On Negation, Completeness and Consistency

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1 Introduction

In this paper we try to understand negation from two different points of view: a syntactical one and a semantic one. Accordingly, we identify two different types of negation. The same connective of a given logic might be of both types, but this might not always be the case.

The syntactical point of view is an abstract one. It characterizes connectives according to the internal ***role*** they have inside a logic, regardless of any meaning they are intended to have (if any). With regard to negation our main thesis is that the availability of what we call below an internal negation is what makes a logic essentially ***multiple-conclusion***.

The semantic point of view, in contrast, is based on the intuitive meaning of a given connective. In the case of negation this is simply the intuition that the negation of a proposition A is true if A is not, and not true if A is true.[1]

[1] We have avoided here the term "false", since we do not want to commit ourselves to the view that A is false precisely when it is not true. Our formulation of the intuition is therefore obviously circular, but this is unavoidable in intuitive informal characterizations of basic connectives and quantifiers.

Like in most modern treatments of logics (see, e.g., [Sc74a,b], [Ha79], [Ga81], [Ur84], [Wo88], [Ep95], [Av91a], [Cl91], [FHV92]), our study of negation will be in the framework of Consequence Relations (CRs). Following [Av91a], we use the following rather general meaning of this term:

Definition.

(1) A ***Consequence Relation*** (CR) on a set of formulas is a binary relation $\vdash$ between (finite) multisets of formulas s.t.:

(I) Reflexivity: $A \vdash A$ for every formula A.

(II) Transitivity, or "Cut": if $\Gamma_1 \vdash \Delta_1, A$ and $A, \Gamma_2 \vdash \Delta_2$, then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$.

(III) Consistency: $\emptyset \not\vdash \emptyset$ (where $\emptyset$ is the empty multiset).

(2) A single-conclusion CR is a CR $\vdash$ such that $\Gamma \vdash \Delta$ only if Δ consists of a single formula.

The notion of a (multiple-conclusion) CR was introduced in [Sc74a] and [Sc74b]. It was a generalization of Tarski's notion of a consequence relation, which was single-conclusion. Our notions are, however, not identical to the original ones of Tarski and Scott. First, they both considered ***sets*** (rather than multisets) of formulas. Second, they impose a third demand on CRs: monotonicity. We shall call a (single-conclusion or multiple-conclusion) CR which satisfies these two extra conditions ***ordinary***. A single-conclusion, ordinary CR will be called ***Tarskian***.[2]

The notion of a "logic" is in practice broader than that of a CR, since usually several CRs are associated with a given logic. Given a logic $\mathcal{L}$ there

[2]What we call a Tarskian CR is exactly Tarski's original notion. In [Av94] we argue at length why the notion of a proof in an axiomatic system naturally leads to *our* notion of single-conclusion CR, and why the further generalization to multiple-conclusion CR is also very reasonable.

are in most cases two major single-conclusion CRs which are naturally associated with it: the external CR $\vdash^e_{\mathcal{L}}$ and the internal CR $\vdash^i_{\mathcal{L}}$. For example, if $\mathcal{L}$ is defined by some axiomatic system AS then $A_1, \cdots, A_n \vdash^e_{\mathcal{L}} B$ iff there exists a proof in AS of B from $A_1, \cdots A_n$ (according to the most standard meaning of this notion as defined in undergraduate textbooks on mathematical logic), while $A_1, \cdots, A_n \vdash^i_{\mathcal{L}} B$ iff $A_1 \to (A_2 \to \cdots \to (A_n \to B) \cdots)$ is a theorem of AS (where $\to$ is an appropriate "implication" connective of the logic). Similarly if $\mathcal{L}$ is defined using a Gentzen-type system G then $A_1, \cdots, A_n \vdash^i_{\mathcal{L}} B$ if the sequent $A_1, \cdots, A_n \Rightarrow B$ is provable in G, while $A_1, \cdots A_n \vdash^e_{\mathcal{L}} B$ iff there exists a proof in G of $\Rightarrow B$ from the assumptions $\Rightarrow A_1, \cdots, \Rightarrow A_n$ (perhaps with cuts). $\vdash^e_{\mathcal{L}}$ is always a Tarskian relation, $\vdash^i_{\mathcal{L}}$ frequently is not. The existence (again, in most cases) of these two CRs should be kept in mind in what follows. The reason is that semantic characterizations of connectives are almost always done w.r.t. Tarskian CRs (and so here $\vdash^e_{\mathcal{L}}$ is usually relevant). This is not the case with syntactical characterizations, and here frequently $\vdash^i_{\mathcal{L}}$ is more suitable.

2 The syntactical point of view

2.1 Classification of basic connectives

Our general framework allows us to give a completely abstract definition, ***independent of any semantic interpretation***, of standard connectives. These characterizations explain why these connectives are so important in almost every logical system.

In what follows $\vdash$ is a fixed CR. All definitions are taken to be relative to $\vdash$ (the definitions are taken from [Av91a]).

We consider two types of connectives. The ***internal*** connectives, which make it possible to transform a given sequent into an equivalent one that has a special required form, and the ***combining*** connectives, which allow us to combine (under certain circumstances) two sequents into one which contains exactly the same information. The most common (and useful) among these are the following connectives:

Internal Disjunction: $+$ is an internal disjunction if for all Γ, Δ, A, B:

$$\Gamma \vdash \Delta, A, B \quad \text{iff} \quad \Gamma \vdash \Delta, A + B \ .$$

Internal Conjunction: $\otimes$ is an internal conjunction if for all Γ, Δ, A, B:

$$\Gamma, A, B \vdash \Delta \quad \text{iff} \quad \Gamma, A \otimes B \vdash \Delta \ .$$

Internal Implication: $\rightarrow$ is an internal implication if for all Γ, Δ, A, B:

$$\Gamma, A \vdash B, \Delta \quad \text{iff} \quad \Gamma \vdash A \rightarrow B, \Delta \ .$$

Internal Negation: $\neg$ is an internal negation if the following two conditions are satisfied by all Γ, Δ and A:

$$(1) \qquad A, \Gamma \vdash \Delta \quad \text{iff} \quad \Gamma \vdash \Delta, \neg A$$
$$(2) \qquad \Gamma \vdash \Delta, A \quad \text{iff} \quad \neg A, \Gamma \vdash \Delta \ .$$

Combining Conjunction: $\wedge$ is a combining conjunction iff for all Γ, Δ, A, B:

$$\Gamma \vdash \Delta, A \wedge B \quad \text{iff} \quad \Gamma \vdash \Delta, A \quad \text{and} \quad \Gamma \vdash \Delta, B \ .$$

Combining Disjunction: $\vee$ is a combining disjunction iff for all Γ, Δ, A, B

$$A \vee B, \Gamma \vdash \Delta \quad \text{iff} \quad A, \Gamma \vdash \Delta \quad \text{and} \quad B, \Gamma \vdash \Delta \ .$$

Note: The combining connectives are called "additives" in Linear logic (see [Gi87]) and "extensional" in Relevance logic. The internal ones correspond, respectively, to the "multiplicative" and the "intensional" connectives.

Several well-known logics can be defined using the above connectives:

LL_m — **Multiplicative Linear Logic** (without the propositional constants): This is the logic which corresponds to the ***minimal*** (multiset) CR which includes all the internal connectives.

LL_{ma} — **Propositional Linear Logic** (without the "exponentials" and the propositional constants): This corresponds to the minimal consequence relation which contains all the connectives introduced above.

R_m — **the Intensional Fragment of the Relevance Logic** ***R*****:**[3] This corresponds to the minimal CR which contains all the internal connectives and is ***closed under contraction***.

R **without Distribution:** This corresponds to the minimal CR which contains all the connectives which were described above and is closed under contraction.

RMI_m — **the Intensional Fragment of the Relevance Logic** RMI:[4] This corresponds to the minimal sets-CR which contains all the internal connectives.

Classical Proposition Logic: This of course corresponds to the minimal ordinary CR which has all the above connectives. Unlike the previous logics there is no difference in it between the combining connectives and the corresponding internal ones.

In all these examples we refer, of course, to the ***internal*** consequence relations which naturally correspond to these logics (In all of them it can be defined by either of the methods described above).

2.2 Internal Negation and Strong Symmetry

Among the various connectives defined above only negation essentially demands the use of multiple-conclusion CRs (even the existence of an internal disjunction does not ***force*** multiple-conclusions, although its existence is triv-

[3]see [AB75] or [Du86].

[4]see [Av90a], [Av90b].

ial otherwise.). Moreover, its existence creates full symmetry between the two sides of the turnstyle. Thus in its presence, closure under any of the structural rules on one side entails closure under the same rule on the other, the existence of any of the binary internal connectives defined above implies the existence of the rest, and the same is true for the combining connectives.

To sum up: internal negation is the connective with which "the hidden symmetries of logic" ([Gi87]) are explicitly represented. We shall call, therefore, any multiple-conclusion CR which possesses it ***strongly symmetric***.

Some alternative characterizations of an internal negation are given in the following easy proposition.

Proposition 1 *The following conditions on $\vdash$ are all equivalent:*

(1) $\neg$ *is an internal negation for* $\vdash$.

(2) $\Gamma \vdash \Delta, A$ *iff* $\Gamma, \neg A \vdash \Delta$

(3) $A, \Gamma \vdash \Delta$ *iff* $\Gamma \vdash \Delta, \neg A$

(4) $A, \neg A \vdash$ *and* $\vdash \neg A, A$

(5) $\vdash$ *is closed under the rules:*

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \qquad\qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} .$$

Our characterization of internal negation and of symmetry has been done within the framework of multiple-conclusion relations. Single-conclusion CRs are, however, more natural. We proceed next to introduce corresponding notions for them.

Definition.

(1) Let $\vdash_{\mathcal{L}}$ be a single-conclusion CR (in a language $\mathcal{L}$), and let $\neg$ be a unary connective of $\mathcal{L}$. $\vdash_{\mathcal{L}}$ is called ***strongly symmetric*** w.r.t. to $\neg$, and $\neg$ is called an ***internal negation*** for $\vdash_{\mathcal{L}}$, if there exists a multiple-conclusion CR $\vdash^*_{\mathcal{L}}$ with the following properties:

(i) $\Gamma \vdash^*_{\mathcal{L}} A$ iff $\Gamma \vdash_{\mathcal{L}} A$

(ii) $\neg$ is an internal negation for $\vdash^*_{\mathcal{L}}$

(2) A single-conclusion CR $\vdash_{\mathcal{L}}$ is called ***essentially multiple-conclusion*** iff it has an internal negation.

Obviously, if a CR $\vdash^*_{\mathcal{L}}$ like in the last definition exists then it is unique. We now formulate sufficient and necessary conditions for its existence.

Theorem 2 *$\vdash_{\mathcal{L}}$ is strongly symmetric w.r.t. $\neg$ iff the following conditions are satisfied:*

(i) $A \vdash_{\mathcal{L}} \neg\neg A$

(ii) $\neg\neg A \vdash_{\mathcal{L}} A$

(iii) *If* $\Gamma, A \vdash_{\mathcal{L}} B$ *then* $\Gamma, \neg B \vdash_{\mathcal{L}} \neg A$.

Proof: The conditions are obviously necessary. Assume, for the converse, that $\vdash_{\mathcal{L}}$ satisfies the conditions. Define: $A_1, \cdots, A_n \vdash^s_{\mathcal{L}} B_1, \cdots, B_k$ iff for every $1 \leq i \leq n$ and $1 \leq j \leq k$:

$$A_1, \cdots, A_{i-1}, \neg B_1, \cdots, \neg B_k, A_{i+1}, \cdots, A_n \vdash \neg A_i$$
$$A_1, \cdots, A_n, \neg B_1, \cdots, \neg B_{j-1}, \neg B_{j+1}, \cdots, \neg B_k \vdash B_j \ .$$

It is easy to check that $\vdash^s_{\mathcal{L}}$ is a CR whenever $\vdash_{\mathcal{L}}$ is a CR (whether single-conclusion or multiple-conclusion), and that if $\Gamma \vdash^s_{\mathcal{L}} A$ then $\Gamma \vdash_{\mathcal{L}} A$. The first two conditions imply (together) that $\neg$ is an internal negation for $\vdash^s_{\mathcal{L}}$ (in particular: the second entails that if $A, \Gamma \vdash^s_{\mathcal{L}} \Delta$ then $\Gamma \vdash^s_{\mathcal{L}} \Delta, \neg A$ and the first that if $\Gamma \vdash^s_{\mathcal{L}} \Delta, A$ then $\neg A, \Gamma \vdash^s_{\mathcal{L}} \Delta$). Finally, the third condition entails that $\vdash^s_{\mathcal{L}}$ is conservative over $\vdash_{\mathcal{L}}$. □

Examples of logics with an internal negation.

1. Classical logic.

2. Extensions of classical logic, like the various modal logics.

3. Linear logic and its various fragments.

4. The various Relevance logics (like R and RM (see [AB75], [Du86], [AB92]) or RMI ([Av90a,b])) and their fragments.

5. The various many-valued logics of Łukasiewicz, as well as Sobociński 3-valued logic ([So52]).

Examples of logics without an internal negation.

1. Intuitionistic logic.

2. Kleene's 3-valued logic and its extension LPF ([Jo86]).

Note: Again, in all these examples above it is the ***internal*** CR which is essentially multiple-conclusion (or not) and has an internal negation. This is true even for classical predicate calculus: There, e.g., $\forall x A(x)$ follows from $A(x)$ according to the ***external*** CR, but $\neg A(x)$ does not follow from $\neg\forall x A(x)$.[5]

All the positive examples above are instances of the following proposition, the easy proof of which we leave to the reader:

Proposition 3 *Let $\mathcal{L}$ be any logic in a language containing $\neg$ and $\rightarrow$. Suppose that the set of valid formulae of $\mathcal{L}$ includes the set of formulae in the language of $\{\neg, \rightarrow\}$ which are theorems of Linear Logic,*[6] *and that it is closed under MP for $\rightarrow$. Then the internal consequence relation of $\mathcal{L}$ (defined using $\rightarrow$ as in the introduction) is strongly symmetric (with respect to $\neg$).*

[5]The internal CR of classical logic has been called the "truth" CR in [Av91a] and was denoted there by $\vdash^t$, while the external one was called the "validity" CR and was denoted by $\vdash^v$. On the propositional level there is no difference between the two.

[6]Here $\neg$ should be translated into linear negation, $\rightarrow$ – into linear implication.

The next two theorems discuss what properties of $\vdash_{\mathcal{L}}$ are preserved by $\vdash^s_{\mathcal{L}}$. The proofs are straightforward.

Theorem 4 *Assume $\vdash_{\mathcal{L}}$ is essentially multiple-conclusion.*

1. *$\vdash^s_{\mathcal{L}}$ is monotonic iff so is $\vdash_{\mathcal{L}}$.*

2. *$\vdash^s_{\mathcal{L}}$ is closed under expansion (the converse of contraction) iff so is $\vdash_{\mathcal{L}}$.*

3. *$\wedge$ is a combining conjunction for $\vdash^s_{\mathcal{L}}$ iff it is a combining conjunction for $\vdash_{\mathcal{L}}$.*

4. *$\rightarrow$ is an internal implication for $\vdash^s_{\mathcal{L}}$ iff it is an internal implication for $\vdash_{\mathcal{L}}$.*

Notes:
1) Because $\vdash^s_{\mathcal{L}}$ is strongly symmetric, Parts (3) and (4) can be formulated as follows: $\vdash^s_{\mathcal{L}}$ has the internal connectives iff $\vdash_{\mathcal{L}}$ has an internal implication and it has the combining connectives iff $\vdash_{\mathcal{L}}$ has a combining conjunction.
2) In contrast, a combining disjunction for $\vdash_{\mathcal{L}}$ is not necessarily a combining disjunction for $\vdash^s_{\mathcal{L}}$. It is easy to see that a necessary and sufficient condition for this to happen is that $\vdash_{\mathcal{L}} \neg(A \vee B)$ whenever $\vdash_{\mathcal{L}} \neg A$ and $\vdash_{\mathcal{L}} \neg B$. An example of an essentially multiple-conclusion system with a combining disjunction which does not satisfy the above condition is RMI of [Av90a,b]. That system indeed does not have a combining conjunction. This shows that a ***single-conclusion*** logic $\mathcal{L}$ with an internal negation and a combining disjunction does not necessarily have a combining conjunction (unless $\mathcal{L}$ is monotonic). The converse situation is not possible, though: If $\neg$ is an internal negation and $\wedge$ is a combining conjunction then $\neg(\neg A \wedge \neg B)$ defines a combining disjunction even in the single-conclusion case.
3) An internal conjunction $\otimes$ for $\vdash_{\mathcal{L}}$ is also not necessarily an internal conjunction for $\vdash^s_{\mathcal{L}}$. We need here the extra condition that if $A \vdash_{\mathcal{L}} \neg B$ then

$\vdash_{\mathcal{L}} \neg(A \otimes B)$. An example which shows that this condition does not necessarily obtain even if $\vdash_{\mathcal{L}}$ is an ordinary CR, is given by the following CR $\vdash_{triv}$:

$$A_1, \cdots, A_n \vdash_{triv} B \quad \text{iff} \quad n \geq 1 \ .$$

It is obvious that $\vdash_{triv}$ is a Tarskian CR and that every unary connective of its language is an internal negation for it, while every binary connective is an internal conjunction. The condition above fails, however, for $\vdash_{triv}$.
4) The last example shows also that $\vdash^{s}_{\mathcal{L}}$ may not be closed under contraction when $\vdash_{\mathcal{L}}$ does, even if $\vdash_{\mathcal{L}}$ is Tarskian. Obviously, $\Gamma \vdash^{s}_{triv} \Delta$ iff $|\Gamma \cup \Delta| \geq 2$. Hence $\vdash^{s}_{triv} A, A$ but $\nvdash^{s}_{triv} A$. The exact situation about contraction is given in the next proposition.

Proposition 5 *If $\vdash_{\mathcal{L}}$ is essentially multiple-conclusion then $\vdash^{s}_{\mathcal{L}}$ is closed under contraction iff $\vdash_{\mathcal{L}}$ is closed under contraction and satisfies the following condition:*

If $A \vdash_{\mathcal{L}} B$ and $\neg A \vdash_{\mathcal{L}} B$ then $\vdash_{\mathcal{L}} B$.

In case $\vdash_{\mathcal{L}}$ has a combining disjunction this is equivalent to:

$$\vdash_{\mathcal{L}} \neg A \vee A \ .$$

Proof: Suppose first that $\vdash_{\mathcal{L}}$ is closed under contraction and satisfies the condition. Assume that $\Gamma \vdash^{s}_{\mathcal{L}} \Delta, A, A$. If either Γ or Δ is not empty then this is equivalent to $\neg A, \neg A, \Gamma^* \vdash_{\mathcal{L}} B$ for some Γ^* and B. Since $\vdash_{\mathcal{L}}$ is closed under contraction, this implies that $\neg A, \Gamma^* \vdash_{\mathcal{L}} B$, and so $\Gamma \vdash^{s}_{\mathcal{L}} \Delta, A$. If both Γ and Δ are empty then we have $\neg A \vdash_{\mathcal{L}} A$. Since also $A \vdash_{\mathcal{L}} A$, the condition implies that $\vdash_{\mathcal{L}} A$, and so $\vdash^{s}_{\mathcal{L}} A$.

For the converse, suppose $\vdash^{s}_{\mathcal{L}}$ is closed under contraction. This obviously entails that so is also $\vdash_{\mathcal{L}}$. Assume now that $A \vdash_{\mathcal{L}} B$ and $\neg A \vdash_{\mathcal{L}} B$. Then $A \vdash^{s}_{\mathcal{L}} B$ and $\vdash^{s}_{\mathcal{L}} B, A$. Applying cut we get that $\vdash^{s}_{\mathcal{L}} B, B$, and so $\vdash^{s}_{\mathcal{L}} B$. It follows that $\vdash_{\mathcal{L}} B$. □

3 The semantic point of view

We turn in this section to the semantic aspect of negation.

3.1 The General Framework

A "semantics" for a logic consists of a set of "models". The main property of a model is that every sentence of a logic is either true in it or not (and not both). The logic is sound with respect to the semantics if the set of sentences which are true in each model is closed under the CR of the logic, and complete if a sentence φ follows (according to the logic) from a set T of assumptions iff every model of T is a model of φ. Such a characterization is, of course, possible only if the CR we consider is Tarskian. ***In this section we assume, therefore, that we deal only with Tarskian CRs.*** For logics like Linear Logic and Relevance logics this means that we consider only the ***external*** CRs which are associated with them (see the Introduction).

Obviously, the essence of a "model" is given by the set of sentences which are true in it. Hence a semantics is, essentially, just a set S of theories. Intuitively, these are the theories which (according to the semantics) provide a full description of a possible state of affairs. Every other theory can be understood as a partial description of such a state, or as an approximation of a full description. Completeness means, then, that a sentence φ follows from a theory T iff φ belongs to every superset of T which is in S (in other words: iff φ is true in any possible state of affairs of which T is an approximation).

Now what constitutes a "model" is frequently defined using some kind of algebraic structures. Which kind (matrices with designated values, possible worlds semantics and so on) varies from one logic to another. It is difficult, therefore, to base a general, uniform theory on the use of such structures. Semantics (= a set of theories!) can also be defined, however, purely syntactically. Indeed, below we introduce several types of syntactically defined semantics which are very natural for ***every*** logic with "negation". Our investigations will be based on these types.

Our description of the notion of a model reveals that ***externally*** it is based on two classical "laws of thought": the law of contradiction and the law of excluded middle. When this external point of view is reflected inside the logic with the help of a unary connective $\neg$ we call this connective a (strong) ***semantic negation***. Its intended meaning is that $\neg A$ should be true precisely when A is not. The law of contradiction means then that only consistent theories may have a model, while the law of excluded middle means that the set of sentences which are true in some given model should be negation-complete. The sets of consistent theories, of complete theories and of normal theories (theories that are both) have, therefore a crucial importance when we want to find out to what degree a given unary connective of a logic can be taken as a semantic negation. Thus complete theories reflect a state of affairs in which the law of excluded middle holds. It is reasonable, therefore, to say that this law semantically obtains for a logic $\mathcal{L}$ if its consequence relation $\vdash_{\mathcal{L}}$ is ***determined*** by its set of complete theories. Similarly, $\mathcal{L}$ (strongly) satisfies the law of contradiction iff $\vdash_{\mathcal{L}}$ is determined by its set of consistent theories, and it semantically satisfies both laws iff $\vdash_{\mathcal{L}}$ is determined by its set of normal theories.

The above characterizations might seem unjustifiably strong for logics which are designed to allow non-trivial inconsistent theories. For such logics the demand that $\vdash_{\mathcal{L}}$ should be determined by its set of normal theories is reasonable only if we start with a consistent set of assumptions (this is called strong ***c***-normality below). A still weaker demand (***c***-normality) is that any consistent set of assumptions should be an approximation of at least one normal state of affairs (in other words: it should have at least one normal extension).

It is important to note that the above characterizations are independent of the existence of any internal reflection of the laws (for example: in the forms $\neg(\neg A \wedge A)$ and $\neg A \vee A$, for suitable $\wedge$ and $\vee$). There might be strong connections, of course, in many important cases, but they are neither necessary nor always simple.

We next define our general notion of semantics in precise terms.

Definition. Let $\mathcal{L}$ be a logic in L and let $\vdash_{\mathcal{L}}$ be its associated CR.

1. A ***setup*** for $\vdash_{\mathcal{L}}$ is a set of formulae in L which is closed under $\vdash_{\mathcal{L}}$. A ***semantics*** for $\vdash_{\mathcal{L}}$ is a nonempty set of setups which does not include the trivial setup (i.e., the set of all formulae).

2. Let S be a semantics for $\vdash_{\mathcal{L}}$. An S-***model*** for a formula A is any setup in S to which A belongs. An S-model of a theory T is any setup in S which is a superset of T. A formula is called S-valid iff every setup in S is a model of it. A formula A S-follows from a theory T ($T \vdash_{\mathcal{L}}^{S} A$) iff every S-model of T is an S-model of A.

Proposition 6 *$\vdash_{\mathcal{L}}^{S}$ is a (Tarskian) consequence relation and $\vdash_{\mathcal{L}} \subseteq \vdash_{\mathcal{L}}^{S}$.*

Notes:

1. $\vdash_{\mathcal{L}}^{S}$ is not necessarily finitary even if $\vdash$ is.

2. $\vdash_{\mathcal{L}}$ is just $\vdash_{\mathcal{L}}^{S(\mathcal{L})}$ where $S(\mathcal{L})$ is the set of all setups for $\vdash_{\mathcal{L}}$.

3. If $S_1 \subseteq S_2$ then $\vdash_{\mathcal{L}}^{S_2} \subseteq \vdash_{\mathcal{L}}^{S_1}$.

Examples:

1. For classical propositional logic the standard semantics consists of the setups which are induced by some valuation in $\{t, f\}$. These setups can be characterized as theories T such that

   (i) $\neg A \in T$ iff $A \notin T$ (ii) $A \wedge B \in T$ iff both $A \in T$ and $B \in T$

   (and similar conditions for the other connectives).

2. In classical predicate logic we can define a setup in S to be any set of formulae which consists of the formulae which are true in some given first-order structure relative to some given assignment. Alternatively we can take a setup to consist of the formulae which are ***valid*** in some given first-order structure. In the first case $\vdash^S = \vdash^t$, in the second $\vdash^S = \vdash^v$, where $\vdash^t$ and $\vdash^v$ are the "truth" and "validity" consequence relations of classical logic (see [Av91a] for more details).

3. In modal logics we can define a "model" as the set of all the formulae which are true in some world in some Kripke frame according to some valuation. Alternatively, we can take a model as the set of all formulae which are valid in some Kripke frame, relative to some valuation. Again we get the two most usual consequence relations which are used in modal logics (see [Av91a] or [FHV92]).

From now on the ***following two conditions will be assumed in all our general definitions and propositions:***

1. The language contains a negation connective $\neg$.

2. For no A are both A and $\neg A$ theorems of the logic.

Definition. Let S be a semantics for a CR $\vdash_{\mathcal{L}}$

1. $\vdash_{\mathcal{L}}$ is strongly complete relative to S if $\vdash_{\mathcal{L}}^S = \vdash_{\mathcal{L}}$.

2. $\vdash_{\mathcal{L}}$ is weakly complete relative to S if for all A, $\vdash_{\mathcal{L}} A$ iff $\vdash_{\mathcal{L}}^S A$.

3. $\vdash_{\mathcal{L}}$ is ***c***-complete relative to S if every consistent theory of $\vdash_{\mathcal{L}}$ has a model in S.

4. $\vdash_{\mathcal{L}}$ is strongly ***c***-complete relative to S if for every A and every ***consistent*** T, $T \vdash_{\mathcal{L}}^S A$ iff $T \vdash_{\mathcal{L}} A$.

Notes:

1. Obviously, strong completeness implies strong c-completeness, while strong c-completeness implies both c-completeness and weak completeness.

2. Strong completeness means that deducibility in $\vdash_{\mathcal{L}}$ is equivalent to semantic consequence in S. Weak completeness means that theoremhood in $\vdash_{\mathcal{L}}$ (i.e., derivability from the empty set of assumptions) is equivalent to semantic validity (= truth in all models). c-completeness means that consistency implies satisfiability. It becomes identity if only consistent sets can be satisfiable, i.e., if $\{\neg A, A\}$ has a model for no A. This is obviously too strong a demand for paraconsistent logics. Finally, strong c-completeness means that if we restrict ourselves to ***normal*** situations (i.e., consistent theories) then $\vdash_{\mathcal{L}}$ and $\vdash_{\mathcal{L}}^{S}$ are the same. This might sometimes be weaker than full strong completeness.

The last definition uses the concepts of "consistent" theory. The next definition clarifies (among other things) the meaning of this notion as we are going to use in it this paper.

Definition. Let $\mathcal{L}$ and $\vdash_{\mathcal{L}}$ be as above. A theory in L ***consistent*** if for no A it is the case that $T \vdash_{\mathcal{L}} A$ and $T \vdash_{\mathcal{L}} \neg A$, ***complete*** if for all A, either $T \vdash_{\mathcal{L}} A$ or $T \vdash_{\mathcal{L}} \neg A$, ***normal*** if it is both consistent and complete. $CS_{\mathcal{L}}$, $CP_{\mathcal{L}}$ and $N_{\mathcal{L}}$ will denote, respectively, the sets of its consistent, complete and normal theories.

Given $\vdash_{\mathcal{L}}$, the three classes, $CS_{\mathcal{L}}$, $CP_{\mathcal{L}}$ and $N_{\mathcal{L}}$, provide 3 different syntactically defined semantics for $\vdash_{\mathcal{L}}$, and 3 corresponding consequence relations $\vdash_{\mathcal{L}}^{CS_{\mathcal{L}}}$, $\vdash_{\mathcal{L}}^{CP_{\mathcal{L}}}$ and $\vdash_{\mathcal{L}}^{N_{\mathcal{L}}}$. We shall henceforth denote these CRs by $\vdash_{\mathcal{L}}^{CS}$, $\vdash_{\mathcal{L}}^{CP}$ and $\vdash_{\mathcal{L}}^{N}$, respectively. Obviously, $\vdash_{\mathcal{L}}^{CS} \subseteq \vdash_{\mathcal{L}}^{N}$ and $\vdash_{\mathcal{L}}^{CP} \subseteq \vdash_{\mathcal{L}}^{N}$. In the rest of this section we investigate these relations and the completeness properties they induce.

Let us start with the easier case: that of $\vdash^{CS}_{\mathcal{L}}$. It immediately follows from the definitions (and our assumptions) that relative to it every logic is strongly c-complete (and so also c-complete and weakly complete). Hence the only completeness notion it induces is the following:

Definition. A logic $\mathcal{L}$ with a consequence relation $\vdash_{\mathcal{L}}$ is strongly consistent if $\vdash^{CS}_{\mathcal{L}} = \vdash_{\mathcal{L}}$.

$\vdash^{CS}_{\mathcal{L}}$ is not a really interesting CR. As the next theorem shows, what it does is just to trivialize inconsistent $\vdash_{\mathcal{L}}$-theories. Strong consistency, accordingly, might not be a desirable property, certainly not a property that any logic with negation should have.

Proposition 7

1. *$T \vdash^{CS}_{\mathcal{L}} A$ iff either T is inconsistent in $\mathcal{L}$ or $T \vdash_{\mathcal{L}} A$. In particular, T is $\vdash^{CS}_{\mathcal{L}}$-consistent iff it is $\vdash_{\mathcal{L}}$-consistent.*

2. *$\mathcal{L}$ is strongly consistent iff $\neg A, A \vdash_{\mathcal{L}} B$ for all A, B (iff T is consistent whenever $T \nvdash A$).*

3. *Let $\mathcal{L}^{CS}$ be obtained from $\mathcal{L}$ by adding the rule: from $\neg A$ and A infer B. Then $\vdash^{CS}_{\mathcal{L}} = \vdash_{\mathcal{L}^{CS}}$. In particular: if $\vdash_{\mathcal{L}}$ is finitary then so is $\vdash^{CS}_{\mathcal{L}}$.*

4. *$\vdash^{CS}_{\mathcal{L}}$ is strongly consistent.*

We turn now to $\vdash^{CP}$ and $\vdash^{N}$. In principle, each provides 4 notions of completeness. We don't believe, however, that considering the two notions of c-consistency is natural or interesting in the framework of $\vdash^{CP}$ (c-completeness, e.g., means there that every consistent theory has a complete extension, but that extension might not be consistent itself). Accordingly we shall deal with the following 6 notions of syntactical completeness.[7]

[7] In [AB75] the term "syntactically complete" was used for what we call below "strongly c-normal".

Definition.

Let $\mathcal{L}$ be a logic and let $\vdash_{\mathcal{L}}$ be its consequence relation.

1. $\mathcal{L}$ is ***strongly complete*** if it is strongly complete relative to CP.

2. $\mathcal{L}$ is ***weakly complete*** if it is weakly complete relative to CP.

3. $\mathcal{L}$ is ***strongly normal*** if it is strongly complete relative to N.

4. $\mathcal{L}$ is ***weakly normal*** if it is weakly complete relative to N.

5. $\mathcal{L}$ is ***c-normal*** if it is c-complete relative to N.

6. $\mathcal{L}$ is ***strongly c-normal*** if it is strongly c-complete relative to N (this is easily seen to be equivalent to $\vdash_{\mathcal{L}}^{N} = \vdash_{\mathcal{L}}^{CS}$).

For the reader's convenience we repeat what these definitions actually mean:

1. $\mathcal{L}$ is strongly complete iff whenever $T \nvdash_{\mathcal{L}} A$ there exists a complete extension T^* of T such that $T^* \nvdash_{\mathcal{L}} A$.

2. $\mathcal{L}$ is weakly complete iff whenever A is not a theorem of $\mathcal{L}$ there exists a complete T^* such that $T^* \nvdash_{\mathcal{L}} A$.

3. $\mathcal{L}$ is strongly normal iff whenever $T \nvdash_{\mathcal{L}} A$ there exists a complete and consistent extension T^* of T such that $T^* \nvdash_{\mathcal{L}} A$.

4. $\mathcal{L}$ is weakly normal iff whenever A is not a theorem of $\mathcal{L}$ there exists a complete and consistent theory T^* such that $T^* \nvdash_{\mathcal{L}} A$.

5. $\mathcal{L}$ is c-normal if every consistent theory of $\mathcal{L}$ has a complete and consistent extension.

6. $\mathcal{L}$ is strongly c-normal iff whenever T is consistent and $T \nvdash_{\mathcal{L}} A$ there exists a complete and consistent extension T^* of T such that $T^* \nvdash_{\mathcal{L}} A$.

Our next proposition provides simpler syntactical characterizations of some of these notions in case $\vdash_{\mathcal{L}}$ is finitary.

Proposition 8 *Assume that $\vdash_{\mathcal{L}}$ is finitary.*

1. *$\mathcal{L}$ is strongly complete iff for all T, A and B:*

$$(*) \quad T, A \vdash_{\mathcal{L}} B \quad \textit{and} \quad T, \neg A \vdash_{\mathcal{L}} B \quad \textit{imply} \quad T \vdash_{\mathcal{L}} B$$

   In case $\mathcal{L}$ has a combining disjunction $\vee$ then $(*)$ is equivalent to the theoremhood of $\neg A \vee A$ (excluded middle).

2. *$\mathcal{L}$ is strongly normal if for all T and A:*

$$(**) \quad T \vdash_{\mathcal{L}} A \quad \textit{iff} \quad T \cup \{\neg A\} \quad \textit{is inconsistent.}$$

3. *$\mathcal{L}$ is strongly c-normal iff $(**)$ obtains for every* consistent T.

4. *$\mathcal{L}$ is c-normal iff for every consistent T and every A either $T \cup \{A\}$ or $T \cup \{\neg A\}$ is consistent.*

Proof: Obviously, strong completeness implies $(*)$. For the converse, assume that $T \nvdash B$. Using $(*)$, we extend T in stages to a complete theory such that $T^* \nvdash B$. This proves part 1. The other parts are straightforward. □

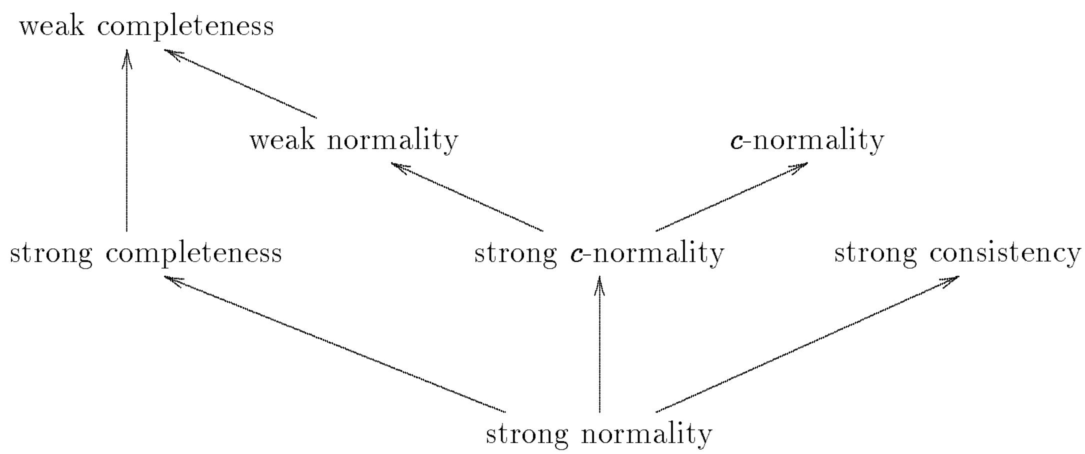
Corollaries:

1. If $\mathcal{L}$ is strongly normal then it is strongly symmetric w.r.t. $\neg$. Moreover: $\vdash^s_{\mathcal{L}}$ is an ordinary multiple-conclsion CR.

2. If $\mathcal{L}$ is strongly symmetric w.r.t. $\neg$ then it is strongly complete iff $\vdash^s_{\mathcal{L}}$ is closed under contraction.

Proof: These results easily follows from the last proposition and Theorems 2, 4 and 5 above. □

In the figure below we display the obvious relations between the seven properties of logics which were introduced here (where an arrow means "contained in"). The next theorem shows that no arrow can be added to it:



Theorem 9 *A logic can be:*

1. *strongly consistent and c-normal without even being weakly complete*
2. *strongly complete and strongly c-normal without being strongly consistent (and so without being strongly normal)*
3. *strongly consistent without being c-normal*
4. *strongly complete, weakly normal and c-normal without being strongly c-normal*
5. *strongly complete and c-normal without being weakly normal*

6. *strongly consistent, c-normal and weakly normal without being strongly c-normal (=strongly normal in this case, because of strong consistency)*

7. *strongly complete without being c-normal.*[8]

Proof: Appropriate examples for 1-6 are given below, respectively, in theorems 12, 18, 32, 19, 34 and the corollary to theorem 19. As for the last part, let $\mathcal{L}$ be the following system in the language of $\{\neg, \rightarrow\}$:[9]

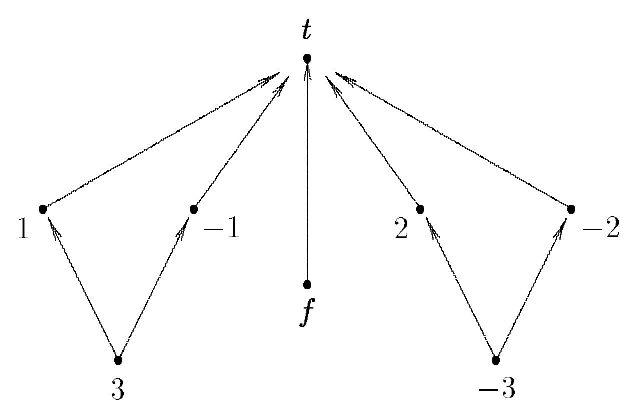
$Ax1$: $A \rightarrow (B \rightarrow A)$

$Ax2$: $A \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$

$Ax3$: $(\neg A \rightarrow B) \rightarrow ((A \rightarrow B) \rightarrow B)$

(MP) $\dfrac{A \quad A \rightarrow B}{B}$.

Obviously, the deduction theorem for $\rightarrow$ holds for this system, since MP is the only rule of inference, and we have $Ax1$ and $Ax2$. This fact, $Ax3$ and proposition 8 guarantee that it is strongly complete. To show that it is not c-normal, we consider the theory $T_0 = \{p \rightarrow q, p \rightarrow \neg q, \neg p \rightarrow r, \neg p \rightarrow \neg r\}$. Obviously, T_0 has no complete and consistent extension. We show that it is consistent nevertheless. For this we use the following structure:



[8]Hence the two standard formulations of the "strong consistency" of classical logic are *not* equivalent in general.

[9]Classical logic is obtained from it by adding $\neg A \rightarrow (A \rightarrow B)$ as axiom (see [Ep95, Ch.2L]).

Define in this structure $\boldsymbol{a} \to \boldsymbol{b}$ as $\boldsymbol{t}$ if $\boldsymbol{a} \leq \boldsymbol{b}$, $\boldsymbol{b}$ otherwise, $\neg\boldsymbol{x}$ as $\boldsymbol{f}$ if $\boldsymbol{x} = \boldsymbol{t}$, $\boldsymbol{t}$ if $\boldsymbol{x} = \boldsymbol{f}$ and $-\boldsymbol{x}$ otherwise. It is not difficult now to show that if $T \vdash A$ in the present logic for some T and A, and $\boldsymbol{v}$ is a valuation in this structure such that $\boldsymbol{v}(B) = \boldsymbol{t}$ for all $B \in T$, then $\boldsymbol{v}(A) = \boldsymbol{t}$. Take now $\boldsymbol{v}(\boldsymbol{p}) = 3$, $\boldsymbol{v}(\boldsymbol{q}) = 1$, $\boldsymbol{v}(\boldsymbol{r}) = 2$. Then $\boldsymbol{v}(B) = \boldsymbol{t}$ for all $B \in T_0$, but obviously there is no A such that $\boldsymbol{v}(A) = \boldsymbol{v}(\neg A) = \boldsymbol{t}$. Hence T_0 is consistent. $\square$

We end this introductory subsection with a characterization of $\vdash_{\mathcal{L}}^{CP}$ and $\vdash_{\mathcal{L}}^{N}$. The proofs are left to the reader.

Proposition 10

1. *$\vdash_{\mathcal{L}}^{CP}$ is strongly complete, and is contained in any strongly complete extension of $\vdash_{\mathcal{L}}$.*
2. *Suppose $\vdash_{\mathcal{L}}$ is finitary. $T \vdash_{\mathcal{L}}^{CP} A$ iff for some $B_1, \ldots, B_n$ $(n \geq 0)$ we have that $T \cup \{B_1^*, \ldots, B_n^*\} \vdash_{\mathcal{L}} A$ for every set $\{B_1^*, \ldots, B_n^*\}$ such that $B_i^* = B_i$ or $B_i^* = \neg B_i$ for all i.*
3. *If $\vdash_{\mathcal{L}}$ is finitary, then so is $\vdash_{\mathcal{L}}^{CP}$.*

Proposition 11

1. *$\vdash_{\mathcal{L}}^{N}$ is strongly normal, and is contained in every strongly normal extension of $\vdash_{\mathcal{L}}$.*
2. *If $\vdash_{\mathcal{L}}$ is finitary then $T \vdash_{\mathcal{L}}^{N} A$ iff for some $B_1, \ldots, B_n$ we have that for all $\{B_1^*, \ldots, B_n^*\}$ where $B_i^* \in \{B_i, \neg B_i\}$ $(i = 1, \ldots, n)$, either $T \cup \{B_1^*, \ldots, B_n^*\}$ is inconsistent or $T \cup \{B_1, \ldots, B_n\} \vdash_{\mathcal{L}} A$*
3. *$\vdash_{\mathcal{L}}^{N}$ is finitary if $\vdash_{\mathcal{L}}$ is.*

3.2 Classical and Intuitionistic Logics

Obviously, classical propositional logic is strongly normal. In fact, most of the proofs of the completeness of classical logic relative to its standard two-valued semantics begin with demonstrating the condition $(**)$ in Proposition 8, and are based on the fact that every complete and consistent theory determines a unique valuation in $\{t, f\}$ - and vice versa. In other words: N here is exactly the usual semantics of classical logic, only it can be characterized also using especially simple algebraic structure (and valuations in it). One can argue that this strong normality ***characterizes*** classical logic. To be specific, it is not difficult to show the following claims:

1. classical logic is the only logic in the language of $\{\neg, \wedge\}$which is strongly normal w.r.t. $\neg$ and for which $\wedge$ is an internal conjunction. Similar claims hold for the $\{\neg, \rightarrow\}$ language, if we demand $\rightarrow$ to be an internal implication and for the $\{\neg, \vee\}$ language, if we demand $\vee$ to be a combining disjunction.

2. Any logic which is strongly normal and has either an internal implication, or an internal conjunction or a combining disjunction contains classical propositional logic.

The next proposition summarizes the relevant facts concerning intuitionistic logic. The obvious conclusion is that although the official intuitionistic negation has some features of negation, it still lacks most. Hence, it cannot be taken as a real negation from our semantic point of view.

Proposition 12 *Intuitionistic logic is strongly consistent and c-normal, but it is not even weakly complete.*

Proof: Strong consistency follows from part 3 of Proposition 7. c-normality follows from part 4 of Proposition 8, since in intuitionistic logic if both $T \cup \{A\}$

and $T \cup \{\neg A\}$ are inconsistent then $T \vdash_H \neg A$ and $T \vdash_H \neg\neg A$, and so T is inconsistent. Finally, $\neg A \vee A$ belongs to every complete setup, but is not intuitionistically valid. $\Box$

Note: Intuitionistic logic and classical logic have exactly the same consistent and complete setups, since any complete intuitionistic theory is closed under the elimination rule of double negation. Hence any consistent intuitionistic theory has a classical two-valued model.

What about fragments (with negation) of Intuitionistic LogicΓ Well, they are also strongly consistent and *c*-normal, by the same proof. Moreover, $((A \to B) \to A) \to A$ is another example of a sentence which belongs to every complete setup (since $A \vdash_H ((A \to B) \to A) \to A$ and $\neg A \vdash_H ((A \to B) \to A) \to A$), but is not provable. The set of theorems of the pure $\{\neg, \wedge\}$ fragment, on the other hand, is identical to that of classical logic, as is well known. This fragment is, therefore, easily seen to be weakly normal. It is still neither strongly complete nor strongly *c*-normal, since $\neg\neg A \vdash_H^{CP} A$. $\Box$

Finally, we note the important fact that classical logic can be viewed as the completion of intuitionistic logic. More precisely:

Proposition 13

1. $\vdash_H^{CS} = \vdash_H$
2. $\vdash_H^{CP} = \vdash_H^N =$ *classical logic.*

Proof:

2. $\vdash_L^{CP} = \vdash_L^N$ whenever L is strongly consistent (i.e., all nontrivial theories are consistent). In the proof of the previous proposition we have seen also that $\vdash_H^{CP} \neg A \vee A$ and $\vdash_H^{CP} ((A \to B) \to A) \to A$. It is well known, however, that by adding either of this schemes to intuitionistic logic we get classical logic. Hence classical logic is contained in $\vdash_H^{CP}$. Since

classical logic is already strongly complete, $\vdash_H^{CP}$ is exactly classical logic. (Note that this is true for any fragment of the language which includes negation.) □

3.3 Linear Logic (LL)

In the next 3 subsections we are going to investigate some known substructural logics ([SD93]). Before doing it we must emphasize again that in this section it is only the external, Tarskian consequence relation of these logics which can be relevant. This consequence relation can very naturally be defined by using the standard Hilbert-type formulations of these logics: $A_1, \ldots, A_n \vdash_{\mathcal{L}}^e B$ ($\mathcal{L} = LL, R, RM, RMI$, etc.) iff there exists an ordinary deduction of B from $A_1, \ldots, A_n$ in the corresponding Hilbert-type system. This definition is insensitive to the exact choice of axioms (or even rules), provided we take all the rules as rules of derivation and not just as rules of proof. In the case of Linear Logic one can use for this the systems given in [Av88] or in [Tr90]. An alternative equivalent definition of the various external CRs can be given using the standard Gentzen-types systems for these logics (in case such exist), as explained in the introduction. Still another characterization in the case of Linear Logic can be given using the phase semantics of [Gi87]: $A_1, \ldots, A_n \vdash_{LL}^e B$ iff B is true in every phase model of $A_1, \ldots, A_n$. In what follows we shall omit the superscript "e" and write just $\vdash_{LL}$, $\vdash_{LL_m}$, etc.

Unlike in [Gi87] we shall take below negation as one of the connectives of the language of linear logic and write $\neg A$ for the negation of A (this corresponds to Girard's A^-). As in [Av88] and in the relevance logic literature, we use arrow ($\rightarrow$) for linear implication.

We show now that linear logic is incomplete with respect to our various notions.

Proposition 14 LL_m (LL_{ma}, LL) ***is not strongly consistent.***

Proposition 15 *LL_m (LL_{ma}, LL) is neither strongly complete nor c-normal.*

Proof: Consider the following theory:

$$T = \{p \to \neg p\,,\ \neg p \to p\}\ .$$

From the characterization of $\vdash_{LL_m}$ given in [Av92] it easily follows that has T been inconsistent then there would be a provable sequent of the form: $\neg p \to p,\ \neg p \to p, \ldots, \neg p \to p, p \to \neg p, \ldots, p \to \neg p \Rightarrow$. But in any cut-free proof of such a sequent the premises of the last applied rule should have an odd number of occurrences of p, which is impossible in a provable sequent of the purely multiplicative linear logic. Hence T is consistent. Obviously, every complete extension of T proves p and $\neg p$ and so is inconsistent. This shows that LL_m is not c-normal. It also shows that p is not provable from T, although it is provable from any complete extension of it, and so LL_m is not strongly complete. □

Proposition 16 *LL_{ma} (and so also LL) is not weakly complete.*

Proof: $\sim A \oplus A$ is not a theorem of linear logic, but it belongs to any complete theory. □

It follows that Linear logic (and its multiplicative-additive fragment) has none of the properties we define in this section. Its negation is therefore not really a negation from our present ***semantic*** point of view.

Our results still leave the possibility that LL_m might be weakly complete or even weakly normal. We conjecture that it is not, but we have no counterexample.

We end this section by giving axiomatizations of $\vdash^{CP}_{LL}$ and $\vdash^{N}_{LL}$.

Proposition 17 *1. Let LL^{CP} be the full Hilbert-type system for linear logic (as given in* [Av88]*) together with the rule: from $!A \to B$ and $!\neg A \to B$ infer B. Then $\vdash^{CP}_{LL} = \vdash_{LL^{CP}}$.*

2. *Let LL^N be LL^{CP} together with the disjunctive syllogism for $\oplus$ (from $\neg A$ and $A \oplus B$ infer B). Then $\vdash^N_{LL} = \vdash_{LL^N}$.*

Proof:

1. The necessitation rule (from A infer $!A$) is one of the rules of LL.[10] It follows therefore that B should belong to any complete setup which contains both $!A \rightarrow B$ and $!\neg A \rightarrow B$. Hence the new rule is valid for $\vdash^{CP}_{LL}$ and $\vdash_{LL^{CP}} \subseteq \vdash^{CP}_{LL}$.

   For the converse, assume $T \vdash^{CP}_{LL} A$. Then there exist $B_1, \ldots, B_n$ like in proposition 10(2). We prove by induction on n that $T \vdash_{LL^{CP}} A$. The case $n = 0$ is obvious. Suppose the claim is true for $n-1$. We show it for n. By the deduction theorem for LL, $!B^*_1, \ldots, !B^*_n \Rightarrow A$ is derivable from T in LL^{CP}.[11] More precisely: $!B^*_1 \otimes !B^*_2 \ldots \otimes !B^*_n \rightarrow A$ is derivable from T for any choice of $B^*_1, \ldots, B^*_n$. Since $!C \otimes !D \leftrightarrow !(C \& D)$ is a theorem of LL, this means that both $!B_n \rightarrow (!(B^*_1 \& \ldots \& B^*_{n-1}) \rightarrow A)$ and $!\neg B_n \rightarrow (!(B^*_1 \& \ldots \& B^*_{n-1}) \rightarrow A)$. By the new rule of LL^{CP} we get therefore that $T \vdash_{LL^{CP}} !(B^*_1 \& \ldots \& B^*_{n-1}) \rightarrow A$, and so $T \vdash_{LL^{CP}} !B^*_1 \otimes !B^*_2 \otimes \ldots \otimes !B^*_{n-1} \rightarrow A$ for all choices of $B^*_1, \ldots, B^*_{n-1}$. An application of the induction hypothesis gives $T \vdash_{LL^{CP}} A$.

2. The proof is similar, only this time we should have (by proposition 11) that $T \cup \{B^*_1, \ldots, B^*_n\}$ is either inconsistent in L^N or proves A there. In both cases it proves $A \oplus -$ in LL^{CP}. The same argument as before will show that $T \vdash_{LL^{CP}} A \oplus -$. Since $\vdash_{LL} \neg -$, one application of the disjunctive syllogism will give $T \vdash_{LL^{CP}} A$. It remains to show that the disjunctive syllogism is valid for $\vdash^N_{LL}$. This is easy, since $\{\neg A,\ A \oplus B,\ \neg B\}$ is inconsistent in LL, and so any complete and consistent extension of $\{\neg A,\ A \oplus B\}$ will necessarily contain B. □

[10] Note again that we are talking here about $\vdash^e_{LL}$!

[11] In fact, at the beginning it is derivable from T in LL, but for the induction to go through we need to assume derivability in LL^{CP} at each step.

3.4 The Standard Relevance Logic R and its Relatives

In this section we investigate the standard relevance logic R of Anderson and Belnap ([AB75], [Du86]) and its various extensions and fragments. Before doing this we should again remind the reader what consequence relation we have in mind: the ordinary one which is associated with the standard Hilbert-type formulations of these logics. As in the case of linear logic, this means that we take both rules of R (MP and adjunction) as rules of derivation and define $T \vdash_R A$ in the most straightforward way.

Let us begin with the purely intensional (=multiplicative) fragment of R: R_m. We state the results for this system, but they hold for all its nonclassical various extensions (by axioms) which are discussed in the literature.

Theorem 18 ***R_m is not strongly consistent, but it is strongly complete and strongly c-normal.***

Proof: It is well-known that R_m is not strongly consistent in our sense. Its main property that we need for the other claims is that $T, A \vdash_{R_m} B$ iff either $T \vdash_{R_m} B$ or $T \vdash_{R_m} A \to B$. The strong completeness of R_m follows from this property by the provability of $(\neg A \to B) \to ((A \to B) \to B)$ and proposition 8(1).

To show strong c-normality, we note first that a theory T is inconsistent in R_m iff $T \vdash_{R_m} \neg(B \to B)$ for some B (because $\vdash_{R_m} \neg B \to (B \to \neg(B \to B)))$. Suppose now that T is consistent and $T \nvdash_{R_m} A$. Were $T \cup \{\neg A\}$ inconsistent then by the same main property and the consistency of T we would have that $T \vdash_{R_m} \neg A \to \neg(B \to B)$ for some B, and so that $T \vdash_{R_m} (B \to B) \to A$ and $T \vdash_{R_m} A$. A contradiction. Hence $T \cup \{\neg A\}$ is consistent and we are done by proposition 8(3). $\square$

The last theorem is the optimal theorem concerning negation that one can expect from a logic which was designed to be paraconsistent. It shows that with respect to normal "situations" (i.e., consistent theories) the negation connective of R_m behaves exactly as in classical logic. The difference, therefore, is mainly w.r.t. inconsistent theories. Unlike classical logic they are not necessarily trivial in R_m. Strong completeness means, though, that excluded middle, at least, can be assumed even in the abnormal situations.

When we come to R as a whole the situation is not as good as for the purely intensional fragments. Strong c-normality is lost. What we do have is the following:

Theorem 19 ***R is strongly complete, c-normal and weakly normal,*** [12] ***but it is neither strongly consistent nor strongly c-normal.***

Proof: Obviously, R is not strongly consistent. It is also well known that $\neg p, p \vee q \nvdash_R q$. Still q belongs to any complete and consistent extension of the (even classically!) consistent theory $\{\neg p, p \vee q\}$, since $\{\neg p, p \vee q, \neg q\}$ is not consistent in R. It follows that R is not strongly c-normal. On the other hand, to any extension $\mathcal{L}$ of R by axiom schemes it is true that if $T, A \vdash_{\mathcal{L}} C$ and $T, B \vdash_{\mathcal{L}} C$, then $T, A \vee B \vdash_{\mathcal{L}} C$ ([AB75]). Since $\vdash_R A \vee \neg A$, this and proposition 8(1) entail that any such extension is strongly complete. Suppose, next, that T is theory and A a formula such that $T \cup \{A\}$ and $T \cup \{\neg A\}$ are inconsistent ($\mathcal{L}$ as above). Then for some B and C it is the case that $T, A \vdash_{\mathcal{L}} \neg B \wedge B$ and $T, \neg A \vdash_{\mathcal{L}} \neg C \wedge C$. It follows that $T, A \vee \neg A \vdash_{\mathcal{L}} (\neg B \wedge B) \vee (\neg C \wedge C)$. Since $A \vee \neg A$ and $\neg[(\neg B \wedge B) \vee (\neg C \wedge C)]$ are both theorems of R, T is inconsistent in $\mathcal{L}$. By proposition 8(4) this shows that any such logic is c-normal. Suppose, finally, that $\nvdash_R A$. Had $\{\neg A\}$ been inconsistent, we would have that for some B, $\neg A \vdash_R \neg B \wedge B$. This, in turn, entails that $A \vee \neg A \vdash_R A \vee (\neg B \wedge B)$, and so that $\vdash_R A \vee (\neg B \wedge B)$. On the other hand, $\vdash_R \neg(\neg B \wedge B)$. By the famous theorem of Meyer and Dunn concerning the admissibility of the disjunctive syllogism in R ([AB75], [Du86]) it would follow, therefore, that $\vdash_R A$, contradicting our assumption. Hence $\{\neg A\}$ is consistent, and so, by the c-normality of R which we have just

[12]Weak normality is proved in [AB75] under the name "syntactical completeness".

proved, it has a consistent and complete extension which obviously does not prove A. This shows that R is weakly normal (the proof for RM is identical). $\square$

Corollary: $\vdash_R^{CS}$ is strongly consistent, c-normal and weakly normal, but it is not strongly c-normal.

Note: A close examination of the proof of the last theorem shows that the properties of R which are described there are shared by many of its relatives (like RM, for example). We have, in fact, the following generalizations:

1. Every extension of R which is not strongly consistent is also not strongly c-normal.

2. Every extension of R by axiom-schemes is both strongly complete and c-normal.

3. Every extension of R by axiom schemes for which the disjunctive syllogism is an admissible rule[13] is weakly normal.

In fact,(1)–(3) are true (with similar proofs) also for many systems weaker than R in the relevance family, like E.

Our results show that $\vdash_R^{CP} = \vdash_R$, but $\vdash_R^N \neq \vdash_R^{CS}$ (since R is not strongly c-normal). Hence $\vdash_R^N$ is a new consequence relation, and we turn next to axiomatize it.

Definition. Let $\mathcal{L}$ be an extension of R by axiom schemes and let $\mathcal{L}^N$ be the system which is obtained from $\mathcal{L}$ by adding to it the disjunctive syllogism (γ) as an extra rule: from $\neg A$ and $A \vee B$ infer B.

Theorem 20 $\vdash_{\mathcal{L}}^N = \vdash_{\mathcal{L}^N}$.

Proof: To show that $\vdash_{\mathcal{L}^N} \subseteq \vdash_{\mathcal{L}}^N$ it is enough to show that $\neg A, A \vee B \vdash_{\mathcal{L}}^N B$. This was already done, in fact, in the proof of the last theorem. For the

[13]See [AB75] and [Du86] for examples and criteria when this is the case.

converse, assume $T \vdash^N_{\mathcal{L}} A$. Since $\mathcal{L}$ is $\boldsymbol{c}$-normal (see last note), $T \cup \{\neg A\}$ cannot be $\mathcal{L}$-consistent. Hence $T \cup \{\neg A\} \vdash_{\mathcal{L}} \neg B \wedge B$ for some B. This entails that $T \vdash_{\mathcal{L}} A \vee (\neg B \wedge B)$ and that $T \vdash_{\mathcal{L}^N} A$ exactly as in the proof of the weak normality of R. □

3.5 The Purely Relevant Logic *RMI*

The purely relevant logic *RMI* was introduced in [Av90a,b]. Proof-theoretically it differs from R in that:

(i) The converse of contraction (or, equivalently, the mingle axiom of *RM*) is valid in it. This is equivalent to the idempotency of the intensional disjunction $+$ (="par" of Girard). In the purely multiplicative fragment RMI_m it means also that assumptions with respect to $\rightarrow$ can be taken as coming in ***sets*** (rather than multisets, as in LL_m or R_m).

(ii) The adjunction rule ($B, C \vdash B \wedge C$) as well as the distribution axiom ($A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$) are accepted only if B and C are "relevant". This relevance relation can be expressed in the logic by the sentence $R^+(A, B) = (A \rightarrow A) + (B \rightarrow B)$, which should be added as an extra premise to adjunction and distribution (this sentence is the counterpart of the "mix" rule of [Gi87]).

We start our investigation with the easier case of RMI_m.

Theorem 21 ***Exactly like R_m, RMI_m is not strongly consistent, but it is both strongly complete and strongly c-normal.***

Proof: Exactly like in the case of R_m. □

Like in classical logic, and unlike the case of R_m, these two main properties of RMI_m are strongly related to simple, intuitive, algebraic semantics. Originally, in fact, RMI_m was designed to correspond to a class of structures which are called in [Av90a] "full relevant disjunctive lattices" (full r.d.l.). A full r.d.l is a structure which results if we take a tree and attach to each node $\boldsymbol{b}$ its own two basic truth-values $\{\boldsymbol{t}_b, \boldsymbol{f}_b\}$. To a leaf $\boldsymbol{b}$ of the tree we ***can*** attach instead a single truth-value I_b which is the negation of itself (its meaning is "both true and false" or "degenerate"). $\boldsymbol{b}$ is called abnormal in this case. Intuitively, the nodes of the tree represent "domains of discourse". Two domains are relevant to each other if they have a common branch, while $\boldsymbol{b}$ being nearer than $\boldsymbol{a}$ to the root on a branch intuitively means that $\boldsymbol{b}$ has a higher "degree of reality" (or higher "degree of significance") than $\boldsymbol{a}$ (we write $\boldsymbol{a} < \boldsymbol{b}$ in this case). The operation of $\neg$ (negation) is defined on a full r.d.l. M in the obvious way, while $+$ (relevant disjunction) is defined as follows: Let $|\boldsymbol{t}_a| = |\boldsymbol{f}_a| = |I_a| = \boldsymbol{a}$, and let $\mathrm{val}(\boldsymbol{t}_b) = \boldsymbol{t}$, $\mathrm{val}(\boldsymbol{f}_b) = \boldsymbol{f}$ and $\mathrm{val}(I_b) = I$. Define $\boldsymbol{x}_{\leq_+}\boldsymbol{y}$ if either $\boldsymbol{x} = \boldsymbol{y}$ or $|\boldsymbol{x}| < |\boldsymbol{y}|$ or $|\boldsymbol{x}| = |\boldsymbol{y}|$ and $\mathrm{val}(\boldsymbol{y}) = \boldsymbol{t}$. $(M, \leq_+)$ is an upper semilattice. Let $\boldsymbol{x} + \boldsymbol{y} = \sup_{\leq_+}(\boldsymbol{x}, \boldsymbol{y})$. An RMI_m-model is a pair $(M, \boldsymbol{v})$ where M is a full r.d.l. and $\boldsymbol{v}$ a valuation in it (which respects the operations). A sentence A is true in a model $(M, \boldsymbol{v})$ if $\mathrm{val}(\boldsymbol{v}(A)) \neq$. Obviously, every model $(M, \boldsymbol{v})$ determines an RMI_m-setup of all the formulae which are true in it. Denote the collection of all these setups by RDL_m.

Proposition 22 $CP_{RMI_m} = RDL_m$

Proof: It is shown in [Av90b] that the Lindenbaum algebra of any complete RMI_m-theory determines a model in which exactly its sentences are true. This implies that $CP_{RMI_m} \subseteq RDL_m$. The converse is obvious from the definitions. □

Corollary: ([Av90b]): RMI_m is sound and complete for the semantics of full r.d.l.s. In other words: $T \vdash_{RMI_m} A$ iff A is true in every model of T.

Proof: Checking soundness is straightforward, while completeness follows from the syntactic strong compleness of RMI_m (theorem 21) and the last theorem. □

The strong $\boldsymbol{c}$-normality of RMI_m also has an interpretation in terms of the semantics of full r.d.l.s. In order to describe it we need first some definitions:

Definition.

1. A full r.d.l is consistent iff for every $\boldsymbol{x}$ in it val($\boldsymbol{x}$) $\in \{\boldsymbol{t}, \boldsymbol{f}\}$ (i.e., the intermediate truth-value I is not used in its construction). This is equivalent to: $\boldsymbol{x} \neq \neg \boldsymbol{x}$ for all $\boldsymbol{x}$.

2. A model $(\boldsymbol{M}, \boldsymbol{v})$ is consistent iff $\boldsymbol{M}$ is consistent.

3. $CRDL_m$ is the collection of the RMI_m-setups which are determined by some consistent model.

Note:

On every tree one can base exactly one consistent full r.d.l. (but in general many inconsistent ones).

Proposition 23 $N_{RMI_m} = CRDL_m$.

Proof: In the construction from [Av90b] which is mentioned in the proof of proposition 22, a complete and consistent theory is easily seen to determine a consistent model. The converse is obvious. □

In view of the last proposition, the strong $\boldsymbol{c}$-normality of RMI_m and its two obvious corollaries (weak normality and $\boldsymbol{c}$-normality) can be reformulated in terms of the algebraic models as follows:

Proposition 24

1. *If T is consistent then $T \vdash_{RMI_m} A$ iff A is true in any consistent model of T.*

2. *$\vdash_{RMI_m} A$ iff A is true in any consistent model.*

3. *Every consistent RMI_m-theory has a consistent model.*

It follows that if we restrict our attention to consistent RMI_m-theories, we can also restrict our semantics to consistent full r.d.l.s, needing, therefore, only the classical two truth-values $\boldsymbol{t}$ and $\boldsymbol{f}$, but not I.

Exactly as in the case of R, when we pass to RMI things become more complicated. Moreover, although we are going to show that RMI has ***exactly*** the same properties as R, the proofs are harder.

Theorem 25 *RMI is strongly complete.*

Proof: The proof is like the one for R given above, since RMI has the relevant properties of R which were used there (see [Av90b]). □

Like in the case of RMI_m, the strong completeness of RMI is directly connected to the semantics of full r.d.l.s. This semantics is extended in [Av90a,b] to the full language by defining the operator $\wedge$ on a full r.d.l. as follows: define $\leq$ on M by: $\boldsymbol{x} \leq \boldsymbol{y}$ iff $\mathrm{val}(\neg\boldsymbol{x}+\boldsymbol{y}) \neq \boldsymbol{f}$. $(M, \leq)$ is a lattice. Let $\boldsymbol{x} \wedge \boldsymbol{y} = \inf_{\leq}(\boldsymbol{x}, \boldsymbol{y})$. The notions of an RMI-model, consistent RMI-model and the truth of a formula A (of the language of RMI) in such models are defined as in the case of RMI_m. The classes of setups RDL and $CRDL$ are also defined like their counterparts in the case of RMI_m. Again we have:

Proposition 26

1. *$CP_{RMI} = RDL$.*

2. *$N_{RMI} = CRDL$.*

Proof: Similar to the proofs of propositions 22 and 23. □

Again, theorem 25 and 26(1) entail the following: result of [Av90b]:

Corollary: RMI is sound and complete for the semantics of full r.d.l.s.

Theorem 27

1. *$\vdash_{RMI} A$ iff A is valid in all the consistent models.*

2. *RMI is weakly normal.*

Proof:

1. Suppose that $\nvdash_{RMI} A$. Then there is a model (M, v) in which A is not true. Let M' be the consistent full r.d.l based on T_M (the tree on which M is based). Let v' be any valuation in M' which satisfies the following conditions: (i) $|v'(P)| = |v(P)|$ for every atomic P, (ii) $v'(P) = v(P)$ whenever $|v(P)|$ is normal in M. It is easy to see that conditions (i) and (ii) are preserved if we replace P by any sentence. In particular $v'(A) = v(A)$ and so A is not valid in the consistent model M'.

2. Immediate from part (1) and proposition 26(2) □

Theorem 28

1. *RMI is c-normal.*

2. *Every consistent RMI-Theory has consistent model.*

Proof: (1) By proposition 8(4) it suffices to prove that if T is consistent and A a sentence then either $T \cup \{A\}$ or $T \cup \{\neg A\}$ is consistent. This is not so easy, however, since like in R, $T \cup \{A\}$ might be inconsistent even if $T \nvdash \neg A$, while unlike in R, (γ) for $\vee$ is ***not*** sound for $\vdash^{N}_{RMI}$.

Suppose then that $T \cup \{A\}$ and $T \cup \{\neg A\}$ are both inconsistent. Since $\neg B, B \vdash_{RMI} \neg(B \rightarrow B)$, this means, by RMI deduction theorem for $\supset$[14] that there exist sentences B and C such that $T \vdash_{RMI} A \supset \neg(B \rightarrow B)$,

[14] See [Av90b]. The connective $\supset$ is defined there by $a \supset b = b \vee (a \rightarrow b)$.

$T \vdash_{RMI} \neg A \supset \neg(C \to C)$. In order to prove that T is inconsistent it is enough therefore to show that the following theory F_0 is inconsistent:

$$F_0 = \{A \supset \neg(B \to B) \ \ , \ \ \neg A \supset \neg(C \to C)\} \ .$$

For this we show that the following sentence φ and its negation are theorems of F_0 (where $\boldsymbol{a} \circ \boldsymbol{b} = \neg(\neg \boldsymbol{a} + \neg \boldsymbol{b})$):

$$\varphi = (B \vee [\neg A \circ R^+ (A + C, B)]) \wedge (C \vee [(A + C) \circ R^+ (A + B, C)]) \ .$$

By the completeness theorem it suffices to show that φ gets a neutral value (I) in every model of F_0. Let (M, v) be such a model, and denote by R the relevance relation between the nodes of the tree on which M is based. It is easy to see that:

a) $|v(A)| \not< |v(B)| \qquad |v(A)| \not< |v(C)|$

b) If $|v(A)| \not R \, |v(B)|$ or if $v(A)$ is designated then $v(B)$ is neutral.

c) If $|v(A)| \not R \, |v(C)|$ or if $v(\neg A)$ is designated then $v(C)$ is neutral.

Denote, for convenience, $v(A)$ by $\boldsymbol{a}$, $v(B)$ by $\boldsymbol{b}$, $v(C)$ by $\boldsymbol{c}$, and the two conjuncts of φ by φ_1 and φ_2 respectively. Then:

(i) If $|\boldsymbol{b}| \not R \, (|\boldsymbol{a}| \vee |\boldsymbol{c}|)$ then $v(\varphi_1) = \boldsymbol{b}$. Also we have then that $|\boldsymbol{c}| \leq |\boldsymbol{a}| \vee |\boldsymbol{c}| < |\boldsymbol{a}| \vee |\boldsymbol{b}| = |\boldsymbol{a} + \boldsymbol{b}|$ (since always $(|\boldsymbol{a}| \vee |\boldsymbol{b}|) \, R \, (|\boldsymbol{a}| \vee |\boldsymbol{c}|)$). Hence $|\boldsymbol{c}| \, R \, |\boldsymbol{a} + \boldsymbol{b}|$ and so $v(\varphi_2) = \boldsymbol{t}_{|a| \vee |b| \vee |c|}$. It follows that $v(\varphi) = \boldsymbol{b}$ and so $v(\varphi)$ is neutral by b) above.

(ii) If $|\boldsymbol{b}| \, R \, (|\boldsymbol{a}| \vee |\boldsymbol{c}|)$ and either $|\boldsymbol{a}| < |\boldsymbol{a}| \vee |\boldsymbol{c}|$ or $\mathrm{val}(\boldsymbol{a}) = \boldsymbol{f}$ then, by a), $|\boldsymbol{b}| \leq |\boldsymbol{a}| \vee |\boldsymbol{c}|$ and either $|\boldsymbol{a}| \not R \, |\boldsymbol{c}|$ or $v(\neg A)$ is designated. Hence $\boldsymbol{c}$ is neutral by c). It follows (since either $|\boldsymbol{a}| \not R \, |\boldsymbol{c}|$ or $\mathrm{val}(\boldsymbol{a}) = \boldsymbol{f}$), that either $|\boldsymbol{a}| \not R \, |\boldsymbol{c}|$ or $|\boldsymbol{c}| < |\boldsymbol{a}|$. In both cases $v(A + C) = \boldsymbol{f}_{|a| \vee |b| \vee |c|}$, $v(\varphi_2) = \boldsymbol{c}$, and $v(\varphi_1) = \boldsymbol{t}_{|a| \vee |b| \vee |c|}$. Hence $v(\varphi) = \boldsymbol{c}$, which is neutral.

(iii) If $|b|\, R\, (|a| \vee |c|)$, $|a| = |a| \vee |c|$ and a is designated then, by a), $|a| = |a| \vee |b| \vee |c|$. If $\mathrm{val}(a) = I$ then also $\mathrm{val}(b) = I$ and $\mathrm{val}(c) = I$, and so $\mathrm{val}(v(\varphi)) = I$. If $\mathrm{val}(a) = t$ then by b) b is neutral and so $|b| < |a|$ ($|a|$ is normal!). Obviously $|c| \leq |a|$ in this case, and so $v(\varphi_1) = b$, $v(\varphi_2) = t_{|a|} = a$ and $v(\varphi) = b$, which is neutral.

(2) Immediate from (1) and proposition 26(2). □

Proposition 29 *RMI is not strongly c-normal.*

Proof: Let ψ_1 and ψ_2 be the two elements of the theory F_0 from the last proof. Let $T = \{\psi_1\}$, $A = \neg\psi_2$. Then T is consistent (even classically!) and A is provable in every consistent and complete extension of T (since F_0 is inconsistent). Hence $T \vdash^N_{RMI} A$. However, $T \nvdash_{RMI} A$ since it is easy to construct a full model of ψ_1 in which $\neg\psi_2$ is not true. (ψ_1 is neutral in this model.) □

Like in the case of R, our results show that $\vdash^N_{RMI}$ is stronger than $\vdash_{RMI}$ and $\vdash^{CS}_{RMI}$. We now construct a formal system for this consequence relation.

Definition. The system $RMIC$ is RMI strengthened by MT for $\supset$:

$$A \supset B\ , \quad \neg B \vdash \neg A\ .$$

Theorem 30

1. *$T \vdash_{RMIC} A$ iff $T \vdash^N_{RMI} A$*
2. *$\vdash_{RMIC} A$ iff $\vdash_{RMI} A$.*

Proof:

1. Obviously, if both $A \supset B$ and $\neg B$ are true in a consistent model (M, v) then so is $\neg A$. Hence if $T \vdash_{RMIC} A$ then $T \vdash^N_{RMI} A$. For the

converse, suppose $T \vdash^N_{RMI} A$. Then by Theorem 26 $T \cup \{\neg A\}$ has no consistent model. This means, by Theorem 28, that $T \cup \{\neg A\}$ is inconsistent. Hence $T \vdash_{RMI} \neg A \supset \neg(B \to B)$ for some B. Since also $\vdash_{RMI} \neg\neg(B \to B)$, we have that $T \vdash_{RMIC} \neg\neg A$, by applying M.T. Hence $T \vdash_{RMIC} A$.

2. Immediate from 1) and theorem 27(2). □

Notes:

1. From 30(2) it is clear that the system *RMI* is closed under M.T. for $\supset$. By applying this rule to theories we can make, however, any inconsistent theory trivial. This resembles the status of (γ) in R and E. Indeed (γ) may be viewed as M.T. for the usual implication as defined in classical logic. A comparison of theorems 30 and 20 deepens the analogy (note that *RMI* is ***not*** an extension of R and 20 fails for it!).

2. Despite 30(2) *RMI* and *RMIC* are totally different even for consistent theories, as we have seen in prop. 29. ***It is important, however, to note that theory T is consistent in RMI iff it is consistent in RMIC.*** This follows easily from theorem 28.

3.6 Three Valued Logics

Like in section 2, we consider here only the 3-valued logic which we call in [Av91b] "natural" (in fact, only those with Tarskian CR). All these logics have the connectives $\{\neg, \wedge, \vee\}$ as defined by Kleene. The weaker ones have only these connectives as primitive. The stronger ones have also an implication connective which reflect their consequence relation.

Suppose the truth-values are $\{t, f, I\}$. t and f correspond to the classical truth values. Hence t is designated, f is not. The 3-valued logics are therefore naturally divided into two main classes: those in which I is not designated, and those in which it is. The first type of logics can be understood as those in which the law of contradiction is valid, but excluded middle is not. The second type – the other way around.

3.6.1 Kleene's basic 3-valued logic

This logic, which we denote by $\boldsymbol{K\ell}$, has only $\boldsymbol{t}$ as designated and $\{\neg, \vee, \wedge\}$ as primitives. It has no valid formula, but it does have a non-trivial consequence relation, defined by the 3-valued semantics. A setup in this semantics is any set of the form $\{A \mid \boldsymbol{v}(A) = \boldsymbol{t}\}$ where $\boldsymbol{v}$ is a 3-valued valuation, and the consequence relation $\vdash_{K\ell}$ is defined by this semantics. A sound and strongly complete Gentzen-type or natural deduction formulations have been given in several places (see, e.g., [BCJ84] or [Av91b]).

The properties of $\vdash_{K\ell}$ which are relevant to the present paper are summarized in the following theorem:

Theorem 31

1. *Like intuitionistic logic, $\vdash_{K\ell}$ is strongly consistent, c-normal but not even weakly complete.*

2. *$\vdash_{K\ell}^{CP}$ is classical logic.*

Proof:

1. Since $\neg A, A \vdash_{K\ell} B$, $\vdash_{K\ell}$ is strongly normal. Since $\vdash_{K\ell}^{CP} A \vee \neg A$ but $\nvdash_{K\ell} A \vee \neg A$, $\vdash_{K\ell}$ is not weakly complete.

   We turn now to $\boldsymbol{c}$-normality. First we need a lemma

   Lemma. *If T has a 3-valued model then it has also a classical, two valued model.*

   Proof of the lemma: It is enough to show that every finite subset of T has a two-valued model (by compactness of classical logic). So let Γ

be a finite set which has a 3-valued model. Since De-Morgan laws and the double-negation laws are valid for the three-valued truth tables, we may assume that all the formulas in Γ are in negation normal form. We prove now the claim by induction on the number of $\wedge$ and $\vee$ in Γ. If all the formulas in Γ are either atomic or negations of atomic formula, then the claim is obvious. If $\Gamma = \Gamma_1 \cup \{A \wedge B\}$ then Γ has a model iff $\Gamma_1 \cup \{A, B\}$ has a model, and so we can apply the induction hypothesis to $\Gamma_1 \cup \{A, B\}$. If $\Gamma = \Gamma_1 \cup \{A \vee B\}$ then Γ has a model iff either $\Gamma_1 \cup \{A\}$ or $\Gamma_1 \cup \{B\}$ has, and we can apply the induction hypothesis to the one which does, getting by this a two-valued model for Γ.

To complete the proof of the theorem, let T be a consistent $\vdash_{K\ell}$-theory. The definitions of consistency and of $\vdash_{K\ell}$ imply in this case that it has some 3-valued model. By the lemma it has also a two-valued model. Let T^* be the set of all the formulae that are true in that two-valued model. Then T^* is a $\vdash_{K\ell}$-setup which is consistent (even classically), complete, and an extension of T.

2. Since $\vdash_{K\ell}^{CP} \neg A \vee A$ and $\neg A \vee C,\ A \vee B \vdash_{K\ell} C \vee B$, it is easy to show, using (for example) Shoenfield's axiomatization of classical logic in [Sh67] that $\vdash_{C\ell} \subseteq \vdash_{K\ell}^{CP}$. The converse is obvious, since $\vdash_{K\ell} \subseteq \vdash_{C\ell}$ and $\vdash_{C\ell}$ is strongly complete (by $\vdash_{C\ell}$ we mean here classical logic). □

3.6.2 $LPF/\mathbf{L}_3$

LPF was developed in [BCJ84] for the VDM Project. As explained in [Av91b], it can be obtained from $\vdash_{K\ell}$ by adding an internal implication $\supset$ so that $T, A \vdash_{LPF} B$ iff $T \vdash_{LPF} A \supset B$. The definition of $\supset$ is: $\boldsymbol{a} \supset \boldsymbol{b} = \boldsymbol{t}$ if $\boldsymbol{a} \neq \boldsymbol{t}$, $\boldsymbol{b}$ if $\boldsymbol{a} = \boldsymbol{t}$. Alternatively one can add to the language Lukasiewicz's implication, or the operator Δ used in [BCJ84]. All these connectives are definable from one another with the help of $\neg, \wedge$ and $\vee$.

Theorem 32

1. $\vdash_{LPF}$ ***is strongly consistent but neither weakly complete nor c-normal.***

2. $\vdash_{LPF}^{CP}$ ***is classical logic.***

Proof:

1. That $\vdash_{LPF}$ is strongly consistent but not weakly normal follows from the corresponding fact for $\vdash_{K\ell}$, since $\vdash_{LPF}$ is a conservative extension of $\vdash_{K\ell}$. As for c-normality, it is enough to note that $\{(A \vee \neg A) \supset B, \neg B\}$ is consistent in LPF (take $v(A) = I$, $v(B) = f$) but obviously has no consistent and complete extension.

2. Again, take any axiomatization of classical logic in the LPF-language and check that all the axioms and rules are valid in $\vdash_{LPF}^{CP}$. □

3.6.3 The Basic Paraconsistent 3-valued logic PAC

This logic, which we call PAC in [Av91b] [15], has the same language (with the same definitions of the connectives) as $\vdash_{K\ell}$. The difference is that here both t and I are designated. A setup in the intended semantics is, therefore, this time a set of the form $\{A \mid v(A) = t \text{ or } v(A) = I\}$,where v is a three-valued valuation. A sound and strongly complete (relative to the 3-valued semantics) Gentzen-type axiomatization is given in [Av91b].[16]

Theorem 33

1. $\vdash_{PAC}$ ***is strongly complete, weakly normal and c-normal. It is neither strongly consistent nor strongly c-normal.***

[15] It is a fragment of several logics which got several names in the literature – see next subsection.

[16] Giving a faithful Hilbert-type system is somewhat a problem here, since the set of valid formulas is identical to that of classical logic, but the consequence relation is not.

2. $\vdash^N_{PAC}$ ***is identical to classical logic.***

Proof:

1. The strong completeness theorem for the Gentzen-type system entails that $\vdash_{PAC}$ is finitary. Hence to show strong syntactical completeness it is enough to show that the condition in 8(1) obtains. This is easy. Weak normality is immediate from the fact that $\vdash_{PAC} A$ iff A is a classical tautology (see [Av91b]) and that $\vdash_{PAC} \subseteq \vdash_{C\ell}$. $\boldsymbol{c}$-normality is proved exactly as for R (it is easy to check that $\vdash_{PAC}$ has all the properties which are used in that proof). It is also easy to check that $\neg \boldsymbol{p}, \boldsymbol{p} \nvdash_{PAC} \boldsymbol{q}$ and that $\{\neg \boldsymbol{p}, \boldsymbol{p} \vee \boldsymbol{q}\}$ is consistent, that $\neg \boldsymbol{p}, \boldsymbol{p} \vee \boldsymbol{q} \vdash^N_{PAC} \boldsymbol{q}$ but $\neg \boldsymbol{p}, \boldsymbol{p} \vee \boldsymbol{q} \nvdash_{PAC} \boldsymbol{q}$ (take $\boldsymbol{v}(\boldsymbol{p}) = I$, $\boldsymbol{v}(\boldsymbol{q}) = \boldsymbol{f}$). Hence $\vdash_{PAC}$ is not strongly $\boldsymbol{c}$-normal and not strongly consistent.

2. Since all classical tautologies are valid in $\vdash_{PAC}$ and MP for classical implication is valid for $\vdash^N_{PAC}$, $\vdash_{C\ell} \subseteq \vdash^N_{PAC}$. The converse is obvious, since $\vdash_{C\ell}$ is strongly $\boldsymbol{c}$-normal and $\vdash_{PAC} \subseteq \vdash_{C\ell}$. □

3.6.4 RM_3/J_3

This logic is obtained from PAC by the addition of certain connectives while keeping the same CR. There are two essential ways that this has been done (independently) in the literature (they were shown equivalent in [Av91b]):

(i) Adding an implication $\rightarrow$, defined as in [So52]. In this way we get the strongest logic in the relevance family: the three-valued extension of RM. It is in this way that this logic arose in the relevance literature. The corresponding matrix is called there M_3 and the logic RM_3. It can be axiomatized by adding to R the axioms $A \rightarrow (A \rightarrow A)$ and $A \vee (A \rightarrow B)$.

(ii) Adding an implication $\supset$, defined by (see [dC74]) $\boldsymbol{a} \supset \boldsymbol{b} = \boldsymbol{t}$ if $\boldsymbol{a} = \boldsymbol{f}$, $\boldsymbol{a} \supset \boldsymbol{b} = \boldsymbol{b}$ otherwise. For this connective the deduction theorem holds. In this form the logic was called J_3 in [DO85] (see also [Ep95]) [17]. It was independently investigated also in [Av86] and in [Ro89]. Strongly complete Hilbert-type formulations with M.P. for $\supset$ as the only rule of inference were given in those papers, and a cut-free Gentzen-type formulation can be found in [Av91b].

In what follows we shall use the neutral name ***Pac**** for the CR of PAC in the extended language. The next theorem shows that the main difference between ***Pac**** and PAC is that ***Pac**** is ***not*** weakly normal.

Theorem 34 *1. Pac* is strongly complete and c-normal. It is neither strongly consistent nor weakly normal.*

2. $\vdash^N_{Pac^*}$ ***is identical to classical logic.***

Proof:

1. Strong completeness and $\boldsymbol{c}$-normality can easily be proved. Since $\vdash_{Pac^*}$ is a conservative extension of $\vdash_{Pac}$, it is not strongly consistent. Finally $\vdash^N_{Pac^*} A \wedge \neg A \supset B$, since $\neg(A \wedge \neg A \supset B) \vdash_{Pac^*} A \wedge \neg A$, but $\nvdash_{Pac^*} A \wedge \neg A \supset B$ (the same argument applies to $(A \wedge \neg A \rightarrow B)$).

2. It is provable in [Du70] that classical logic is the only proper extension of RM_3 in the language of $\{\neg, \vee, \wedge, \rightarrow\}$ (from the point of view of theoremhood). Since we have just seen that the set of valid sentences in $\vdash^N_{Pac^*}$ is such a proper extension, and since MP for $\rightarrow$ is valid for it, $\vdash^N_{Pac^*}$ should be identical to $\vdash_{C\ell}$ (in this language). The same argument works for the $\{\neg, \vee, \wedge, \supset\}$ language using the results of [Av86].

[17][DO85] and [Ep95] consider a language with more connectives, but we shall not treat them here.

Alternatively, it is not difficult to show that by adding $\neg A \wedge A \rightarrow B$ to the Hilbert-type formulation of RM_3 or $\neg A \wedge A \supset B$ to that of J_3 we get classical logic in the corresponding languages. $\Box$

4 Conclusion

We have seen two different aspects of negation. From our two points of view the major conclusions are:

- The negation of classical logic is a perfect negation from both syntactical and semantic points of view.

- Next come the intensional fragments of the standard relevance logics (R_m, RMI_m, RM_m). Their negation is an internal negation for their associated internal CR. Relative to the external one, on the other hand, it has the optimal properties one may expect a semantic negation to have in a paraconsistent logic. In the full systems (R, RMI, RM) the situation is similar, though less perfect (from the semantic point of view). It is even less perfect for the 3-valued paraconsistent logic.

- The negation of Linear Logic is a perfect internal negation w.r.t. its associated internal CR. It is not, however, a negation from the semantic point of view. The same applies to Łukasiewicz 3-valued logic.

- The negations of intuitionistic logic and of Kleen's 3-valued logic are not really negations from the two points of view presented here.

In addition we have seen that within our general semantic framework, any consequence relation which is not strongly normal naturally induces one or more derived consequence relations in which its negation better deserves this name. We gave sound and complete axiomatic systems for these derived relations for all the substructural logics we have investigated.

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