Non-deterministic Semantics for Families of Paraconsistent Logics

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Abstract

We investigate four large families of paraconsistent logics. Two are obtained from the positive fragment of classical logic (with or without a bottom element) by adding to it the classical $(\Rightarrow \neg)$ rule, as well as various standard Gentzen-type rules for combinations of negation with other connectives. Two others are similarly obtained from the positive fragment of intuitionistic logic (with or without a bottom element). We provide for all the systems simple semantics which is based on non-deterministic three-valued structures, and prove soundness and completeness for all of them. We show how the semantics can be used for proving interesting proof-theoretical properties of some of these systems. We also determine what version of the cut-elimination theorem obtains in each case. Among the 6144 logics included in these families there are famous paraconsistent logics like $CluN, CLuNs, C_{min}, C_\omega, PI^*$ and J_3.

1 Introduction

In paraconsistent logics we want to reject inferences of the form:

$$\neg p, p \vdash q$$

Where p and q are distinct atomic formulas. Intuitively this means that we would like to allow situations in which somehow both p and $\neg p$ are considered to be true, while q is false. This naturally leads to two different ways in which a proposition φ may be true. φ may be *consistently true* (or absolutely true), and φ may be *inconsistently true* (or contradictory, or paradoxical). In the first case φ is true and its negation is false, while in the second case both φ and its negation are true. Since on the other hand we would like to retain at least the validity of the law of excluded middle ELM (in order for our negation to at least partially deserve this name), this intuition may formally be reflected by the use of *three* truth values: t (for "consistently true"), $\top$ (for "inconsistently true"), and f (for "not true"). Given some meta-notion of truth and falsity, we accordingly expect a valuation v in $\{t, f, \top\}$ to satisfy:

- $v(\varphi) = t$ if φ is true and $\neg\varphi$ is false

- $v(\varphi) = \top$ if φ is true and $\neg\varphi$ is true
- $v(\varphi) = f$ if φ is false (and so $\neg\varphi$ is true, by ELM)

Given the truth value of φ, what do these principles tell us about the truth-value of its negation? Well, it is easy to see that they dictate the following derived principles (and nothing stronger):

- If $v(\varphi) = t$ then $v(\neg\varphi) = f$
- If $v(\varphi) = \top$ then $v(\neg\varphi) \in \{t, \top\}$
- If $v(\varphi) = f$ then $v(\neg\varphi) \in \{t, \top\}$

It follows that the truth-value of φ does not fully determine the truth-value of $\neg\varphi$. Hence *nondeterministic* semantics seems to be appropriate here.

In this paper we explore the application of these ideas for large families of paraconsistent logics. We concentrate on logics which are easily and naturally defined by using Gentzen-type systems with various standard, very common, rules for negation. The differences between the 6144 different systems we investigate are with respect to:

The underlying logic : We consider four possibilities: positive classical logic, positive classical logic with a bottom element (the falsehood constant **ff**), positive intuitionistic logic, and positive intuitionistic logic with a bottom element (again, the falsehood constant **ff**).

The rules for negation : We include the ($\Rightarrow \neg$) rule (which corresponds to ELM) in *all* the systems we consider (including the intuitionistic ones). In addition, our systems may have various rules for combining negation with other connectives (taken from a list given below).

We provide simple non-deterministic semantics for all the systems we consider, and prove for all of them soundness and completeness with respect to this semantics. We use this semantics for showing various properties of the systems, including their being conservative over their underlying logics, and an appropriate version of the cut-elimination theorem in each case.

2 Preliminaries

In what follows p, q, r denote atomic formulas, $A, B, C, \psi, \varphi, \phi$ denote arbitrary formulas, and Γ, Δ denote finite sets of formulas. A sequent has the form $\Gamma \Rightarrow \Delta$. Following tradition, we write Γ, φ and Γ, Δ for $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \Delta$ (respectively). By a *logic* we shall mean a pair $\langle \mathcal{L}, \vdash \rangle$, in which $\mathcal{L}$ is a propositional language, and $\vdash$ is a consequence relation on $\mathcal{L}$.

2.1 The Standard Positive Logics

THE SYSTEM LK^+

Axioms: $A \Rightarrow A$

Structural Rules: Cut, Weakening

Logical Rules:

$$(\supset\Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \quad (\Rightarrow\supset)$$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \quad (\Rightarrow \wedge)$$

$$(\vee \Rightarrow) \quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \quad (\Rightarrow \vee)$$

THE SYSTEM LK: This is the system obtained from LK^+ by adding to it the following axiom:

$$\mathbf{ff} \Rightarrow$$

THE SYSTEMS LJ^+ and LJ: These are the systems which are obtained from LK^+ and LK (respectively) by weakening the $(\Rightarrow\supset)$ rule to:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

Notes:

1. LK is a standard Gentzen-type calculus for propositional classical logic, while LK^+ is its purely positive fragment. The system LJ is the propositional fragment of a well-known (see [Tak75]) multiple-conclusion version of Gentzen's original sequent calculus for intuitionistic logic, while LJ^+ is its purely positive fragment. The systems are sound and complete for these logics and admit cut-elimination.

2. In both LK and LJ it is possible to define the usual negation connective of the corresponding logics by letting $\sim\varphi =_{Df} \varphi \supset \mathbf{ff}$ (for intuitionistic logic this is in fact the common procedure). We shall nevertheless take all four systems as "positive" logics, since our principal goal is to investigate the systems which are obtained from them by adding to their language an independent, paraconsistent negation connective $\neg$. In the case of LK and LJ this would mean that we are extending the propositional classical and intuitionistic logics with an extra connective, not definable in their language.

2.2 Standard Rules for Negation and Corresponding Systems

The two standard Gentzen-type rules for classical negation are:

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \quad (\Rightarrow \neg)$$

Now the rule $(\neg \Rightarrow)$ forces any contradiction to entail every formula. This rule should therefore be rejected in the framework of paraconsistent logics. In order for a connective $\neg$ to still be called "negation", paraconsistent logics usually retain the other rule (corresponding to LEM, the law of excluded middle). This choice leads to the following four basic paraconsistent systems:

Definition 2.1 The systems PLK, $PLK^{\mathbf{ff}}$, PLJ, and $PLJ^{\mathbf{ff}}$ are obtained from LK^+, LK, LJ^+, and LJ (respectively) by enriching their language with the unary connective $\neg$, and adding $(\Rightarrow \neg)$ to their sets of rules.

Instead of $(\neg \Rightarrow)$ (and sometimes instead of both classical rules of negation) many paraconsistent logics and relevance logics employ rules for introducing combinations of negation with other connectives. The most common rules used for this task are the following:

$$(\neg\neg \Rightarrow) \quad \frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A} \quad (\Rightarrow \neg\neg)$$

$$(\neg \supset \Rightarrow) \quad \frac{A, \neg B, \Gamma \Rightarrow \Delta}{\neg(A \supset B), \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \supset B)} \quad (\Rightarrow \neg \supset)$$

$$(\neg\vee \Rightarrow) \quad \frac{\Gamma, \neg A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg A \qquad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)} \quad (\Rightarrow \neg\vee)$$

$$(\neg\wedge \Rightarrow) \quad \frac{\Gamma, \neg A \Rightarrow \Delta \qquad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg A, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \quad (\Rightarrow \neg\wedge)$$

Now in the original formulation of Gentzen ([Gen69]) the rules $(\wedge \Rightarrow)$ and $(\Rightarrow \vee)$ were split into two rules, each with only one side formula. To make our investigation finer we do the same here to $(\neg\vee \Rightarrow)$, $(\Rightarrow \neg\wedge)$ and $(\neg \supset\Rightarrow)$. So instead of these three rules we will consider the following six:

$$(\neg \supset\Rightarrow)_1 \quad \frac{A, \Gamma \Rightarrow \Delta}{\neg(A \supset B), \Gamma \Rightarrow \Delta} \qquad\qquad \frac{\neg B, \Gamma \Rightarrow \Delta}{\neg(A \supset B), \Gamma \Rightarrow \Delta} \quad (\neg \supset\Rightarrow)_2$$

$$(\neg\vee \Rightarrow)_1 \quad \frac{\Gamma, \neg A \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad\qquad \frac{\Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \quad (\neg\vee \Rightarrow)_2$$

$$(\Rightarrow \neg\wedge)_1 \quad \frac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \quad (\Rightarrow \neg\wedge)_2$$

Definition 2.2 Let NR be the union of the following sets of rules:

$$\{(\neg\neg \Rightarrow), (\Rightarrow \neg\neg), (\Rightarrow \neg \supset), (\Rightarrow \neg\vee), (\neg\wedge \Rightarrow)\}$$

$$\{(\neg \supset\Rightarrow)_1, (\neg \supset\Rightarrow)_2, (\neg\vee \Rightarrow)_1, (\neg\vee \Rightarrow)_2, (\Rightarrow \neg\wedge)_1, (\Rightarrow \neg\wedge)_2\}$$

For $\mathbf{L} \in \{PLK, PLK^{\mathbf{ff}}, PLJ, PLJ^{\mathbf{ff}}\}$ and $S \subseteq NR$ we denote by $\mathbf{L}(S)$ the system which is obtained from $\mathbf{L}$ by adding to it the rules in S.

Historical Notes: Some of the systems introduced in Definition 2.2 have already been studied in the literature. PLK itself and $PLK(NR)$ were introduced in [Bat80], where they were called PI and PI^s (respectively). Their names have later been changed by Batens to $CLuN$ and $CLuNs$, respectively (see e.g. [Bat00]). $PLK(NR)$ was independently introduced (together with the 3-valued deterministic semantics described below) in [Avr86, Avr91, Roz89]. In [Avr91] it was called PAC (this name is adopted in [CM02]). $PLK^{\mathbf{ff}}(NR)$ was originally introduced in [Sch60]. Later it was reintroduced (together with its 3-valued deterministic semantics) in [DdC70, D'o85], and was called there J_3 (see also [Eps90]). In [CM02] it is called $LFI1$. The system $PLK(\{(\neg\neg \Rightarrow)\})$ was studied under the name C_{min} in [CM99]. $PLK(\{(\Rightarrow \neg\neg), (\Rightarrow \neg\vee)\})$ was again introduced in [Bat80] (under the name PI^*). $PLJ(\{(\neg\neg \Rightarrow)\})$ is Raggio's formulation (in [Rag68]) of da Costa famous system C_ω (see [dC74]).

2.3 Corresponding Hilbert-type Systems

Some of the systems mentioned above (like C_ω and C_{min}) have originally been introduced using Hilbert-type systems. Such systems can easily be given for every system $\mathbf{L}(S)$. We start with some standard Hilbert-type system for $\mathbf{L}$ (having MP as the only rule of inference), and add to it the axiom $\varphi \vee \neg\varphi$, together with the axioms which correspond to the negation rules in S. Here is the list of axioms that correspond to our 11 rules (details are left for the reader):

$(\neg\neg \Rightarrow)$: $\neg\neg\varphi \supset \varphi$

$(\Rightarrow \neg\neg)$: $\varphi \supset \neg\neg\varphi$

$(\neg \supset\Rightarrow)_1$: $\neg(\varphi \supset \psi) \supset \varphi$

$(\neg \supset\Rightarrow)_2$: $\neg(\varphi \supset \psi) \supset \neg\psi$

$(\Rightarrow \neg \supset)$: $(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$

$(\neg\vee \Rightarrow)_1$: $\neg(\varphi \vee \psi) \supset \neg\varphi$

$(\neg\vee \Rightarrow)_2$: $\neg(\varphi \vee \psi) \supset \neg\psi$

$(\Rightarrow \neg\vee)$: $(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$

$(\neg\wedge \Rightarrow)$: $\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$

$(\Rightarrow \neg\wedge)_1$: $\neg\varphi \supset \neg(\varphi \wedge \psi)$

$(\Rightarrow \neg\wedge)_2$: $\neg\psi \supset \neg(\varphi \wedge \psi)$

2.4 Nondeterministic Matrices

Our main semantical tool in what follows will be the following generalization from [AL01, AL03] of the concept of a matrix:

Definition 2.3

1. A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$, where:

   (a) $\mathcal{T}$ is a non-empty set of *truth values*.

   (b) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{T}$.

   (c) For every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{T}^n$ to $2^{\mathcal{T}} - \{\emptyset\}$.

   We say that $\mathcal{M}$ is *(in)finite* if so is $\mathcal{T}$.

2. Denote by $\mathcal{F}$ the set of formulas of $\mathcal{L}$. A *valuation* in an Nmatrix $\mathcal{M}$ is a function $v : \mathcal{F} \to \mathcal{T}$ that satisfies the following condition for every n-ary connective $\diamond$ of $\mathcal{L}$ and $\psi_1, \ldots, \psi_n \in \mathcal{F}$:

$$v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$$

3. A valuation v in an Nmatrix $\mathcal{M}$ is a *model* of (or *satisfies*) a formula ψ in $\mathcal{M}$ (notation: $v \models^{\mathcal{M}} \psi$) if $v(\psi) \in \mathcal{D}$. v is a *model* in $\mathcal{M}$ of a set Γ of formulas (notation: $v \models^{\mathcal{M}} \Gamma$) if it satisfies every formula in Γ.

4. $\vdash_{\mathcal{M}}$, the consequence relation induced by the Nmatrix $\mathcal{M}$, is defined as follows:

   $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every v such that $v \models^{\mathcal{M}} \Gamma$ there exists $\varphi \in \Delta$ such that $v \models^{\mathcal{M}} \varphi$

5. A logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is *sound* for an Nmatrix $\mathcal{M}$ (where $\mathcal{L}$ is the language of $\mathcal{M}$) if $\vdash \subseteq \vdash_{\mathcal{M}}$. $\mathbf{L}$ is *complete* for $\mathcal{M}$ if $\vdash \supseteq \vdash_{\mathcal{M}}$. $\mathcal{M}$ is *characteristic* for $\mathbf{L}$ if $\mathbf{L}$ is both sound and complete for it (i.e.: if $\vdash = \vdash_{\mathcal{M}}$).

Note: We shall identify an ordinary (deterministic) matrix with an Nmatrix the functions in $\mathcal{O}$ of which always return singletons.

Definition 2.4 Let $\mathcal{M}_1 = \langle \mathcal{T}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{T}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$. $\mathcal{M}_2$ is called a *refinement* of $\mathcal{M}_1$ if $\mathcal{T}_1 = \mathcal{T}_2$, $\mathcal{D}_1 = \mathcal{D}_2$, and $\widetilde{\diamond}_{\mathcal{M}_1}(\vec{x}) \subseteq \widetilde{\diamond}_{\mathcal{M}_2}(\vec{x})$ for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\vec{x} \in \mathcal{T}^n$.

The following proposition can easily be proved:

Proposition 2.5 *If $\mathcal{M}_2$ is a refinement of $\mathcal{M}_1$ then $\vdash_{\mathcal{M}_1} \subseteq \vdash_{\mathcal{M}_2}$. Hence if $\mathbf{L}$ is sound for $\mathcal{M}_1$ then $\mathbf{L}$ is also sound for $\mathcal{M}_2$.*

3 The Classical Case

3.1 The General Semantics

Classical Logic has of course the semantics of the usual two-valued deterministic matrix. This semantics can however easily be generalized as follows.

Definition 3.1 1. Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes that of LK^+. We say that $\mathcal{M}$ is *suitable* for LK^+ if the following conditions are satisfied:

- If $a \in \mathcal{D}$ and $b \in \mathcal{D}$ then $a \tilde{\wedge} b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ then $a \tilde{\wedge} b \subseteq \mathcal{T} - \mathcal{D}$
- If $b \notin \mathcal{D}$ then $a \tilde{\wedge} b \subseteq \mathcal{T} - \mathcal{D}$

- If $a \in \mathcal{D}$ then $a \tilde{\vee} b \subseteq \mathcal{D}$
- If $b \in \mathcal{D}$ then $a \tilde{\vee} b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \tilde{\vee} b \subseteq \mathcal{T} - \mathcal{D}$

- If $a \notin \mathcal{D}$ then $a \tilde{\supset} b \subseteq \mathcal{D}$
- If $b \in \mathcal{D}$ then $a \tilde{\supset} b \subseteq \mathcal{D}$
- If $a \in \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \tilde{\supset} b \subseteq \mathcal{T} - \mathcal{D}$

2. Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes the language of LK. We say that $\mathcal{M}$ is *suitable* for LK if it is suitable for LK^+, and the following condition is satisfied:

   - $\tilde{\mathbf{ff}} \subseteq \mathcal{T} - \mathcal{D}$

Theorem 3.2 *LK (LK^+) is sound for any Nmatrix $\mathcal{M}$ which is suitable for it. Moreover: it is complete for the relevant fragment of $\mathcal{M}$.*

Proof: We leave the easy proof for the reader. ■

Note: A more general formulation of the last theorem is that an appropriate model for LK is a triple $\langle \mathcal{T}, \mathcal{D}, v \rangle$, where $\emptyset \subset \mathcal{D} \subset \mathcal{T}$, and v is a valuation in $\mathcal{T}$ satisfying:

- $v(\varphi \wedge \psi) \in \mathcal{D}$ iff $v(\varphi) \in \mathcal{D}$ and $v(\psi) \in \mathcal{D}$
- $v(\varphi \vee \psi) \in \mathcal{D}$ iff $v(\varphi) \in \mathcal{D}$ or $v(\psi) \in \mathcal{D}$
- $v(\varphi \supset \psi) \in \mathcal{D}$ iff $v(\varphi) \notin \mathcal{D}$ or $v(\psi) \in \mathcal{D}$
- $v(\mathbf{ff}) \notin \mathcal{D}$

Convention: For convenience, we use henceforth the same symbol for a connective and for a corresponding nondeterministic operation in a given Nmatrix. We shall also denote by the same symbol (usually $\mathcal{O}$) the set of connectives of a language $\mathcal{L}$ and the corresponding set of operations of an Nmatrix for $\mathcal{L}$.

We turn now to Nmatrices for paraconsistent negation which are based on the basic three truth values described in the introduction.

Definition 3.3 Let $\mathcal{M}_P$ ($\mathcal{M}_P^{\mathbf{ff}}$) be the unique Nmatrix $\langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ for the language $\{\neg, \wedge, \vee, \supset\}$ ($\{\neg, \wedge, \vee, \supset, \mathbf{ff}\}$) which satisfies the following conditions:

- $\mathcal{T} = \{t, \top, f\}$
- $\mathcal{D} = \{t, \top\}$
- $\mathcal{M}_P$ ($\mathcal{M}_P^{\mathbf{ff}}$) is suitable for LK^+ (LK)
- $\diamond(x, y) \in \{\mathcal{D}, \{f\}\}$ for $x, y \in \mathcal{T}$ and $\diamond \in \{\wedge, \vee, \supset\}$
- $\neg t = \{f\} \quad \neg f = \neg \top = \mathcal{D}$

For the reader's convenience, here are the indeterministic truth tables of $\mathcal{M}_P$ (for $\mathcal{M}_P^{\mathbf{ff}}$ one just has to add the condition $\widetilde{\mathbf{ff}} = \{f\}$, i.e: that $v(\mathbf{ff}) = f$ for any legal valuation v).

$\vee$	**f**	$\top$	**t**
f	f	$t, \top$	$t, \top$
$\top$	$t, \top$	$t, \top$	$t, \top$
t	$t, \top$	$t, \top$	$t, \top$

$\wedge$	**f**	$\top$	**t**
f	f	f	f
$\top$	f	$t, \top$	$t, \top$
t	f	$t, \top$	$t, \top$

$\supset$	**f**	$\top$	**t**
f	$t, \top$	$t, \top$	$t, \top$
$\top$	f	$t, \top$	$t, \top$
t	f	$t, \top$	$t, \top$

$\neg$	**f**	$\top$	**t**
	$t, \top$	$t, \top$	f

Note: It is straightforward to see that a valuation v in $\mathcal{T} = \{t, \top, f\}$ is $\mathcal{M}_P$-legal iff it satisfies the following four classical conditions:

$\wedge$: $v(\varphi \wedge \psi) = f$ iff $v(\varphi) = f$ or $v(\psi) = f$

$\vee$: $v(\varphi \vee \psi) = f$ iff $v(\varphi) = f$ and $v(\psi) = f$

$\supset$: $v(\varphi \supset \psi) = f$ iff $v(\varphi) \neq f$ and $v(\psi) = f$

$\neg$: $v(\neg\psi) = f$ iff $v(\psi) = t$

v is a legal $\mathcal{M}_P^{\mathbf{ff}}$-valuation iff in addition $v(\mathbf{ff}) = f$.

Proposition 3.4 *PLK^+ and PLK are sound for $\mathcal{M}_P$ and $\mathcal{M}_P^{\mathbf{ff}}$ (respectively).*

Proof: This follows from Theorem 3.2 and the fact that $\neg f \subseteq \mathcal{D}$. ■

Notes:

1. By proposition 2.5, PLK^+ and PLK are sound also for any refinement of $\mathcal{M}_P$ and $\mathcal{M}_P^{\mathbf{ff}}$ (respectively). On the other hand these matrices are *not* the most indeterministic matrices based on $\mathcal{T} = \{t, \top, f\}$ for which PLK^+ and PLK are sound. From the proof of Proposition 3.4 it is clear that we could have taken $\neg t = \neg\top = \mathcal{T}$. This would have left us with no difference between t and $\top$, and with semantics which is equivalent to the two-valued nondeterministic semantics for these logics presented in [AL03] (and implicit already in [Bat00] and elsewhere). However, the difference between t and $\top$ will be crucial below for the various extensions of PLK^+ and PLK with other negation rules.

2. We shall later see that PLK^+ and PLK are also complete for $\mathcal{M}_P$ and $\mathcal{M}_P^{\mathbf{ff}}$ (respectively).

3.2 The Effects of the Negation Rules

We turn now to the effects of the various negation rules. We shall show that to each of them corresponds a condition which leads to a certain refinement of $\mathcal{M}_P$ (or $\mathcal{M}_P^{\mathbf{ff}}$). These conditions are independent of each other, but never contradict each other. To see how these conditions are obtained, take $(\neg \supset\Rightarrow)_2$ as an example. This rule is equivalent to the validity of $\neg(\varphi \supset \psi) \vdash \neg\psi$. It means therefore that we should have $v(\neg(\varphi \supset \psi)) = f$ in case $v(\neg\psi) = f$. By the truth tables of $\mathcal{M}_P$, this means that $v(\varphi \supset \psi)$ should be t in case $v(\psi) = t$. This is therefore the condition that corresponds to this rule, and it turns 3 possible indeterministic choices in $\mathcal{M}_P$ (or $\mathcal{M}_P^{\mathbf{ff}}$) to deterministic ones. Similar analysis can be done to the other ten rules. The resulting list of conditions is listed in the next Definition.

Definition 3.5 *The refining conditions induced by the negation rules:*

$C(\neg\neg\Rightarrow)$: $\neg f = \{t\}$

$C(\Rightarrow\neg\neg)$: $\neg\top = \{\top\}$

$C(\neg\supset\Rightarrow)_1$: $f \supset x = \{t\}$

$C(\neg\supset\Rightarrow)_2$: $x \supset t = \{t\}$

$C(\Rightarrow \neg \supset)$: $t \supset \top = \top \supset \top = \{\top\}$

$C(\neg\vee \Rightarrow)_1$: $t \vee x = \{t\}$

$C(\neg\vee \Rightarrow)_2$: $x \vee t = \{t\}$

$C(\Rightarrow \neg\vee)$: $f \vee \top = \top \vee f = \top \vee \top = \{\top\}$

$C(\neg\wedge \Rightarrow)$: $t \wedge t = \{t\}$

$C(\Rightarrow \neg\wedge)_1$: $\top \wedge t = \top \wedge \top = \{\top\}$

$C(\Rightarrow \neg\wedge)_2$: $t \wedge \top = \top \wedge \top = \{\top\}$

Definition 3.6

1. For $S \subseteq NR$ let $C(S) = \{C(r) \mid r \in S\}$
2. For $S \subseteq NR$ let $\mathcal{M}_P[S]$ and $\mathcal{M}_P^{\mathbf{ff}}[S]$ be the weakest refinements of $\mathcal{M}_P$ and $\mathcal{M}_P^{\mathbf{ff}}$ (respectively) in which the conditions in $C(S)$ are satisfied.

The following proposition can now easily be proved:

Proposition 3.7 *If $S \subseteq NR$ then $PLK(S)$ ($PLK^{\mathbf{ff}}(S)$) is sound for $\mathcal{M}_P[S]$ ($\mathcal{M}_P^{\mathbf{ff}}[S]$).*

We simultaneously prove now the completeness of the 2^{12} systems considered in Proposition 3.7 and the cut-elimination theorem for them.

Theorem 3.8 *Let $S \subseteq NR$, and assume that $\Gamma \Rightarrow \Delta$ does not have a cut free proof in $PLK(S)$ ($PLK^{\mathbf{ff}}(S)$). Then $\Gamma \not\vdash_{\mathcal{M}_P[S]} \Delta$ ($\Gamma \not\vdash_{\mathcal{M}_P^{\mathbf{ff}}[S]} \Delta$). In other words: there is a valuation v in $\mathcal{M}_P[S]$ ($\mathcal{M}_P^{\mathbf{ff}}[S]$) such that $v(\varphi) \in \{t, \top\}$ if $\varphi \in \Gamma$, while $v(\psi) = f$ if $\psi \in \Delta$.*

Proof: We prove the case of $PLK(S)$ (the proof in the case of $PLK^{\mathbf{ff}}(S)$ is almost identical).

Call a sequent $\Gamma \Rightarrow \Delta$ *saturated* if it closed under the inverses of the rules of $PLK(S)$. More precisely: $\Gamma \Rightarrow \Delta$ is saturated if it satisfies the following conditions:

1. If $\varphi \wedge \psi \in \Gamma$ then $\varphi, \psi \in \Gamma$
2. If $\varphi \wedge \psi \in \Delta$ then $\varphi \in \Delta$ or $\psi \in \Delta$
3. If $\varphi \vee \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$

4. If $\varphi \vee \psi \in \Delta$ then $\varphi, \psi \in \Delta$

5. If $\varphi \supset \psi \in \Gamma$ then $\varphi \in \Delta$ or $\psi \in \Gamma$

6. If $\varphi \supset \psi \in \Delta$ then $\varphi \in \Gamma$ and $\psi \in \Delta$

7. If $\neg\varphi \in \Delta$ then $\varphi \in \Gamma$

8. If $(\neg\neg \Rightarrow) \in S$ and $\neg\neg\varphi \in \Gamma$ then $\varphi \in \Gamma$

9. If $(\Rightarrow \neg\neg) \in S$ and $\neg\neg\varphi \in \Delta$ then $\varphi \in \Delta$

10. If $(\neg \supset\Rightarrow)_1 \in S$ and $\neg(\varphi \supset \psi) \in \Gamma$ then $\varphi \in \Gamma$

11. If $(\neg \supset\Rightarrow)_2 \in S$ and $\neg(\varphi \supset \psi) \in \Gamma$ then $\neg\psi \in \Gamma$

12. If $(\Rightarrow \neg \supset) \in S$ and $\neg(\varphi \supset \psi) \in \Delta$ then $\varphi \in \Delta$ or $\neg\psi \in \Delta$

13. If $(\neg\wedge \Rightarrow) \in S$ and $\neg(\varphi \wedge \psi) \in \Gamma$ then $\neg\varphi \in \Gamma$ or $\neg\psi \in \Gamma$

14. If $(\Rightarrow \neg\wedge)_1 \in S$ and $\neg(\varphi \wedge \psi) \in \Delta$ then $\neg\varphi \in \Delta$

15. If $(\Rightarrow \neg\wedge)_2 \in S$ and $\neg(\varphi \wedge \psi) \in \Delta$ then $\neg\psi \in \Delta$

16. If $(\neg\vee \Rightarrow)_1 \in S$ and $\neg(\varphi \vee \psi) \in \Gamma$ then $\neg\varphi \in \Gamma$

17. If $(\neg\vee \Rightarrow)_2 \in S$ and $\neg(\varphi \vee \psi) \in \Gamma$ then $\neg\psi \in \Gamma$

18. If $(\Rightarrow \neg\vee) \in S$ and $\neg(\varphi \vee \psi) \in \Delta$ then $\neg\varphi \in \Delta$ or $\neg\psi \in \Delta$

Assume now that $\Gamma \Rightarrow \Delta$ does not have a cut free proof in $PLK(S)$. It is a standard matter to prove that $\Gamma \Rightarrow \Delta$ can be extended to a saturated sequent $\Gamma^* \Rightarrow \Delta^*$ (where $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$) which also has no cut free proof in $PLK(S)$. We may assume therefore that $\Gamma \Rightarrow \Delta$ is already saturated. We inductively define now a refuting valuation v of $\Gamma \Rightarrow \Delta$ as follows:

- $v(\varphi) = f$ iff one of the following conditions is satisfied:

  f1 $\varphi \in \Delta$

  f2 $\varphi = \neg\psi$ and $v(\psi) = t$

  f3 $\varphi = \psi_1 \wedge \psi_2$ and $v(\psi_1) = f$ or $v(\psi_2) = f$

  f4 $\varphi = \psi_1 \vee \psi_2$ and $v(\psi_1) = v(\psi_2) = f$

f5 $\varphi = \psi_1 \supset \psi_2$ and $v(\psi_1) \neq f$ while $v(\psi_2) = f$

- $v(\varphi) = t$ iff one of the following conditions is satisfied:

  t1 $\neg\varphi \in \Delta$

  t2 $\varphi = \neg\psi$, $(\neg\neg \Rightarrow) \in S$, and $v(\psi) = f$

  t3 $\varphi = \psi_1 \wedge \psi_2$, $(\neg\wedge \Rightarrow) \in S$ and $v(\psi_1) = v(\psi_2) = t$

  t4 $\varphi = \psi_1 \vee \psi_2$, $(\neg\vee \Rightarrow)_1 \in S$ and $v(\psi_1) = t$

  t5 $\varphi = \psi_1 \vee \psi_2$, $(\neg\vee \Rightarrow)_2 \in S$ and $v(\psi_2) = t$

  t6 $\varphi = \psi_1 \supset \psi_2$, $(\neg \supset\Rightarrow)_1 \in S$ and $v(\psi_1) = f$

  t7 $\varphi = \psi_1 \supset \psi_2$, $(\neg \supset\Rightarrow)_2 \in S$ and $v(\psi_2) = t$

- $v(\varphi) = \top$ in any other case.

We prove now by induction on the complexity of φ that $v(\varphi)$ is well defined (i.e.: if one of the conditions f1-f5 is satisfied then none of the conditions t1-t7 is satisfied), that if $\varphi \in \Gamma$ then $v(\varphi) \neq f$, and that if $\neg\varphi \in \Gamma$ then $v(\varphi) \neq t$. First note that since $\Gamma \Rightarrow \Delta$ has no cut-free proof, $\Gamma \cap \Delta = \emptyset$. By the definition of a saturated sequent this entails that if φ satisfies t1 then it cannot satisfy f1 (since if φ satisfies t1 then $\varphi \in \Gamma$). With these two observations in the background, the induction proceeds as follows:

$\varphi = p$ (p atomic): In this case only conditions are f1 and t1 are relevant. Since these conditions cannot be satisfied together, $v(\varphi)$ is well defined. Moreover: if $\varphi \in \Gamma$ then f1 is not satisfied and so $v(\varphi) \neq f$, while if $\neg\varphi \in \Gamma$ then t1 is not satisfied and so $v(\varphi) \neq t$.

$\varphi = \neg\psi$: The relevant conditions in this case are f1, f2, t1, and t2. f2 trivially contradicts t2. We check the remaining cases:

- If φ satisfies f1 then $\psi \in \Gamma$. Hence by induction hypothesis $v(\psi) \neq f$ and so t2 is not satisfied.
- Suppose φ satisfies t1. Then $\varphi \in \Gamma$. Hence $\neg\psi \in \Gamma$, and so $v(\psi) \neq t$ by induction hypothesis. It follows that f2 is not satisfied.

It follows that $v(\varphi)$ is well defined. Moreover, our argument shows that if $\varphi \in \Gamma$ then neither f1 nor f2 are satisfied, and so $v(\varphi) \neq f$. Suppose next that $\neg\varphi = \neg\neg\psi \in \Gamma$. Then t1 is not satisfied. If $(\neg\neg \Rightarrow) \notin S$ then so is t2. Otherwise $\psi \in \Gamma$, and so $v(\psi) \neq f$ by induction hypothesis. Hence again t2 is not satisfied. It follows that $v(\varphi)$ cannot be t in this case.

$\varphi = \psi_1 \wedge \psi_2$: The relevant conditions in this case are f1, f3, t1, and t3. f3 and t3 trivially contradict each other (by the induction hypothesis). We check the remaining two cases:

- If φ satisfies f1 then either $\psi_1 \in \Delta$ or $\psi_2 \in \Delta$ (since $\Gamma \Rightarrow \Delta$ is saturated). Hence by induction hypothesis either $v(\psi_1) = f$ or $v(\psi_2) = f$ (and both are well-defined), and so t3 is not satisfied.
- Suppose φ satisfies t1. Then $\varphi \in \Gamma$. Hence both $\psi_1 \in \Gamma$ and $\psi_2 \in \Gamma$. By induction hypothesis both $v(\psi_1) \neq f$ and $v(\psi_2) \neq f$, and so f3 is not satisfied.

It follows that $v(\varphi)$ is well defined. Moreover, our argument shows that if $\varphi \in \Gamma$ then neither f1 nor f3 are satisfied, and so $v(\varphi) \neq f$. Suppose next that $\neg\varphi \in \Gamma$. Then t1 is not satisfied. If $(\neg\wedge \Rightarrow) \notin S$ then so is t3. Otherwise either $\neg\psi_1 \in \Gamma$ or $\neg\psi_2 \in \Gamma$ (since $\Gamma \Rightarrow \Delta$ is saturated), and so by induction hypothesis either $v(\psi_1) \neq t$ or $v(\psi_2) \neq t$. Hence again t3 is not satisfied. It follows that $v(\varphi)$ cannot be t in this case.

$\varphi = \psi_1 \vee \psi_2$: The relevant conditions in this case are f1, f4, t1, t4, and t5. f4 trivially contradicts both t4 and t5. We check the remaining cases:

- If φ satisfies f1 then both $\psi_1 \in \Delta$ and $\psi_2 \in \Delta$ (since $\Gamma \Rightarrow \Delta$ is saturated). Hence by induction hypothesis both $v(\psi_1) = f$ and $v(\psi_2) = f$ (and both are well-defined). Hence t4 and t5 cannot be satisfied.
- Suppose φ satisfies t1. Then $\varphi \in \Gamma$. Hence either $\psi_1 \in \Gamma$ or $\psi_2 \in \Gamma$. By induction hypothesis either $v(\psi_1) \neq f$ or $v(\psi_2) \neq f$, and so f4 is not satisfied.

It follows that $v(\varphi)$ is well defined. Moreover, our argument shows that if $\varphi \in \Gamma$ then neither f1 nor f4 are satisfied, and so $v(\varphi) \neq f$. Suppose next that $\neg\varphi \in \Gamma$. Then t1 is not satisfied. If $(\neg\vee \Rightarrow)_1 \notin S$ then so is t4. Otherwise $\neg\psi_1 \in \Gamma$, and so by induction hypothesis $v(\psi_1) \neq t$. Hence again t4 is not satisfied. That t5 is not satisfied is shown similarly. It follows that $v(\varphi)$ cannot be t in this case.

$\varphi = \psi_1 \supset \psi_2$: We leave this case to the reader.

We have shown that v is well defined, and that if $\varphi \in \Gamma$ then $v(\varphi) \neq f$ (and so $v(\varphi) \in \mathcal{D}$ in this case). Note also that if $\varphi \in \Delta$ then $v(\varphi) = f$ by definition of v. Hence v refutes $\Gamma \Rightarrow \Delta$, and it remains to prove only that v is a legal valuation. For this we need first to show that v is $\mathcal{M}_P$-legal, and for this it suffices to show that it satisfies the four conditions listed in the Note that follows Definition 3.3. Now the "if" parts of these conditions directly follow from the definition of v (conditions f2-f5). We show the "only if" parts.

- Assume $v(\neg\psi) = f$. By definition, this is possible only if $\neg\psi$ satisfies either condition f2 or condition f1. If it satisfies f1 then by condition t1 $v(\psi) = t$, and this is of course also true if $\neg\psi$ satisfies condition f2.
- Assume $v(\varphi \wedge \psi) = f$. By definition, this is possible only if $\varphi \wedge \psi$ satisfies either condition f3 or condition f1. If it satisfies f1 then either $\varphi \in \Delta$ or $\psi \in \Delta$ (since $\Gamma \Rightarrow \Delta$ is saturated). Hence in this case either $v(\varphi) = f$ or $v(\psi) = f$, and this is of course true also if $\varphi \wedge \psi$ satisfies condition f3.
- The cases of $\varphi \vee \psi$ and $\varphi \supset \psi$ are similar to the case of $\varphi \wedge \psi$.

We next show that v satisfies the various conditions induced by the rules of S. For the left introduction rules this is immediate from the definition of v (conditions t2-t7). We prove now the conditions induced by the right introduction rules.

$C(\Rightarrow \neg\neg)$: Assume $(\Rightarrow \neg\neg) \in S$, and $v(\psi) = \top$. Then ψ does not satisfy t1, and so $\neg\psi$ does not satisfy f1. Obviously it does not satisfy f2-f5 either, and so $v(\neg\psi) \neq f$. Assume now that $\neg\neg\psi \in \Delta$. Then $\psi \in \Delta$ (since $\Gamma \Rightarrow \Delta$ is saturated and $(\Rightarrow \neg\neg) \in S$). Hence $v(\psi) = f$, contradicting $v(\psi) = \top$. It follows that $\neg\psi$ does not satisfy t1. Obviously it does not satisfy t2-t7 either, and so $v(\neg\psi) \neq t$. The only possibility that remains is that $v(\neg\psi) = \top$.

$C(\Rightarrow \neg \supset)$: Assume $(\Rightarrow \neg \supset) \in S$, and $v(\varphi) \neq f$, $v(\psi) = \top$. Assume further that $\varphi \supset \psi \in \Delta$. Then $\varphi \in \Gamma$ and $\psi \in \Delta$ (since $\Gamma \Rightarrow \Delta$ is saturated). This entails that $v(\psi) = f$, contradicting $v(\psi) = \top$. Hence $\varphi \supset \psi$ does not satisfy f1. Obviously it does not satisfy f2-f5 either, and so $v(\varphi \supset \psi) \neq f$. Assume next that $\neg(\varphi \supset \psi) \in \Delta$. Then $\varphi \in \Delta$ or $\neg\psi \in \Delta$ (since $\Gamma \Rightarrow \Delta$ is saturated and $(\Rightarrow \neg \supset) \in S$). Hence $v(\varphi) = f$ or $v(\psi) = t$, contradicting our assumptions concerning $v(\varphi)$ and $v(\psi)$. It follows that $\varphi \supset \psi$ does not satisfy t1. Obviously it does not satisfy t2-t7 either, and so $v(\varphi \supset \psi) \neq t$. The only possibility that remains is that $v(\varphi \supset \psi) = \top$.

We leave the cases of $C(\Rightarrow \neg\vee)$, $C(\Rightarrow \neg\wedge)_1$, and $C(\Rightarrow \neg\wedge)_2$ to the reader. ■

Corollary 3.9 *If $S \subseteq NR$ then $PLK(S)$ ($PLK^{\mathbf{ff}}(S)$) is sound and complete with respect to $\mathcal{M}_P[S]$ ($\mathcal{M}_P^{\mathbf{ff}}[S]$).*

Corollary 3.10 *If $S \subseteq NR$ then $PLK(S)$ and $PLK^{\mathbf{ff}}(S)$ admit cut-elimination.*

Note: In particular, it follows from Corollary 3.10 that the above Gentzen-type systems for C_{min}, PI^* and $LFI1$ admit cut-elimination.

3.3 Some Applications

In this subsection we apply our soundness and completeness results for deriving interesting properties of some of the systems considered above. Our main tool will be the following Definition and simple Lemma, the trivial proof of which we leave to the reader:

Definition 3.11 Let $\mathcal{L}$ be a propositional language, and let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. A *semivaluation* in $\mathcal{M}$ is any function $v' : \mathcal{F}' \to \mathcal{T}$ such that $\mathcal{F}'$ a set of formulas of $\mathcal{L}$ which is closed under subformulas, and v' respects $\mathcal{M}$ (in the sense that if $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'$ is then $v'(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$).

Lemma 3.12 *Any semivaluation can be extended to a valuation v in $\mathcal{M}$.*

Corollary 3.13 *$PLK(S)$ and $PLK^{\mathbf{ff}}(S)$ are decidable for every $S \subseteq NR$.*

Proof: Let $\Gamma \Rightarrow \Delta$ be a sequent of the language of $PLK(S)$. Let $\mathcal{F}'$ be the set of all subformulas of formulas in $\Gamma \Rightarrow \Delta$. To decide whether $\Gamma \Rightarrow \Delta$ is provable in $PLK(S)$, it suffices by Lemma 3.12 (together with the soundness and completeness of $PLK(S)$ with respect to $\mathcal{M}_P[S]$) to check whether every $v' : \mathcal{F}' \to \{t, f, \top\}$ which is a semivaluation in $\mathcal{M}_P[S]$ has the property that either $v'(\varphi) = f$ for some $\varphi \in \Gamma$, or $v'(\varphi) \neq f$ for some $\varphi \in \Delta$. Since the number of such semivaluations is finite, this is a decision procedure. The proof for $PLK^{\mathbf{ff}}(S)$ is similar. ∎

Theorem 3.14 *Let S_L be the set of the left introduction rules of NR.*

1. *If φ is not a subformula of ψ then $\neg\psi \nvdash_{PLK(S_L)} \neg\varphi$.*

2. *Two formulas $\neg\varphi$ and $\neg\psi$ can be equivalent in $PLK(S_L)$ (in the sense that $\neg\varphi \vdash_{PLK(S_L)} \neg\psi$ and $\neg\psi \vdash_{PLK(S_L)} \neg\varphi$) only if they are identical.*

Proof: The second part is immediate from the first. To show the first, let $\mathcal{F}$ be the set of subformulas of $\{\neg\varphi, \neg\psi\}$. Define a function v' on $\mathcal{F}$ by assigning t to φ, f to $\neg\varphi$, and $\top$ to all other formulas of $\mathcal{F}$. It is easy to verify that v' is a semivaluation of $PLK(S_L)$. By Lemma 3.12 v' can be extended to a valuation v in $\mathcal{M}_P[S_L]$. Now v is a model of $\neg\psi$ in $\mathcal{M}_P[S_L]$ which is not a model of $\neg\varphi$. ■

Theorem 3.15 *Let $S_0 = NR - \{(\Rightarrow \neg\neg), (\Rightarrow \neg\wedge)_1, (\Rightarrow \neg\wedge)_2\}$. Then No formula of the form $\neg\varphi$ is provable in $PLK(S_0)$ (or any subsystem of $PLK(S_0)$.*

Proof: We prove first the following Lemma: If Γ is a finite set of formulas of the language of $PLK(S_0)$, then there is a valuation v in $\mathcal{M}_P[S_0]$ which assigns a designated element to all the formulas of Γ, and assigns t to at least one of them.

To prove the lemma it suffices by Lemma 3.12 to show that if $\mathcal{F}'$ is the set of subformulas of Γ then there exists a semivaluation v' defined on $\mathcal{F}'$ which has the desired property. We construct such v' by an induction on the total sum of connectives in Γ. If this sum is 0 (i.e. if all formulas in Γ are atomic) then the existence of an appropriate v' is trivial. Otherwise pick some $\varphi \in \Gamma$ which is not a subformula of any other formula in Γ, and let $\mathcal{F}'' = \mathcal{F}' - \{\varphi\}$. Then $\mathcal{F}''$ is closed under subformulas. Construct now v' as follows:

$\varphi = \neg\psi$: Let v'' assign $\top$ to all formulas of $\mathcal{F}''$. It is easy to see that v'' is a semivaluation. Extend v'' to v' with domain $\mathcal{F}'$ by letting $v'(\varphi) = t$. Since $(\Rightarrow \neg\neg) \notin S$, v' is a semivaluation in $\mathcal{M}_P[S_0]$, and it obviously has the required properties.

$\varphi = \psi_1 \vee \psi_2$ **or** $\varphi = \psi_1 \wedge \psi_2$: Let $\Gamma' = \Gamma \cup \{\psi_1, \psi_2\} - \varphi$. By induction hypothesis there is a semivaluation v'' in $\mathcal{M}_P[S_0]$ with domain $\mathcal{F}''$ such that v'' assigns a designated element to all the formulas of Γ', and assigns t to at least one of them. Extend v'' to v' with domain $\mathcal{F}'$ by letting $v'(\varphi) = t$ if either $v''(\psi_1) = t$ or $v''(\psi_2) = t$, $v'(\varphi) = \top$ otherwise. Since neither $(\Rightarrow \neg\wedge)_1$ nor $(\Rightarrow \neg\wedge)_2$ is in S_0, v' is a semivaluation in $\mathcal{M}_P[S_0]$, and it obviously has the required properties.

$\varphi = \psi_1 \supset \psi_2$: Let $\Gamma' = \Gamma \cup \{\psi_2\} - \varphi$, and let $\mathcal{F}'''$ be the set of subformulas of Γ'. By induction hypothesis there is a semivaluation v''' in $\mathcal{M}_P[S_0]$ with domain $\mathcal{F}'''$ such that v''' assigns a designated element to all the formulas of Γ', and assigns t to at least one of them. By Lemma 3.12 v''' can be extended to a semivaluation v'' with domain $\mathcal{F}''$. Extend v'' further to v' with domain $\mathcal{F}'$ by letting $v'(\varphi) = t$ if $v''(\psi_2) = t$, $v'(\varphi) = \top$ otherwise. It is easily seen that v' is a semivaluation in $\mathcal{M}_P[S_0]$ as desired.

The proof of the Theorem itself is now easy. Let φ be a formula of $PLK(S_0)$. By the lemma there is a valuation v in $\mathcal{M}_P[S_0]$ such that $v(\varphi) = t$. Hence $v(\neg\varphi) = f$, and so $\neg\varphi$ is not valid in $\mathcal{M}_P[S_0]$. It follows that $\neg\varphi$ is not provable in $PLK(S_0)$. ■

Note: The second part of Theorem 3.14 and Theorem 3.15 have been proved in [CM99] for the much weaker system C_{min}.

Proposition 3.16 *Theorem 3.15 cannot be improved: If no formula of the form $\neg\varphi$ is provable in $PLK(S)$ then $S \subseteq S_0$.*

Proof: This follows from the following (easily established) facts:

- If $(\Rightarrow \neg\neg) \in S$ then $\vdash_{PLK(S)} \neg\neg(\varphi \supset \varphi)$.
- If $(\Rightarrow \neg\wedge)_1 \in S$ then $\vdash_{PLK(S)} \neg(\varphi \wedge \neg\varphi)$.
- If $(\Rightarrow \neg\wedge)_2 \in S$ then $\vdash_{PLK(S)} \neg(\neg\varphi \wedge \varphi)$.

4 The Intuitionistic Case

4.1 The General semantics

The previous section was is devoted to paraconsistent extensions of positive classical logics.[1] It seems however that positive *intuitionistic* logic is a better starting point for investigating negation. The valid sentences of this fragment are all intuitively correct. Positive classical logic, in contrast, includes counterintuitive tautologies like $(A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)$ or $((A \supset B) \supset A) \supset A$. Moreover: the classical natural deduction rules for the positive connectives ($\wedge, \vee$ and $\supset$) define the intuitionistic positive logic, not the classical one. It is only with the aid of the classical rules for the classical negation that one can prove the counterintuitive positive tautologies mentioned above.

Now it is well known that it is impossible to conservatively add to intuitionistic positive logic a negation which is both explosive (i.e.: $\neg A, A \vdash B$ for all A, B) and for which LEM is valid. With such an addition we get classical logic. The intuitionists reject indeed LEM, retaining the explosive nature of negation (which is defined using the constant **ff** and implication). In this section we investigate conservative extensions of intuitionistic (positive) logic (with or without **ff**) with a paraconsistent negation for which LEM *is* valid.

As in the case of LK, we start with generalizing the standard, two-valued semantics of LJ^+ (or LJ). Recall that this semantics is provided by the class of all frames of the form $\mathcal{W} = \langle W, \leq, v \rangle$,

[1]By this it follows the survey [CM02].

where $\langle W, \leq \rangle$ is a nonempty partially ordered set (of "worlds"), and $v : W \times \mathcal{F} \to \mathcal{T}$ (where $\mathcal{F}$ is the set of formulas of the language) which satisfies the following conditions:

1. If $y \geq x$ and $v(x, \varphi) = t$ then $v(y, \varphi) = t$.[2]

2.
   - $v(x, \varphi \wedge \psi) = f$ iff $v(x, \varphi) = f$ or $v(x, \psi) = f$
   - $v(x, \varphi \vee \psi) = f$ iff $v(x, \varphi) = f$ and $v(x, \psi) = f$
   - $v(x, \mathbf{ff}) = f$ (in case **ff** is in the language).

3. $v(x, \varphi \supset \psi) = t$ iff $v(y, \psi) = t$ for every $y \geq x$ such that $v(y, \varphi) = t$

Obviously, if $\mathcal{W} = \langle W, \leq, v \rangle$ is a frame, then for every $x \in W$ the function $\lambda\varphi.v(x, \varphi)$ behaves like an ordinary classical valuation with respect to all the connectives except $\supset$. The treatment of $\supset$ is indeed what distinguishes between (positive) classical and intuitionistic logics. This observation leads to the following nondeterministic generalization of Kripke frames for intuitionistic logic:

Definition 4.1 Let $\mathcal{L}$ be a propositional language which has $\supset$ as one of its connectives, and let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. Denote by $\mathcal{F}$ be the set of formulas of $\mathcal{L}$. An $\mathcal{M}$-*frame* for $\mathcal{L}$ is a triple $\mathcal{W} = \langle W, \leq, v \rangle$ such that:

1. $\langle W, \leq \rangle$ is a nonempty partially ordered set

2. $v : W \times \mathcal{F} \to \mathcal{T}$ satisfies:
   - The persistence condition: if $y \geq x$ and $v(x, \varphi) \in \mathcal{D}$ then $v(y, \varphi) \in \mathcal{D}$
   - For every $x \in W$, $\lambda\varphi.v(x, \varphi)$ is an $\mathcal{M}$-valuation.
   - $v(x, \varphi \supset \psi) \in \mathcal{D}$ iff $v(y, \psi) \in \mathcal{D}$ for every $y \geq x$ such that $v(y, \varphi) \in \mathcal{D}$

We say that a formula φ is *true* in a world $x \in W$ of a frame $\mathcal{W}$ if $v(x, \varphi) \in \mathcal{D}$. A sequent $\Gamma \Rightarrow \Delta$ is *valid* in $\mathcal{W}$ if for every $x \in W$ there is either $\varphi \in \Gamma$ such that φ is not true in x, or $\psi \in \Delta$ such that ψ is true in x.

Note: Obviously, if $\mathcal{M}_1$ is a refinement of $\mathcal{M}_2$, then any $\mathcal{M}_1$-frame is also an $\mathcal{M}_2$-frame.

Definition 4.2 1. Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes the language of LJ^+. We say that $\mathcal{M}$ is *suitable* for LJ^+ if the following conditions are satisfied:

[2] For the language of LJ it suffices to demand this for atomic formulas only. It is then possible to prove that every formula has this property. This does not remain true for the nondeterministic generalizations we present below.

- If $a \in \mathcal{D}$ and $b \in \mathcal{D}$ then $a \wedge b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ then $a \wedge b \subseteq \mathcal{T} - \mathcal{D}$
- If $b \notin \mathcal{D}$ then $a \wedge b \subseteq \mathcal{T} - \mathcal{D}$

- If $a \in \mathcal{D}$ then $a \vee b \subseteq \mathcal{D}$
- If $b \in \mathcal{D}$ then $a \vee b \subseteq \mathcal{D}$
- If $a \notin \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \vee b \subseteq \mathcal{T} - \mathcal{D}$

- If $b \in \mathcal{D}$ then $a \supset b \subseteq \mathcal{D}$
- If $a \in \mathcal{D}$ and $b \notin \mathcal{D}$ then $a \supset b \subseteq \mathcal{T} - \mathcal{D}$

2. Let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language which includes the language of LJ. We say that $\mathcal{M}$ is *suitable* for LJ if it is suitable for LJ^+, and the following condition is satisfied:

   - $\mathbf{ff} \subseteq \mathcal{T} - \mathcal{D}$

Note: An Nmatrix which is suitable for LJ^+ (LJ) is also suitable for LK^+ (LK) iff it satisfies just one more condition: that if $a \notin \mathcal{D}$ then $a \supset b \subseteq \mathcal{D}$.

Theorem 4.3 *Assume $\mathcal{W}$ is an $\mathcal{M}$-frame where $\mathcal{M}$ is suitable for LJ^+ (LJ). Then any sequent which is provable in LJ^+ (LJ) is valid in $\mathcal{W}$.*

Proof: Again we leave the easy proof to the reader. ■

Note: A more general formulation of the last theorem is that an appropriate model for LJ is a tuple $\langle \mathcal{T}, \mathcal{D}, W, \leq, v \rangle$, where $\emptyset \subset \mathcal{D} \subset \mathcal{T}$, $\langle W, \leq \rangle$ is a nonempty partially ordered set, and $v : W \times \mathcal{F} \to \mathcal{T}$ is a valuation which satisfies the following conditions:

- If $y \geq x$ and $v(x, \varphi) \in \mathcal{D}$ then $v(y, \varphi) \in \mathcal{D}$
- $v(x, \varphi \wedge \psi) \in \mathcal{D}$ iff $v(x, \varphi) \in \mathcal{D}$ and $v(x, \psi) \in \mathcal{D}$
- $v(x, \varphi \vee \psi) \in \mathcal{D}$ iff $v(x, \varphi) \in \mathcal{D}$ or $v(x, \psi) \in \mathcal{D}$
- $v(x, \mathbf{ff}) \notin \mathcal{D}$
- $v(x, \varphi \supset \psi) \in \mathcal{D}$ iff $v(y, \psi) \in \mathcal{D}$ for every $y \geq x$ such that $v(y, \varphi) \in \mathcal{D}$

We turn now to $\mathcal{M}$-frames for paraconsistent negation where $\mathcal{M}$ is an Nmatrix which is based on our basic three truth values.

Definition 4.4 Let $\mathcal{M}_{IP}$ ($\mathcal{M}^{\mathbf{ff}}_{IP}$) be the unique Nmatrix $\langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ for the language $\{\neg, \wedge, \vee, \supset\}$ ($\{\neg, \wedge, \vee, \supset, \mathbf{ff}\}$) which satisfies the following conditions:

- $\mathcal{T} = \{t, \top, f\}$
- $\mathcal{D} = \{t, \top\}$
- $\mathcal{M}_{IP}$ ($\mathcal{M}^{\mathbf{ff}}_{IP}$) is suitable for LJ^+ (LJ)
- $\diamond(x, y) \in \{\mathcal{T}, \mathcal{D}, \{f\}\}$ for $x, y \in \mathcal{T}$ and $\diamond \in \{\wedge, \vee, \supset\}$
- $f \supset f = \mathcal{T}$
- $\neg t = \{f\} \quad \neg f = \neg \top = \mathcal{D}$

Note: The only difference between $\mathcal{M}_{IP}$ and $\mathcal{M}_P$ (or $\mathcal{M}^{\mathbf{ff}}_{IP}$ and $\mathcal{M}^{\mathbf{ff}}_P$) is in the truth table for $\supset$. For $\mathcal{M}_{IP}$ (and $\mathcal{M}^{\mathbf{ff}}_{IP}$) this table is:

$\supset$	**f**	$\top$	**t**
f	$t, \top, f$	$t, \top$	$t, \top$
$\top$	f	$t, \top$	$t, \top$
t	f	$t, \top$	$t, \top$

Proposition 4.5 *PLJ ($PLJ^{\mathbf{ff}}$) is sound for every $\mathcal{M}_{IP}$-frame ($\mathcal{M}^{\mathbf{ff}}_{IP}$-frame).*

Proof: This follows from Theorem 4.3 and the fact that $\neg f \subseteq \mathcal{D}$. ■

Proposition 4.6 *In the definition of an $\mathcal{M}_{IP}$-frame (or $\mathcal{M}^{\mathbf{ff}}_{IP}$-frame) the persistence condition (see Definition 4.1) can be replaced by the following* monotonicity condition*:*

- If $x \leq y$ then $v(x, \varphi) \leq_k v(y, \varphi)$

where $\leq_k$ on $\{t, f, \top\}$ is defined by: $t \leq_k \top$, $f \leq_k \top$ (and $a \leq_k a$).

Proof: The monotonicity condition trivially implies the persistence condition. For the converse we should show that if $x \leq y$ and $v(y, \varphi) \in \{t, f\}$ then $v(x, \varphi) = v(y, \varphi)$. This is immediate from the persistence condition in case $v(y, \varphi) = f$. Suppose that $v(y, \varphi) = t$. Then $v(y, \neg\varphi) = f$ (since v respects $\mathcal{M}_{IP}$). Hence $v(x, \neg\varphi) = f$. Since v respects the operations of $\mathcal{M}_{IP}$, this is possible only if $v(x, \varphi) = t$. ■

4.2 The Effects of the Negation Rules

We are interested in this section in extensions of positive intuitionistic logic with a paraconsistent negation. However, one should be cautious when rules for negations are added to positive intuitionistic logic. Thus the addition of both $(\neg \Rightarrow)$ and $(\Rightarrow \neg)$ to LJ^+ is not conservative: the positive fragment of the resulting logic is equivalent to LK^+, not to LJ^+. Accordingly, our first problem is: what combinations of our 11 rules for negation can be added to PLJ (or $PLJ^{\mathbf{ff}}$) so that the resulting system is conservative over LJ^+ (or LJ)? In this section this problem is solved with the help of nondeterministic frames.

We start by following what we have done in the classical case.

Definition 4.7 For $S \subseteq (NR)$ let $\mathcal{M}_{IP}[S]$ and $\mathcal{M}^{\mathbf{ff}}_{IP}[S]$ be the weakest refinements of $\mathcal{M}_{IP}$ and $\mathcal{M}^{\mathbf{ff}}_{IP}$ (respectively) in which the conditions in $C(S)$ are satisfied.

The following soundness theorem can again easily be proved:

Proposition 4.8 *If $S \subseteq NR$ then $PLJ(S)$ is sound for $\mathcal{M}_{IP}[S]$-frames, and $PLJ^{\mathbf{ff}}(S)$ is sound for $\mathcal{M}^{\mathbf{ff}}_{IP}[S]$-frames.*

For itself proposition 4.8 does not have much value. Thus it does not guarantee that $PLJ(S)$ is conservative over LJ^+. It is not even constructively usable for showing non-provability in some $PLJ(S)$ (or $PLJ^{\mathbf{ff}}(S)$). The reason is that a valuation is an infinite object, and it is not clear that a partial description would suffice. Let for example $W = \{a, b\}$ with $a < b$, and define $v'(a,p) = v'(a,q) = v'(b,q) = f$, $v'(b,p) = \top$. Then v' respects the monotonicity condition, but if $(\neg \supset\Rightarrow)_1 \in S$ then there is no extension v of v' such that $\langle W, \leq v\rangle$ is an $\mathcal{M}_{IP}[S]$-frame (since $v(a, p \supset q)$ should be f according to the definition of an $\mathcal{M}_{IP}$-frame, while according to $C(\neg \supset\Rightarrow)_1$ it should be t. We next show that $C(\neg \supset\Rightarrow)_1$ is the only rule that causes such problems.

Definition 4.9 $INR =_{Df} NR - \{(\neg \supset\Rightarrow)_1\}$

Definition 4.10 Let $\mathcal{L}$ be a propositional language which has $\supset$ as one of its connectives, and let $\mathcal{M} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O}\rangle$ be an Nmatrix for $\mathcal{L}$. Denote by $\mathcal{F}$ be the set of formulas of $\mathcal{L}$. An $\mathcal{M}$-*semiframe* is a triple $\mathcal{W} = \langle W, \leq, v'\rangle$ such that:

1. $\langle W, \leq\rangle$ is a nonempty partially ordered set.

2. $v' : W \times \mathcal{F}' \to \mathcal{T}$ satisfies:

- $\mathcal{F}'$ is a subset of $\mathcal{F}$ which is closed under subformulas.
- The monotonicity condition: if $y \geq x$ and $\varphi \in \mathcal{F}'$, then $v(x, \varphi) \leq_k v(y, \varphi)$.
- v' respects $\mathcal{M}$: If $\diamond(\psi_1, \ldots, \psi_n) \in \mathcal{F}'$, then $v'(x, \diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(v(x, \psi_1), \ldots, v(x, \psi_n))$.
- If $\varphi \supset \psi \in \mathcal{F}'$ then $v'(x, \varphi \supset \psi) \in \mathcal{D}$ iff $v'(y, \psi) \in \mathcal{D}$ for every $y \geq x$ such that $v'(y, \varphi) \in \mathcal{D}$.

Theorem 4.11 *Let $S \subseteq INR$, and let $\langle W, \leq, v' \rangle$ (where $v' : W \times \mathcal{F}' \to \mathcal{T}$) be an $\mathcal{M}_{\mathcal{IP}}[S]$-semiframe ($\mathcal{M}_{\mathcal{IP}}{}^{\mathbf{ff}}[S]$-semiframe). Then there exists an $\mathcal{M}_{\mathcal{IP}}[S]$-frame ($\mathcal{M}_{\mathcal{IP}}{}^{\mathbf{ff}}[S]$-frame) $\mathcal{W} = \langle W, \leq, v \rangle$ such that v extends v'.*

Proof: We imitate as far as possible the construction used in the proof of Theorem 3.8. The required v is inductively defined as follows:

- $v(x, \varphi) = f$ iff one of the following conditions is satisfied:

  f1 $\varphi \in \mathcal{F}'$, and $v'(x, \varphi) = f$

  f2 $\varphi = \neg\psi$ and $v(x, \psi) = t$

  f3 $\varphi = \psi_1 \wedge \psi_2$ and $v(x, \psi_1) = f$ or $v(x, \psi_2) = f$

  f4 $\varphi = \psi_1 \vee \psi_2$ and $v(x, \psi_1) = v(x, \psi_2) = f$

  f5 $\varphi = \psi_1 \supset \psi_2$ and there exists $y \geq x$ such that $v(y, \psi_1) \neq f$ while $v(y, \psi_2) = f$

- $v(x, \varphi) = t$ iff one of the following conditions is satisfied:

  t1 $\varphi \in \mathcal{F}'$, and $v'(x, \varphi) = t$

  t2 $\varphi = \neg\psi$, $(\neg\neg \Rightarrow) \in S$, and $v(x, \psi) = f$

  t3 $\varphi = \psi_1 \wedge \psi_2$, $(\neg\wedge \Rightarrow) \in S$ and $v(x, \psi_1) = v(x, \psi_2) = t$

  t4 $\varphi = \psi_1 \vee \psi_2$, $(\neg\vee \Rightarrow)_1 \in S$ and $v(x, \psi_1) = t$

  t5 $\varphi = \psi_1 \vee \psi_2$, $(\neg\vee \Rightarrow)_2 \in S$ and $v(x, \psi_2) = t$

  t6 $\varphi = \psi_1 \supset \psi_2$, $(\neg \supset\Rightarrow)_2 \in S$ and $v(x, \psi_2) = t$

- $v(x, \varphi) = \top$ in any other case.

We prove first by a simultaneous induction on the complexity of φ that if $\varphi \in \mathcal{F}'$ then $v(x,\varphi) = v'(x,\varphi)$, that $\lambda x.v(x,\varphi)$ is $\leq_k$-monotonic, and that $v(x,\varphi)$ is well defined (i.e.: if one of the conditions f1-f5 is satisfied for x and φ then none of the conditions t1-t6 is satisfied for them).

$v(x,\varphi)$ **is well-defined:** Since $\langle W, \leq, v' \rangle$ is an $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$-semiframe, it is obvious that if φ and x satisfy f1 then they cannot satisfy any of the conditions t1-t6, and if they satisfy t1 then they cannot satisfy any of the conditions f1-f5. It is also very easy to see that with the exception of the pair f5/t6, no condition from f2-f5 can be in conflict with a condition from t2-t6. Assume therefore that x and φ satisfy $t6$. Then $\varphi = \psi_1 \supset \psi_2$, and $v(x,\psi_2) = t$. By induction hypothesis on ψ_2, $v(y,\psi_2) = t$ for all $y \geq x$. Hence x and φ do not satisfy $f5$ in this case.[3]

v **is an extension of** v'**:** It is trivial that $v(x,\varphi) = v'(x,\varphi)$ if $\varphi \in \mathcal{F}'$ and $v'(x,\varphi) = f$ or $v'(x,\varphi) = t$. It is also easy to check that since $\langle W, \leq, v' \rangle$ is an $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$-semiframe, the induction hypothesis concerning the subformulas of φ implies that none of the conditions f1-f5, t1-t6 can be satisfied for φ and x in case $\varphi \in \mathcal{F}'$ and $v'(x,\varphi) = \top$. Hence $v(x,\varphi) = \top = v'(x,\varphi)$ in this case as well.

$\lambda x.v(x,\varphi)$ **is** $\leq_k$**-monotonic:** We show that if $y \geq x$ and $v(y,\varphi) = f$ then $v(x,\varphi) = f$, while if $v(y,\varphi) = t$ then $v(x,\varphi) = t$. Assume e.g. that $v(y,\varphi) = f$. Then y and φ satisfy one of the condition f1-f5. Our assumptions concerning v', the transitivity of $\leq$ and the induction hypothesis concerning the subformulas of φ together imply that x and φ satisfy the same condition in case $y \geq x$. Hence $v(x,\varphi) = f$ too. The argument in case $v(y,\varphi) = t$ is similar.

We next show that for every $x \in W$, $v(x,\varphi)$ satisfies the constraints imposed by $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$. This follows from our assumptions concerning v' in case $\varphi \in \mathcal{F}'$. On the other hand the definition of v entails that in case $\varphi \notin \mathcal{F}'$, $v(x,\varphi) = f$ iff this is dictated by $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$, and also $v(x,\varphi) = t$ iff this is dictated by $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$. This immediately entails that also $v(x,\varphi) = \top$ in this case if (but not only if!) this is dictated by$\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$.

To complete the proof it remains to show that $v(x,\psi_1 \supset \psi_2) = f$ iff there exists $y \geq x$ such that $v(y,\psi_1) \neq f$ while $v(y,\psi_2) = f$. Well, the "if" part follows from the definition of v (condition f5). The converse follows from our assumptions concerning v' in case $\psi_1 \supset \psi_2 \in \mathcal{F}'$, and from the definition of v in case $\psi_1 \supset \psi_2 \notin \mathcal{F}'$. ■

The following is an easy corollary of Theorem 4.11:

[3]Note that the argument breaks down if $(\neg \supset \Rightarrow)_1 \in S$, since $C(\neg \supset \Rightarrow)_1$ may be in conflict with condition f5!

Theorem 4.12 *If $S \subseteq INR$ then $PLJ(S)$ is a conservative extension of LJ^+ (and $PLJ^{\mathbf{ff}}(S)$ is a conservative extension of LJ).*

Proof: We do the case of LJ^+ (the case of LJ is identical). Let $\mathcal{F}'$ be the set of $\neg$-free formulas of $\mathcal{L}$. Obviously, $\mathcal{F}'$ is closed under subformulas. Let s be a sequent consisting of formulas from $\mathcal{F}'$, and assume that s is not provable in LJ^+. We show that it is also not provable in $PLJ(S)$. By the completeness of LJ^+ relative to ordinary Kripke frames, there is an ordinary Kripke frame $\langle W, \leq, u\rangle$ in which s is not valid. Define a partial valuation $v' : W \times \mathcal{F}' \to \{t, f, \top\}$ by:

$$v'(x,\varphi) = \begin{cases} \top & u(x,\varphi) = t \\ f & u(x,\varphi) = f \end{cases}$$

It is easy to check that $\langle W, \leq, v'\rangle$ is an $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$-semiframe (since a formula φ might be forced by $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$ to be assigned t only if it either involves $\neg$ or if some subformula of φ is assigned t.[4]). By Theorem 4.11 there exists an $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ such that v extends v'. Obviously s is not valid in $\mathcal{W}$, and so $PLJ(S) \not\vdash s$ by Theorem 4.8. ■

Our next proposition shows that Theorem 4.12 cannot be improved:

Proposition 4.13 *If $(\neg \supset\Rightarrow)_1 \in S$ then $PLJ(S)$ is not conservative over LJ^+. In fact, every valid sequent of LK^+ is provable in it.*

Proof: The sequent $\varphi \supset \psi, \varphi$ can be proved in $PLJ(\{(\neg \supset\Rightarrow)_1\}$ as follows:

$$\dfrac{\dfrac{\varphi \supset \psi \Rightarrow \varphi \supset \psi}{\Rightarrow \varphi \supset \psi, \neg(\varphi \supset \psi)} \qquad \dfrac{\varphi \Rightarrow \varphi}{\neg(\varphi \supset \psi) \Rightarrow \varphi}}{\Rightarrow \varphi \supset \psi, \varphi}$$

Now it is well known that this sequent is not provable in LJ^+, and that by adding it to LJ^+ we get a system which is equivalent to LK^+. ■

From now on we shall concentrate on systems of the form $PLJ(S)$ (or $PLJ^{\mathbf{ff}}(S)$) where $S \subseteq INR$. We shall formulate and prove our results only for the extensions of PLJ. Completely analogous results for extensions of $PLJ^{\mathbf{ff}}$ can easily be formulated and proved (using almost identical arguments).

[4]Note that again this is not true if $(\neg \supset\Rightarrow)_1 \in S$, as an examination of $C(\neg \supset\Rightarrow)_1$ quickly reveals.

4.3 Completeness and Cut-elimination

In Section 3 we prove simultaneously the completeness of our various extensions of (positive) classical logic and the cut-elimination theorem for these systems. We are going to do now something similar for our extensions of (positive) intuitionistic logic. However, here things are more complicated because of the following two facts:

The cut-elimination theorem fails: It is easy to see that the sequent $\Rightarrow p, q \supset \neg p$ is provable in PLJ using a cut on $\neg p$, but no cut-free proof for it exists even in $PLJ(INR)$.

Even analytic cuts are not always sufficient: A standard, quite satisfactory, substitute for full cut-elimination when the latter fails is to allow only *analytic* cuts. These are cuts in proofs in which the cut-formula is a subformula of the endsequent of the proof. Unfortunately, not for every $S \subseteq INR$ it is the case that non-analytic cuts can be eliminate in $PLJ(S)$. Two rules are problematic from this point of view:

- If $(\Rightarrow \neg \supset) \in S$ then the sequent $\Rightarrow p, q \supset \neg(q \supset p)$ is provable using a non-analytic cut on $\neg p$. However, there is no proof in $PLJ(S)$ of this sequent in which all cuts are analytic. To see this, consider the following sequent s:

  $$q \supset p \Rightarrow p, q, \neg(q \supset p), q \supset \neg(q \supset p)$$

  The two two sides of s are disjoint, and it is easy to check that in any proof of a (provable) subsequent of s in which all cuts are on subformulas of s, one of the premises of the last inference is also a subsequent of s with disjoint sides. Hence no subsequent of s has a proof of this sort, and so any proof of a subsequent of s includes non-analytic cuts.

- If $(\Rightarrow \neg\vee) \in S$ then the sequent $\Rightarrow p, q \supset (\neg r \supset \neg(p \vee r))$ is provable using a non-analytic cut on $\neg p$. However, there is no proof in $PLJ(S)$ of this sequent in which all cuts are analytic. To see this, consider the following sequent s:

  $$r, p \vee r \Rightarrow p, q, \neg r, \neg(p \vee r), \neg r \supset \neg(p \vee r), q \supset (\neg r \supset \neg(p \vee r))$$

  Again it is easy to check that in any proof of a subsequent of s in which all cuts are on subformulas of s, one of the premises of the last inference is a subsequent of s with disjoint sides. Hence any proof of a subsequent of s includes non-analytic cuts.

Besides proving the completeness of our various conservative extensions of LJ^+, this subsection has two other main goals. One is to determine exactly in which of these systems one can eliminate non-analytic cuts. The other is to present a satisfactory substitute (and use it to prove completeness of *all* the systems) in case this is impossible. We start with the second goal.

Definition 4.14 Let s be a sequent. A cut is called s-analytic if the cut formula is a subformula of some formula of s. A proof in a Gentzen-type system is called s-analytic iff all cuts in it are s-analytic. A proof of a sequent s is called analytic if it is s-analytic.

Definition 4.15

1. $nsf(\varphi)$ is inductively defined as follows:
   (a) If φ is atomic then $nsf(\varphi) = \{\varphi\}$
   (b) If $\varphi = \neg p$ (p atomic) then $nsf(\varphi) = \{p, \neg p\}$
   (c) If $\varphi \in \{\psi_1 \wedge \psi_2, \psi_1 \vee \psi_2, \psi_1 \supset \psi_2\}$ then $nsf(\varphi) = \{\varphi\} \cup nsf(\psi_1) \cup nsf(\psi_2)$
   (d) If $\varphi = \neg\neg\psi$ then $nsf(\varphi) = \{\varphi\} \cup nsf(\neg\psi)$
   (e) If $\varphi \in \{\neg(\psi_1 \wedge \psi_2), \neg(\psi_1 \vee \psi_2), \neg(\psi_1 \supset \psi_2)\}$ then $nsf(\varphi) = \{\varphi\} \cup nsf(\neg\psi_1) \cup nsf(\neg\psi_2)$
2. ψ is called an *n-subformula* of φ if $\psi \in nsf(\varphi)$. It is called an *n-subformula* of a sequent s iff it is an n-subformula of some formula in s.
3. A cut in a proof is called *s-n_analytic* if it is done on some n-subformula of s. A proof is called *s-n_analytic* if all cuts in it are s-n_analytic.
4. A proof of a sequent s is called *n_analytic* if it is s-n_analytic.

Note that every subformula of a formula or a sequent is an n-subformula of that formula or sequent, and that an n-subformula of an n-subformula of φ is itself an n-subformula of φ. Note also that every n-subformula of a formula φ is either a subformula of φ, or a negation of such a subformula. Moreover: proper n-subformulas of φ are always simpler than φ, and are "internal" to φ (in the sense that if φ is written in Polish notation than any n-subformula of φ is obtained from it by omitting some of the symbols of φ, without changing the order of the remaining ones). Now a crucial observation here is that in a system $PLJ(S)$ even cut-free proofs do not have the usual subformula property, but it is easy to see that they have the *n-subformula property.* This property is kept if *n-analytic* cuts are allowed. Hence in the present context a very satisfactory substitute for full cut-elimination is the possibility to eliminate cuts which are not n-analytic.

Theorem 4.16 *Let $S \subseteq INR$ and assume that the sequent s does not have an n_analytic proof in $PLJ(S)$. Then there exists an $\mathcal{M}_{\mathcal{IP}}[S]$-frame in which s is not valid.*

Proof: Let $\mathcal{F}'$ be the set of subformulas of s, $\mathcal{F}''$ — the set of s-n_subformulas of s, and let W be the set of all sequents which do not have an s-n_analytic proof in $PLJ(S)$, and the union of their two sides is $\mathcal{F}''$. Obviously, if $\Gamma \Rightarrow \Delta$ does not have an s-n_analytic proof in a Gentzen-type system G, and ψ is an n_subformula of s, then either $\psi, \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta, \psi$ does not have an s-n_analytic proof in G. It follows that any sequent which consists of elements of $\mathcal{F}''$ and has no s-n_analytic proof in $PLJ(S)$, can be extended to an element of W. In particular s itself is a subsequent of some sequent $\Gamma_0 \Rightarrow \Delta_0 \in W$. Define now a partial order $\leq$ on W as follows: $\Gamma_1 \Rightarrow \Delta_1 \leq \Gamma_2 \Rightarrow \Delta_2$ if $\Gamma_1 \subseteq \Gamma_2$ (iff $\Delta_2 \subseteq \Delta_1$, since $\Gamma_1 \cup \Delta_1 = \Gamma_2 \cup \Delta_2$).

Finally, define inductively $v' : W \times \mathcal{F}' \to \mathcal{T}$ as follows:

- $v'(\Gamma \Rightarrow \Delta, \varphi) = f$ iff $\varphi \in \Delta$.
- $v'(\Gamma \Rightarrow \Delta, \varphi) = t$ iff one of the following conditions is satisfied:

  t1 $\neg\varphi \in \Delta$

  t2 $\varphi = \neg\psi$, $(\neg\neg \Rightarrow) \in S$, and $v'(\Gamma \Rightarrow \Delta, \psi) = f$

  t3 $\varphi = \psi_1 \wedge \psi_2$, $(\neg\wedge \Rightarrow) \in S$ and $v'(\Gamma \Rightarrow \Delta, \psi_1) = v'(\Gamma \Rightarrow \Delta, \psi_2) = t$

  t4 $\varphi = \psi_1 \vee \psi_2$, $(\neg\vee \Rightarrow)_1 \in S$ and $v'(\Gamma \Rightarrow \Delta, \psi_1) = t$

  t5 $\varphi = \psi_1 \vee \psi_2$, $(\neg\vee \Rightarrow)_2 \in S$ and $v'(\Gamma \Rightarrow \Delta, \psi_2) = t$

  t6 $\varphi = \psi_1 \supset \psi_2$, $(\neg \supset\Rightarrow)_2 \in S$ and $v'(\Gamma \Rightarrow \Delta, \psi_2) = t$

- $v'(\Gamma \Rightarrow \Delta, \varphi) = \top$ in any other case.

We prove now by induction on the complexity of φ that v' is well defined (i.e. if φ and $\Gamma \Rightarrow \Delta$ satisfy one of the conditions t1-t6 then $\varphi \notin \Delta$).

t1 : Since $\Gamma \Rightarrow \Delta$ has no s-n_analytic proof, If $\neg\varphi \in \Delta$ then $\varphi \notin \Delta$.

t2 : Suppose $\varphi = \neg\psi$ and $v'(\Gamma \Rightarrow \Delta, \psi) = f$. Then $\psi \in \Delta$, and so $\varphi \notin \Delta$ (since $\Gamma \Rightarrow \Delta$ has no s-n_analytic proof).

t3 : Suppose $\varphi = \psi_1 \wedge \psi_2$ and $v'(\Gamma \Rightarrow \Delta, \psi_1) = v'(\Gamma \Rightarrow \Delta, \psi_2) = t$. By induction hypothesis, $\psi_1 \notin \Delta$ and $\psi_2 \notin \Delta$. Since ψ_1 and ψ_2 are in $\mathcal{F}'$, $\{\psi_1, \psi_2\} \subseteq \Gamma$. Hence $\varphi \notin \Delta$ (otherwise $\Gamma \Rightarrow \Delta$ would have a cut-free proof).

t4-t6 : we leave the proof to the reader.

We next prove by induction that if $v'(\Gamma \Rightarrow \Delta, \varphi) = t$ then $\neg\varphi, \Gamma \Rightarrow \Delta$ has an s-analytic (in fact, cut-free) proof.

φ and $\Gamma \Rightarrow \Delta$ satisfy t1: This case is trivial.

φ and $\Gamma \Rightarrow \Delta$ satisfy t2: Then $\psi \in \Delta$. Now $\neg\varphi \Rightarrow \psi$ has a cut-free proof in this case, since $\varphi = \neg\psi$ and $(\neg\neg \Rightarrow) \in S$. Hence $\neg\varphi, \Gamma \Rightarrow \Delta$ also has cut-free proof.

φ and $\Gamma \Rightarrow \Delta$ satisfy t3: By induction hypothesis, $\neg\psi_1, \Gamma \Rightarrow \Delta$ and $\neg\psi_2, \Gamma \Rightarrow \Delta$ have cut-free proofs. Using $(\neg\wedge \Rightarrow)$ (which is in S in this case), we get a cut-free proof of $\neg\varphi, \Gamma \Rightarrow \Delta$.

We leave the other three cases to the reader.

We show now that $\langle W, \leq, v' \rangle$ is an $\mathcal{M}_{\mathcal{IP}}[S]$-semiframe.

$\mathcal{F}'$ is a subset of $\mathcal{F}$ closed under subformulas: This is obvious.

The monotonicity condition: Let $\Gamma_1 \Rightarrow \Delta_1 \leq \Gamma_2 \Rightarrow \Delta_2$. Assume that $v'(\Gamma_2 \Rightarrow \Delta_2, \varphi) = f$. Then $\varphi \in \Delta_2$, and so $\varphi \in \Delta_1$ by the definition of $\leq$. It follows that $v'(\Gamma_1 \Rightarrow \Delta_1, \varphi) = f$. The proof that if $v'(\Gamma_2 \Rightarrow \Delta_2, \varphi) = t$ then $v'(\Gamma_1 \Rightarrow \Delta_1, \varphi) = t$ is by induction on the complexity of φ. If $v'(\Gamma_2 \Rightarrow \Delta_2, \varphi) = t$ because φ and $\Gamma_2 \Rightarrow \Delta_2$ satisfy t1, then $v'(\Gamma_1 \Rightarrow \Delta_1, \varphi) = t$ because in this case also φ and $\Gamma_1 \Rightarrow \Delta_1$ satisfy t1 (since $\Delta_2 \subseteq \Delta_1$). If $v'(\Gamma_2 \Rightarrow \Delta_2, \varphi) = t$ because they satisfy one of t2-t6, then the claim easily follows from the induction hypothesis and the already shown fact that if $v'(\Gamma_2 \Rightarrow \Delta_2, \psi) = f$ then $v'(\Gamma_1 \Rightarrow \Delta_1, \psi) = f$.

v' respects the condition concerning $\supset$: Let $\varphi \supset \psi \in \mathcal{F}'$ and $\Gamma \Rightarrow \Delta \in W$. Assume first that there exists $\Gamma_1 \Rightarrow \Delta_1 \in W$ such that $\Gamma \Rightarrow \Delta \leq \Gamma_1 \Rightarrow \Delta_1$, and $v'(\Gamma_1 \Rightarrow \Delta_1, \varphi) \neq f$ while $v'(\Gamma_1 \Rightarrow \Delta_1, \psi) = f$. Then $\varphi \in \Gamma_1$ and $\psi \in \Delta_1$. since $\varphi, \varphi \supset \psi \Rightarrow \psi$ has a cut-free proof, $\varphi \supset \psi \notin \Gamma_1$, and so $\varphi \supset \psi \in \Delta_1$. Since $\Delta_1 \subseteq \Delta$, also $\varphi \supset \psi \in \Delta$, and so $v'(\Gamma \Rightarrow \Delta, \varphi \supset \psi) = f$.

For the converse, assume that $v'(\Gamma \Rightarrow \Delta, \varphi \supset \psi) = f$. Then $\varphi \supset \psi \in \Delta$. It follows that $\Gamma, \varphi \Rightarrow \psi$ is a sequent which consists of elements of $\mathcal{F}''$, and does not have an s-n_analytic proof. Hence it can be extended to a sequent $\Gamma_1 \Rightarrow \Delta_1$ in W. Obviously, $\Gamma \Rightarrow \Delta \leq \Gamma_1 \Rightarrow \Delta_1$, and $v'(\Gamma_1 \Rightarrow \Delta_1, \varphi) \neq f$ while $v'(\Gamma_1 \Rightarrow \Delta_1, \psi) = f$.

v' **respects** $\mathcal{M}_{\mathcal{IP}}[S]$: We prove first that if $\varphi \in \mathcal{F}'$ is a complex formula and $\Gamma \Rightarrow \Delta \in W$ then $v'(\Gamma \Rightarrow \Delta, \varphi) = f$ iff this should be the case according to $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$ (and the definition of a semiframe). This has already been proved in case $\varphi = \psi_1 \supset \psi_2$. We prove the other cases:

- Suppose $\varphi = \neg\psi \in \mathcal{F}'$. Then also $\psi \in \mathcal{F}'$. If $v'(\Gamma \Rightarrow \Delta, \varphi) = f$ then $\varphi \in \Delta$, and so (by condition t1), $v'(\Gamma \Rightarrow \Delta, \psi) = t$. For the converse, assume that $v'(\Gamma \Rightarrow \Delta, \psi) = t$. Then $\varphi, \Gamma \Rightarrow \Delta$ has a cut-free proof (by what we have proved above). Hence $\varphi \notin \Gamma$. Since $\varphi \in \mathcal{F}'$, this implies that $\varphi \in \Delta$, and so $v'(\Gamma \Rightarrow \Delta, \varphi) = f$.
- Suppose $\varphi = \psi_1 \wedge \psi_2 \in \mathcal{F}'$. Assume first that $v'(\Gamma \Rightarrow \Delta, \psi_1) = f$. Then $\psi_1 \in \Delta$. Hence $\varphi \notin \Gamma$ (since $\psi_1 \wedge \psi_2 \Rightarrow \psi_1$ has a cut-free proof), and so $\varphi \in \Delta$. It follows that $v'(\Gamma \Rightarrow \Delta, \varphi) = f$ in this case. The proofs that if $v'(\Gamma \Rightarrow \Delta, \psi_2) = f$ then $v'(\Gamma \Rightarrow \Delta, \varphi) = f$, and that if $v'(\Gamma \Rightarrow \Delta, \psi_1) \neq f$ and $v'(\Gamma \Rightarrow \Delta, \psi_2) \neq f$ then $v'(\Gamma \Rightarrow \Delta, \varphi) \neq f$ are similar.
- The proof that if $\varphi = \psi_1 \vee \psi_2 \in \mathcal{F}'$, then $v'(\Gamma \Rightarrow \Delta, \varphi) = f$ iff either $v'(\Gamma \Rightarrow \Delta, \psi_1) = f$ or $v'(\Gamma \Rightarrow \Delta, \psi_2) = f$, is left to the reader.

We prove now that v' respects the conditions in $C(S)$. This is true by definition of v' (conditions t2-t6) for $S \cap \{C(\neg\neg \Rightarrow), C(\neg\wedge \Rightarrow), C(\neg\vee \Rightarrow)_1, C(\neg\vee \Rightarrow)_2, C(\neg \supset \Rightarrow)_2\}$.

$C(\Rightarrow \neg\neg)$**:** Assume $(\Rightarrow \neg\neg) \in S$, $v'(\Gamma \Rightarrow \Delta, \psi) = \top$, and $\neg\psi \in \mathcal{F}'$. Then $\psi \in \mathcal{F}'$, and $\psi \notin \Delta$. Hence $\psi \in \Gamma$. Since $\psi \Rightarrow \neg\neg\psi$ has a cut-free proof in case $(\Rightarrow \neg\neg) \in S$, $\neg\neg\psi \notin \Delta$. It follows that $\Gamma \Rightarrow \Delta$ and $\neg\psi$ do not satisfy t1. Obviously they do not satisfy t2-t6 either, and so $v'(\Gamma \Rightarrow \Delta, \neg\psi) \neq t$. That in this case $v'(\Gamma \Rightarrow \Delta, \neg\psi) \neq f$ has already been proved. It follows that $v'(\Gamma \Rightarrow \Delta, \neg\psi) = \top$.

$C(\Rightarrow \neg \supset)$**:** Assume that $(\Rightarrow \neg \supset) \in S$, $v'(\Gamma \Rightarrow \Delta, \varphi) \neq f$, $v'(\Gamma \Rightarrow \Delta, \psi) = \top$, and that $\varphi \supset \psi \in \mathcal{F}'$. Then $\varphi \in \mathcal{F}'$, $\varphi \notin \Delta$, $\psi \in \mathcal{F}'$, $\psi \notin \Delta$, and $\neg\psi \notin \Delta$. Hence $\varphi \in \Gamma$. Assume that $\neg(\varphi \supset \psi) \in \Delta$. Then $\neg(\varphi \supset \psi) \in \mathcal{F}''$. Hence also $\neg\psi \in \mathcal{F}''$. It followed that $\neg\psi \in \Gamma \cup \Delta$, and since $\neg\psi \notin \Delta$, $\neg\psi \in \Gamma$. But $\varphi, \neg\psi \Rightarrow \neg(\varphi \supset \psi)$ has a cut-free proof in case $(\Rightarrow \neg \supset) \in S$, and so $\Gamma \Rightarrow \Delta$ has such a proof too, contradicting $\Gamma \Rightarrow \Delta \in W$. It follows that $\neg(\varphi \supset \psi) \notin \Delta$ and so $\Gamma \Rightarrow \Delta$ and $\varphi \supset \psi$ do not satisfy t1. Obviously they do not satisfy t2-t6 either, and so $v'(\Gamma \Rightarrow \Delta, \varphi \supset \psi) \neq t$. That in this case $v'(\Gamma \Rightarrow \Delta, \varphi \supset \psi) \neq f$ has already been proved. It follows that $v'(\Gamma \Rightarrow \Delta, \varphi \supset \psi) = \top$.

Again we leave the cases of $C(\Rightarrow \neg\vee)$, $C(\Rightarrow \neg\wedge)_1$, and $C(\Rightarrow \neg\wedge)_2$ to the reader.

We have shown that $\langle W, \leq, v' \rangle$ is an $\mathcal{M}_{\mathcal{IP}}[S]$-semiframe. Therefore by Theorem 4.11 there exists an $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$-frame $\mathcal{W} = \langle W, \leq, v \rangle$ such that v extends v'. Now it is easy to see that our original sequent s is false in the world $\Gamma_0 \Rightarrow \Delta_0$ of $\mathcal{W}$ (of which s is a subsequent). Hence s is not valid in the $\mathcal{M}_{\mathcal{IP}}[\mathcal{S}]$-frame $\mathcal{W}$. ∎

Corollary 4.17 *If $S \subseteq INR$ then $PLJ(S)$ is sound and complete for $\mathcal{M}_{\mathcal{IP}}[S]$-frames.*

Note: In the special case of da Costa's C_ω Corollary 4.17 provides illuminating semantics which is much simpler than the Kripke-type semantics given in [Baa86] and the bivaluations semantics of [Lop86]. Here is a compact description of this semantics for the reader particularly interested in C_ω: a Kripke-type frame for C_ω is a triple $\langle W, \leq, v \rangle$ such that $\langle W, \leq \rangle$ is a nonempty partially ordered set, and $v : W \times \mathcal{F} \to \{t, f, \top\}$ is a valuation which satisfies the following conditions:

- If $x \leq y$ then $v(x, \varphi) \leq_k v(y, \varphi)$
- $v(x, \varphi \wedge \psi) = f$ iff $v(x, \varphi) = f$ or $v(x, \psi) = f$
- $v(x, \varphi \vee \psi) = f$ iff $v(x, \varphi) = f$ and $v(x, \psi) = f$
- $v(x, \varphi \supset \psi) = f$ iff there exists $y \geq x$ such that $v(y, \varphi) \neq f$ while $v(y, \psi) = f$
- $v(x, \neg\varphi) = f$ iff $v(x, \varphi) = t$
- If $v(x, \varphi) = f$ then $v(x, \neg\varphi) = t$

It is interesting to note that by changing the last clause to an "iff" we get an adequate semantics for the system obtained from C_ω by adding the axiom $\varphi \supset \neg\neg\varphi$ (which corresponds to the rule $(\Rightarrow \neg\neg)$. In both cases a frame is of course a model of a formula φ if $v(x, \varphi) \neq f$ for every $x \in W$.

Corollary 4.18 *If s has a proof in $PLJ(S)$ $(S \subseteq INR)$ then s has there a n_analytic proof.*

We show finally that n_analytic cuts can actually be replaced by strictly *analytic* ones in all of the systems $PLJ(S)$ except those which contain at least one of the two problematic rules noted at the beginning of this subsection.

Theorem 4.19 *Let $S \subseteq INR - \{(\Rightarrow \neg \supset), (\Rightarrow \neg\vee)\}$. If s has a proof in $PLJ(S)$ then s has there an analytic proof.*

Proof: Assume that the sequent s does not have an analytic proof in $PLJ(S)$. Again it suffices to show that there exists an $\mathcal{M}_{\mathcal{IP}}[S]$-semiframe in which s is not valid.

Let $\mathcal{F}'$ and $\mathcal{F}''$ be as in the proof of Theorem 4.16. Call a sequent $\Gamma \Rightarrow \Delta$ *s-acceptable* if $\Gamma, \Delta \subseteq \mathcal{F}''$, and there are sets $\Delta^0, \ldots, \Delta^k$ such that $\Delta = \bigcup_{i=0}^{k} \Delta^i$, $\Delta^0 \subseteq \mathcal{F}'$, and for every $i > 0$ and φ, $\varphi \in \Delta^i$ iff $\varphi \notin \bigcup_{j<i} \Delta^j$, and it satisfies one of the following conditions:

1. $(\Rightarrow \neg\neg) \in S$ and $\neg\neg\varphi \in \Delta^{i-1}$.

2. $(\Rightarrow \neg\wedge)_1 \in S$ and there exist ψ_1, ψ_2 such that $\varphi = \neg\psi_1$ and $\neg(\psi_1 \wedge \psi_2) \in \Delta^{i-1}$.

3. $(\Rightarrow \neg\wedge)_2 \in S$ and there exist ψ_1, ψ_2 such that $\varphi = \neg\psi_2$ and $\neg(\psi_1 \wedge \psi_2) \in \Delta^{i-1}$.

Let W be the set of all sequents which are s-acceptable, contain all the formulas of $\mathcal{F}'$, and have no s-analytic proof.

Lemma 4.20 Let $s' = \Gamma' \Rightarrow \Delta'$ be a sequent which has no s-analytic proof, and such that $\Gamma' \subseteq \mathcal{F}''$, $\Delta' \subseteq \mathcal{F}'$. Then s' is a subsequent of some sequent in W.

Proof of Lemma 4.20: Since s' has no s-analytic proof, one can first add to it in stages all the formulas of $\mathcal{F}'$, so that the resulting sequent $s'' = \Gamma'' \Rightarrow \Delta''$ still has no s-analytic proof. Let $\Delta^0 = \Delta''$. Since only formulas from $\mathcal{F}'$ have been added to s', $\Delta^0 \subseteq \mathcal{F}'$. Define Δ^i for $i > 0$ by letting $\varphi \in \Delta^i$ iff $\varphi \notin \bigcup_{j<i} \Delta^j$, and it satisfies one of the conditions 1-3 above. Then $\Delta_i \subseteq \mathcal{F}''$ for all i. Since $\mathcal{F}''$ is finite, There is k such that $\Delta_k = \emptyset$. For this k $\Gamma'' \Rightarrow \bigcup_{j<k} \Delta^j$ is obviously an element of W with the required properties.

Define now $\leq$ on W by: $\Gamma_1 \Rightarrow \Delta_1 \leq \Gamma_2 \Rightarrow \Delta_2$ if either $\Gamma_1 = \Gamma_2$ and $\Delta_1 = \Delta_2$, or Γ_1 is a proper subset of Γ_2.

Lemma 4.21 If $\Gamma_1 \Rightarrow \Delta_1 \leq \Gamma_2 \Rightarrow \Delta_2$ then $\Delta_2 \subseteq \Delta_1$.

Proof of Lemma 4.21: Let $\Delta_2 = \bigcup_{i=0}^{k} \Delta_2^i$, where $\Delta_2^0, \ldots, \Delta_2^k$ are as in the definition of an s-acceptable sequent. We prove by induction on i that if $\varphi \in \Delta_2^i$ then $\varphi \in \Delta_1$.

For the base case, assume that $\varphi \in \Delta_2^0$. Then $\varphi \in \mathcal{F}'$. Hence $\varphi \in \Gamma_1 \cup \Delta_1$. Suppose that $\varphi \in \Gamma_1$. Then also $\varphi \in \Gamma_2$ (by definition of $\leq$), and so $\varphi \in \Gamma_2 \cap \Delta_2$. This contradicts the fact that $\Gamma_2 \Rightarrow \Delta_2$ has no s-analytic proof. It follows that $\varphi \in \Delta_1$.

For the induction step assume that $\Delta_2^i \subseteq \Delta_1$, and let $\varphi \in \Delta_2^{i+1}$. By definition there are three cases to consider. Assume, e.g., that $(\Rightarrow \neg\wedge)_1 \in S$ and there exist ψ_1, ψ_2 such that $\varphi = \neg\psi_1$ and

$\neg(\psi_1 \wedge \psi_2) \in \Delta_2^i$ (the other cases are treated similarly). By induction hypothesis, $\neg(\psi_1 \wedge \psi_2) \in \Delta_1$. Since $\Gamma_1 \Rightarrow \Delta_1$ is s-acceptable and $(\Rightarrow \neg\wedge)_1 \in S$, also $\varphi = \neg\psi_1 \in \Delta_1$.[5].

We now define $v' : W \times \mathcal{F}' \to \mathcal{T}$ exactly as we did in the proof of Theorem 4.16. Then we prove that v' is well defined, and that if $v'(\Gamma \Rightarrow \Delta, \varphi) = t$ then $\neg\varphi, \Gamma \Rightarrow \Delta$ has an s-analytic proof (the proof of these facts are again identical to those given in the proof of Theorem 4.16).

We now show that $\langle W, \leq, v' \rangle$ is an $\mathcal{M}_{\mathcal{IP}}[S]$-semiframe.

The monotonicity condition: The proof is almost identical to that given in in the proof of Theorem 4.16. The only difference is that Lemma 4.21 should be used when facts of the form $\Delta_2 \subseteq \Delta_1$ are needed.

v' respects the condition concerning $\supset$: Let $\varphi \supset \psi \in \mathcal{F}'$ and $\Gamma \Rightarrow \Delta \in W$. Then $\psi \in \mathcal{F}'$. Assume that $v'(\Gamma \Rightarrow \Delta, \varphi \supset \psi) = f$. We show that there exists $\Gamma_1 \Rightarrow \Delta_1 \in W$ such that $\Gamma_1 \Rightarrow \Delta_1 \geq \Gamma \Rightarrow \Delta$, and $v'(\Gamma_1 \Rightarrow \Delta_1, \varphi) \neq f$ while $v'(\Gamma_1 \Rightarrow \Delta_1, \psi) = f$. Well, from our assumption it follows that $\varphi \supset \psi \in \Delta$. Since $\psi \Rightarrow \varphi \supset \psi$ has a cut-free proof, $\psi \notin \Gamma$. Hence $\psi \in \Delta$ (since $\psi \in \mathcal{F}'$). If $\varphi \in \Gamma$ then $v'(\Gamma \Rightarrow \Delta, \varphi) \neq f$ while $v'(\Gamma \Rightarrow \Delta, \psi) = f$. Hence we can take in this case $\Gamma_1 = \Gamma$ and $\Delta_1 = \Delta$. If $\varphi \notin \Gamma$ consider $\Gamma, \varphi \Rightarrow \psi$. This sequent has no s-analytic proof (otherwise $\Gamma \Rightarrow \Delta$ would have one), contains only formulas from $\mathcal{F}''$, and its left-hand side contains only formulas from $\mathcal{F}'$. By Lemma 4.20 it can therefore be extended to a sequent $\Gamma_1 \Rightarrow \Delta_1 \in W$. Since $\varphi \notin \Gamma$, Γ is a proper subset of Γ_1, and so $\Gamma \Rightarrow \Delta \leq \Gamma_1 \Rightarrow \Delta_1$. Obviously $v'(\Gamma_1 \Rightarrow \Delta_1, \varphi) \neq f$ while $v'(\Gamma_1 \Rightarrow \Delta_1, \psi) = f$, as required.

The converse of what we have just proven is shown exactly as in the proof of Theorem 4.16.

v' respects $\mathcal{M}_{\mathcal{IP}}[S]$: The proofs that $v'(\Gamma \Rightarrow \Delta, \varphi) = f$ iff it should, and that $v'(\Gamma \Rightarrow \Delta, \varphi) = t$ in any case it should, are exactly as in the proof of Theorem 4.16. It remains to prove that $v'(\Gamma \Rightarrow \Delta, \varphi) = \top$ whenever it should. By the constraints on S, there are only three cases to consider here:

- Assume $(\Rightarrow \neg\neg) \in S$, $v'(\Gamma \Rightarrow \Delta, \psi) = \top$, and $\neg\psi \in \mathcal{F}'$. The proof that in this case $v'(\Gamma \Rightarrow \Delta, \neg\psi) = \top$ is again identical to the one given in the proof of Theorem 4.16.
- Assume that $(\Rightarrow \neg\wedge)_1 \in S$, $v'(\Gamma \Rightarrow \Delta, \varphi) = \top$, $v'(\Gamma \Rightarrow \Delta, \psi) \neq f$, and $\varphi \wedge \psi \in \mathcal{F}'$. We should show that $v'(\Gamma \Rightarrow \Delta, \varphi \wedge \psi) = \top$. Well, that $v'(\Gamma \Rightarrow \Delta, \varphi \wedge \psi) \neq f$ in such a

[5]This is the step in the proof which will fail if one of the two problematic rules is added.

case was already proven (since v' assigns the value f only if it should). We show now that $v'(\Gamma \Rightarrow \Delta, \varphi \wedge \psi) \neq t$ as well. Suppose otherwise. Since t2-t6 obviously fail in this case, this can happen only if $\neg(\varphi \wedge \psi) \in \Delta$. But since $(\Rightarrow \neg\wedge)_1 \in S$ and $\Gamma \Rightarrow \Delta$ is s-acceptable, this implies that $\neg\varphi \in \Delta$ too. It follows that $v'(\Gamma \Rightarrow \Delta, \varphi) = t$ by definition of v', contradicting $v'(\Gamma \Rightarrow \Delta, \varphi) = \top$.

- An argument similar to the previous one applies in the case where $(\Rightarrow \neg\wedge)_2 \in S$, $v'(\Gamma \Rightarrow \Delta, \psi) = \top$, $v'(\Gamma \Rightarrow \Delta, \varphi) \neq f$, and $\varphi \wedge \psi \in \mathcal{F}'$.

Together with Lemma 4.20, the fact that $\langle W, \leq, v' \rangle$ is an $\mathcal{M}_{\mathcal{IP}}[S]$-semiframe easily entails the Theorem (see the proof of Theorem 4.16). ■

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