NEGATION: TWO POINTS OF VIEW

Arnon Avron
Department of Computer Science
Sackler Faculty of Exact Sciences
Tel Aviv University, Ramat Aviv, Israel 69978

1 Introduction

In this paper we look at negation from two different points of view: a syntactical one and a semantical one. Accordingly, we identify two different types of negation. The same connective of a given logic might be of both types, but this might not always be the case.

The syntactical point of view is an abstract one. It characterizes connectives according to the internal *role* they have inside a logic, regardless of any meaning they are intended to have (if any). With regard to negation our main thesis is that the availability of what we call below an internal negation is what makes a logic essentially *multiple-conclusion*.

The semantic point of view, in contrast, is based on the intuitive meaning of a given connective. In the case of negation this is simply the intuition that the negation of a proposition A is true if A is not, and not true if A is true.[1]

Like in most modern treatments of logics (see, e.g., [Sc74], [Ha79], [Ga81], [Ur84], [Wo88], [Ep90], [Av91a], [Cl91], [FHV92]), our study of negation will

[1]We have avoided here the term "false", since we do not want to commit ourselves to the view that A is false precisely when it is not true. Our formulation of the intuition is therefore obviously circular, but this is unavoidable in intuitive informal characterizations of basic connectives and quantifiers.

be in the framework of Consequence Relations (CRs). Following [Av91a], we use the following rather general meaning of this term:

Definition.

(1) A *Consequence Relation* (CR) on a set of formulas is a binary relation $\vdash$ between (finite) multisets of formulas s.t.:

(I) Reflexivity: $A \vdash A$ for every formula A.

(II) Transitivity, or "Cut": if $\Gamma_1 \vdash \Delta_1, A$ and $A, \Gamma_2 \vdash \Delta_2$, then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$.

(III) Consistency: $\emptyset \not\vdash \emptyset$ (where $\emptyset$ is the empty multiset).

(2) A single-conclusion CR is a CR $\vdash$ such that $\Gamma \vdash \Delta$ only if Δ consists of a single formula.

The notion of (multiple-conclusion) CR was introduced in [Sc74a] and [Sc74b]. It was a generalization of Tarski's notion of a consequence relation, which was single-conclusion. Our notions are, however, not identical to the original ones of Tarski and Scott. First, they both considered *sets* (rather than multisets) of formulas. Second, they impose a third demand on CRs: monotonicity. We shall call a (single-conclusion or multiple-conclusion) CR which satisfies these two extra conditions *ordinary*. A single-conclusion, ordinary CR will be called *Tarskian*.[2]

The notion of a "logic" is in practice broader then that of a CR, since usually several CRs are associated with a given logic.[3] Given a logic $\mathcal{L}$ there are in most cases two major single-conclusion CRs which are naturally

[2]What we call a Tarskian CR is exactly Tarski's original notion. In [Av94] we argue at length why the notion of a proof in an axiomatic system naturally leads to *our* notion of single-conclusion CR, and why the further generalization to multiple-conclusion CR is also very reasonable.

[3]This is true even about classical logic: see [Av91a] or [Av94], which contains many other examples (see also section 3 below).

associated with it: the external $\vdash^e_{\mathcal{L}}$ and the internal $\vdash^i_{\mathcal{L}}$. For example, if $\mathcal{L}$ is defined by some axiomatic system AS then $A_1, \cdots, A_n \vdash^e_{\mathcal{L}} B$ iff there exists a proof in AS of B from $A_1, \cdots A_n$ (according to the most standard meaning of this notion as defined in undergraduate textbooks on mathematical logic), while $A_1, \cdots, A_n \vdash^i_{\mathcal{L}} B$ iff $A_1 \to (A_2 \to \cdots \to (A_n \to B)\cdots)$ is a theorem of AS (where $\to$ is an appropriate "implication" connective of the logic). Similarly if $\mathcal{L}$ is defined using a Gentzen-type system G then $A_1, \cdots, A_n \vdash^i_{\mathcal{L}} B$ if the sequent $A_1, \cdots, A_n \Rightarrow B$ is provable in G, while $A_1, \cdots A_n \vdash^e_{\mathcal{L}} B$ iff there exists a proof in G of $\Rightarrow B$ from the assumptions $\Rightarrow A_1, \cdots, \Rightarrow A_n$ (perhaps with cuts). $\vdash^e_{\mathcal{L}}$ is always a Tarskian relation, $\vdash^i_{\mathcal{L}}$ frequently not. The existence (again, in most cases) of these two CRs should be kept in mind in what follows. The reason is that semantical characterizations of connectives (in particular of negation in this work) is almost always done w.r.t. Tarskian CRs (and so here $\vdash^e_{\mathcal{L}}$ is usually relevant). This is not the case with syntactical characterizations, and here frequently $\vdash^i_{\mathcal{L}}$ is more suitable.[4]

A final note: in order to give the global picture, we have omitted almost all proofs. Most of them are straightforward anyway. Those which are not, are (or will be) given elsewhere.

2 The syntactical point of view

2.1 Classification of basic connectives

Our general framework allows us to give a completely abstract definition, *independent of any semantical interpretation*, of standard connectives. These characterizations explain why these connectives are so important in almost every logical system.

In what follows $\vdash$ is a fix CR. All definitions are taken to be relative to

[4]I have first introduced the notations $\vdash^i$ and $\vdash^e$ in [Av88] with respect to Linear Logic. The distinction between $\vdash^i_{LL}$ and $\vdash^e_{LL}$ will be of importance also in this paper.

$\vdash$ (the definitions are taken from [Av91a]).

We consider two types of connectives. The first, which we call *internal* connectives, makes it possible to transform a given sequent to an equivalent one that has a special required form. The second, which we call *combining* connectives allows us to combine (under certain circumstances) two sequents into one which contain exactly the same information.

The most common (and useful) connectives are the following:

Internal Disjunction: $+$ is an internal disjunction if for all Γ, Δ, A, B:

$$\Gamma \vdash \Delta, A, B \quad \text{iff} \quad \Gamma \vdash \Delta, A + B \ .$$

Internal Conjunction: $\otimes$ is an internal conjunction if for all Γ, Δ, A, B:

$$\Gamma, A, B \vdash \Delta \quad \text{iff} \quad A \otimes B \vdash \Delta \ .$$

Internal Implication: $\rightarrow$ is an internal implication if for all Γ, Δ, A, B:

$$\Gamma, A \vdash, B, \Delta \quad \text{iff} \quad \Gamma \vdash A \rightarrow B, \Delta \ .$$

Internal Negation: $\neg$ is an internal negation if the following two conditions are satisfied by all Γ, Δ and A:

$$\begin{array}{lccc} (1) & A, \Gamma \vdash \Delta & \text{iff} & \Gamma \vdash \Delta, \neg A \\ (2) & \Gamma \vdash \Delta, A & \text{iff} & \neg A, \Gamma \vdash \Delta \ . \end{array}$$

Combining Conjunction: We call a connective $\wedge$ a combining conjunction iff for all Γ, Δ, A, B:

$$\Gamma \vdash \Delta, A \wedge B \quad \text{iff} \quad \Gamma \vdash \Delta, A \quad \text{and} \quad \Gamma \vdash \Delta, B \ .$$

Combining Disjunction: We call a connective $\vee$ a combining disjunction iff for all Γ, Δ, A, B

$$A \vee B, \Gamma \vdash \Delta \quad \text{iff} \quad A, \Gamma \vdash \Delta \quad \text{and} \quad B, \Gamma \vdash \Delta \ .$$

Note: The combining connectives are called "additives" in Linear logic (see [Gi87]) and "extensional" in Relevance logic. The internal ones correspond, respectively, to the "multiplicatives" and the "intensional" connectives.

Several well-known logics can be defined using the above connectives:

Multiplicative Linear Logic: This is the logic which corresponds to the *minimal* (multiset) CR which includes all the internal connectives.

Propositional Linear Logic: (without the "exponentials" and the propositional constants). This corresponds to the minimal consequence relation which contains all the connectives introduced above.

$R_{\simeq}$ **the Intensional Fragment of the Relevance Logic** R**:**[5] This corresponds to the minimal CR which contains all the internal connectives and is *closed under contraction.*

R **without Distribution:** This corresponds to the minimal CR which contains all the connectives which were described above and is closed under contraction.

RMI$_{\simeq}$**:**[6] This corresponds to the minimal sets-CR which contains all the internal connectives.

Classical Proposition Logic: This of course corresponds to the minimal ordinary CR which has all the above connectives. Unlike the previous logics there is no difference in it between the combining connectives and the corresponding internal ones.

2.2 Internal negation and strong symmetry

Among the various connectives defined above only negation essentially demands the use of multiple-conclusion CRs (even the existence of an internal disjunction does not force multiple-conclusions, although its existence is trivial otherwise.). Moreover, its existence creates full symmetry between the two

[5]see [AB75] or [Du86].
[6]see [Av90a], [Av90b].

sides of the turnstyle. Thus in its presence, closure under any of the structural rules on one side entails closure under the same rule on the other, the existence of any of the binary internal connectives defined above implies the existence of the rest, and the same is true for the combining connectives.

To sum up: internal negation is the connective with which "the hidden symmetries of logic" [Gi87] are explicitly represented. We shall call, therefore, any multiple-conclusion CR which possesses it *strongly symmetrical.*

Some alternative characterizations of internal negation are given in the following proposition.

Proposition 1 *The following conditions on* $\vdash$ *are all equivalent:*

(1) $\neg$ *is an internal negation for* $\vdash$.

(2) $\Gamma \vdash \Delta, A$ *iff* $\Gamma, \neg A \vdash \Delta$

(3) $A, \Gamma \vdash \Delta$ *iff* $\Gamma \vdash \Delta, \neg A$

(4) $A, \neg A \vdash$ *and* $\vdash \neg A, A$

(5) $\vdash$ *is closed under the rules:*

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \qquad\qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ .$$

Our characterization of internal negation and of symmetry has been done within the framework of multiple-conclusion relations. Single-conclusion CRs are, however, more natural. We proceed next to introduce corresponding notions for them.

Definition.

(1) Let $\vdash_{\mathcal{L}}$ be a single-conclusion CR (in a language $\mathcal{L}$), and let $\neg$ be a unary connective of $\mathcal{L}$. $\vdash_{\mathcal{L}}$ is called strongly symmetric w.r.t. to $\neg$, and $\neg$ is called an internal negation for $\vdash_{\mathcal{L}}$ if there exists a multiple-conclusion CR $\vdash_{\mathcal{L}}^{*}$ with the following properties:

(i) $\Gamma \vdash_{\mathcal{L}}^{*} A$ iff $\Gamma \vdash A$

(ii) $\neg$ is an internal negation for $\vdash_{\mathcal{L}}^{*}$

(2) A single-conclusion CR $\vdash_{\mathcal{L}}$ is called *essentially multiple-conclusion* iff it has an internal negation.

Obviously, if a CR $\vdash^*_{\mathcal{L}}$ like in the last definition exists then it is unique. We now formulate sufficient and necessary conditions for its existence.

Theorem 2 *$\vdash_{\mathcal{L}}$ is strongly symmetric w.r.t. $\neg$ iff the following conditions are satisfied:*

(i) $A \vdash_{\mathcal{L}} \neg\neg A$

(ii) $\neg\neg A \vdash_{\mathcal{L}} A$

(iii) *If* $\Gamma, A \vdash_{\mathcal{L}} B$ *then* $\Gamma, \neg B \vdash_{\mathcal{L}} \neg A$.

Proof: The conditions are obviously necessary. Assume, for the converse, that $\vdash_{\mathcal{L}}$ satisfies the conditions. Define: $A_1, \cdots, A_n \vdash^s_{\mathcal{L}} B_1, \cdots, B_k$ iff for every $1 \leq i \leq n$ and $1 \leq j \leq k$:

$$A_1, \cdots, A_{i-1}, \neg B_1, \cdots, \neg B_k, A_{i+1}, \cdots, A_n \vdash \neg A_i$$
$$A_1, \cdots, A_n, \neg B_1, \cdots, -B_{j-1}, \neg B_{j+1}, \cdots, \neg B_k \vdash B_j \ .$$

It is easy to check that $\vdash^s_{\mathcal{L}}$ is a CR whenever $\vdash_{\mathcal{L}}$ is a CR (whether single-conclusion or multiple-conclusion), and that if $\Gamma \vdash^s_{\mathcal{L}} A$ then $\Gamma \vdash_{\mathcal{L}} A$. The first two conditions imply (together) that $\neg$ is an internal negation for $\vdash^s_{\mathcal{L}}$ (in particular: the second entails that if $A, \Gamma \vdash^s_{\mathcal{L}} \Delta$ then $\Gamma \vdash^s_{\mathcal{L}} \Delta, \neg A$ and the first that if $\Gamma \vdash^s_{\mathcal{L}} \Delta, A$ then $\neg A, \Gamma \vdash^s_{\mathcal{L}} \Delta$). Finally, the third condition entails that $\vdash^s_{\mathcal{L}}$ is conservative over $\vdash_{\mathcal{L}}$. □

Proposition 3 *Let $\mathcal{L}$ be any logic in a language containing $\neg$ and $\rightarrow$. Suppose that the set of valid formulae of $\mathcal{L}$ includes the set of formulae in the $\{\neg, \rightarrow\}$ language which are theorems of Linear Logic,*[7] *and that it is closed under MP for $\rightarrow$. Then the internal consequence relation of $\mathcal{L}$ (defined using $\rightarrow$ as in the introduction) is strongly symmetrical (with respect to $\neg$).*

[7] Here $\neg$ should be translated into linear negation, $\rightarrow$ – into linear implication.

Examples.

1. Classical logic.

2. Extensions of classical logic, like the various modal logics.

3. Linear logic and its various fragments.

4. The various Relevance logics (like R and RM (see [AB75], [Du86], [AB92]) or RMI ([Av90])) and their fragments.

5. The various many-valued logics of Lukasiewicz.

All the systems above have, therefore, an internal negation. A major system which does not have one is intuitionistic logic. Other examples (positive and negative) will be encountered below.

Note. In all these logics it is the *internal* CR which is essentially multiple-conclusion and has an internal negation.[8] This is true even for classical predicate calculus: There, e.g. $\forall x A(x)$ follows from $A(x)$ according to the *external* CR, but $\neg A(x)$ does not follow from $\neg \forall x A(x)$.[9]

We next discuss what properties of $\vdash_{\mathcal{L}}$ are preserved by $\vdash^s_{\mathcal{L}}$.

Theorem 4 *Assume $\vdash_{\mathcal{L}}$ is essentially multiple-conclusion.*

1. *$\vdash^s_{\mathcal{L}}$ is monotonic iff so is $\vdash_{\mathcal{L}}$.*

2. *$\vdash^s_{\mathcal{L}}$ is closed under expansion (= the converse of contraction) iff so is $\vdash_{\mathcal{L}}$.*

[8]The definition of this internal CR depends on the choice of the implication connective. However, the same CR is obtained from the standard Gentzen-type formulations of these logics (and most of them have one) by the method described in the introduction.

[9]The internal CR of classical logic has been called the "truth" CR in [Av91a] and was denoted by $\vdash^t$, while the external one was called the "validity" CR and was denoted by $\vdash^v$. On the propositional level there is no difference between the two.

3. *$\wedge$ is a combining conjunction for $\vdash^s_{\mathcal{L}}$ iff it is a combining conjunction for $\vdash_{\mathcal{L}}$.*

4. *$\rightarrow$ is an internal implication for $\vdash^s_{\mathcal{L}}$ iff it is an internal implication for $\vdash_{\mathcal{L}}$.*

Notes.
1) Because $\vdash^s_{\mathcal{L}}$ has a symmetrical negation, Parts (3) and (4) can be formulated as follows: $\vdash^s_{\mathcal{L}}$ has the internal connectives iff $\vdash_{\mathcal{L}}$ has an internal implication and it has the combining connectives iff $\vdash_{\mathcal{L}}$ has a combining conjunction.
2) In contrast, a combining disjunction for $\vdash_{\mathcal{L}}$ is not necessarily a combining disjunction for $\vdash^s_{\mathcal{L}}$. It is easy to see that a necessary and sufficient condition for this to happen is that $\vdash_{\mathcal{L}} \neg(A \vee B)$ whenever $\vdash_{\mathcal{L}} \neg A$ and $\vdash_{\mathcal{L}} \neg B$. An example of an essentially multiple-conclusion system with a combining disjunction which does not satisfy the above condition is RMI of [Av90]. That system indeed does not have a combining conjunction. This shows that a *single-conclusion* logic $\mathcal{L}$ with an internal negation and combining disjunction does not necessarily have a combining conjunction (unless $\mathcal{L}$ is monotonic). The converse situation is not possible, though: If $\neg$ is an internal negation and $\wedge$ is a combining conjunction then $\neg(\neg A \wedge \neg B)$ defines a combining disjunction even in the single-conclusion case.
3) An internal conjunction $\otimes$ for $\vdash_{\mathcal{L}}$ is also not necessarily an internal conjunction for $\vdash^s_{\mathcal{L}}$. We need the extra condition that if $A \vdash_{\mathcal{L}} \neg B$ then $\vdash_{\mathcal{L}} \neg(A \otimes B)$. An example which shows that this condition does not necessarily obtain even if $\vdash_{\mathcal{L}}$ is an ordinary CR, is given by the following CR $\vdash_{triv}$:

$$A_1, \cdots, A_n \vdash_{triv} B \quad \text{iff} \quad n \geq 1 \ .$$

It is obvious that $\vdash_{triv}$ is a Tarskian CR and that every unary connective of its language is a symmetrical negation for it, while every binary connective

is an internal conjunction. The condition above fails, however, for $\vdash_{triv}$.
4) The last example shows also that $\vdash^s_{\mathcal{L}}$ may not be closed under contraction when $\vdash_{\mathcal{L}}$ does, even if $\vdash_{\mathcal{L}}$ is Tarskian. Obviously, $\Gamma \vdash^s_{triv} \Delta$ iff $|\Gamma \cup \Delta| \geq 2$. Hence $\vdash^s_{triv} A, A$ but $\nvdash^s_{triv} A$. The exact situation about contraction is given in the next proposition.

Proposition 5 *If $\vdash_{\mathcal{L}}$ is essentially multiple-conclusion then $\vdash^s_{\mathcal{L}}$ is closed under contraction iff $\vdash_{\mathcal{L}}$ is closed under contraction and satisfies the following condition:*

$$\text{If } A \vdash_{\mathcal{L}} B \text{ and } \neg A \vdash_{\mathcal{L}} B \text{ then } \vdash_{\mathcal{L}} B.$$

In case $\vdash_{\mathcal{L}}$ has a combining disjunction this is equivalent to:

$$\vdash_{\mathcal{L}} \neg A \vee A \ .$$

Note. From the syntactical point of view, therefore, the law of excluded middle is just an internal representation of the structural law of contraction!

2.3 Weak internal negation and symmetry

The strong symmetry conditions are really strong. We now consider what happens if we relax them.

We start with some general observations (part of which have already been made in the proof of Theorem 2, others are generalizations of results of the previous subsection):[10]

Proposition 6 (1) *If $\neg$ is a unary connective of $\vdash_{\mathcal{L}}$ then $\vdash^s_{\mathcal{L}}$, as defined in the proof of Theorem 2 is a (multiple-conclusion) CR. Moreover:*

(i) *If $\Gamma \vdash^s_{\mathcal{L}} A$ then $\Gamma \vdash_{\mathcal{L}} A$.*

(ii) *$\vdash^s_{\mathcal{L}} A$ iff $\vdash_{\mathcal{L}} A$ (in other words: $\vdash^s_{\mathcal{L}}$ and $\vdash_{\mathcal{L}}$ have the same set of valid sentences, and differ "only" w.r.t. their consequence relations).*

[10] Propositions 7, 8 and 10 are from [Av91b].

(2) $\vdash^s_{\mathcal{L}}$ *is a conservative extension of* $\vdash_{\mathcal{L}}$ *iff condition (iii) of Theorem 2 obtains.*

$\vdash^s_{\mathcal{L}}$ is the natural CR which is induced by trying to view the connective $\neg$ of $\vdash_{\mathcal{L}}$ as negation. Accordingly we define:

Definition.

1. A unary connective $\neg$ of $\vdash_{\mathcal{L}}$ is called (weakly) symmetrical if it is an internal negation of $\vdash^s_{\mathcal{L}}$.
2. If $\neg$ is symmetrical then we call $\vdash^s_{\mathcal{L}}$ the *symmetrical version* of $\vdash_{\mathcal{L}}$.

Proposition 7 *$\neg$ is symmetrical in $\vdash_{\mathcal{L}}$ if the first two conditions of Theorem 2 are satisfied ($A \vdash_{\mathcal{L}} \neg\neg A$ and $\neg\neg A \vdash_{\mathcal{L}} A$).*

Definition.

1. A combining conjunction $\wedge$ for $\vdash_{\mathcal{L}}$ is called symmetrical if $\vdash_{\mathcal{L}}$ is closed under the rules:
   $$\frac{\Gamma, \neg A \vdash_{\mathcal{L}} \Delta \quad \Gamma, \neg B \vdash_{\mathcal{L}} \Delta}{\Gamma, \neg(A \wedge B) \vdash_{\mathcal{L}} \Delta} \qquad \frac{\Gamma \vdash_{\mathcal{L}} \Delta, \neg A}{\Gamma \vdash_{\mathcal{L}} \Delta, \neg(A \wedge B)} \quad \frac{\Gamma \vdash_{\mathcal{L}} \Delta, \neg B}{\Gamma \vdash_{\mathcal{L}} \Delta, \neg(A \wedge B)}$$
2. A combining disjunction $\vee$ for $\vdash_{\mathcal{L}}$ is called symmetrical if $\vdash_{\mathcal{L}}$ is closed under the dual rules.

Proposition 8 *A symmetrical combining conjunction (disjunction) for $\vdash_{\mathcal{L}}$ is a combining conjunction (disjunction) for $\vdash^s_{\mathcal{L}}$.*

Proposition 9 (1) *$\vdash^s_{\mathcal{L}}$ is monotonic iff $\vdash_{\mathcal{L}}$ is monotonic and $\neg A, A \vdash_{\mathcal{L}} B$ for every A, B.*
(2) *$\vdash^s_{\mathcal{L}}$ is closed under expansion iff $\vdash_{\mathcal{L}}$ is closed under expansion, and for all A: $A, \neg A \vdash_{\mathcal{L}} A$ and $A, \neg A \vdash_{\mathcal{L}} \neg A$ (in particular, if $\vdash_{\mathcal{L}}$ is monotonic then $\vdash^s_{\mathcal{L}}$*

is closed under expansion).

(3) *(a) If $\vdash_{\mathcal{L}}$ is Tarskian with a symmetrical combining disjunction $\vee$ then $\vdash^s_{\mathcal{L}}$ is closed under contraction iff $\vdash_{\mathcal{L}} \neg A \vee A$ for all A.*

(b) If $\vdash_{\mathcal{L}}$ is Tarskian and condition (iii) of Theorem 2 is satisfied (and so $\vdash^s_{\mathcal{L}}$ is a conservative extension of $\vdash_{\mathcal{L}}$) then $\vdash^s_{\mathcal{L}}$ is closed under contraction iff for all Γ, A, B: if $\Gamma, A \vdash_{\mathcal{L}} B$ and $\Gamma, \neg A \vdash_{\mathcal{L}} B$ then $\Gamma \vdash_{\mathcal{L}} B$.

Note. The conditions in the definitions of symmetrical conjunction and disjunction were formulated for arbitrary CRs since $\vdash^s_{\mathcal{L}}$ is defined (and has all the properties described so far in this subsection) even in case $\vdash_{\mathcal{L}}$ is multiple-conclusion.

We next turn our attention to the problem of having an internal implication for $\vdash^s_{\mathcal{L}}$. If $\rightarrow$ is such a connective then $\vdash^s_{\mathcal{L}} A \rightarrow B$ iff $A \vdash^s_{\mathcal{L}} B$ iff $A \vdash_{\mathcal{L}} B$ and $\neg B \vdash_{\mathcal{L}} \neg A$. Suppose now that $\vdash_{\mathcal{L}}$ has an internal implication $\supset$ and a combining conjunction $\wedge$. Then the last two conditions are together equivalent to $\vdash_{\mathcal{L}} (A \supset B) \wedge (\neg B \supset \neg A)$. This, in turn, is equivalent to $\vdash^s_{\mathcal{L}} (A \supset B) \wedge (\neg B \supset \neg A)$. Hence the last formula provides an obvious candidate for defining $\rightarrow$.

Proposition 10 *Suppose $\wedge$ is a symmetrical combining conjunction for $\vdash_{\mathcal{L}}$, $\supset$ is an internal implication for $\vdash_{\mathcal{L}}$ and $\vdash_{\mathcal{L}}$ is closed under the following rules:*

$$\frac{\Gamma, A, \neg B \vdash_{\mathcal{L}} \Delta}{\Gamma, \neg(A \supset B) \vdash_{\mathcal{L}} \Delta} \quad \frac{\Gamma_1 \vdash_{\mathcal{L}} \Delta_1, A \quad \Gamma_2 \vdash_{\mathcal{L}} \Delta_2, \neg B}{\Gamma_1, \Gamma_2 \vdash_{\mathcal{L}} \Delta_1, \Delta_2, \neg(A \supset B)} \ .$$

(These two rules will be called below the symmetry conditions for implication.) Define:

$$A \rightarrow B =_{Df} (A \supset B) \wedge (\neg B \supset \neg A) \ .$$

Then $\rightarrow$ is an internal implication for $\vdash^s_{\mathcal{L}}$.

The various propositions of this section naturally lead to several interesting systems which have symmetrical negation. First, by collecting the various conditions above on $\neg, \vee$ and $\wedge$ we get the following basic system BS:

Axioms:

$$A \Rightarrow A \ .$$

Rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \ \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \ \frac{\Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \ \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \ \frac{\Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}$$

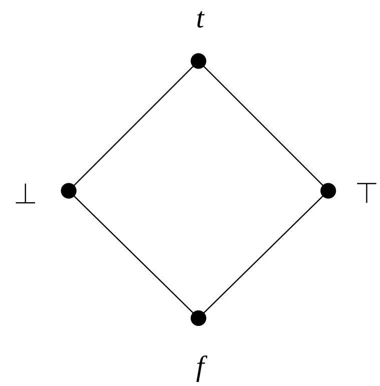
It is easy to see that only sequents of the form $A \Rightarrow B$ are provable in BS and that BS admits cut-elimination. Moreover: BS is essentially multiple-conclusion since it satisfies condition (iii) of Theorem 2.

Another interesting fact about BS is:

Proposition 11 $\vdash^s_{BS} = LL_a$ *(the purely additive fragment of Linear Logic).*

The next step is to extend $\vdash_{BS}$ to an ordinary CR by adding the structural rules. It does not really matter here if we add them on both sides (getting an ordinary multiple-conclusion CR) or only on the l.h.s. (getting a Tarskian CR), since we get the same single-conclusion fragment in both cases, and so the same symmetrical version. Let us call the resulting system FDE. FDE is not a conservative extension of BS since $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$ is provable in it, but not in BS. It is well known that $\vdash_{FDE} A_1, \cdots, A_n \Rightarrow$

$B_1, \cdots, B_m$ iff $A_1 \wedge A_2 \wedge \cdots \wedge A_n \rightarrow B_1 \vee B_2 \vee \cdots \vee B_m$ is a "first-degree-entailment" of the standard relevance logics like R (see [AB75], [Du86]). Moreover FDE has the following 4-valued characteristic matrix:



where $\neg t = f$, $\neg f = t$, $\neg - = -$, $\neg \top = \top$, $\vee$ and $\wedge$ are the lattice operations and $D = \{t, \top\}$ is the set of the designated values. In fact $\vdash_{FDE} \Gamma \Rightarrow \Delta$ iff whenever v is a valuation in this matrix s.t. $v(A) \in D$ for every $A \in \Gamma$, we have $v(B) \in D$ for some $B \in \Delta$.

What can we say about $\vdash^s_{FDE}\Gamma$ According to the above propositions it is closed under expansion, but not under contraction or weakening. It has $\neg$ as an internal negation and $\wedge, \vee$ as combining conjunction and disjunction, respectively. Another important property is the following semantic characterization.

Proposition 12 $\vdash^s_{FDE} \Gamma \Rightarrow \Delta$ *if for every valuation v in the above four-valued matrix, either $v(A) = f$ for some $A \in \Gamma$, or $v(B) = t$ for some $B \in \Delta$, or $v(A) = \top$ for every $A \in \Gamma \cup \Delta$ or $v(A) = v(B) = -$ for two different occurrences of formulae A, B of Γ, Δ.*

Proposition 9 suggests two natural methods of extending FDE. The first is to add to it the axioms $\neg A, A \Rightarrow B$. This corresponds, in the multiple-conclusion version, to adding $\neg A, A \Rightarrow$ and the structural rules on the right.

(Again the multiple-conclusion version is cut-free and a conservative extension of the Tarskian one.) The resulting system is, in fact, exactly Kleene's 3-valued logic (of $\{t, f, -\}$) and so has been called Kl in [Av91b]. By Proposition 9 $\vdash^s_{Kl}$ is monotonic, but not closed under contraction. It is shown in [Av91b] that $A_1, \cdots, A_n \vdash^s_{Kl} B$ iff $A_1 \to (A_1 \to \cdots \to (A_n \to B)$ is valid in Lukasiewicz' 3-valid logic L_3.

The second natural addition to FDE is by the axioms $\Rightarrow \neg A \vee A$. In the multiple-conclusion case this corresponds to adding $\Rightarrow \neg A, A$ as axioms and the structural rules on the right (again we get a conservative, cut-free version). This time the resulting logic, Pac, is sound and complete w.r.t. the 3-valued logic of $\{t, f, \top\}$ (also known as J_3-see [dC74], [DO85], [Ep90]). It has the same set of valid formulae as classical logic, but it is *paraconsistent* ($\neg p, p \nvdash q$). $\vdash^s_{Pac}$ is this time closed under contraction and its converse, but not under weakening. It corresponds to the $\{\neg, \vee, \wedge\}$-fragment of the 3-valued logic RM_3 ([AB75]) in the same way as $\vdash^s_{Kl}$ corresponds to Lukasiewicz L_3 (see [Av91b]).

By making both additions we get, of course, classical logic.

Things get more complicated when we add to the language a symmetrical implication. Thus by adding to BS the rules:

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}$$

$$\frac{\Gamma, A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \supset B)}$$

we get a system, BSI, which does *not* have property (iii) of Theorem 2, and not only sequents of the form $A \Rightarrow B$ are provable in it. BSI is still only single-conclusion though. As for $\vdash^s_{BSI}$, the best we can tell about it at present is that its $\{\neg, \vee, \wedge, \to\}$-fragment (where $A \to B = (A \supset B) \wedge (\neg B \supset \neg A)$, as above) is at least as strong as the multiplicative-additive fragment of Linear Logic (without the propositional constants).

A more significant change is made when we add to BSI the standard structural rules. Here it does matter whether we do it on both sides or only on the l.h.s., since the single-conclusion fragment of the system BL which we get by the first option is a proper extension of the system N^- which we get by the second one. In fact, the purely positive fragment of BL is identical to that of classical logic, while that of N^- – to the corresponding intuitionistic fragment.[11]

Semantically, BL corresponds to the logic we get from $\{t, f, \top, -\}$ if we define $a \supset b$ to be t if $a \notin D$, b otherwise (see [AA94]). N^-, on the other hand corresponds to Kripke-style structures which are based on this four-valued logic (see, e.g., [Wa93]). Both systems admit cut-elimination.

It follows from the propositions above that the symmetrical versions of $\vdash_{BL}$ and $\vdash_{N^-}$ ($\vdash^s_{BL}$ and $\vdash^s_{N^-}$) are neither monotonic nor closed under contraction, but they have all the internal and combining connectives (the internal implication is again $\rightarrow$ as defined above). The $\{\neg, \wedge, \vee, \rightarrow\}$ fragment of $\vdash^s_{N^-}$ is at least as strong as (and might be identical to) the multiplicative-additive fragment of Linear Logic, strengthened by the expansion rule and the distribution axiom (i.e. R where contraction is replaced by its converse). For $\vdash^s_{BL}$, on the other hand, we have exactly the same semantic characterization as given in Proposition 12.

By adding $\neg A, A \Rightarrow B$ as axioms to BL (or, alternatively, $\neg A, A \Rightarrow$) we again get the 3-valued logic of $\{t, f, -\}$, with the above definition of $\supset$. This is exactly the system LPF of [BCJ84] (see also [Jo86], [Av91b]). By adding the same axiom to N^- we get N (Nelson's strong system of constructive negation). Semantically, N corresponds to Kripke-style structures which are based on this 3-valued logic (see, e.g., [Wa93]). The symmetrical versions of both systems are now monotonic, but still not closed under contraction.

[11] BL was introduced, under a different name, in [Av91b]. It is investigated and shown to be the logic of logical bilattices in [AA96] (see also [AA94]). N^- is Nelson's weak system of constructive negation. This system and the full system N (see below) were independently introduced by Nelson (see [AN84]) and Kutschera ([Ku69]). See [Wa93] for details on both systems.

$\vdash^s_{LPF}$ is shown in [Av91b] to be identical to Lukasiewicz' 3-valued logic. Its internal implication $\rightarrow$ is, in fact, *exactly* Lukasiewicz' implication. $\vdash^s_N$ *might* correspond to the substructural system BCK of Grishin (see [OK85] and [SD93] for descriptions and references).

In contrast to what happens when we add $\neg A, A \Rightarrow B$ to N^- and BL, when we add $\Rightarrow \neg A \vee A$ to both we do get equivalent systems (this is due to the fact that $\neg(A \supset B) \vee (A \supset B) \vdash_{N^-} ((A \supset B) \supset A) \supset A$, and so we get the full classical positive fragment). It is more natural, therefore, to work here within the multiple-conclusion version, where by adding $\Rightarrow \neg A, A$ instead we get an equivalent cut-free formulation. The resulting logic is this time the logic of $\{t, f, \top\}$ (again, with the above definition of $\supset$). This logic was introduced independently in [DO85], [Av86] and [Ro89]. In [DO85] it is called J_3 (see also [Ep90]). Its most important property is that is is a maximal paraconsistent logic in its language (see [Av86]), and the strongest in the family of the paraconsistent logics of da-Costa ([dC74]). Its symmetrical version $\vdash^s_{J_3}$ is this time closed under contraction and its converse, but it is not monotonic. In [Av91b] it is shown that it is identical to RM_3-the unique 3-valued extension of RM, and the strongest logic in the family of relevant and semirelevant logics. Its internal implication $\rightarrow$ is this time exactly the Sobociński implication [So52].

Again by making both types of additions to BL or to N^- we get classical propositional logic.

3 The semantic point of view

We turn in this section to the semantic aspect of negation.

A "semantics" for a logic consists of a set of "models". The main property of a model is that every sentence of a logic is either true in it or not (and not both). The logic is sound with respect to the semantics if the set of sentences which are true in each model is closed under the CR of the logic, and complete if a sentence φ follows (according to the logic) from a set T of assumptions iff every model of T is a model of φ. Such a characterization is, of course, possible only if the CR we consider is Tarskian. *In this section we assume, therefore, that we deal only with Tarskian CRs.* For logics like Linear Logic and Relevance logics this means that we consider only the *external* CRs which are associated with them (see the Introduction).

Obviously, the essence of a "model" is given by the set of sentences which are true in it. Hence a semantics is, essentially, just a set S of theories. Intuitively, these are the theories which (according to the semantics) provide a full description of a possible state of affairs. Every other theory can be understood as a partial description of such a state, or as an approximation of a full description. Completeness means, then, that a sentence φ follows from a theory T iff φ belongs to every superset of T which is in S (in other words: iff φ is true in any possible state of affairs of which T is an approximation).

Now what constitutes a "model"" is frequently defined using some kind of algebraic structures. Which kind (matrices with designated values, possible worlds semantics and so on) varies from one logic to another. It is difficult, therefore, to base a general, uniform theory on the use of such structures. Semantics (= a set of theories!) can also be defined, however, purely syntactically. Indeed, below we introduce several types of syntactically defined semantics which are very natural for *every* logic with "negation". Our investigations will be based on these types.

Our description of the notion of a model reveals that *externally* it is based on two classical "laws of thought": the law of contradiction and the law of excluded middle. When this external point of view is internally reflected inside the logic with the help of a unary connective $\neg$ we call this connective a (strong) *semantic negation*. Its intended meaning is that $\neg A$ should be true precisely when A is not. The law of contradiction internally means then that only consistent theories may have a model, while the law of excluded middle internally means that the set of sentences which are true in some given model should be negation-complete. The sets of consistent theories, of complete theories and of normal theories (theories that are both) have, therefore a crucial importance when we want to find out to what degree a given unary connective of a logic can be taken as a semantic negation. Thus complete theories reflect a state of affairs in which the law of excluded middle holds. It is reasonable, therefore, to say that this law semantically obtains for a logic L if its consequence relation $\vdash_L$ is *determined* by its set of complete theories. Similarly, L (strongly) satisfies the law of contradiction iff $\vdash_L$ is determined by its set of consistent theories, and it semantically satisfies both laws iff $\vdash_L$ is determined by its set of normal theories.

The above characterizations might seem unjustifiably strong for logics which are designed to allow non-trivial inconsistent theories. For such logics the demand that $\vdash_{\mathcal{L}}$ should be determined by its set of normal theories is reasonable only if we start with a consistent set of assumptions (this is called strong c-normality below). A still weaker demand (c-normality) is that any consistent set of assumptions should be an approximation of at least one normal state of affairs (in other words: it should have at least one normal extension).

It is important to note that the above characterizations are independent of the existence of any internal reflection of the laws (for example: in the forms $\neg(\neg A \wedge A)$ and $\neg A \vee A$, for suitable $\wedge$ and $\vee$). There might be strong connections, of course, in many important cases, but they are neither necessary nor always simple.

We next define our general notion of semantics in precise terms.

Definition. Let $\mathcal{L}$ be a logic in L and let $\vdash_{\mathcal{L}}$ be its associated (Tarskian) CR.

1. A *setup* for $\vdash_{\mathcal{L}}$ is a set of formulae in L which is closed under $\vdash_{\mathcal{L}}$. A *semantics* for $\vdash_{\mathcal{L}}$ is a nonempty set of setups which does not include the trivial setup (i.e., the set of all formulae).

2. Let S be a semantics for $\vdash_{\mathcal{L}}$. An S-*model* for a formula A is any setup in S to which A belongs. An S-model of a theory T is any setup in S which is a superset of T. A formula is called S-valid iff every setup in S is a model of it. A formula A S-follows from a theory T ($T \vdash_{\mathcal{L}}^{S} A$) iff every S-model of T is an S-model of A.

Proposition 13 $\vdash_{\mathcal{L}}^{S}$ *is a consequence relation and* $\vdash_{\mathcal{L}} \subseteq \vdash_{\mathcal{L}}^{S}$.

Note.

1. $\vdash_{\mathcal{L}}^{S}$ is not necessarily finitary even if $\vdash$ is.

2. $\vdash_{\mathcal{L}}$ is just $\vdash_{\mathcal{L}}^{S^*}$ where S^* is the set of all setups.

3. If $S_1 \subseteq S_2$ then $\vdash_{\mathcal{L}}^{S_2} \subseteq \vdash_{\mathcal{L}}^{S_1}$.

Examples:

1. For classical propositional logic the standard semantics consists of the setups which are induced by some valuation in $\{t, f\}$. These setups can be characterized as theories T such that

   $$(i) \quad \neg A \in T \;\text{ iff }\; A \notin T \quad (ii) \quad A \wedge B \in T \;\text{ iff both }\; A \in T \text{ and } B \in T$$

   (and similar conditions for the other connectives).

2. In classical predicate logic we can define a setup in S to be any set of formulae which consists of the formulae which are true in some given first-order structure relative to some given assignment. Alternatively we can take a setup to consist of the formulae which are *valid* in some given first-order structure. In the first case $\vdash^S = \vdash^t$, in the second $\vdash^S = \vdash^v$, where $\vdash^t$ and $\vdash^v$ are the "truth" and "validity" consequence relations of classical logic (see [Av91a] for more details).

¿From now on the *following two conditions will be assumed in all our general definitions and propositions:*

1. The language contains a negation connective $\neg$.
2. For no A are both A and $\neg A$ theorems of the logic.

Definition. Let S be a semantics for a CR $\vdash_{\mathcal{L}}$

1. $\vdash_{\mathcal{L}}$ is strongly complete relative to S if $\vdash_{\mathcal{L}}^{S} = \vdash_{\mathcal{L}}$.
2. $\vdash_{\mathcal{L}}$ is weakly complete relative to S if for all A, $\vdash_{\mathcal{L}} A$ iff $\vdash_{\mathcal{L}}^{S} A$.
3. $\vdash_{\mathcal{L}}$ is c-complete relative to S if every consistent theory of $\vdash_{\mathcal{L}}$ has a model in S.
4. $\vdash_{\mathcal{L}}$ is strongly c-complete relative to S if for every A and every *consistent* T, $T \vdash_{\mathcal{L}}^{S} A$ iff $T \vdash_{\mathcal{L}} A$.

Notes:

1. Obviously, strong completeness implies strong c-completeness, while strong c-completeness implies both c-completeness and weak completeness.
2. Strong completeness means that deducibility in $\vdash_{\mathcal{L}}$ is equivalent to semantical consequence in S. Weak completeness means that theoremhood in $\vdash_{\mathcal{L}}$ (i.e., derivability from the empty set of assumptions) is equivalent to semantical validity (= truth in all models). c-completeness means that consistency implies satisfiability. It becomes identity if only consistent sets can be satisfiable, i.e., if $\{\neg A, A\}$ has a model for no A. This is obviously too strong a demand for paraconsistent logics. Finally, strong c-completeness means that if we restrict ourselves to *normal* situations (i.e., consistent theories) then $\vdash_{\mathcal{L}}$ and $\vdash_{\mathcal{L}}^{S}$ are the same. This might sometimes be weaker than full strong completeness.

The last definition uses the concepts of "consistent" theory. The next definition clarifies (among other things) the meaning of this notion as we are going to use in this paper.

Definition. Let $\mathcal{L}$ and $\vdash_{\mathcal{L}}$ be as above. A theory in L *consistent* if for no A it is the case that $T \vdash_{\mathcal{L}} A$ and $T \vdash_{\mathcal{L}} \neg A$, *complete* if for all A, either $T \vdash_{\mathcal{L}} A$ or $T \vdash_{\mathcal{L}} \neg A$, *normal* if it is both consistent and complete. CS, CP and N will denote, respectively, the sets of all consistent, complete and normal theories.

Given $\vdash_{\mathcal{L}}$, the three classes, CS, CP and N, provide 3 different syntactically defined semantics for $\vdash_{\mathcal{L}}$, and 3 corresponding consequence relations $\vdash_{\mathcal{L}}^{CS}$, $\vdash_{\mathcal{L}}^{CP}$ and $\vdash_{\mathcal{L}}^{N}$ such that $\vdash_{\mathcal{L}}^{CS}\subseteq\vdash_{\mathcal{L}}^{N}$ and $\vdash_{\mathcal{L}}^{CP}\subseteq\vdash_{\mathcal{L}}^{N}$. Accordingly, we get several notions of syntactical completeness of $\vdash_{\mathcal{L}}$. In the rest of this section we investigate these relations and the completeness properties they induce.

Let us start with the easier case: that of $\vdash_{\mathcal{L}}^{CS}$. It immediately follows from the definitions (and our assumptions) that relative to it every logic is strongly c-complete (and so also c-complete and weakly complete). Hence the only completeness notion it induces is the following:

Definition. A logic $\mathcal{L}$ with a consequence relation $\vdash_{\mathcal{L}}$ is strongly consistent if $\vdash_{\mathcal{L}}^{CS}=\vdash_{\mathcal{L}}$.

Proposition 14 (1) *$T \vdash_{\mathcal{L}}^{CS} A$ iff either T is inconsistent in $\mathcal{L}$ or $T \vdash_{\mathcal{L}} A$. In particular, T is $\vdash_{\mathcal{L}}^{CS}$-consistent iff it is $\vdash_{\mathcal{L}}$-consistent, and for a $\vdash_{\mathcal{L}}$-consistent T, $T \vdash_{\mathcal{L}}^{CS} A$ iff $T \vdash_{\mathcal{L}} A$.*
(2) *$\mathcal{L}$ is strongly consistent iff $\neg A, A \vdash_{\mathcal{L}} B$ for all A, B (iff T is consistent whenever $T \nvdash A$).*

We next turn our attention to $\vdash_{\mathcal{L}}^{CP}$ and $\vdash_{\mathcal{L}}^{N}$:
Definition. Let $\mathcal{L}$ be a logic and $\vdash_{\mathcal{L}}$ its consequence relation.

1. $\mathcal{L}$ is *strongly (syntactically) complete* if it is strongly complete relative to CP.

2. $\mathcal{L}$ is *weakly (syntactically) complete* if it is weakly complete relative to CP.

3. $\mathcal{L}$ is *strongly normal* if it is strongly complete relative to N.

4. $\mathcal{L}$ is *weakly normal* if it is weakly complete relative to N.

5. $\mathcal{L}$ is *c-normal* if it is c-complete relative to N.

6. $\mathcal{L}$ is *strongly c-normal* if it is strongly c-complete relative to N (this is easily seen to be equivalent to $\vdash_{\mathcal{L}}^{N} = \vdash_{\mathcal{L}}^{CS}$).

For the reader's convenience we review what these definitions actually mean:

Proposition 15 (1) *$\mathcal{L}$ is strongly complete iff whenever $T \nvdash_{\mathcal{L}} A$ there exists a complete extension T^* of T such that $T^* \nvdash_{\mathcal{L}} A$.*
(2) *$\mathcal{L}$ is weakly complete iff whenever A is not a theorem of $\mathcal{L}$ there exists a complete T^* such that $T^* \nvdash_{\mathcal{L}} A$.*
(3) *$\mathcal{L}$ is strongly normal iff whenever $T \nvdash_{\mathcal{L}} A$ there exists a complete and consistent extension T^* of T such that $T^* \nvdash_{\mathcal{L}} A$.*
(4) *$\mathcal{L}$ is weakly normal iff whenever A is not a theorem of $\mathcal{L}$ there exists a complete and consistent theory T^* such that $T^* \nvdash_{\mathcal{L}} A$.*
(5) *$\mathcal{L}$ is c-normal if every consistent theory of $\mathcal{L}$ has a complete and consistent extension.*
(6) *$\mathcal{L}$ is strongly c-normal iff whenever T is consistent and $T \nvdash_{\mathcal{L}} A$ there exists a complete and consistent extension T^* of T such that $T^* \nvdash_{\mathcal{L}} A$.*

Proposition 16 *$If \mathcal{L} is finitary then \mathcal{L}$ is strongly complete iff for all T, A and B:*

$$(*) \quad T, A \vdash_{\mathcal{L}} B \quad \textit{and} \quad T, \neg A \vdash_{\mathcal{L}} B \quad \textit{imply} \quad T \vdash_{\mathcal{L}} B \ .$$

In case $\mathcal{L}$ has a combining disjunction $\vee$ so that $T, A \vee B \vdash_{\mathcal{L}} C$ iff both $T, A \vdash_{\mathcal{L}} C$ and $T, B \vdash_{\mathcal{L}} C$ then $(*)$ is equivalent to the theoremhood of $\neg A \vee A$.

Propositions 14(2), 16 and 9 reveal the following interesting connections between $\vdash^s_{\mathcal{L}}$ of the previous section and some of the semantic notions introduced here:

Proposition 17 *Let $\vdash_{\mathcal{L}}$ be Tarskian.*
(1) *$\vdash_{\mathcal{L}}$ is strongly consistent iff $\vdash^s_{\mathcal{L}}$ is monotonic.*
(2) *If $\vdash^s_{\mathcal{L}}$ is a conservative extension of $\vdash_{\mathcal{L}}$ or if $\vdash_{\mathcal{L}}$ has a combining disjunction then $\vdash_{\mathcal{L}}$ is strongly complete iff $\vdash^s_{\mathcal{L}}$ is closed under contraction.*
(3) *Under the assumption in (2), $\vdash_{\mathcal{L}}$ is strongly normal iff $\vdash^s_{\mathcal{L}}$ is ordinary.*

In Figure 1 we display the obvious relations between the seven properties of logics which we introduce above (where an arrow means "contained in"). In [Av91] it is shown that no arrow can be added to it:

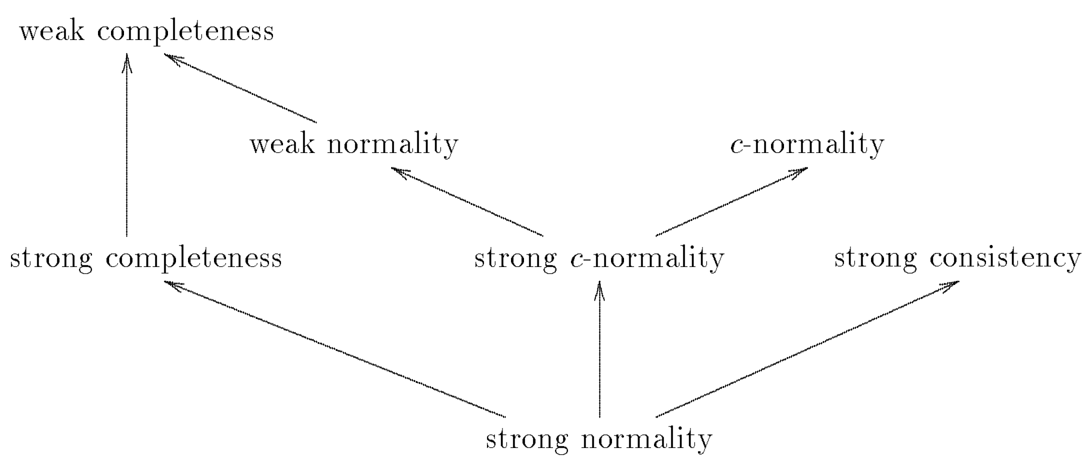


Figure 1

The next theorem summarizes the related properties of the main logics studied in this paper. For proofs we refer the reader to [Av9Γ]. It should be emphasized that for Linear Logic, relevance logics, etc. only the associated *external* CR is considered, since the notion of semantic negation makes sense only for Tarskian CRs.

Theorem 18 *1. Classical logic is strongly normal.*

2. *The intensional ("multiplicative") fragment of the standard relevance logics (like $R_{\simeq}$, $RMI_{\simeq}$, $RM_{\simeq}$) is strongly complete and strongly c-normal, but not strongly consistent.*

3. *The logics R, RMI and RM from the relevance family are strongly complete, c-normal and weakly normal. They are neither strongly c-normal nor strongly consistent. The same properties are shared by the $\{\neg, \vee, \wedge\}$-fragment of the three-valued logic J_3.*

4. *The 3-valued logic J_3 (together with the implication connectives $\supset$ or $\rightarrow$) is strongly complete and c-normal. It is not even weakly complete though (and not strongly consistent).*

5. *Intuitionistic logic and Kleene's 3-valued logic are strongly consistent and c-normal, but not even weakly complete.*

6. *The Logics N and LPF are strongly consistent but lack all the other properties of Figure 1.*

7. *Linear Logic (i.e. $\vdash^{e}_{LL}$ and its various fragments) and N^{-} lack all the properties of Figure 1.*

4 Conclusion

We have seen two different aspects of negation. From our two points of view the major conclusions are:

- The negation of classical logic is a perfect negation from both syntactical and semantic points of view.

- Next come the intensional fragments of the standard relevance logics ($R_{\simeq}, RMI_{\simeq}, RM_{\simeq}$). Their negation is an internal negation for their associated internal CR. Relative to the external one, on the other hand, it has the optimal properties one may expect a semantic negation to have in a paraconsistent logic. In the full systems (R, RMI, RM) the situation is similar, though less perfect (from the semantic point of view).

- The negation of Linear Logic is a perfect internal negation w.r.t. its associated internal CR. It is not, in any sense, a negation from the semantic point of view.

- The negation of intuitionistic logic is not really a negation from either point of view.

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