Non-deterministic Multiple-valued Structures

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Abstract

The ordinary concept of a multiple-valued matrix is generalized by introducing non-deterministic matrices (Nmatrices), in which non-deterministic computations of truth-values are allowed. It is shown that some important logics for reasoning under uncertainty can be characterized by finite Nmatrices (and so they are decidable), although they have only infinite characteristic ordinary (deterministic) matrices. It is further shown that there is no n such that the use of finite Nmatrices for characterizing logics can be reduced to those with less than n truth-values. A generalized compactness theorem that applies to all finite Nmatrices is then proved. Finally, a strong connection is established between the admissibility of the cut rule in canonical Gentzen-type propositional systems, non-triviality of such systems, and the existence of sound and complete non-deterministic two-valued semantics for them. This connection is used for providing a complete solution for the old "Tonk" problem of Prior.

1 Introduction

The classical principle of assigning truth value to formulas is deterministic compositionality: the truth value of a complex formula is uniquely determined by the truth values of its subformulas. An agent acting in the real world often has however only incomplete or imprecise knowledge to guide its decisions. This knowledge may even be inconsistent. When this is the case an alternative approach might be needed. One possible such alternative is to borrow the idea of *non-deterministic* computations from automata and computability theory, and apply it for assigning truth-values to complex formulas. This approach has indeed (implicitly) been used in [Bat98] for handling inconsistent data. This was done, however, in an ad-hoc way. The purpose of this paper is to provide a general framework and theoretical foundations for this approach. Our basic idea is to introduce a natural generalization of the logical concept of a matrix (or multiple-valued structure).

In this generalization the value that a valuation assigns to a complex formula can be chosen non-deterministically from a certain nonempty set of options. We call therefore these structures *non-deterministic matrices* (*Nmatrices*, in short).

In addition to their obvious potential (which admittedly has yet to be demonstrated and exploited) for reasoning under uncertainty and for specification and verification of non-deterministic programs, Nmatrices have considerable practical technical applications. Thus it is well known that every "decent" propositional logic can be characterized semantically using a multi-valued matrix ([LS58]). However, there are important propositional logics which have only infinite characteristic matrices (examples will be given below). Such characteristic matrices are frequently of little help in providing decision procedures for their logics, or in getting real insight into them. Our generalization of the concept of a matrix allows us to replace in many cases an infinite characteristic matrix for a given propositional logic by a characteristic *finite* structure that automatically provides a decision procedure. Another application given below is in proof theory: we use Nmatrices for proving a very general cut-elimination theorem for propositional Gentzen-type systems.

The structure of the rest of this paper is as follows: In section 2 we review some basic concepts related to logics and matrices. In section 3 we introduce Nmatrices and their associated logics. We then present two examples of the use of our framework for handling negated formulas in cases where classical logic is impractical (especially in the face of possible contradictions). Both examples are taken from the literature, and they have corresponding cut-free Gentzen-type calculi. We show that both have no finite characteristic (deterministic) matrix, but one has a two-valued characteristic Nmatrix, while the other a three-valued characteristic Nmatrix (but not a two-valued one). We show also that in general, no logic which has a characteristic two-valued Nmatrix with at least one proper non-deterministic operation can have a finite characteristic matrix. Then we show that for every n there are logics which have characteristic n-valued Nmatrices, but no characteristic Nmatrix with fewer than n elements. Finally we prove in that section a general compactness theorem that applies to every logic which has a finite characteristic Nmatrix. Section 4 is devoted to two-valued Nmatrices and their applications. We establish there a strong connection between canonical Gentzen-type systems and such Nmatrices. By canonical propositional Gentzen-type systems we mean systems which have (in addition to the standard axioms and structural rules) only pure logical rules which have the subformula property, introduce exactly one occurrence of a connective in their conclusion, and no other occurrence of any connective is mentioned anywhere else in their formulation. We show that such a system is not trivial iff it has a characteristic two-valued Nmatrix, and that in such a case it admits cut elimination. We show also how the two-valued nondeterministic semantics can be used to transform such a system into one which has for each connective at most one introduction rule for each side. As a side effect we get a complete characterization of the cases in which a given set of canonical Gentzen-type introduction rules for a connective define the semantics of that connective (providing by this a complete solution for the old "Tonk" problem of Prior, raised in [Pri60]). We conclude the paper in the last section with remarks on possible applications and directions for further research.

2 Preliminaries

In what follows $\mathcal{L}$ is a *countable* [1] propositional language, $\mathcal{W}$ is its set of wffs, p, q, r denote atomic formulas, ψ, φ, ϕ denote arbitrary formulas (of $\mathcal{L}$), and Γ, Δ denote sets of formulas. $Fv(X)$ denotes the set of atomic formulas occurring in X.

Definition 2.1

1. A *Tarskian consequence relation* (*tcr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ and formulas of $\mathcal{L}$ that satisfies the following conditions:

s-R	*strong reflexivity*:	if $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$.
M	*monotonicity*:	if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \varphi$.
C	*Transitivity (cut)*:	if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$.

2. [Sco74a, Sco74b] A *Scott consequence relation* (*scr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ that satisfies the following conditions:

s-R	*strong reflexivity*:	if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$.
M	*monotonicity*:	if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$.
C	*Transitivity (cut)*:	if $\Gamma \vdash \psi, \Delta$ and $\Gamma', \psi \vdash \Delta'$ then $\Gamma, \Gamma' \vdash \Delta, \Delta'$.

[1] This requirement is not essential, but it is convenient.

Conventions: Let $\vdash$ be a tcr. Below $\Gamma \vdash \{\varphi\}$ means that $\Gamma \vdash \varphi$, and $\Gamma \vdash$ (or $\Gamma \vdash \emptyset$) means that $\Gamma \vdash \varphi$ for every $\varphi \in \mathcal{W}$. Provided that the cardinality of Δ is at most 1, these two conventions will allow us to use statements of the form $\Gamma \vdash \Delta$ even in case $\vdash$ is a tcr.

Definition 2.2 An scr or tcr $\vdash$ for $\mathcal{L}$ is *structural* (or substitution-invariant) if for every uniform $\mathcal{L}$-substitution σ and every Γ and Δ, if $\Gamma \vdash \Delta$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$. $\vdash$ is *finitary* if the following condition holds for all $\Gamma, \Delta \subseteq \mathcal{W}$: if $\Gamma \vdash \Delta$ then there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash \Delta'$. $\vdash$ is *consistent* (or *non-trivial*) if there exist non-empty Γ and Δ s.t. $\Gamma \not\vdash \Delta$. $\vdash$ is *strongly uniform* if it satisfies the following condition: if $\bigcup_{i \in I} \Gamma_i \vdash \bigcup_{i \in I} \Delta_i$ and $Fv(\Gamma_i \cup \Delta_i) \cap Fv(\Gamma_j \cup \Delta_j) = \emptyset$ for $i \neq j$, then there exists i_0 such that $\Gamma_{i_0} \vdash \Delta_{i_0}$.

Proposition 2.3 *There are exactly four inconsistent finitary scrs and two finitary tcrs in any given language.*

Proof: We prove the case of scrs. For this observe first that any inconsistent finitary scr $\vdash$ has the following two properties:

1. If $\Gamma_0 \vdash$ for some Γ_0 then $\Gamma \vdash$ for every nonempty Γ.
2. If $\vdash \Delta_0$ for some Δ_0 then $\vdash \Delta$ for every nonempty Δ.

We show the first property as an example. By Monotonicity, it trivially holds if Γ_0 is empty. Assume therefore that it is not. Then there exist $\psi_1, \ldots, \psi_n \in \Gamma_0$ such that $\psi_1, \ldots, \psi_n \vdash$ (because $\vdash$ is finitary). Since $\vdash$ is inconsistent, $\Gamma \vdash \psi_i$ for every nonempty Γ. By applying the cut rule n times we get therefore that $\Gamma \vdash$ for every nonempty Γ.

From these two properties it immediately follows that either $\Gamma \vdash \Delta$ iff both Γ and Δ are non-empty, or $\Gamma \vdash \Delta$ iff Γ is non-empty, or $\Gamma \vdash \Delta$ iff Δ is non-empty, or $\Gamma \vdash \Delta$ for all Γ and Δ. All these possibilities indeed define inconsistent finitary scrs, which are different from each other. ∎

Note: The four inconsistent finitary scrs and the two inconsistent finitary tcrs should be considered trivial. We exclude them therefore from our definition of a *logic*:

Definition 2.4 A (Tarskian) propositional *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language and $\vdash$ is an scr (a tcr) for $\mathcal{L}$ which is structural and consistent. $\langle \mathcal{L}, \vdash \rangle$ is finitary if $\vdash$ is finitary, and uniform if $\vdash$ is uniform.

The most standard and general method for defining (Tarskian) propositional logics is by using a many-valued *matrices*:

Definition 2.5 [2] A *matrix* for $\mathcal{L}$ is a tuple $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{T}$ is a non-empty set of *truth values*, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{T}$ (its *designated values*),[3] and for every n-ary ($n \geq 0$) connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{\diamond}$ from $\mathcal{T}^n$ to $\mathcal{T}$. A *valuation* in $\mathcal{S}$ is a function $v : \mathcal{W} \to \mathcal{T}$ that satisfies the condition: if $\diamond$ is an n-ary connective, and $\psi_1, \ldots, \psi_n \in \mathcal{W}$, then $v(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$. v *satisfies* a formula ψ in $\mathcal{S}$ ($v \models^{\mathcal{S}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model of* Γ in $\mathcal{S}$ ($v \models^{\mathcal{S}} \Gamma$) if it satisfies every formula in Γ. φ *follows* from Γ in $\mathcal{S}$ (notation: $\Gamma \vdash^1_{\mathcal{S}} \varphi$) if $v \models^{\mathcal{S}} \varphi$ for every model v of Γ in $\mathcal{S}$. Δ *follows* from Γ in $\mathcal{S}$ (notation: $\Gamma \vdash_{\mathcal{S}} \Delta$) if for every model v of Γ in $\mathcal{S}$, $v \models^{\mathcal{S}} \varphi$ for some $\varphi \in \Delta$ ($\vdash_{\mathcal{S}}$ and $\vdash^1_{\mathcal{S}}$ are respectively the scr and tcr induced by $\mathcal{S}$). We say that $\mathcal{S}$ is *finite* if so is $\mathcal{T}$. If $\vdash$ is an scr (tcr) then $\mathcal{S}$ is called a *characteristic matrix* for $\vdash$ if $\vdash = \vdash_{\mathcal{S}}$ ($\vdash = \vdash^1_{\mathcal{S}}$). $\mathcal{S}$ is a *weakly-characteristic matrix* for $\vdash$ if for all ψ, $\vdash \psi$ iff $\vdash_{\mathcal{S}} \psi$.

Theorem 2.6 *Let $\mathcal{S}$ be a matrix for a language $\mathcal{L}$. Then $\langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is a uniform propositional logic, and $\langle \mathcal{L}, \vdash^1_{\mathcal{S}} \rangle$ is a uniform Tarskian propositional logic.*

Proof: That $\vdash^1_{\mathcal{S}}$ and $\vdash_{\mathcal{S}}$ are consistent follows from our demand that the set $\mathcal{D}$ should be both non-empty and a proper subset of $\mathcal{T}$. The proofs of the other conditions are easy. ■

A converse of Theorem 2.6 in the Tarskian case also holds:

Theorem 2.7 [LS58, Wój88, Urq01] [4] *Every uniform Tarskian propositional logic has a characteristic matrix.*

3 General Non-deterministic Matrices

3.1 The Concept of a Non-deterministic Matrix

We present now our generalization of the concept of a matrix [5].

[2]See e.g. [Urq01].

[3]Hence we can always assume that $\mathcal{T}$ includes the two classical values t and f, where $t \in \mathcal{D}$ and $f \notin \mathcal{D}$.

[4]This is Wójcicki's generalization of the famous theorem of Łos and Suszko. We are grateful to Josep Maria Font for clarifying the whole subject for us, and for other useful comments.

[5]It has been pointed out to us by João Marcos that a particular case of this idea, using a similar name, has been used in [CE98]. We take here the opportunity to thank him for bringing this to our attention.

Definition 3.1

1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$, where:

   (a) $\mathcal{T}$ is a non-empty set of *truth values.*

   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{T}$.

   (c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{\diamond}$ from $\mathcal{T}^n$ to $2^{\mathcal{T}} - \{\emptyset\}$.

   We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{T}$.

2. A *valuation* in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{W} \to \mathcal{T}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \dots, \psi_n \in \mathcal{W}$:

$$v(\diamond(\psi_1, \dots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \dots, v(\psi_n))$$

3. A valuation v in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model* in $\mathcal{M}$ of a set Γ of formulas (notation: $v \models^{\mathcal{M}} \Gamma$) if it satisfies every formula in Γ.

4. $\vdash^1_{\mathcal{M}}$ and $\vdash_{\mathcal{M}}$, the tcr and scr induced by the Nmatrix $\mathcal{M}$, are defined as follows:

   $\Gamma \vdash^1_{\mathcal{M}} \varphi$ if $v \models^{\mathcal{M}} \varphi$ for every v such that $v \models^{\mathcal{M}} \Gamma$

   $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every v such that $v \models^{\mathcal{M}} \Gamma$ there exists $\varphi \in \Delta$ such that $v \models^{\mathcal{M}} \varphi$

5. An Nmatrix $\mathcal{M}$ is *characteristic* for a tcr (scr) $\vdash$ if $\vdash = \vdash^1_{\mathcal{M}}$ ($\vdash = \vdash_{\mathcal{M}}$). $\mathcal{M}$ is a *weakly-characteristic* for $\vdash$ if for all ψ, $\vdash \{\psi\}$ iff $\vdash^1_{\mathcal{M}} \{\psi\}$. [6]

6. Two Nmatrices $\mathcal{M}_1$ and $\mathcal{M}_2$ for $\mathcal{L}$ are *equivalent* if $\vdash^1_{\mathcal{M}_1} = \vdash^1_{\mathcal{M}_2}$.

Note: We shall identify an ordinary (deterministic) matrix $\langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ with the Nmatrix $\langle \mathcal{T}, \mathcal{D}, \mathcal{O}' \rangle$, where $\mathcal{O}' = \{\lambda \bar{x} \in \mathcal{T}^{n(\diamond)}\{\tilde{\diamond}(\bar{x})\} \mid \tilde{\diamond} \in \mathcal{O}\}$. Obviously, The definitions of $\vdash_{\mathcal{M}}$ and $\models^{\mathcal{M}}$ for an arbitrary Nmatrix $\mathcal{M}$ are generalizations of this particular case.

[6]Note that $\vdash_{\mathcal{M}} \{\psi\}$ iff $\vdash^1_{\mathcal{M}} \{\psi\}$.

Proposition 3.2 *If $\mathcal{M}$ is an Nmatrix for $\mathcal{L}$ then $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a uniform logic, and $\langle \mathcal{L}, \vdash^1_{\mathcal{M}} \rangle$ is a uniform Tarskian logic.*

Proof: We show the case of $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$. It is easy to show strong reflexivity, monotonicity, transitivity, and uniformity. For structurality, note that for every substitution σ and every valuation v in $\mathcal{M}$, $v' = v \circ \sigma$ is also a valuation in $\mathcal{M}$: if $\diamond$ is an n-ary connective, and $\widetilde{\diamond} \in \mathcal{O}$ is its interpretation in $\mathcal{M}$, then $v'(\diamond(\psi_1, \ldots, \psi_n)) = v(\sigma(\diamond(\psi_1, \ldots, \psi_n))) = v(\diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))) \in \widetilde{\diamond}(v(\sigma(\psi_1)), \ldots, v(\sigma(\psi_n))) = \widetilde{\diamond}(v'(\psi_1), \ldots, v'(\psi_n))$. Now suppose $\Gamma \vdash_{\mathcal{M}} \Delta$. If v is a model of $\sigma(\Gamma)$ then $v' = v \circ \sigma$ is a valuation in $\mathcal{M}$, and for all ψ, $v' \models^{\mathcal{M}} \psi$ iff $v \models^{\mathcal{M}} \sigma(\psi)$. Therefore $v' \models^{\mathcal{M}} \Gamma$. Hence $v' \models^{\mathcal{M}} \phi$ for some $\phi \in \Delta$, and so $v \models^{\mathcal{M}} \sigma(\phi)$. Thus $\sigma(\Gamma) \vdash_{\mathcal{M}} \sigma(\Delta)$. ∎

Corollary 3.3 *Every Nmatrix has an equivalent matrix.*

Proof: This follows from Proposition 3.2 and Theorem 2.7. ∎

Note: Although every Nmatrix has an equivalent matrix, there are many finite Nmatrices for which there are no equivalent *finite* matrices. A particularly important class of examples for this phenomenon is provided by the following Theorem:

Theorem 3.4 *Let $\mathcal{M}$ be a two-valued Nmatrix which has at least one proper nondeterministic operation. Then $\mathcal{M}$ has no equivalent finite characteristic matrix. If in addition $\mathcal{M}$ includes the classical implication, then $\vdash_{\mathcal{M}}$ has no finite weakly-characteristic matrix.*

Proof: Let $\mathcal{M}$ be a two-valued proper Nmatrix. Then there is some n-ary connective $\diamond$ and some $x_1, \ldots, x_n \in \{t, f\}$ such that $\widetilde{\diamond}(x_1, \ldots, x_n) = \{t, f\}$.

Assume first that there is such a tuple for which $x_i = t$ for some i. We may assume without a loss in generality that $x_n = t$. Let $p_1, \ldots, p_n$ be some atomic formulas and define $\psi_0 = p_n$ and $\psi_{i+1} = \diamond(p_1, \ldots, p_{n-1}, \psi_i)$ for all $i \geq 0$. Let $\mathcal{S}$ be some m-valued matrix for the same language. Then for any valuation v in $\mathcal{S}$, $v(\psi_m) = v(\psi_k)$ for some $k < m$. Indeed, if v assigns to all the m formulas $\psi_0, \ldots, \psi_{m-1}$ different truth values then ψ_m is also assigned one of these values (since there are only m truth values). Otherwise, there are $0 \leq i < j < m$ such that $v(\psi_i) = v(\psi_j)$. Since $\mathcal{S}$ is a matrix, $v(\psi_m) = v(\psi_{m-j+i})$. It follows that $\psi_0, \ldots, \psi_{m-1} \vdash_{\mathcal{S}} \psi_m$. Consider now the $\mathcal{M}$-valuation v such that $v(p_i) = x_i$

for all $1 \leq i \leq n$, $v(\psi_i) = t$ for all $0 \leq i \leq m-1$ and $v(\psi_m) = f$ (this is a legal valuation, since $\tilde{\diamond}(x_1, \ldots, x_n) = \{t, f\}$, and $x_n = t$). This valuation shows that $\psi_0, \ldots, \psi_{m-1} \not\vdash_{\mathcal{M}} \psi_m$, and so $\mathcal{M}$ and $\mathcal{S}$ are not equivalent.

Assume now that $\langle f, \ldots, f \rangle$ is the only tuple for which $\tilde{\diamond}(x_1, \ldots, x_n) = \{t, f\}$. We may assume that $n = 1$ (otherwise define $\diamond'(\varphi)$ to be $\diamond(\varphi, \ldots, \varphi)$, and use $\diamond'$). So $\tilde{\diamond}(f) = \{t, f\}$, and either $\tilde{\diamond}(t) = \{f\}$ or $\tilde{\diamond}(t) = \{t\}$. To shorten the presentation of the proof, we assume also that $\diamond$ is the only connective of the language.

Case 1: $\tilde{\diamond}(t) = \{f\}$. Then $p, \diamond p \vdash^1_{\mathcal{M}} q$ (since $\{p, \diamond p\}$ has no model in $\mathcal{M}$). It follows that if $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ is a finite matrix such that $\vdash^1_{\mathcal{M}} \subseteq \vdash^1_{\mathcal{S}}$ then $\tilde{\diamond}(a) \notin \mathcal{D}$ if $a \in \mathcal{D}$. Hence if $a \in \mathcal{T}$ then $\tilde{\diamond}^k(a) \notin \mathcal{D}$ for arbitrary large k. It follows that if $\mathcal{T} - \mathcal{D} = \{f_1, \ldots, f_l\}$ then there exist positive integers $n_1, n_2, \ldots, n_l$ such that $n_i - n_{i+1} \geq 2$ and $\tilde{\diamond}^{n_i}(f_i) \notin \mathcal{D}$ for all $1 \leq i \leq l$ (where we let $n_{l+1} = 0$). Hence $\diamond^{n_1}(p), \ldots, \diamond^{n_l}(p) \vdash^1_{\mathcal{S}} p$. It is easy however to see that by defining $v(\varphi) = t$ iff $\varphi \in \{\diamond^{n_1}(p), \ldots, \diamond^{n_l}(p)\}$ we get a legal valuation in $\mathcal{M}$ which is a model of $\{\diamond^{n_1}(p), \ldots, \diamond^{n_l}(p)\}$ but not a model of p. Hence $\vdash^1_{\mathcal{M}} \neq \vdash^1_{\mathcal{S}}$.

Case 2: $\tilde{\diamond}(t) = \{t\}$. Then $p \vdash^1_{\mathcal{M}} \diamond p$. Hence if $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ is a finite matrix such that $\vdash^1_{\mathcal{M}} \subseteq \vdash^1_{\mathcal{S}}$ then $\tilde{\diamond}(a) \in \mathcal{D}$ if $a \in \mathcal{D}$. It follows that for every $a \in \mathcal{T} - \mathcal{D}$, either $\diamond^n(a) \notin \mathcal{D}$ for all $n \geq 0$, or there is $n_a \geq 1$ such that $\diamond^n(a) \in \mathcal{D}$ for all $n \geq n_a$. Let k be the maximum of the n_a's for the a's of the second type (and 0 if no such a exists). Then $\diamond^{k+1}(p) \vdash^1_{\mathcal{S}} \diamond^k(p)$. It is easy however to see that by defining $v(p) = v(\diamond p) = \ldots = v(\diamond^k(p)) = f$ and $v(\diamond^{k+1}(p)) = t$ we get a legal valuation in $\mathcal{M}$ which is a model of $\diamond^{k+1}(p)$ but not of $\diamond^k(p)$. Hence $\vdash^1_{\mathcal{M}} \neq \vdash^1_{\mathcal{S}}$.

Assume next that $\mathcal{M}$ is a two-valued proper Nmatrix which includes the classical implication. Then $\vdash_{\mathcal{M}} \psi$ for every instance of an implicational classical tautology, and there is some n-ary connective $\diamond$ and $x_1, \ldots, x_n \in \{t, f\}$ such that $\tilde{\diamond}(x_1, \ldots, x_n) = \{t, f\}$. Let $p_1, \ldots, p_n$ be some atomic formulas and define again $\psi_0 = p_n$ and $\psi_{i+1} = \diamond(p_1, \ldots, p_{n-1}, \psi_i)$ for all $i \geq 0$. Again if $\mathcal{S}$ is an m-valued matrix for the same language, then for any valuation v in $\mathcal{S}$, $v(\psi_m) = v(\psi_k)$ for some $k < m$. Hence for every $\mathcal{S}$-valuation v there is some $k < m$ such that v assigns the same truth-value to the formulas $\psi_0 \supset (\psi_1 \supset \ldots (\psi_{m-1} \supset \psi_m) \ldots)$ and $\psi_0 \supset (\psi_1 \supset \ldots (\psi_{m-1} \supset \psi_k) \ldots)$. The latter is an instance of a purely implicational classical tautology, and so v assigns it the value t. Hence the former is valid in $\mathcal{S}$. Similarly, $\psi_m \supset \psi_{m-1} \vee \ldots \vee \psi_0$ is valid in $\mathcal{S}$ (we use here the fact that disjunction is definable in classical logic from implication, since $\varphi \vee \psi$ is equivalent there to $(\varphi \supset \psi) \supset \psi$). It is easy, however, to refute the first in $\mathcal{M}$ if $x_n = t$ by employing the same valuation used above. The second, on the other hand, is refutable in case $x_n = f$ by any $\mathcal{M}$-valuation v such that $v(p_i) = x_i$ for all $1 \leq i \leq n$, $v(\psi_i) = f$ for all $0 \leq i \leq m-1$ and $v(\psi_m) = t$. ■

Note: Theorem 3.4 is not valid in general for Nmatrices with more than two elements. For example: let $\mathcal{M}$ be an Nmatrix such that $\tilde{\diamond}(x_1, \ldots, x_n) \in \{\mathcal{D}, \mathcal{T} - \mathcal{D}\}$ for any connective $\diamond$ of its language, and for every $x_1, \ldots, x_n \in \mathcal{T}$. Then obviously $\vdash_{\mathcal{M}}$ (and so $\vdash^1_{\mathcal{M}}$) has a characteristic two-valued (deterministic) matrix.

3.2 Examples

Let $\mathcal{L}_{\text{cl}} = \{\wedge, \vee, \supset, \neg\}$. Let **HCL**$^+$ be some standard Hilbert-type system which is sound and strongly complete for the positive fragment of classical propositional logic, and has MP as the only rule of inference. Let **GCL**$^+$ be some standard Gentzen-type system for that fragment. We present first two extensions of **HCL**$^+$ and **GCL**$^+$ which have finite characteristic Nmatrices, but no finite characteristic (deterministic) matrices.

3.2.1 Batens' Logic CLuN

The logic **CLuN** from [BCK99] is the minimal logic in $\mathcal{L}_{\text{cl}}$ that contains the positive fragment of classical logic and in which $\emptyset \vdash \psi, \neg\psi$ for all ψ. Obviously, a corresponding Gentzen-type system **GCLuN** is obtained from **GCL**$^+$ by adding to it the rule $[\Rightarrow \neg]$ of classical logic (from $\Gamma, \varphi \Rightarrow \Delta$ infer $\Gamma \Rightarrow \Delta, \neg\varphi$). It is also easy to see that a corresponding Hilbert-type system **HCLuN** for it is obtained from **HCL**$^+$ by adding to it the law of excluded middle, $\psi \vee \neg\psi$, as an axiom schema.

We show now that **CLuN** has a simple characteristic two-valued Nmatrix [7].

Definition 3.5 The Nmatrix $\mathcal{M}_r$ for $\mathcal{L}_{\text{cl}}$ has $\{t, f\}$ as the set of truth values and t as the designated value. The interpretation of the connectives is given by:

- If $\diamond \in \{\wedge, \vee, \supset\}$ then $\tilde{\diamond}(x_1, x_2) = \{g_\diamond(x_1, x_2)\}$ (where $g_\diamond$ denotes the corresponding classical boolean operation).

- $\tilde{\neg} f = \{t\}$, $\tilde{\neg} t = \{t, f\}$

[7] This Nmatrix was implicitly introduced, and the corresponding soundness and completeness theorem (Proposition 3.6 below) proved, in [BCK99].

Proposition 3.6 $\vdash_{CLuN} = \vdash_{\mathcal{M}_r}$

Proof: we show that $\vdash_{\mathbf{GCLuN}} = \vdash_{\mathcal{M}_r}$. The soundness part is straightforward. We prove completeness (together with cut-elimination) by showing that if $\Gamma \Rightarrow \Delta$ does not have a cut-free proof in **GCLuN** then $\Gamma \not\vdash_{\mathcal{M}_r} \Delta$. For this extend first $\Gamma \Rightarrow \Delta$ to a sequent $\Gamma^* \Rightarrow \Delta^*$ with the following properties:

1. $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$.
2. $\Gamma^* \Rightarrow \Delta^*$ does not have a cut-free proof in **GCLuN**.
3. If $\psi \supset \phi \in \Gamma^*$ then $\psi \in \Delta^*$ or $\phi \in \Gamma^*$. If $\psi \supset \phi \in \Delta^*$ then $\psi \in \Gamma^*$ and $\phi \in \Delta^*$. Similar conditions obtain for $\psi \wedge \phi$ and $\psi \vee \phi$.
4. If $\neg\psi \in \Delta^*$ then $\psi \in \Gamma^*$.

Now define a valuation $v : \mathcal{W}_{\mathrm{cl}} \to \{t, f\}$ as follows:

- For atomic p, $v(p) = t$ iff $p \in \Gamma^*$.
- $v(\psi_1 \diamond \psi_2) = g_\diamond(v(\psi_1), v(\psi_2))$ in case $\diamond \in \{\wedge, \vee, \supset\}$.
- $v(\neg\psi) = t$ iff $v(\psi) = f$ or $\neg\psi \in \Gamma^*$.

v is obviously a legal valuation in $\mathcal{M}_r$. We now show by induction on the complexity of formulas in $\Gamma^* \cup \Delta^*$ that if $\phi \in \Gamma^*$ then $v(\phi) = t$, and if $\phi \in \Delta^*$ then $v(\phi) = f$.

- Assume ϕ is atomic. If $\phi \in \Gamma^*$ then $v(\phi) = t$ by definition. If $\phi \in \Delta^*$ then $\phi \notin \Gamma^*$ by property 2 of $\Gamma^* \Rightarrow \Delta^*$. Hence $v(\phi) = f$.
- Assume $\phi = \psi_1 \diamond \psi_2$ for $\diamond \in \{\wedge, \vee, \supset\}$. Then the claim easily follows from property 3 of $\Gamma^* \Rightarrow \Delta^*$, and the induction hypothesis for ψ_1 and ψ_2.
- Assume $\phi = \neg\psi$. If $\phi \in \Gamma^*$ then $v(\phi) = v(\neg\psi) = t$ by definition of v. If $\phi \in \Delta^*$ then $\psi \in \Gamma^*$ by property 4 of $\Gamma^* \Rightarrow \Delta^*$. Hence $v(\psi) = t$ by the induction hypothesis. In addition, property 2 of $\Gamma^* \Rightarrow \Delta^*$ and the fact that $\neg\psi \in \Delta^*$ together imply that $\neg\psi \notin \Gamma^*$. It follows by definition of v that $v(\phi) = v(\neg\psi) = f$ in this case.

By property 1 of $\Gamma^* \Rightarrow \Delta^*$, v is a countermodel of $\Gamma \Rightarrow \Delta$. Hence $\Gamma \not\vdash_{\mathcal{M}_r} \Delta$. ∎

Note: By theorem 3.4, **CLuN** has no finite weakly characteristic (ordinary) matrix.

3.2.2 The Paraconsistent System C_{min}

The paraconsistent system C_{min} is used in [CM02] (see there for origins and extensive list of references) as the basis for their "Taxonomy of C-systems". A Hilbert-type axiomatization of this system is obtained from **HCLuN** by adding to it the law of double negation elimination, $\neg\neg\varphi \supset \varphi$, as an axiom schema. We show now that this system has a simple characteristic *three*-valued Nmatrix.

Definition 3.7 The Nmatrix $\mathcal{M}_c$ for $\mathcal{L}_{\text{cl}}$ have $\{t, f, \top\}$ as the set of truth values and $\mathcal{D} = \{t, \top\}$ as the set of designated values. The interpretations of the connectives are:

- $\tilde{\supset}(x_1, x_2) = \begin{cases} \mathcal{D} & \text{if } x_1 = f \text{ or } x_2 \in \mathcal{D} \\ \{f\} & \text{otherwise} \end{cases}$
- $\tilde{\vee}(x_1, x_2) = \begin{cases} \mathcal{D} & \text{if } x_1 \in \mathcal{D} \text{ or } x_2 \in \mathcal{D} \\ \{f\} & \text{otherwise} \end{cases}$
- $\tilde{\wedge}(x_1, x_2) = \begin{cases} \mathcal{D} & \text{if } x_1 \in \mathcal{D} \text{ and } x_2 \in \mathcal{D} \\ \{f\} & \text{otherwise} \end{cases}$
- $\tilde{\neg} x = \begin{cases} \{f\} & x = t \\ \{t\} & x = f \\ \mathcal{D} & x = \top \end{cases}$

Theorem 3.8 $\vdash_{C_{min}} = \vdash^1_{\mathcal{M}_c}$ [8]

Proof: The soundness of C_{min} with respect to $\mathcal{M}_c$ is easy. We prove completeness. So let $\mathbf{T}$ be a theory and φ_0 a sentence such that $\mathbf{T} \nvdash_{C_{min}} \varphi_0$. We construct a model of $\mathbf{T}$ in $\mathcal{M}_c$ which is not a model of φ_0. For this extend $\mathbf{T}$ to a maximal theory $\mathbf{T}^*$ such that $\mathbf{T}^* \nvdash_{C_{min}} \varphi_0$. $\mathbf{T}^*$ has the following properties:

1. $\psi \notin \mathbf{T}^*$ iff $\psi \supset \varphi_0 \in \mathbf{T}^*$.
2. If $\psi \notin \mathbf{T}^*$ then $\psi \supset \varphi \in \mathbf{T}^*$ for every sentence φ of $\mathcal{L}_{\text{cl}}$.
3. $\varphi \vee \psi \in \mathbf{T}^*$ iff either $\varphi \in \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.
4. $\varphi \wedge \psi \in \mathbf{T}^*$ iff both $\varphi \in \mathbf{T}^*$ and $\psi \in \mathbf{T}^*$.
5. $\varphi \supset \psi \in \mathbf{T}^*$ iff either $\varphi \notin \mathbf{T}^*$ or $\psi \in \mathbf{T}^*$.

[8] A completely different 3-valued Nmatrix was implicitly shown to be characteristic for C_{min} in [CM99].

6. For every sentence φ of $\mathcal{L}_{\rm cl}$ either $\varphi \in \mathbf{T}^*$ or $\neg\varphi \in \mathbf{T}^*$.

7. If $\neg\neg\varphi \in \mathbf{T}^*$ then $\varphi \in \mathbf{T}^*$.

The proofs of properties 1–5 are exactly as in the case of $\mathbf{HCL}^+$: Property 1 follows from the deduction theorem (which is obviously valid for C_{min}) and the maximality of $\mathbf{T}^*$. Property 2 is proved first for φ_0 using 1 and the tautology $((\varphi_0 \supset \varphi) \supset \varphi_0) \supset \varphi_0$. It then follows for all $\psi \notin \mathbf{T}^*$ by 1. Properties 3–5 are easy corollaries of 1, 2, and the maximality of $\mathbf{T}^*$. Finally property 6 is immediate from property 3 and the axiom of excluded middle, while property 7 follows from the axiom of double negation.

Define now a valuation v in $\mathcal{M}_c$ as follows:

$$v(\psi) = \begin{cases} f & \psi \notin \mathbf{T}^* \\ t & \psi \in \mathbf{T}^*, \neg\psi \notin \mathbf{T}^* \\ \top & \psi \in \mathbf{T}^*, \neg\psi \in \mathbf{T}^* \end{cases}$$

Note that $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$. We use this to prove that v is a legal valuation, i.e.: v respects the interpretations of the connectives in $\mathcal{M}_c$. That this is the case for the positive connectives easily follows from properties 3–5 of $\mathbf{T}^*$. We prove next the case of $\neg$:

- Assume $v(\psi) = f$. Then $\psi \notin \mathbf{T}^*$. Hence $\neg\psi \in \mathbf{T}^*$ and $\neg\neg\psi \notin \mathbf{T}^*$ by properties 6 and 7 of $\mathbf{T}^*$. Hence $v(\neg\psi) = t$.

- Assume $v(\psi) = t$. By definition, this implies that $\neg\psi \notin \mathbf{T}^*$. Hence $v(\neg\psi) = f$.

- Assume $v(\psi) = \top$. By definition, this implies that $\neg\psi \in \mathbf{T}^*$. Hence $v(\neg\psi) \in \mathcal{D}$.

Since $v(\psi) \in \mathcal{D}$ iff $\psi \in \mathbf{T}^*$, $v(\psi) \in \mathcal{D}$ for every $\psi \in \mathbf{T}$, while $v(\varphi_0) \notin \mathcal{D}$. Hence v is a model of $\mathbf{T}$ which is not a model of $v(\varphi_0)$. ∎

Proposition 3.9 *C_{min} has no finite weakly-characteristic matrix.*

Proof: Let $\mathcal{M}$ be some m-valued matrix for $\mathcal{L}_{\rm cl}$ in which all classical positive tautologies are valid. Since $\mathcal{M}$ is deterministic, $v(\neg^m p) \in \{v(p), v(\neg p), \ldots, v(\neg^{m-1} p)\}$ for every valuation v in $\mathcal{M}$ (see the proof of theorem 3.4). Like in that proof of theorem 3.4, this entails that $\psi = p \wedge \neg p \wedge \ldots \wedge \neg^{m-1} p \supset \neg^m p$ is valid in $\mathcal{M}$. However, ψ can be refuted in $\mathcal{M}_c$ by assigning $\top$ to $p, \neg p, \ldots, \neg^{m-2} p$, t to $\neg^{m-1} p$ and f to $\neg^m p$. Hence $\not\vdash_{C_{min}} \psi$. ∎

Proposition 3.10 *C_{min} has no weakly characteristic two-valued Nmatrix. Moreover: the classical two-valued matrix is the only Nmatrix for which C_{min} is weakly sound.*

Proof: Let $\mathcal{M}$ be a two-valued Nmatrix for which C_{min} is weakly sound. It is easy to see (see the proof of Proposition 3.11 below) that the interpretations of the positive connectives in $\mathcal{M}$ should be identical to the classical ones. The validity of the law of excluded middle forces therefore the condition $\tilde{\neg} f = \{t\}$. This leave us with just three possible candidates, and among them the law of double negation is valid only in the classical matrix. ■

3.2.3 Using Arbitrary Number of Values

From Proposition 3.10 it follows that the use of finite Nmatrices cannot be reduced to the two-valued case. We show now that two-valued Nmatrices are not even sufficient for weakly-characterizing all propositional logics which have a characteristic finite *matrix.*

Proposition 3.11 *Łukasiewicz's three-valued logic $\vdash_l^3$ does not have a weakly characteristic two-valued Nmatrix.*

Proof: Suppose that $\mathcal{M}$ is a two-valued Nmatrix such that $\vdash_{\mathcal{M}} \psi$ iff $\vdash_l^3 \psi$. Without a loss in generality, we may assume that the two truth values are t and f, where only t is designated. Let $\tilde{\supset}$ be the interpretations of $\supset$ in $\mathcal{M}$. Since $\vdash_l^3 p \supset p$, $v(p \supset p) = t$ for all v in $\mathcal{M}$. Hence $\tilde{\supset}(t,t) = \tilde{\supset}(f,f) = \{t\}$. Since $\vdash_l^3 p \supset (p \supset p)$ and $\tilde{\supset}(f,f) = \{t\}$, $\tilde{\supset}(f,t) = \{t\}$. Now $\tilde{\supset}(t,f) \neq \{t\}$, or else $\tilde{\supset}(x,y) = \{t\}$ for all $x, y \in \{t,f\}$, and so $p \supset q$ would be $\vdash_l^3$-valid, which is not the case. $\tilde{\supset}(t,f)$ is also not $\{t,f\}$, or else $p \supset ((p \supset q) \supset q)$ (which is $\vdash_l^3$-valid) would not be valid (a countermodel: $v(p) = t$, $v(q) = f$, $v(p \supset q) = t$, $v((p \supset q) \supset q) = f$, $v(p \supset ((p \supset q) \supset q)) = f$). Hence $\tilde{\supset}(t,f) = \{f\}$. It follows that $\tilde{\supset}$ is the classical implication, and so $\vdash_{\mathcal{M}} \neq \vdash_l^3$. ■

We next show that in general there is no n such that the use of finite Nmatrices can be limited to those with less than n truth-values.

Definition 3.12 For all $n \geq 2$, $\vdash_r^n$ ($\vdash_l^n$) is the minimal logic in $\mathcal{L}_{\text{cl}}$ that contains the classical positive logic and in which $\emptyset \vdash_r^n \psi, \neg\psi, \ldots, \neg^{n-1}\psi$ ($\psi, \neg\psi, \ldots, \neg^{n-1}\psi \vdash_l^n \emptyset$). [9]

[9] Note that $CluN$ is just $\vdash_r^2$.

It is easy to see that a Hilbert-type calculus $\mathbf{H}^n_r$ for $\vdash^n_r$ can be obtained from $\mathbf{HCL}^+$ by adding to it the axiom $[\neg\, ^n_r]$: $\psi \vee \neg\psi \vee \ldots \vee \neg^{n-1}\psi$. Similarly, a calculus $\mathbf{H}^n_l$ for $\vdash^n_l$ is obtained from $\mathbf{HCL}^+$ by adding to it the axiom $[\neg\, ^n_l]$: $(\psi \wedge \neg\psi \wedge \ldots \wedge \neg^{n-1}\psi) \supset \phi$. A cut-free Gentzen-type calculus $\mathbf{G}^n_r$ for $\vdash^n_r$ is obtained from $\mathbf{GCL}^+$ by adding to it the rule:

$$[\Rightarrow \neg\, ^n] \qquad \frac{\Gamma_0, \psi \Rightarrow \Delta_0 \quad \Gamma_1, \neg\psi \Rightarrow \Delta_1 \quad \ldots \quad \Gamma_{n-2}, \neg^{n-2}\psi \Rightarrow \Delta_{n-2}}{\Gamma_0, \ldots, \Gamma_{n-2} \Rightarrow \Delta_0, \ldots, \Delta_{n-2}, \neg^{n-1}\psi}$$

Similarly, $\mathbf{G}^n_l$ is obtained from $\mathbf{GCL}^+$ by adding to it the rule:

$$[\neg \Rightarrow\, ^n] \qquad \frac{\Gamma_0 \Rightarrow \Delta_0, \psi \quad \Gamma_1 \Rightarrow \Delta_1, \neg\psi \quad \ldots \quad \Gamma_{n-2} \Rightarrow \Delta_{n-2}, \neg^{n-2}\psi}{\Gamma_0, \ldots, \Gamma_{n-2}, \neg^{n-1}\psi \Rightarrow \Delta_0, \ldots, \Delta_{n-2}}$$

We next introduce finite characteristic Nmatrices for $\vdash^n_r$ and $\vdash^n_l$:

$\mathcal{M}^n_r$: Let $\mathcal{T}_t = \{0\}$ and $\mathcal{T}_f = \{1, \ldots, n-1\}$. $\mathcal{M}^n_r$ has $\mathcal{T} = \mathcal{T}_t \cup \mathcal{T}_f$ as the set of truth values, and $\mathcal{T}_t$ as the set of designated values. The interpretation of the connectives is as follows (where $\diamond \in \{\wedge, \vee, \supset\}$, and $g_\diamond$ treats all values in $\mathcal{T}_t$ as the classical t and all values in $\mathcal{T}_f$ as the classical f):

$$\tilde{\diamond}(x_1, x_2) = \begin{cases} \mathcal{T}_t & \text{if } g_\diamond(x_1, x_2) = t \\ \mathcal{T}_f & \text{otherwise} \end{cases}$$

$$\tilde{\neg} x = \begin{cases} \{x-1\} & \text{if } x > 0 \\ \mathcal{T} & \text{if } x = 0 \end{cases}$$

$\mathcal{M}^n_l$: This is defined exactly as $\mathcal{M}^n_r$, only this time $\mathcal{T}_t = \{1, \ldots, n-1\}$ and $\mathcal{T}_f = \{0\}$.

Proposition 3.13 $\vdash_{\mathbf{H}^n_r} = \vdash_{\mathcal{M}^n_r}$ *and* $\vdash_{\mathbf{H}^n_l} = \vdash_{\mathcal{M}^n_l}$.

Proof: We first show the claim for $\mathbf{H}^n_r$. Soundness is easy to verify. For completeness, assume that $\mathbf{T} \nvdash_{\mathbf{H}^n_r} \psi$. Extend $\mathbf{T}$ to a maximal set $\mathbf{T}^*$, such that $\mathbf{T}^* \nvdash_{\mathbf{H}^n_r} \psi$. Like in the proof of Theorem 3.8, $\mathbf{T}^*$ has the following properties:

1. $\phi \wedge \tau \in \mathbf{T}^*$ iff $\phi \in \mathbf{T}^*$ and $\tau \in \mathbf{T}^*$

2. $\phi \vee \tau \in \mathbf{T}^*$ iff $\phi \in \mathbf{T}^*$ or $\tau \in \mathbf{T}^*$

3. $\phi \supset \tau \in \mathbf{T}^*$ iff $\phi \notin \mathbf{T}^*$ or $\tau \in \mathbf{T}^*$

Define a function $v : \mathcal{W}_{\rm cl} \rightarrow \mathcal{T}$ by letting $v(\phi)$ be the first i $(0 \leq i \leq n-1)$ such that $\neg^i\phi \in \mathbf{T}^*$ (such i exists by 2 above and the axiom $[\neg\ {}^n_r]$). Then $v(\phi) \in \mathcal{T}_t$ (i.e.: $v(\phi) = 0$) iff $\phi \in \mathbf{T}^*$. Therefore 1-3 above entail that v respects the interpretations of $\wedge$, $\vee$, and $\supset$. Obviously, it also respects the interpretation of $\neg$. Hence v is a valuation in $\mathcal{M}^n_r$. By definition, v is a model of $\mathbf{T}^*$ (and so of $\mathbf{T}$) in $\mathcal{M}^n_r$ which is not a model of ψ.

The claim for $\mathbf{H}^n_l$ is shown similarly. The only difference is that we define $v(\phi)$ to be the first i $(0 \leq i \leq n-1)$ such that $\neg^i\phi \notin \mathbf{T}^*$ (and use $[\neg\ {}^n_l]$ to justify this definition). ■

Proposition 3.14 *Every characteristic Nmatrix for $\vdash^n_r$ or $\vdash^n_l$ has at least n truth values.*

Proof: Suppose that $\mathcal{M}$ is an Nmatrix with less than n truth values and that $\vdash_{\mathcal{M}} \subseteq \vdash^n_r$. Using $\mathcal{M}^n_r$, it is easy to verify that for all $n \geq 2$, $\not\vdash^n_r p, \neg p \ldots \neg^{n-2}p$. It follows that there is a valuation v in $\mathcal{M}$ such that none of $v(p), v(\neg p), \ldots, v(\neg^{n-2}p)$ is designated. Hence there must be $0 \leq i < j \leq n-2$ such that $v(\neg^i p) = v(\neg^j p)$ (since there are $n-1$ formulas here, while $\mathcal{M}$ has at most $n-2$ nondesignated truth values). It is possible therefore to define an $\mathcal{M}$-valuation v' such that:

$$v'(\neg^k p) = \begin{cases} v(\neg^k p) & \text{if } k \leq j \\ v(\neg^{k-j+i} p) & \text{if } j < k < n \end{cases}$$

This v' refutes $\vdash_{\mathcal{M}} p, \neg p, \ldots, \neg^{n-1}p$. Hence $\vdash_{\mathcal{M}}$ is not the same as $\vdash^n_r$ – a contradiction.

The claim for $\vdash^n_l$ is shown similarly, using the fact that $p, \neg p \ldots \neg^{n-2}p \not\vdash^n_l \emptyset$. ■

Note: It is easy to see that the proof in the case of $\vdash^n_l$ applies also to its Tarskian fragment.

3.3 The General Compactness Theorem

We now show that the classical compactness theorem is valid for any finite Nmatrix (generalizing by this a similar result for finite matrices which has been proved in [SS71, SS78]).

Theorem 3.15 *If $\mathcal{M}$ is a finite Nmatrix then $\vdash_{\mathcal{M}}$ (and so also $\vdash^1_{\mathcal{M}}$) is finitary.*

For the proof we need two lemmas.

Lemma 3.16 *Let $\vdash$ be an scr. Call a pair $\langle\Gamma, \Delta\rangle$* compact *(w.r.t. $\vdash$) if there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash \Delta'$. Then for every pair $\langle\Gamma, \Delta\rangle$ which is not compact there exist $\Gamma^* \supseteq \Gamma$ and $\Delta^* \supseteq \Delta$ such that $\Gamma^* \cup \Delta^* = \mathcal{W}$, and $\langle\Gamma^*, \Delta^*\rangle$ is also not compact.*

Proof: Let $\psi_1, \psi_2, \ldots$ be some ordering of the formulas of $\mathcal{W}$. Define:

$\langle \Gamma_0, \Delta_0 \rangle = \langle \Gamma, \Delta \rangle$;

$$\langle \Gamma_{n+1}, \Delta_{n+1} \rangle = \begin{cases} \langle \Gamma_n \cup \{\psi_{n+1}\},\ \Delta_n \rangle & \text{if } \langle \Gamma_n \cup \{\psi_{n+1}\},\ \Delta_n \rangle \text{ is not compact} \\ \langle \Gamma_n,\ \Delta_n \cup \{\psi_{n+1}\} \rangle & \text{otherwise} \end{cases}$$

Note that since $\vdash$ is closed under cut, $\langle \Gamma_n \cup \{\psi_{n+1}\},\ \Delta_n \rangle$ and $\langle \Gamma_n,\ \Delta_n \cup \{\psi_{n+1}\} \rangle$ cannot both be compact in case $\langle \Gamma_n, \Delta_n \rangle$ is not compact. Hence for all n, $\langle \Gamma_n, \Delta_n \rangle$ is not compact, and $\psi_n \in \Gamma_n \cup \Delta_n$. It follows that $\Gamma^* = \bigcup \{\Gamma_n \mid n \geq 0\}$ and $\Delta^* = \bigcup \{\Delta_n \mid n \geq 0\}$ have the desired properties. ■

Lemma 3.17 *Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be a finite Nmatrix. Suppose that for all $x \in \mathcal{T}$ there is a unary connective J_x in the language, such that for all $y \in \mathcal{T}$, $\widetilde{J_x}(y) \subseteq \mathcal{D}$* if *$y = x$, and $\widetilde{J_x}(y) \subseteq \mathcal{T} - \mathcal{D}$* if *$y \neq x$. Then $\vdash_{\mathcal{M}}$ is finitary.*

Proof: It can be easily verified that the following claims hold for every formula ψ:

(1) If $x, y \in \mathcal{T}$ and $x \neq y$ then $J_x(\psi), J_y(\psi) \vdash_{\mathcal{M}} \emptyset$.

(2) If $\mathcal{T} = \{t_1, \ldots, t_n\}$ then $\emptyset \vdash_{\mathcal{M}} J_{t_1}(\psi), \ldots, J_{t_n}(\psi)$.

(3) If $x \in \mathcal{D}$ then $J_x(\psi) \vdash_{\mathcal{M}} \psi$.

(4) If $x \in \mathcal{T} - \mathcal{D}$ then $J_x(\psi), \psi \vdash_{\mathcal{M}} \emptyset$.

Suppose now that there are Γ and Δ such that $\Gamma \vdash_{\mathcal{M}} \Delta$ but $\langle \Gamma, \Delta \rangle$ is not compact w.r.t. $\vdash_{\mathcal{M}}$. Using Lemma 3.16, extend the pair $\langle \Gamma, \Delta \rangle$ to a pair $\langle \Gamma^*, \Delta^* \rangle$ such that:

(5) $\langle \Gamma^*, \Delta^* \rangle$ is not compact, and $\Gamma^* \cup \Delta^* = \mathcal{W}$.

Now (1), (2), and (5) imply that for every ψ, there is exactly one $x \in \mathcal{T}$ such that $J_x(\psi) \in \Gamma^*$. Let $v(\psi)$ be this single x. Then:

6) $J_{v(\psi)}(\psi) \in \Gamma^*$ for every ψ.

We show now that if $\psi_0 = \diamond(\psi_1, \ldots, \psi_n)$ then $v(\psi_0) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ (so v is an $\mathcal{M}$-valuation). Assume this is not the case. Then no valuation in $\mathcal{M}$ can assign the value $v(\psi_i)$ to ψ_i for all $0 \leq i \leq n$. Thus $J_{v(\psi_0)}(\psi_0), \ldots, J_{v(\psi_n)}(\psi_n) \vdash_{\mathcal{M}} \emptyset$, and therefore $\langle \Gamma^*, \Delta^* \rangle$ is compact by (6). This contradicts (5).

It remains to show that v refutes $\Gamma^* \vdash_{\mathcal{M}} \Delta^*$. Assume $\psi \in \Gamma^*$. By (6), $J_{v(\psi)}(\psi) \in \Gamma^*$. Hence $v(\psi) \in \mathcal{D}$ by (4) and (5). Similarly, if $\psi \in \Delta^*$ then $v(\psi) \notin \mathcal{D}$ by (6) and (3). ■

Proof of Theorem 3.15: Let $\mathcal{M}$ be a finite Nmatrix for $\mathcal{L}$. Extend $\mathcal{L}$ to a language $\mathcal{L}'$ by adding to $\mathcal{L}$ a new unary connective J_x for every truth value x of $\mathcal{M}$. Let $\mathcal{M}'$ be an extension of $\mathcal{M}$ to $\mathcal{L}'$ that satisfies the conditions of Lemma 3.17 (such an extension obviously exists, since $\mathcal{D}$ is a nonempty, proper subset of $\mathcal{T}$). Suppose now that $\Gamma \vdash_{\mathcal{M}} \Delta$. Obviously, $\Gamma \vdash_{\mathcal{M}'} \Delta$ as well. Hence $\langle \Gamma, \Delta \rangle$ is compact w.r.t. $\vdash_{\mathcal{M}'}$ by Lemma 3.17. Since Γ and Δ are in $\mathcal{L}$, This easily entails that $\langle \Gamma, \Delta \rangle$ is also compact w.r.t. $\vdash_{\mathcal{M}}$. ■

Theorem 3.15 applies of course to finite matrices. On the other hand it was proved in [SS78] that the compactness theorem fails for some infinite matrix (even if its set of designated values is required to be finite). Hence Theorem 3.15 cannot be generalized for arbitrary Nmatrices.

4 Two-valued N-matrices and Canonical Gentzen Systems

This section is devoted to an investigation of a special (but important) type of Nmatrices: the two-valued ones. We show that it has close connections with a particularly important special type of propositional Gentzen-type systems: the canonical ones. [10]

Definition 4.1

1. A Gentzen-type system **G** is *standard* if its set of axioms includes the standard axioms $\Gamma, \psi \Rightarrow \psi, \Delta$ and it has all the standard structural rules (including cut). [11]

2. Let **G** be a standard Gentzen-type system. The scr $\vdash_{\mathbf{G}}$ which is induced by **G** is defined by: $\Gamma \vdash_{\mathbf{G}} \Delta$ iff there exist finite $\Gamma_0 \subseteq \Gamma, \Delta_0 \subseteq \Delta$ such that the sequent $\Gamma_0 \Rightarrow \Delta_0$ is provable in **G** (note that in case Γ and Δ are finite this is equivalent to the provability in **G** of $\Gamma \Rightarrow \Delta$).

3. A standard Gentzen-type system **G** is consistent if $\vdash_{\mathbf{G}}$ is consistent.

Conventions: From now on by a "calculus" we shall mean a standard Gentzen-type calculus, and Γ and Δ will denote *finite* sets of formulas.

In an ideal Gentzen-type system (of which the usual systems for classical logic provide the principal examples) every logical rule should be an introduction rule for one connective, it should introduce exactly one occurrence of that connective in its conclusion, and no other

[10] Most of the results of this section has first been published in [AL01].

[11] This means that we can take Γ, Δ in a sequent $\Gamma \Rightarrow \Delta$ to be finite *sets* of formulas.

occurrence of that connective or any other connective should be mentioned anywhere else in its formulation. Moreover: the rule should be *pure* (i.e., there should be no side conditions limiting its application), and its side formulas should be immediate subformulas of the principal formula. The next definition formulates this idea in exact terms, and provides a method for describing such rules.

Definition 4.2

1. A *canonical rule* of arity n is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C$, where $m \geq 0$, C is either $\diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ or $\Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ for some connective $\diamond$ (of arity n), and for all $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a nonempty clause such that $\Pi_i, \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.[12]

2. An *application* of a canonical rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} \;/\; \diamond(p_1, \ldots, p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \Delta}$$

   where Π_i^* and Σ_i^* are obtained from Π_i and Σ_i (respectively) by substituting ψ_j for p_j (for all $1 \leq j \leq n$), and Γ, Δ are any sets of formulas. An application of a canonical rule with a conclusion of the form $\Rightarrow \diamond(p_1, \ldots, p_n)$ is defined similarly.

Note: While sequents are written in a metalanguage for $\mathcal{L}$ (which includes the extra symbol $\Rightarrow$), a canonical rule is formulated in a meta-meta language of $\mathcal{L}$ (which includes one further extra symbol: /).

Example 4.3 *The two usual introduction rules for the classical conjunction can be formulated as follows:* $\{p_1, p_2 \Rightarrow\ \} \;/\; p_1 \wedge p_2 \Rightarrow$ *and* $\{\Rightarrow p_1\ ,\ \Rightarrow p_2\} \;/\; \Rightarrow p_1 \wedge p_2$. *Applications of these rules have the form:*

$$\frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}$$

Definition 4.4 A standard calculus is called *canonical* if in addition to the standard axioms and the standard structural rules it has only canonical logical rules.

[12] By a clause we mean a sequent which consists of atomic formulas only. When propositional clauses are written in this way, resolution and cut amount to the same thing. $\{p_1, p_2, \ldots, p_n\}$ are, recall, the first n atomic formulas.

For every canonical Gentzen-type system **G**, the relation $\vdash_{\mathbf{G}}$ (see Definition 4.1) defined by **G** is obviously a structural (and finitary) scr. However, in order to ensure that $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ is a *logic*, we need to impose some constraints on the set of rules of **G**. The following definition provides a constructive equivalent of the consistency condition:

Definition 4.5 A canonical calculus **G** is called *coherent*, if for every two rules of **G** of the form $S_1 / \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ and $S_2 / \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$, the set of clauses $S_1 \cup S_2$ is classically inconsistent (and so the empty clause can be derived from it using cuts).

Example 4.6 The two classical rules for conjunction described in Example 4.3 form a coherent set of rules. Here $S_1 = \{p_1, p_2 \Rightarrow\ \}$, $S_2 = \{\ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$ and so $S_1 \cup S_2$ is the classically inconsistent set $\{p_1, p_2 \Rightarrow\ ,\ \ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$.

Theorem 4.7 *Let* **G** *be a canonical Gentzen-type System in a language* $\mathcal{L}$. *The following conditions concerning* **G** *are equivalent:*

1. **G** *admits cut-elimination.*
2. **G** *is consistent (i.e.* $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ *is a logic).*
3. **G** *is coherent.*
4. *There is a 2-Nmatrix* $\mathcal{M}$ *such that* $\vdash_{\mathbf{G}} \Gamma \Rightarrow \Delta$ *iff every valuation* v *in* $\mathcal{M}$ *satisfies* $\Gamma \Rightarrow \Delta$ *(i.e.* $v(\psi) = t$ *for some* $\psi \in \Delta$, *or* $v(\psi) = f$ *for some* $\psi \in \Gamma$*).*

Proof: By a *2-Nmatrix* we shall mean below an Nmatrix in which $\mathcal{T} = \{t, f\}$ and $\mathcal{D} = \{t\}$. Obviously, any two-valued Nmatrix is isomorphic to some 2-Nmatrix. Hence we concentrate in what follows on 2-Nmatrices.

1⇒2: In canonical systems, *clauses* which are not axioms can be proved only by using cuts on non-atomic formulas. Thus, if a canonical calculus admits cut elimination it must be consistent.

2⇒3: Suppose there are two rules $S_1/\diamond(p_1, \ldots, p_n) \Rightarrow$ and $S_2/ \Rightarrow \diamond(p_1, \ldots, p_n)$ such that $S_1 \cup S_2$ is classically consistent. Then there is a classical valuation v that satisfies $S_1 \cup S_2$. Let $\Pi' = \{p_i \mid 1 \leq i \leq n,\ v(p_i) = t\}$ and $\Sigma' = \{p_i \mid 1 \leq i \leq n,\ v(p_i) = f\}$. Let $S'_j = \{\Pi, \Pi' \Rightarrow \Sigma, \Sigma' \mid \Pi \Rightarrow \Sigma \in S_j\}$ for $j = 1, 2$. S'_1 and S'_2 are sets of standard axioms (because v satisfies $\Pi \Rightarrow \Sigma$, there is some $p_i \in \Sigma$ such that $v(p_i) = t$ or some $p_i \in \Pi$ such that $v(p_i) = f$. In the former case, $p_i \in \Pi'$, and in the latter case, $p_i \in \Sigma'$). By applying the first rule on $\mathcal{S}'_1$ we obtain $\Pi', \diamond(p_1, \dots, p_n) \Rightarrow \Sigma'$ and by applying the second rule on $\mathcal{S}'_2$ we obtain $\Pi' \Rightarrow \Sigma', \diamond(p_1, \dots, p_n)$. By cut, $\Pi' \Rightarrow \Sigma'$ is provable. Since $\Pi' \Rightarrow \Sigma'$ is a clause, $\Pi' \cap \Sigma' = \emptyset$, and the calculus is structural, $p \Rightarrow q$ is provable for all $p \neq q$. The structurality of the calculus and the closure under weakening entail that $\Gamma \Rightarrow \Delta$ is provable for every non-empty Γ and Δ. Hence the system is not consistent.

4⇒2: This is trivial.

It remains to prove that 3 implies both 4 and 1. So assume henceforth that **G** is a coherent canonical calculus. We start by transforming **G** to an equivalent canonical coherent calculus $\mathbf{G}^F$ with the following properties:

(i) $\vdash_{\mathbf{G}} \; = \; \vdash_{\mathbf{G}^F}$.

(ii) Every sequent that has a cut-free proof in $\mathbf{G}^F$ has such a proof also in **G**.

(iii) Every rule of $\mathbf{G}^F$ of arity n has the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq n}/C$, where $\Pi_i \cup \Sigma_i = \{p_i\}$ for every $1 \leq i \leq n$.

The transition of **G** into $\mathbf{G}^F$ is done in three stages:

Stage 1: Obtain $\mathbf{G}_1$ by replacing any rule of **G** of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}/C$ by the set of all rules $\{\Pi'_i \Rightarrow \Sigma'_i\}_{1 \leq i \leq m}/C$ that satisfy the two conditions:

- for all $1 \leq i \leq m$, $\Pi'_i \subseteq \Pi_i$ and $\Sigma'_i \subseteq \Sigma_i$.
- for all $1 \leq i \leq m$, $|\Pi'_i| + |\Sigma'_i| = 1$.

It is easy to see that $\mathbf{G}_1$ is also coherent, and that every (cut-free) proof in **G** can be simulated by a (cut-free) proof in $\mathbf{G}_1$ and vice versa. Note also that the number of premises of a given rule of $\mathbf{G}_1$ may be smaller (but not greater) than the number of premises of the rule in **G** from which it was obtained (since premises come in *sets*).

Stage 2: Obtain $\mathbf{G}_2$ from $\mathbf{G}_1$ by discarding all rules for which both $\Rightarrow p$ and $p \Rightarrow$ are premises (for some atomic p). Since any application of such a rule can be simulated using cuts and weakenings, $\mathbf{G}_2$ is equivalent to $\mathbf{G}_1$ (and so to $\mathbf{G}$). Obviously, $\mathbf{G}_2$ is still coherent. Moreover: any cut-free proof in $\mathbf{G}_2$ is also a cut-free proof in $\mathbf{G}_1$, and so can be simulated by a cut-free proof in $\mathbf{G}$. Note also that in $\mathbf{G}_2$ every rule of arity n has at most n premises.

Stage 3: Obtain $\mathbf{G}^F$ from $\mathbf{G}_2$ by by replacing any rule R of $\mathbf{G}_2$ of arity n by the set of all rules of the required form which have the same conclusion as R, and their sets of premises include that of R. Obviously, every (cut-free) proof in $\mathbf{G}^F$ can be simulated by a (cut-free) proof in $\mathbf{G}_2$ (and so in $\mathbf{G}$). To show that $\mathbf{G}^F$ is still coherent and is equivalent to $\mathbf{G}_2$ (and so to $\mathbf{G}$) it suffices to show that every rule $R = \{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/C$ is equivalent to the following pair of rules:

$$R' = \{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} \cup \{p_j \Rightarrow \ \} \ / \ C$$

$$R'' = \{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m} \cup \{\ \Rightarrow p_j\} \ / \ C$$

Indeed, the result of applying R to $\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1\leq i\leq m}$ (see the notations in Definition 4.2) can be achieved by applying first R' to this set together with the standard axiom $\Gamma, \psi_j \Rightarrow \Delta, \psi_j$, and then applying R'' to the resulting sequent together with the original set of premises.

We next use $\mathbf{G}^F$ to define a 2-Nmatrix for which $\mathbf{G}$ is sound and complete. First we need some notations. Let $-t = f$, $-f = t$, $ite(t, A, B) = A$, $ite(f, A, B) = B$, and let Φ, a^x (where Φ might be empty) denote $ite(x, \Phi \cup \{a\}, \Phi)$. Note that in this notation every rule of $\mathbf{G}^F$ has the form $\{p_i^{-x_i} \Rightarrow p_i^{x_i}\}_{1\leq i\leq n} \ / \ \diamond(p_1, \ldots, p_n)^{-y} \Rightarrow \diamond(p_1, \ldots, p_n)^y$ for some n-ary connective $\diamond$ and $x_1, \ldots, x_n, y \in \{t, f\}$. We'll denote a rule of this form by $[\diamond(x_1, \ldots, x_n) : y]$.

Definition 4.8 $\mathcal{M}_{\mathbf{G}}$, the Nmatrix defined by the coherent canonical calculus $\mathbf{G}$, is the 2-Nmatrix in which for each n-ary connective $\diamond$ and every $x_1, \ldots, x_n, y \in \{t, f\}$ we have:

$$\tilde{\diamond}(x_1, \ldots, x_n) = \begin{cases} \{y\} & \text{if } [\diamond(x_1, \ldots, x_n) : y] \text{ is a rule of } \mathbf{G}^F \\ \{t, f\} & \text{otherwise} \end{cases}$$

(This is well-defined, since we assume that $\mathbf{G}$ (and hence $\mathbf{G}^F$) is coherent.)

It is not difficult to see that $\mathbf{G}$ is sound for $\mathcal{M}_{\mathbf{G}}$. Indeed, since $\mathbf{G}$ and $\mathbf{G}^F$ are equivalent, it suffices to show that every rule of $\mathbf{G}^F$ is sound for $\mathcal{M}_{\mathbf{G}}$. So assume that v is a valuation in $\mathcal{M}_{\mathbf{G}}$ which satisfies all the premises $\{\Gamma, \psi_i^{-x_i} \Rightarrow \Delta, \psi_i^{x_i}\}_{1 \leq i \leq n}$ of an application of the rule $[\diamond(x_1, \ldots, x_n) : y]$ of $\mathbf{G}^F$. Then either v satisfies $\Gamma \Rightarrow \Delta$, or else $v(\psi_i) = x_i$ for all $1 \leq i \leq n$, and since $\tilde{\diamond}(x_1, \ldots, x_n) = \{y\}$, necessarily $v(\psi) = y$. In both cases v satisfies $\Gamma, \psi^{-y} \Rightarrow \Delta, \psi^y$.

Given the soundness of $\mathbf{G}$ for $\mathcal{M}_{\mathbf{G}}$, we prove now that 3 implies both 4 and 1 (ending by this the proof of Theorem 4.7) by showing that if $\Gamma \Rightarrow \Delta$ does not have a cut-free proof in $\mathbf{G}$ (and so it does not have a cut-free proof in $\mathbf{G}^F$ either) then $\Gamma \not\vdash_{\mathcal{M}_{\mathbf{G}}} \Delta$. For this extend first $\Gamma \Rightarrow \Delta$ to a sequent $\Gamma^* \Rightarrow \Delta^*$ with the following properties:

1. $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$.

2. $\Gamma^* \Rightarrow \Delta^*$ does not have a cut-free proof in $\mathbf{G}^F$.

3. For every rule $[\diamond(x_1, \ldots, x_n) : y]$ of $\mathbf{G}^F$, if $\diamond(\psi_1, \ldots, \psi_n) \in ite(y, \Delta^*, \Gamma^*)$ then for some $1 \leq i \leq n$, $\psi_i \in ite(x_i, \Delta^*, \Gamma^*)$.

This extension is possible, because if $\Gamma' \Rightarrow \Delta'$ does not have a cut-free proof in $\mathbf{G}^F$, and $\diamond(\psi_1, \ldots, \psi_n) \in ite(y, \Delta', \Gamma')$, then for some $1 \leq i \leq n$, the sequent $\Gamma', \psi_i^{-x_i} \Rightarrow \Delta', \psi_i^{x_i}$ does not have a cut-free proof in $\mathbf{G}^F$ (because otherwise by adding an application of $[\diamond(x_1, \ldots, x_n) : y]$ to the proofs of these sequents we obtain a cut-free proof in $\mathbf{G}^F$ for $\Gamma', \psi^{-y} \Rightarrow \Delta', \psi^y$ – which is exactly $\Gamma' \Rightarrow \Delta'$).

The refuting valuation of $\Gamma \Rightarrow \Delta$ is now defined as follows:

- For atomic q, $v(q) = t$ iff $q \in \Gamma^*$.
- $v(\diamond(\psi_1, \ldots, \psi_n)) = \begin{cases} t & \text{if } \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n)) = \{t\} \text{ or} \\ & \quad [\tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n)) = \{t, f\} \text{ and } \diamond(\psi_1, \ldots, \psi_n) \in \Gamma^*] \\ f & \text{otherwise} \end{cases}$

v is obviously a legal $\mathcal{M}_{\mathbf{G}}$-valuation. We now show by induction on the complexity of a formula $\psi \in \Gamma^* \cup \Delta^*$ that $\psi \in ite(v(\psi), \Gamma^*, \Delta^*)$. In other words: if $\psi \in \Gamma^*$ then $v(\psi) = t$, and if $\psi \in \Delta^*$ then $v(\psi) = f$.

- Assume ψ is atomic. If $\psi \in \Gamma^*$ then $v(\psi) = t$ by definition. If $\psi \in \Delta^*$ then $\psi \notin \Gamma^*$ by property 2 of $\Gamma^* \Rightarrow \Delta^*$. Hence $v(\psi) = f$.

- Let $\psi = \diamond(\psi_1, \ldots, \psi_n)$ and let $x_i = v(\psi_i)$ for $1 \leq i \leq n$.

  Assume $\psi \in \Gamma^*$, but $v(\psi) = f$. According to the definition of v, this can happen only if $\tilde{\diamond}(x_1, \ldots, x_n) = \{f\}$. It follows that $\psi \in ite(v(\psi), \Delta^*, \Gamma^*)$ in this case. Hence $\psi_i \in ite(x_i, \Delta^*, \Gamma^*)$ for some $1 \leq i \leq n$ by property 3 of $\Gamma^* \Rightarrow \Delta^*$ and the fact that by Definition 4.8, $\tilde{\diamond}(x_1, \ldots, x_n) = \{f\}$ in $\mathcal{M}$ only if $[\diamond(x_1, \ldots, x_n) : f]$ is a rule of **G**. It follows that $\psi_i \in \Gamma^* \cup \Delta^*$, and so also $\psi_i \in ite(x_i, \Gamma^*, \Delta^*)$ by the induction hypothesis. Hence $\psi_i \in \Gamma^* \cap \Delta^*$, contradicting property 2 of $\Gamma^* \Rightarrow \Delta^*$.

  Now assume $\psi \in \Delta^*$, but $v(\psi) = t$. According to the definition of v, there are two possibilities here:

  1. $\tilde{\diamond}(x_1, \ldots, x_n) = \{t\} = \{v(\psi)\}$. We get from this a contradiction like in the previous case.
  2. $\tilde{\diamond}(x_1, \ldots, x_n) = \{t, f\}$ and $\psi \in \Gamma^*$. Since $\psi \in \Delta^*$ as well, this contradicts property 2 of $\Gamma^* \Rightarrow \Delta^*$.

By property 1 of $\Gamma^* \Rightarrow \Delta^*$ and what we have just proved, v is a model of Γ in $\mathcal{M}_{\mathbf{G}}$ which does not satisfy any element of Δ. Hence $\Gamma \not\vdash_{\mathcal{M}_{\mathbf{G}}} \Delta$. ■

Example 4.9 Suppose a coherent canonical calculus **G** has the following rule for the ternary connective $\diamond$:

$$\{p_1, p_2 \Rightarrow \;,\; p_1 \Rightarrow p_2 \;,\; p_3 \Rightarrow p_2\} \;/\; \diamond(p_1, p_2, p_3) \Rightarrow$$

The first stage of the process leading to $\mathbf{G}^F$ produces from this rule the following rules:

$$\begin{array}{rc}
(1) & \{p_1 \Rightarrow \;,\; p_3 \Rightarrow \;\} \;/\; \diamond(p_1, p_2, p_3) \Rightarrow \\
(2), (4) & \{p_1 \Rightarrow \;,\; \Rightarrow p_2\} \;/\; \diamond(p_1, p_2, p_3) \Rightarrow \\
(3) & \{p_1 \Rightarrow \;,\; \Rightarrow p_2 \;,\; p_3 \Rightarrow \;\} \;/\; \diamond(p_1, p_2, p_3) \Rightarrow \\
(5) & \{p_2 \Rightarrow \;,\; p_1 \Rightarrow \;,\; p_3 \Rightarrow \;\} \;/\; \diamond(p_1, p_2, p_3) \Rightarrow \\
(6) & \{p_2 \Rightarrow \;,\; p_1 \Rightarrow \;,\; \Rightarrow p_2\} \;/\; \diamond(p_1, p_2, p_3) \Rightarrow \\
(7) & \{p_2 \Rightarrow \;,\; \Rightarrow p_2 \;,\; p_3 \Rightarrow \;\} \;/\; \diamond(p_1, p_2, p_3) \Rightarrow \\
(8) & \{p_2 \Rightarrow \;,\; \Rightarrow p_2\} \;/\; \diamond(p_1, p_2, p_3) \Rightarrow
\end{array}$$

In the second stage (6),(7),(8) are discarded (and of course also (4). It is easy to see that (3),(5) are in fact redundant and can be discarded as well, but this is not important here). Finally, in the third stage we get the following rules of $\mathbf{G}^F$:

$$\{p_1 \Rightarrow \ , \ p_2 \Rightarrow \ , \ p_3 \Rightarrow \} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow$$

$$\{p_1 \Rightarrow \ , \ \Rightarrow p_2 \ , \ p_3 \Rightarrow \} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow$$

$$\{p_1 \Rightarrow \ , \ \Rightarrow p_2 \ , \ \Rightarrow p_3\} \ / \ \diamond(p_1, p_2, p_3) \Rightarrow$$

The validity of these rules means that in $\mathcal{M}_{\mathbf{G}}$ we have:

$$\widetilde{\diamond}(f, f, f) = \widetilde{\diamond}(f, t, f) = \widetilde{\diamond}(f, t, t) = \{f\}$$

Corollary 4.10 *Let* **G** *be a coherent canonical calculus. Then either* **G** *defines a logic which is a fragment of classical logic, or it has no finite characteristic matrix.*

Proof: By Theorem 4.7, **G** is sound and complete for some 2-Nmatrix $\mathcal{S}$. If $\mathcal{S}$ includes only deterministic connectives (i.e.: connectives which return singletons for every combination of truth values), then **G** has a characteristic two-valued *matrix*, and so it is a fragment of classical logic. Otherwise $\mathcal{S}$ has at least one proper non-deterministic operation, and hence its logic has no finite characteristic matrix by Theorem 3.4. ■

The next theorem is a sort of converse to Theorem 4.7:

Theorem 4.11 *Every 2-Nmatrix $\mathcal{M}$ has a sound and complete coherent canonical calculus which for every connective has at most one introduction rule on the left, and at most one introduction rule on the right.*

Proof: Let $\mathbf{G}(\mathcal{M})$ be the canonical calculus which for any n-ary connective $\diamond$ has the following rules (where $\widetilde{\diamond}$ is the interpretation of $\diamond$ in $\mathcal{M}$):

$$[\diamond \Rightarrow] \quad \{\{p_i \mid x_i = t\} \Rightarrow \{p_i \mid x_i = f\}\}_{t \in \widetilde{\diamond}(x_1, \dots, x_n)} \ / \ \diamond(p_1, \dots, p_n) \Rightarrow$$

$$[\Rightarrow \diamond] \quad \{\{p_i \mid x_i = t\} \Rightarrow \{p_i \mid x_i = f\}\}_{f \in \widetilde{\diamond}(x_1, \dots, x_n)} \ / \ \Rightarrow \diamond(p_1, \dots, p_n)$$

Note that if $t \in \widetilde{\diamond}(x_1, \dots, x_n)$ for all $x_1, \dots, x_n$, then the first rule can be discarded, while if $\widetilde{\diamond}(x_1, \dots, x_n) = \{f\}$ for all $x_1, \dots, x_n$ then that rule does not have any premises, i.e. it is a non-standard axiom (this type of axioms is permitted in canonical systems!). Similarly, if $f \in \widetilde{\diamond}(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n$ then the second rule can be discarded, while if $\widetilde{\diamond}(x_1, \ldots, x_n) = \{t\}$ for all $x_1, \ldots, x_n$ then that rule does not have any premises.

The soundness of $\mathbf{G}(\mathcal{M})$ is easy to verify. Take for example $[\diamond \Rightarrow]$. To show its soundness, assume that $\Gamma, \psi_1^{x_1}, \ldots, \psi_n^{x_n} \vdash_{\mathcal{M}} \Delta, \psi_1^{-x_1}, \ldots, \psi_n^{-x_n}$ for all $x_1, \ldots, x_n$ such that $t \in \widetilde{\diamond}(x_1, \ldots, x_n)$, and let v be a model of $\Gamma \cup \{\diamond(\psi_1, \ldots, \psi_n)\}$ in $\mathcal{M}$. Then there are $y_1, \ldots, y_n \in \{t, f\}$ such that $t \in \widetilde{\diamond}(y_1, \ldots, y_n)$ and $v(\psi_i) = y_i$ for all i. Since $\Gamma, \psi_1^{y_1}, \ldots, \psi_n^{y_n} \vdash_{\mathcal{M}} \Delta, \psi_1^{-y_1}, \ldots, \psi_n^{-y_n}$ by assumption, it follows that v is a model of one of the elements of Δ. The soundness of $[\Rightarrow \diamond]$ is proved similarly, while the proof of completeness is similar to that given in the proof of Theorem 4.7. ■

Corollary 4.12 *Every coherent canonical calculus has an equivalent canonical calculus in which every connective has at most one introduction rule for each side.*

Example 4.13 Suppose we have the following interpretation for a binary connective $\diamond$, which makes it a very close relative of the classical conjunction:

$$\widetilde{\diamond}(t,t) = \{t\}, \quad \widetilde{\diamond}(t,f) = \{t,f\}, \quad \widetilde{\diamond}(f,t) = \widetilde{\diamond}(f,f) = \{f\}$$

The corresponding two rules as given in the proof of the last theorem are:

$$[\diamond \Rightarrow] \quad \{p_1, p_2 \Rightarrow \ , \ p_1 \Rightarrow p_2\} \ / \ \diamond(p_1, p_2) \Rightarrow$$

$$[\Rightarrow \diamond] \quad \{p_1 \Rightarrow p_2 \ , \ p_2 \Rightarrow p_1 \ , \ \Rightarrow p_1, p_2\} \ / \ \Rightarrow \diamond(p_1, p_2)$$

It is interesting to note that these rules can be simplified using the resolution calculus. Consider e.g. the set of premises of the second rule. Its closure under resolution [13] is:

$$\{p_1 \Rightarrow p_2 \ , \ p_2 \Rightarrow p_1 \ , \ \Rightarrow p_1, p_2 \ , \ \Rightarrow p_1 \ , \ \Rightarrow p_2 \ , \ p_1 \Rightarrow p_1 \ , \ p_2 \Rightarrow p_2\}$$

We now discard the last two clauses in this set, because they are tautologies. We discard the original three clauses as well, since they are subsumed by $\Rightarrow p_1$ and $\Rightarrow p_2$. It is not difficult to show that this process always leads from one rule into an equivalent one. After applying it to the first rule too, we are left with the following simpler set of rules:

$$[\diamond \Rightarrow]' \quad \{p_1 \Rightarrow \ \} \ / \ \diamond(p_1, p_2) \Rightarrow$$

[13] Note that in the propositional case resolution is just another name for cut.

$$[\Rightarrow \diamond]' \quad \{\Rightarrow p_1\ ,\ \ \Rightarrow p_2\}\ /\ \ \Rightarrow \diamond(p_1, p_2)$$

Note that both rules are standard introduction rules for conjunction.

4.1 On Tonk and the Meaning of Propositional Connectives

There is a long tradition, starting from [Gen69], according to which the meaning of a connective is determined by the introduction and elimination rules which are associated with it (See e.g. [Hod86] and [Sun86] for discussions and references). The supporters of this thesis usually have in mind Natural Deduction systems of an ideal type. In this type of "ideal" systems each connective has its own introduction and elimination rules, which should be pure (in the sense of [Avr91]), mention that connective exactly once, and no other connective should be involved. Unfortunately, already the handling of classical negation requires rules which do not meet these conditions. This problem was solved by Gentzen himself by moving to what was called below "canonical Gentzen-type systems", where instead of introduction and elimination rules there are left introduction rules and right introduction rules.

The thesis according to which the meaning of a connective is given by its introduction (and "elimination") rules was strongly challenged by Prior in [Pri60]. In that paper he introduced his famous "Tonk" connective T, which has two rules of the "ideal" type. The introduction rule allows to infer $\varphi T \psi$ from φ. The elimination rule allows to infer ψ from $\varphi T \psi$. In the presence of Tonk every formula can be derived from any other formula, making the "logic" trivial, and the consequence relation inconsistent (in our terminology).

Prior's paper has made it clear that not every combination of "ideal" introduction and elimination rules can be used for defining a connective. Some constraints should be imposed on the set of rules. The usual solution has been that the introduction and elimination rules should precisely "match", in the sense that the elimination rules should be derived in some sense from the introduction rules (see e.g. [Bel62] and [Sun86]). From our results above it follows that this condition is too strong. What should be required from the set of rules is only *coherence*, which is an absolute (and minimal) condition for non-triviality. Tonk's rules indeed do not meet this condition. in the framework of canonical Gentzen-type systems its rules are translated into the following pair of rules: $\{p_1 \Rightarrow\ \}\ /\ p_1 T p_2 \Rightarrow$ and $\{\ \Rightarrow p_2\}\ /\ \Rightarrow p_1 T p_2$. This pair is not coherent, since the set

$\{p_1 \Rightarrow\ ,\ \Rightarrow p_2\}$ is a classically consistent set of clauses. It is no wonder therefore that the resulting calculus is inconsistent!

On the other hand every coherent set of canonical rules does define a unique nondeterministic connective on {t,f} according to Theorem 4.7. Hence our work provides strong evidence for Gentzen's thesis.

5 Conclusion and Further Research

We have introduced a non-deterministic generalization of many-valued semantics. This generalization enjoys the main advantages of ordinary finite-valued semantics (like implying decidability and compactness), but has a wider range of applicability. We have also shown that on the level of propositional languages there exists a deep connection between the possibility to eliminate cuts in a given canonical Gentzen-type system and the existence of a two-valued characteristic Nmatrix for it.

Some obvious directions for further research:

- To extend the ideas and results of the present paper to first-order languages.
- To develop general proof systems for n-valued Nmatrices, possibly along some of the lines in [BFS00] and [Häh99].
- To explore the use of Nmatrices for reasoning under uncertainty (especially paraconsistent systems, the study of which has provided the original motivation for introducing Nmatrices).
- To investigate the connections between nondeterminism in logic and in computer science. One obvious possible application might be in verification of nondeterministic programs. Another, less immediate, might be in providing logical models for nondeterministic computations. It is well known that systems for intuitionistic logic provide such models for sequential computations via the Curry-Howard isomorphism (see e.g. [GLT88]). The most known attempt to provide an analogous model for parallel programs is Linear Logic ([Gir87]), but it is certainly not sufficient, especially for nondeterministic programs and systems. Logics which are based on Nmatrices might provide new insight into this problem.

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References

[AL01] Arnon Avron and Iddo Lev, "Canonical propositional gentzen-type systems", in *Proc. of the 1st International Joint Conference on Automated Reasoning (IJCAR 2001)* (R. Goré, A Leitsch, and T. Nipkow, eds.), no. 2083 in Lecture notes in AI, pp. 529–544, Springer Verlag, 2001.

[Avr91] Arnon Avron, "Simple consequence relations", *Information and Computation*, vol. 92, no. 1, pp. 105–139, 1991.

[Bat98] Diderik Batens, "Inconsistency-adaptive logics", in *Essays Dedicated to the Memory of Helena Rasiowa* (Ewa Orlowska, ed.), Springer, Heidelberg, New York, 1998.

[BCK99] Diderik Batens, Kristof De Clercq, and Natasha Kurtonina, "Embedding and interpolation for some paralogics. The propositional case." (to appear in *Reports on Mathematical Logic*), 1999.

[Bel62] Nuel. D. Belnap, "Tonk, plonk and plink", *Analysis*, vol. 22, pp. 130–134, 1962.

[BFS00] Matthias Baaz, Christian Fermüller, and Gernot Salzer, "Automated deduction for many-valued logics", in *Handbook of Automated Reasoning* (A. Robinson and A. Voronkov, eds.), Elsevier Science Publishers, 2000.

[CE98] J. M. Crawford and D. W. Etherington, "A non-deterministic semantics for tractable inference", in *Proc. of the 15th International Conference on Artificial Intelligence and the 10th Conference on Innovative Applications of Artificial Intelligence*, pp. 286–291, MIT Press, Cambridge, 1998.

[CM99] W. A. Carnielli and J. Marcos, "Limits for paraconsistent calculi", *Notre Dame Journal of Formal Logic*, vol. 40, pp. 375–390, 1999.

[CM02] W. A. Carnielli and J. Marcos, "A taxonomy of c-systems", in *Paraconsistency — the logical way to the inconsistent* (W. A. Carnielli, M. E. Coniglio, and I. L. M. D'ottaviano, eds.), Lecture notes in pure and applied Mathematics, pp. 1–94, Marcell Dekker, 2002.

[Gen69] Gerhard Gentzen, "Investigations into logical deduction", in *The Collected Works of Gerhard Gentzen* (M. E. Szabo, ed.), pp. 68–131, North Holland, Amsterdam, 1969.

[Gir87] J. Y. Girard, "Linear logic", *Theoretical Computer Science*, vol. 50, pp. 1–101, 1987.

[GLT88] J. Y. Girard, Y. Lafont, and P. Taylor, *Proofs and Types.* No. 7 in Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 1988.

[Häh99] R. Hähnle, "Tableaux for multiple-valued logics", in *Handbook of Tableau Methods* (Marcello D'Agostino, D. M. Gabbay, Reiner Hähnle, and Joachim Posegga, eds.), pp. 529–580, Kluwer Publishing Company, 1999.

[Hod86] Wilfrid Hodges, "Elementary predicate logic", in *Handbook of Philosophical Logic*, vol. I, ch. 1, pp. 1–131, D. Reidel Publishing company, 1986.

[LS58] J. Łos and R. Suszko, "Remarks on sentential logics", *Indagationes Mathematicae*, vol. 20, pp. 177–183, 1958.

[Pri60] A. N. Prior, "The runabout inference ticket", *Analysis*, vol. 21, pp. 38–9, 1960.

[Sco74a] Dana S. Scott, "Completeness and axiomatization in many-valued logics", in *Proc. of the Tarski symposium*, vol. XXV of *Proc. of Symposia in Pure Mathematics*, (Rhode Island), pp. 411–435, American Mathematical Society, 1974.

[Sco74b] Dana S. Scott, "Rules and derived rules", in *Logical theory and semantical analysis* (Soren Stenlund, ed.), pp. 147–161, Reidel, Dordrecht, 1974.

[SS71] D. J. Shoesmith and T. J. Smiley, "Deducibility and many-valuedness", *Journal of Symbolic Logic*, vol. 36, pp. 610–622, 1971.

[SS78] D. J. Shoesmith and T. J. Smiley, *Multiple-Conclusion Logic.* Cambridge University Press, 1978.

[Sun86] Göran Sundholm, "Proof theory and meaning", in *Handbook of Philosophical Logic*, vol. III, ch. 8, pp. 471–506, D. Reidel Publishing company, 1986.

[Urq01] Alasdair Urquhart, "Many-valued logic", in *Handbook of Philosophical Logic* (Dov M. Gabbay and Franz Guenthner, eds.), vol. 2, pp. 249–295, Kluwer Academic Publishers, second ed., 2001.

[Wój88] R. Wójcicki, *Theory of Logical Calculi: Basic Theory of Consequence Operations.* Kluwer, 1988.