A NOTE ON THE STRUCTURE OF BILATTICES

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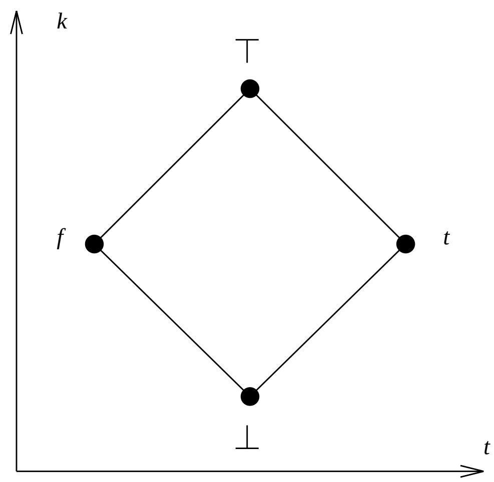
The notion of a bilattice was first introduced by Ginsburg (see [Gin]) as a general framework for a diversity of applications (such as truth maintenance systems, default inferences and others). The notion was further investigated and applied for various purposes by Fitting (see [Fi1]-[Fi6]). The main idea behind bilattices is to use structures in which there are two (partial) order relations, having different interpretations. The two relations should, of course, be connected somehow in order for the mathematical structure to be useful. It is not clear, however, what this connection should be. Ginsberg, for example, has made the connection through an extra operation of negation. Fitting, on the other hand, has investigated connections in the form of conditions on the structure (such as being *interlaced* – see below). These conditions are independent of the existence, or even the possibility to define, operations like Ginsberg's negation.* Fitting defines, accordingly, notions like "an interlaced bilattice", "a distributive bilattice", "a bilattice with negation" and others. He does not provide, however, any definition of the notion of bilattice itself (without an extra modifier). I was unable to find anywhere, in fact, a definition which will cover all the structures which were called "bilattice" in the literature.

Despite the last statement, there is something which seems to be common to all the *finite* bilattices which were used or investigated (and practically only such bilattices were suggested for actual use). All of them can be represented by a kind of a Hasse

* In [Fi5] there is an example of an interlaced bilattice in which no operation of negation as defined by Ginsberg is available.

diagram, which viewed bottom-up, represents one of two order relations while viewed from left to right, represents the other (see examples in Figs. 1 and 2 below). This type of representation seemed, somehow, to be natural, but there has not been (as far as I know) an attempt to show that it can always be used.



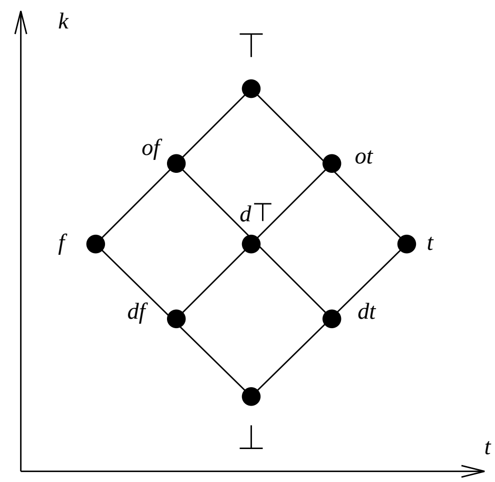


Figure 1: $FOUR$ and $NINE$

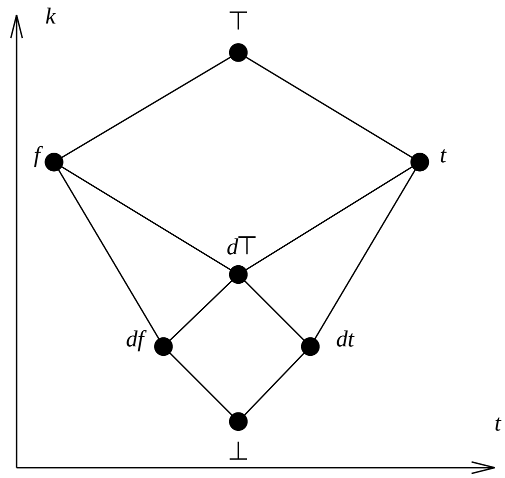


Figure 2: $DEFAULT$

The present note has two purposes. The first is to suggest in the finite case a general definition of "a bilattice" which will cover all the agreed upon particular cases. The second – to show that every such bilattice can be represented by a diagram as described above. This will show, I believe, that the suggested definition is adequate. At the same time it will justify this general method of representation.

Definition 1 [Fi1]. A prebilattice is a structure $B = \langle B, \leq_t, \leq_k \rangle$ such that B is a non-empty set containing at least two elements, and both $\langle B, \leq_t \rangle$ and $\langle B, \leq_k \rangle$ are (complete) lattices.

Notation. Following Fitting, we shall use $\wedge$ and $\vee$ for the lattice operations which correspond to $\leq_t$, and $\otimes$ and $\oplus$ for those that correspond to $\leq_k$.

Definition 2 [Fi1]. A prebilattice is *interlaced* if each of the four operations $\wedge, \vee, \otimes$ and $\oplus$ is monotonic with respect to both $\leq_t$ and $\leq_k$.

Examples. $FOUR$ and $NINE$ (see Fig. 1) are interlaced. $DEFAULT$ (Fig. 2) is not.*

The notion of a prebilattice has been used by Fitting as a general framework for the study of bilattices. In order to investigate representability by graphs it is useful to consider even more general notions.

Definition 3. A *biposet* is a structure $B = \langle B, \leq_t, \leq_k \rangle$ in which both $\langle B, \leq_t \rangle$ and $\langle B, \leq_k \rangle$ are posets.

Notation. $a <_t^1 b$ ($a <_k^1 b$) will mean that b is an immediate $\leq_t$ ($\leq_k$) successor of a. $a <_t b$ ($a <_k b$) will mean that $a \leq_t b$ ($a \leq_k b$) but $a \neq b$.

Definition 4.

(a) A finite biposet B is *graphically representable* if there exists a finite graph $G = \langle V, E \rangle$ of points in R^2 and a bijection f of B onto V such that:

  (1) There is an edge between two vertices x_1, x_2 in V iff $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are related by either $<_t^1$ or $<_k^1$ (i.e. if one of them is an immediate successor of the other according to either $\leq_t$ or $\leq_k$).

  (2) If $(a_1, b_1) \in V$, $(a_2, b_2) \in V$ and the two points are connected by an edge then $a_1 \neq a_2$ and $b_1 \neq b_2$.

  (3) $x <_t y$ iff there exists a (possibly empty) sequence of points $(a_1, b_1), \ldots, (a_{n-1}, b_{n-1})$ in V such that if $f(x) = (a_0, b_0)$ and $f(y) = (a_n, b_n)$ then $a_0 < a_1 < \cdots a_{n-1} < a_n$ and there is an edge between (a_i, b_i) and (a_{i+1}, b_{i+1}) for $i = 0, 1, \ldots, n-1$.

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* $FOUR$ is due to Belnap ([Be1], [Be2]), $DEFAULT$ – to Ginsberg ([Gin]).

(4) $x <_k y$ iff there exists a (possibly empty) sequence of points $(a_1, b_1), \ldots, (a_{n-1}, b_{n-1})$ in V s.t. if $f(x) = (a_0, b_0)$ and $f(y) = (a_n, b_n)$ then $b_0 < b_1 < \cdots b_{n-1} < b_n$ and there is an edge between (a_i, b_i) and (a_{i+1}, b_{i+1}) for $i = 0, \ldots, n-1$.

(b) B is *precisely representable* if in place of (1) we have:

(1)* $\{x_1, x_2\} \in E$ iff $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are related by *both* $<_t^1$ and $<_k^1$.

An example. $FOUR$, $NINE$ and $DEFAULT$ are all graphically representable, as Figures 1 and 2 show. $FOUR$ and $NINE$ are *precisely* representable (see Fig.1). $DEFAULT$ is not. (In Fig. 2 there is an edge between dt and t although $dt <_k t$ but $\neg(dt <_k^1 t)$. It will follow from Theorem 2 below that no other precise representation exists.)

Theorem 1. *A finite biposet B is representable iff the following two conditions are satisfied:*

(i) If $x <_t^1 y$ then $x <_k y$ or $y <_k x$.

(ii) If $x <_k^1 y$ then $x <_t y$ or $y <_t x$.

Proof: The necessity of the conditions is easy. Suppose, for example, that $a <_t^1 b$. Then by condition (1) of Definition 1, there is an edge between $f(x) = (a_0, b_0)$ and $f(b) = (a_1, b_1)$. By condition (2), $b_0 \neq b_1$ and so $b_0 < b_1$ or $b_1 < b_0$. Hence, by condition 4 (with $n = 0$), $x <_k y$ or $y <_k x$. For the converse, assume that conditions (i) and (ii) obtain. Let $B = \{x_1, x_2, \ldots, x_m\}$. We construct a set $V \subseteq R^2$ and bijection f from B onto V so that:

(*) If $f(x_i) = (a_i, b_i)$, $f(x_j) = (a_j, b_j)$ and $x_i <_t x_j$ ($x_i <_k x_j$) then $a_i < a_j$ ($b_i < b_j$). Moreover if $x_i <_t^1 x_j$ ($x_i <_k^1 x_j$) then $a_i \neq a_j$ and $b_i \neq b_j$.

The construction is by induction on m. The case $m = 1$ is trivial. Suppose we have a construction for $\{x_1, \ldots, x_m\}$. Let $A_t = \{i \mid x_i <_t x_{m+1}\}$, $B_t = \{i \mid x_{m+1} <_t x_i\}$. Suppose $f(x_i) = (a_i, b_i)$ for $1 \leq i \leq m$. By induction hypothesis, $a_i < a_j$ for every $i \in A_t$, $j \in B_t$. Let a_{m+1} be some number between $\max_{i \in A_t}\{a_i\}$ and $\min_{i \in B_t}\{a_i\}$ so that $a_{m+1} \neq a_i$ whenever $x_{m+1} <_k^1 x_i$ or $x_i <_k^1 x_{m+1}$. (The existence of such a number is due to the finiteness of m.) Choose b_{m+1} in a similar way (using $\leq_k$ instead of $\leq_t$). The choice of (a_{m+1}, b_{m+1}) can obviously be done so that this point does not belong to $\{f(x_1), \ldots, f(x_m)\}$. Add (a_{m+1}, b_{m+1}) to V and let $f(x_{m+1}) = (a_{m+1}, b_{m+1})$. Obviously, condition (*) is preserved.

Now connect $f(x_i)$ and $f(x_j)$ by an edge iff x_i and x_j are related by either $\leq_t^1$ or

$\leq_k^1$. We claim that the resulting $\langle V, E\rangle$ (together with f) is a graphical representation of B. It is trivial that f is a bijection of B onto V and that the first two conditions in the definition of a graphical representation are satisfied. The proofs of the other two conditions are practically identical. We prove here condition (3).

Assume first that $x <_t y$. Since B is finite, this entails that there are $z_1, \ldots, z_{n-1} \in B$ s.t. $x <_t^1 z_1 <_t^1 \cdots <_t^1 z_{n-1} <_t^1 y$. Let $f(z_i) = (a_i, b_i)$. From the construction of V it follows that $(a_1, b_1), \ldots, (a_{n-1}, b_{n-1})$ are points as required.

For the converse, assume $f(x) = (a_0, b_0)$, $f(y) = (a_n, b_n)$ and that there exist points $(a_1, b_1), \ldots, (a_{n-1}, b_{n-1})$ in V s.t. $a_0 < a_1 < \cdots < a_{n-1} < a_n$ and there is an edge between (a_i, b_i) and (a_{i+1}, b_{i+1}) for $0 \leq i \leq n-1$. This means, by definition, that $z_i = f^{-1}(a_i, b_i)$ and $z_{i+1} = f^{-1}(a_{i+1}, b_{i+1})$ are related by either $<_t^1$ or $<_k^1$. By condition (ii) of the theorem this entails that z_i and z_{i+1} are related by $<_t$. It is impossible that $z_{i+1} <_t z_i$, since this implies, by $(*)$, that $a_{i+1} < a_i$. Hence $z_i <_t z_{i+1}$. It follows that $x = z_0 <_t z_1 <_t \cdots <_t z_{n-1} <_t z_n = y$, and so $x <_t y$. Ξ

Theorem 2. *A finite biposet B is precisely representable if the following two conditions obtain:*

(i) If $x <_t^1 y$ then $x <_k^1 y$ or $y <_k^1 x$.

(ii) If $x <_k^1 y$ then $x <_t^1 y$ or $y <_k^1 x$.

Proof: Obviously (i) and (ii) imply, respectively, conditions (i) and (ii) of Theorem 1. Hence we can apply the same construction as in the proof of Theorem 1. It is immediate from the definitions and (i), (ii), that the representation we get is precise. Ξ

We now suggest the following definition of bilattices in the finite case:

Definition 5.

(a) A finite bilattice is a finite prebilattice which satisfies conditions (i) and (ii) of Theorem 1.

(b) A precise (finite) bilattice is a (finite) prebilattice which satisfies conditions (i) and (ii) of Theorem 2.

If we adopt these definitions we can reformulate the two theorems above as follows:

Corollary.

(a) *A finite prebilattice is a bilattice if it is graphically representable.*

(b) *A finite prebilattice is precise if it is precisely representable.*

A connection with Fitting's notions is given by the following theorem:

Theorem 3. *Every interlaced prebilattice B is precise.**

Proof: Assume, e.g., that $a <_t^1 b$. Then $a \leq_t b$ and so (since B is interlaced) $a = a \oplus a \leq_t a \oplus b \leq_t b \oplus b = b$. Hence $a \leq_t a \oplus b \leq_t b$. Since b is a $\leq_t$ – successor of a, this means that either $a \oplus b = a$ or $a \oplus b = b$. Since $a \neq b$ this entails that either $b <_k a$ or $a <_k b$. Assume, e.g. that the first case holds, i.e. $b <_k a$. We show that a is a $\leq_k$– successor of b. Let c satisfy: $b \leq_k c \leq_k a$. The fact that B is interlaced implies now that $a = b \wedge a \leq_k c \wedge a \leq_k a \wedge a = a$ ($b \wedge a = a$ since $a \leq_t b$!). Hence $c \wedge a = a$ and $a \leq_t c$. On the other hand, from $b \leq_k c \leq_k a$ follows also that $b = b \vee b \leq_k c \vee b \leq a \vee b = b$. Hence $c \vee b = b$ and so $c \leq_t b$. We get $a \leq_t c \leq_t b$. Since $a <_t^1 b$, this means that $c = a$ or $c = b$. To sum up: we have shown that if $b \leq_k c \leq_k a$ then $c = b$ or $c = a$. This means that a is a $\leq_k$-successor of b (since $a \neq b$ and we are assuming $b \leq_k a$).

The other case, $a <_k b$, is similar and is left to the reader. Ξ

Corollary. *Every finite, interlaced bilattice is precisely representable in R^2.*

Conjecture. Every finite and precise bilattice is interlaced.

If the last conjecture is true then the notion of an interlaced bilattice is a natural generalization of the idea of a precisely representable (pre) bilattice, a generalization which can be applied to arbitrary prebilattices, not necessarily finite ones (in infinite lattices the $<^1$ relation does not determine $<$ and may even be empty. This happens, for example, when the order relation is dense). It would be nice to have a similar generalization for the more general class of finite, representable prebilattice. Such a generalization can serve as an adequate definition of the notion of a bilattice. One natural possibility is the following:

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* it is worth mentioning here that in [Fi1] it is shown that every distributive bilattice is interlaced.

Definition 6. A bilattice in the strong sense is a prebilattice which satisfies:

(i) If $a \leq_t b$ then $a \leq_t a \otimes b \leq_t b$ and $a \leq_t a \oplus b \leq_t b$.

(ii) If $a \leq_k b$ then $a \leq_k a \wedge b \leq_k b$ and $a \leq_k a \vee b \leq_k b$.

An example. $FOUR$, $NINE$ and $DEFAULT$ are all bilattices in the strong sense.

Obviously, every interlaced bilattice is a bilattice in the strong sense, and every finite bilattice in the strong sense satisfies the two conditions in Theorem 1 and so is graphically representable. None of the two converses is true. In Figure 3 we have an example of a finite bilattice in the strong sense which is not precise (and so – not interlaced). In Figure 4 we have an example of a representable prebilattice which is not a bilattice in the strong sense.

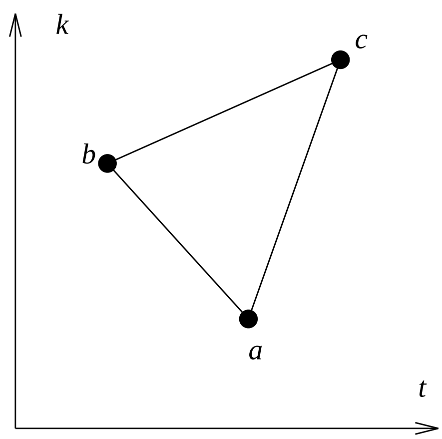

Figure 3

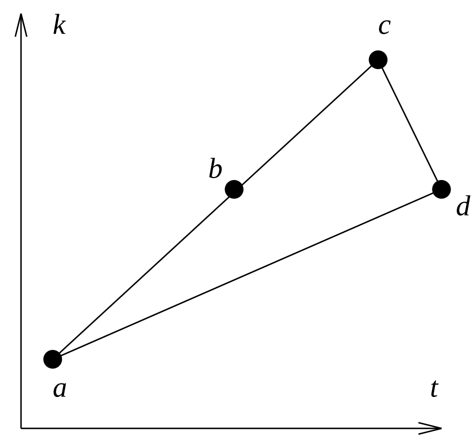

Figure 4

It is not clear that we want a prebilattice as in Figure 4 to count as a bilattice, and so Definition 6 *might* be a good candidate for a definition of this notion.

Another possibility is the following:

Definition 7. A prebilattice is called a bilattice in the weak sense if it satisfies:

(i) If $a \leq_t b$ then these exists $a \leq_t c \leq_t b$ such that $(a \leq_k c \ \vee \ c \leq_k a) \ \wedge \ (b \leq_k c \ \vee c \ \leq_k b)$.

(ii) If $a \leq_k b$ then there exists $a \leq_k c \leq_k b$ such that $(a \leq_t c \ \vee \ c \leq_t a) \ \wedge \ (b \leq_t c \ \vee \ c \leq_t b)$.

Obviously, a bilattice in the strong sense is a bilattice in the weak sense. The inverse, however, fails. Figure 4 again represents a counter-example. Also every bilattice in the weak sense trivially satisfies the conditions in Theorem 1 and so is graphically representable.

Conjecture. In the finite case the converse also holds: every graphically representable prebilattice is a bilattice in the weak sense.

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