rightarrow \Delta)$ iff either $(p \leq q) \in (\Gamma \Rightarrow \Delta)$ or $(p < q) \in (\Gamma \Rightarrow \Delta)$.*

[13] Note that in this paper t is *not* an official symbol of the language of LC. Here, however, it is used in the metalanguage.

– *A sequence* $q_1, \ldots, q_l$ *(where* q_i *is either atomic or* t*) is called a* strictly increasing *sequence for* $\Gamma \Rightarrow \Delta$ *if* $(q_j \triangleleft q_{j+1}) \in (\Gamma \Rightarrow \Delta)$ *for* $1 \leq j \leq l-1$, *and either* $(q_i < q_{i+1}) \in (\Gamma \Rightarrow \Delta)$ *for some* $1 \leq i \leq l-1$, *or* $q_1 = t$, $q_l = f$.

The axioms of the system GLC_{RS} [14] *are those clauses for which there exists a strictly increasing sequence* $q_1 \ldots q_l$ *such that either* $q_1 = q_l$, *or* $q_1 = t$, *or* $q_l = f$.

As we noted above, the system GLC_{RS} is sound and complete for LC, and the cut-elimination theorem obtains for it. The proofs given to this in [AK01] can easily be extended to proofs of the corresponding *strong* versions of these facts by applying the method used in this paper. We omit the details.

To get a system GLC^k_{RS} with similar properties for the *finite* k-valued Gödel-Dummett logic, one needs to add as axioms all basic sequents s having a strictly increasing sequence $q_1 \ldots q_l$ such that for at least k different q_i's, $(q_i < q_{i+1}) \in s$.

Instead of enriching the set of axioms, an alternative approach presented in [AK01] is to employ special analytic simplification rules. Those needed for the infinite-valued LC are:

Transitivity: $$\frac{\Gamma, p \to q, q \to r, p \to r \Rightarrow \Delta}{\Gamma, p \to q, q \to r \Rightarrow \Delta}$$

Left maximality: $$\frac{\Gamma, p \to q, p, q \Rightarrow \Delta}{\Gamma, p \to q, p \Rightarrow \Delta}$$

Right maximality: $$\frac{\Gamma \Rightarrow \Delta, q \to p, p}{\Gamma \Rightarrow \Delta, q \to p}$$

Linearity: $$\frac{\Gamma, p \to q \Rightarrow \Delta, q \to p}{\Gamma \Rightarrow \Delta, q \to p}$$

Minimality of f: $$\frac{\Gamma, p \to f, f \to p \Rightarrow \Delta}{\Gamma, p \to f \Rightarrow \Delta}$$

Note: All the *logical* rules of the systems described in this subsection have what might be called the *semi-subformula property*: written in Polish notation, every formula in their premises either appears in their conclusion or is obtained from some formula there by deleting some of its symbols. This is not very different from the usual subformula property. The simplification rules, in contrast, do not have this property. Nevertheless, each of them is still *analytic* in the sense that its possible premises (for any potential conclusion) are determined by its conclusion, they are finite in number, and can be found effectively. These rules are close in nature to *analytic cuts* (see section 7.3), but they are simpler.

7.2 The Four-valued Logic with a Single Designated Element

The matrix $\mathcal{G}_4$ is a four-valued matrix in which $D = \{t\}$. The corresponding system GLC^4_{RS} can therefore be used as a basis for an appropriate Gentzen-type system for the universal four-valued logic in which there is exactly one designated truth-value. The main problem in doing so is that the set of connectives of this

[14] In [AK01] this version with new axioms is called GLC^*_{RS}.

matrix is not functionally complete. It can however be shown that by adding to L_{LC} the connective $\neg$ as well as the propositional constants $\top$ and $\bot$, and by giving these new connectives the same interpretations they have in $\mathcal{M}_4$, one does obtain a language with a functionally complete set of connectives. Let us call the corresponding matrix $\mathcal{M}_4^{\{t\}}$ [15]. To get a sound and complete system for $\mathcal{M}_4^{\{t\}}$, all we need to do is to add to GLC^4_{RS} appropriate rules and axioms for the new connectives. Now GLC^4_{RS} is based on the idea of using invertible rules for reducing any sequent to a set of "clauses" in the sense of GLC_{RS}. The connective $\neg$ should therefore be treated here exactly as $\vee$ and $\wedge$ (and so its role here is completely different from the one it has in GM_4). In other words: we should find for it a full set of decomposition rules (i.e.: invertible sound rules which make it possible to get rid of it in backward reasoning). It is not difficult to show that the following six rules constitute such a set:

$$(\neg\Rightarrow)\quad \frac{\Gamma, A \to f \Rightarrow \Delta}{\Gamma, \neg A \Rightarrow \Delta}$$

$$(\Rightarrow\neg)\quad \frac{\Gamma \Rightarrow \Delta, A \to f}{\Gamma \Rightarrow \Delta, \neg A}$$

$$(\neg\to\Rightarrow)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta \quad \Gamma, A \to B \Rightarrow \Delta, A \to f}{\Gamma, (\neg A) \to B \Rightarrow \Delta}$$

$$(\Rightarrow\neg\to)\quad \frac{\Gamma, A \to f \Rightarrow \Delta, B \quad \Gamma \Rightarrow \Delta, A, A \to B}{\Gamma \Rightarrow \Delta, (\neg A) \to B}$$

$$(\to\neg\Rightarrow)\quad \frac{\Gamma, A \to f \Rightarrow \Delta \quad \Gamma, B \to f \Rightarrow \Delta \quad \Gamma, A \to B \Rightarrow \Delta, B}{\Gamma, A \to \neg B \Rightarrow \Delta}$$

$$(\Rightarrow\to\neg)\quad \frac{\Gamma, B \Rightarrow \Delta, A \to f \quad \Gamma \Rightarrow \Delta, A \to B, B \to f}{\Gamma \Rightarrow \Delta, A \to \neg B}$$

Note: These rules for $\neg$ are sound and invertible also in $\mathcal{G}_\omega$, provided the interpretation of $\neg$ is given by:

$$\neg a = \begin{cases} f & a = t \\ t & a = f \\ a & \text{otherwise} \end{cases}$$

It follows that by adding these rules to GLC_{RS} and GLC^k_{RS} we get systems which are strongly sound and complete for the corresponding extended matrices, and in which the strong cut elimination theorem obtains.

Returning to $\mathcal{M}_4^{\{t\}}$, we should take into account also the fact that its language includes the constants $\top$ and $\bot$. Their presence is not relevant to the

[15] Note that the interpretations of $\vee$ and $\wedge$ in $\mathcal{M}_4^{\{t\}}$ is *not* identical to interpretations they have in $\mathcal{M}_4$!

decomposition rules, but *is* relevant to the question what "clauses" (i.e.: basic sequents) should be taken as axioms (because now the set of atomic formulas includes these two new constants). It is not difficult to see that what we should do is to add to the axioms of GLC^4_{RS} all basic sequents s for which there exists a strictly increasing sequence $q_1 \dots q_l$ satisfying one of the following conditions:

1. $q_1 = \top$ and there are at least two different i's such that $(q_i < q_{i+1}) \in s$
2. $q_l = \bot$ and there are at least two different i's such that $(q_i < q_{i+1}) \in s$
3. $q_1 = \bot$ and there are at least three different i's such that $(q_i < q_{i+1}) \in s$
4. $q_l = \top$ and there are at least three different i's such that $(q_i < q_{i+1}) \in s$

By a straightforward adaption of the proofs given in [AK01] one can show now that the resulting system $GM_4^{\{t\}}$ is strongly sound and complete for $\mathcal{M}_4^{\{t\}}$, and that the strong cut elimination theorem obtains for it.

7.3 The Modal $S5$

In this subsection we present one important example of the use of an impure rule: the standard Gentzen-type system for the modal logic $S5$. Like most other modal logics, $S5$ has a well-known (particularly simple) possible-worlds semantics: A frame for $S5$ is a pair $\langle W, v\rangle$ where W is a nonempty set (of "possible-worlds") and v is a function which assigns to each sentence in the language of $S5$ a subset of W, so that $v(A \vee B) = v(A) \cup v(B)$, $v(A \wedge B) = v(A) \cap v(B)$, $v(\neg A) = W - v(A)$, and $v(\Box A) = W$ if $v(A) = W$, $\emptyset$ otherwise. A sentence A is *true* in a world $w \in W$ if $w \in v(A)$, and *valid* in W if it true in all worlds of W (i.e.: if $v(A) = W$). It is obvious that a frame may be viewed as a pair of a matrix and a valuation in it. In fact, a slight generalization of this semantical characterization of $S5$ had already been given in [Scr51] (much before the Kripke-style semantics of this system was discovered). This many-valued semantics of $S5$ is given in the next definition.

Definition 18. *Let* $L_\Box = \{\neg, \vee, \wedge, \Box\}$.

A matrix $\mathcal{A} = \langle A, D_A, O_A\rangle$ *for* $L_\Box$ *is called an* $S5$-matrix *if:*

- $\{0, 1\} \subseteq A$ $(0 \neq 1)$
- $D_A = \{1\}$
- $\langle A, \vee, \wedge, \neg, 1, 0\rangle$ *is a Boolean Algebra*
- $\Box a = \begin{cases} 1 & a = 1 \\ 0 & a \neq 1 \end{cases}$

It has been proved in [Scr51] that (the Hilbert-type formulation of) $S5$ is sound and complete for the class of $S5$-matrices. Moreover: if a sentence φ in the language of $S5$ is not a theorem of $S5$, then it is refutable in some *finite* structure of this type (with at most 2^{2^n} elements, where n is the number of atomic formulas occurring in φ). All these finite matrices are embeddable in one denumerable matrix, which is characteristic for $S5$. Hence $S5$ is many-valued logic according to Definition 10[16].

We present now $GS5$, the standard Gentzen-type system for $S5$ (originally introduced in [OM57]). This system is obtained from $GCPL$ by adding to it the following two simple rules:

$$(\Box \Rightarrow) \ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \Box\varphi} \ (\Rightarrow \Box)$$

There is, however, a side condition on the application of $(\Rightarrow \Box)$, which makes this rule impure: all the side formulas (i.e., the formulas of Γ and Δ) should begin with $\Box$ (an equivalent version demands them only to be *essentially modal*, in the sense that each occurrence of an atomic formula should be within the scope of a $\Box$).

Definition 19.

1. *Let s be the sequent $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$. φ_s, the standard interpretation of s, is the sentence $\neg A_1 \vee \ldots \vee \neg A_n \vee B_1 \vee \ldots \vee B_k$.*
2. *A* model *of a sequent s in the language $L_\Box$ is a pair $\langle \mathcal{A}, v\rangle$ where $\mathcal{A}$ is an $S5$-matrix, and v is a valuation in $\mathcal{A}$ such that $v(\varphi_s) = 1$ (in particular: $\langle \mathcal{A}, v\rangle$ is a model of $\Rightarrow \varphi$ iff it is a model of φ).*
3. *A sequent s follows in $S5$ from a set S of sequents if every model of S is also a model of s.*

Theorem 9. Strong Soundness of $GS5$: *If a sequent s has a proof in $GS5$ from a set S of sequents then it follows in $S5$ from S.*

We leave the easy proof to the reader. As usual, we want next to turn to a simultaneous proof for $GS5$ of strong completeness and strong cut elimination. There is a problem, though: the cut elimination theorem is *not* valid for $GS5$. However, we have a satisfactory substitute: the possibility to eliminate *non-analytic* cuts:

Definition 20. *A cut in a proof of a sequent s from a set S of sequents is called analytic if the cut formula is a subformula of some formula in $S \cup \{s\}$.*

Theorem 10. Strong Completeness and Analytic Cut-elimination for $GS5$: *A sequent s follows in $S5$ from $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ iff it has a proof in $GS5$ from S in which every cut is done either on a formula in $\bigcup_{i=1}^n \Gamma_i \cup \bigcup_{i=1}^n \Delta_i$, or on a subformula of the form $\Box A$ of some formula in $S \cup \{s\}$.*

Proof: Again the proof follows to a great extent that of Theorem 2. Given S we define first an S-proof as a proof which is permitted by the formulation of the theorem. We then define when a a sequent $\Gamma^* \Rightarrow \Delta^*$ is S-saturated like in the proof of Theorem 2, but with two more conditions:

[16] As far as I know, this is not true for other famous Modal Logics.Modal Logics.

(A) If $\Box\varphi \in \Gamma^*$ then $\varphi \in \Gamma^*$.
(B) If $\Box\varphi$ is a subformula of a formula in $\Gamma^* \cup \Delta^*$ then $\Box\varphi \in \Gamma^* \cup \Delta^*$.

It is easy to prove that if a sequent s has no S-proof then it can be extended to an S-saturated sequent consisting only of subformulas of formulas in $S \cup \{s\}$. It remains therefore to show that if $s = \Gamma^* \Rightarrow \Delta^*$ is S-saturated then there is a model of S which is not a model of s. For this let W be the set of all S-saturated sequents $\Gamma \Rightarrow \Delta$ which consist only of subformulas of formulas in s, and which satisfy the following condition for every formula A:

$$(*) \quad \Box A \in \Gamma \ \text{ iff } \ \Box A \in \Gamma^*$$

(by condition (B) this implies that also $\Box A \in \Delta$ iff $\Box A \in \Delta^*$). W is obviously finite, and $P(W)$ is a Boolean Algebra with the usual operations of union, intersection etc. This Boolean Algebra can be extended to an S-matrix in a unique way. Let $\mathcal{A}$ be this S-matrix. Define a valuation v in it by:

$$v(p) = \{\Gamma \Rightarrow \Delta \in W \mid p \in \Gamma\}$$

We show next by induction on complexity of formulas that for every $\Gamma \Rightarrow \Delta \in W$ and every sentence φ, if $\varphi \in \Gamma$ then $\Gamma \Rightarrow \Delta \in v(\varphi)$, while if $\varphi \in \Delta$ then $\Gamma \Rightarrow \Delta \notin v(\varphi)$. Most of the induction steps are straightforward. We do here the case $\varphi = \Box\psi$:

- Assume $\Box\psi \in \Gamma$. Then by the definition of W, $\Box\psi$ belongs to the antecedents of all the sequents in W. Hence by condition (A) so does ψ itself. It follows that $v(\psi) = W$, and so also $v(\Box\psi) = W$. In particular: $\Gamma \Rightarrow \Delta \in v(\Box\psi)$.
- Assume $\Box\psi \in \Delta$. Let $\Box(\Gamma^*)$ and $\Box(\Delta^*)$ be the sets of boxed formulas in Γ^* and Δ^* (respectively). The sequent $\Box(\Gamma^*) \Rightarrow \Box(\Delta^*), \psi$ does not have an S-proof (since otherwise so would have $\Gamma^* \Rightarrow \Delta^*$, using an application of $(\Rightarrow \Box)$ and weakenings), and so it can be extended to an S-saturated sequent $\Gamma^\# \Rightarrow \Delta^\#$, consisting only of subformulas of formulas in $\Gamma^* \Rightarrow \Delta^*$. Since $\Gamma^* \Rightarrow \Delta^*$ satisfies condition (B), $\Gamma^\# \Rightarrow \Delta^\#$ cannot have new boxed formulas in addition to those in $\Box(\Gamma^*) \Rightarrow \Box(\Delta^*), \psi$, and so $\Gamma^\# \Rightarrow \Delta^\#$ is in W. By induction hypothesis we have therefore that $\Gamma^\# \Rightarrow \Delta^\# \notin v(\psi)$, and so $v(\Box\psi) = \emptyset$. In particular: $\Gamma \Rightarrow \Delta \notin v(\Box\psi)$.

It can easily be seen that $\langle \mathcal{A}, v\rangle$ is a model of $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$ iff for every $s \in W$, either $s \in v(B_i)$ for some i, or $s \notin v(A_j)$ for some j. Hence what we have just shown imply that $\langle \mathcal{A}, v\rangle$ is a model of S which is not a model of $\Gamma^* \Rightarrow \Delta^*$ (the proof that this entails the validity in $\langle \mathcal{A}, v\rangle$ of the sequents in S is similar to the way this is done in the proof of Theorem 2).

Corollary 9. Strong Completeness of $GS5$:

1. *A sequent s follows in $S5$ from a set S of sequents iff $S \vdash_{GS5} s$.*
2. *A formula φ is valid in $S5$ iff $\vdash_{GS5} \Rightarrow \varphi$.*

Corollary 10. *(***Strong Analytic Cut-elimination for $GS5$:**

1. *A sequent s follows in* $GS5$ *from* $S = \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$ *iff it has a proof in* $GS5$ *from* S *in which every cut is done either on a formula in* $\bigcup_{i=1}^{n} \Gamma_i \cup \bigcup_{i=1}^{n} \Delta_i$*, or on a subformula of the form* $\Box A$ *of some formula in* $S \cup \{s\}$.
2. $\vdash_{GS5} s$ *iff there exists a proof in* $GS5$ *of* s *in which all the cuts are analytic and are performed on formulas of the form* $\Box\psi$.

This corollary suffices for the subformula property, and for developing a good tableaux system for $S5$.

Note: $\vdash_{GS5}$, the standard consequence relation defined by $GS5$, is different from that induced by the many-valued semantics described above (for example: $\varphi \vdash_{S5} \Box\varphi$ according to that semantics, but the sequent $\varphi \Rightarrow \Box\varphi$ is not provable in $GS5$). What we do have is that $\Gamma \vdash_{GS5} \Delta$ iff $\varphi_{\Gamma \Rightarrow \Delta}$ is valid according to this semantics.[17]

7.4 The Infinite-valued Purely Relevant Logic

Substructural logics ([SHD93]) are logics defined by Gentzen-type systems with an irregular set of structural rules (the "regular" set being the one consisting of the standard structural rules, weakening, and cut). The official consequence relation associated with a logic of this kind is therefore not a Scott consequence relation in the sense of Definition 1. Its set of valid formulas may still correspond, however, to some many-valued logic. In subsection 5.3 we have already seen an example in which this happens with a 3-valued logic. We now shortly review another example, this time with an infinite-valued logic.

The system $GRMI_m$ is obtained from $GCPL$ by simply deleting the weakening rule[18] (or from GRM_m of subsection 5.3 by deleting the mingle rule. Note that again we assume that the two sides of a sequent consists of *sets*. In other words: although we do not have weakening, the expansion rule is still available). It is easy to prove, using Gentzen's original purely syntactic method, that cut-elimination still obtains for this system. Moreover: a strong form of the interpolation theorem holds for it: $\Rightarrow \varphi \rightarrow \psi$ is provable only if there exists an interpolant ϕ *containing only atomic formulas common to* φ *and* ψ [19]. This puts the corresponding logic, RMI_m, in the family of Relevance logics (in which framework it

[17] $\vdash_{GS5}$ corresponds to the "truth" consequence relation of modal logic, while the Tarskian consequence relation between singleton sequents which is induced by $GS5$ corresponds to its "validity" consequence relation (see [Avr91b] for these notions). The fact that these CRs are different is due to the impurity of the rules of $GS5$.

[18] In the literature on relevance logic (including [Avr84], from which the results below are taken) the symbols $\sim$, $\rightarrow$, $\circ$, $+$ are used instead of $\neg$, $\supset$, $\wedge$, $\vee$ (respectively). We use therefore these symbols below, except that instead of $\circ$ we use $\otimes$, which is more common today (recall that we did the same in section 5.3).

[19] Unlike in classical logic, such atomic formulas necessarily exist in case $\Rightarrow \varphi \rightarrow \psi$ is provable in $GRMI_m$. This is called the "variable sharing property" in the literature on relevance logics ([AB75,AB92,Dun86]), and it is the most characteristic feature of these logics.

was originally introduced and investigated). In [Avr84] it was proved that it is also weakly many-valued according to Definition 10. The corresponding matrix, called A_ω in [Avr84], is the following:

Truth Values: $\{t, f, I_1, I_2, I_3 \ldots\}$.

Designated Values: All elements except f.

Operations: $\sim t = f$, $\sim f = t$, $\sim I_j = I_j$ $(1 \leq j < \infty)$.

$$a \to b = \begin{cases} t & a = f \text{ or } b = t \\ I_j & a = b = I_j \\ f & \text{otherwise .} \end{cases}$$

$$a + b = \sim a \to b \qquad a \otimes b = \sim (a \to \sim b)$$

Like G_ω, A_ω can be viewed as the "limit" of its finite substructures. Indeed, let A_n be the substructure of A_ω consisting of $\{t, f, I_1, \ldots, I_n\}$. Then we have:

Theorem 11. *A formula φ is valid in A_ω (or in all of the finite structures A_n) iff $\vdash_{GRMI_m} \Rightarrow \varphi$. Moreover, if φ contains n atomic formulas and $\not\vdash_{GRMI_m} \Rightarrow \varphi$ then there is a valuation which refutes φ already in A_n.*

It follows that the logic of the matrix A_ω is weakly many-valued according to Definition 10, and that $GRMI_m$ is sound and weakly complete relative to it.

Notes:

1. Again we have here only *weak* completeness. Indeed, $GRMI_m$ is not strongly complete for A_ω (thus $A \otimes B \vdash_{A_\omega} A$, but the corresponding sequent is not provable in $GRMI_m$).
2. In [Avr97] it is proved that the logic of the matrix A_ω is in fact many-valued according to Definition 10 (not only weakly so).
3. This is the only case dealt with in this survey in which the present known proofs of the completeness of the system and of the cut elimination theorem for it are completely separated. It is an interesting challenge to find a simultaneous proof of both in this case as well.

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