A Formula-Preferential Base for Paraconsistent and Plausible Reasoning Systems

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Abstract

We provide a general framework for constructing natural consequence relations for paraconsistent and plausible nonmonotonic reasoning. The framework is based on preferential systems whose preferences are based on the satisfaction of formulas in models. We show that these natural preferential systems that were originally designed for paraconsistent reasoning fulfill a key condition (stopperedness or smoothness) from the theoretical research of nonmonotonic reasoning. Consequently, the nonmonotonic consequence relations that they induce fulfill the desired conditions of plausible consequence relations. Hence our frameword encompasses different types of preferential systems that were developed from different motivations of paraconsistent reasoning and non-monotonic reasoning, and reveals an important link between them.

1 Introduction

For a long time the research efforts on paraconsistency and on nonmonotonic reasoning were separated. The former research dealt with the question of how to prevent the inference of every fact from an inconsistent source of knowledge, and how to isolate inconsistent parts of the knowledge and yet work in the usual way with the consistent parts. The latter dealt with the question of how to "jump to conclusions" based on partial knowledge of the domain (this is needed since having complete knowledge is often unrealistic), and how to revise previous "hasty" conclusions in the face of new and fuller information.

However, in recent years the formal connections between these two areas have begun to be revealed. It is only natural that such a connection would exist, because conclusions that are drawn based on partial knowledge may contradict new and more reliable information, and each new piece of information may contradict previous information and hence force us to revise some of our knowledge. As the famous example goes, if we conclude that Tweety can fly based on the sole fact that it is a bird, the new piece of information that Tweety is a penguin and penguins cannot fly forces us not only to revise previous conclusions but also to deal with the fact that we now have a contradiction in our knowledge.

Both goals of handling contradictions and reasoning nonmonotonically require some selection between alternatives: which parts of the knowledge to retain and which to discard or change. A central tool in both fields has been *preferential systems*, meaning that only a subset of the models should be relevant for making inferences from a theory. These models are the most preferred ones according to some criterion.

In the research on paraconsistency, preferential systems were used for constructing logics which are paraconsistent but stronger than substructural paraconsistent logics. The preferences in these systems were defined in different ways. Some were based on checking which abnormal formulas (such as $\psi \wedge \neg\psi$) are satisfied in the models of a theory (see e.g. [Priest, 1991; Batens, 1998]). Others were based on preferences between the truth values that are assigned to formulas (see e.g. [Kifer and Lozinskii, 1992; Arieli and Avron, 2000a]).

Preferential systems were also used for providing semantics for nonmonotonic consequence relations (see e.g. [Shoham, 1987; Kraus *et al.*, 1990; Makinson, 1994]). It was discovered, however, that in order for them to fulfill all the desired theoretical properties that plausible nonmonotonic relations should have (see e.g. [Lehmann, 1992]), preferential systems need to satisfy a further condition called stopperedness or smoothness. The problem is that this condition is usually not easy to verify.

In this paper we provide a general framework for constructing natural consequence relations for paraconsistent and plausible nonmonotonic reasoning. The main technique is using preferential systems in which the preference between models is made according to a certain set of formulas which are satisfied in them. The framework is obtained by a generalization of some preferential systems that were used for constructing useful paraconsistent consequence relations, and encompasses also other systems. Moreover, these natural preferential systems that were originally designed for paraconsistent reasoning fulfill the stopperedness condition as well, and hence have also the desired theoretical properties of nonmonotonic consequence relations.

As we said, the theoretical research on nonmonotonic reasoning and the research on paraconsistent reasoning have been conducted separateley at first. Nevertheless, formula-preferential systems, which are a generalization of methods used in the latter, solve a key issue in the former, and help to bridge the gap between the two directions of research and to combine them under a unified framework. This provides strong evidence for their important rule in non-classical reasoning.

The structure of the rest of this paper is as follows. In section 2 we review some basic concepts related to multi-valued monotonic logics, including the concept of a non-deterministic matrix. In section 3 we briefly review one direction of the research in nonmonotonic reasoning, and show how stoppered preferential systems provide semantics for plausible relations. In section 4 we present our main framework of formula-preferential systems and prove that under a natural condition they fulfill the stopperedness condition. We bring several examples of systems from the literature and show how they can be constructed in our framework. In section 5 we review another technique that was used in preferential systems for paraconsistent reasoning, namely the use of preferences between truth values. We show that these systems can be simulated by formula-preferential systems. We conclude the paper with some remarks and directions for further research.

2 Preliminaries

2.1 Consequence relations and semantic structures

In what follows $\mathcal{L}$ is a language, $\mathcal{W}$ is its set of wffs, ψ, ϕ, τ denote arbitrary formulas (of $\mathcal{L}$), and Γ, Δ denote sets of formulas. When the language is propositional, $\mathcal{A}$ denotes its set of propositional variables, and p, q, r denote such variables.

In this paper the non-monotonic consequence relations that we shall use will be based (in a way to be defined below) on underlying monotonic multiple-conclusion consequence relations. The intuitive idea of such a relation $\vdash$ is that $\Gamma \vdash \Delta$ holds true iff either one of the elements of Δ is true or one of the elements of Γ is false. This will be precisely defined in Definition 2.3.

Definition 2.1

1. [Scott, 1974a; 1974b] A (*Scott*) *consequence relation* (*scr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ that satisfies the following conditions:

   s-R *strong reflexivity*:
   if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$
   M *monotonicity*:
   if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$
   C *cut*:
   if $\Gamma \vdash \psi, \Delta$ and $\Gamma', \psi \vdash \Delta'$ then $\Gamma, \Gamma' \vdash \Delta, \Delta'$

2. An scr $\vdash$ is *finitary* if the following condition holds for all $\Gamma, \Delta \subseteq \mathcal{W}$: if $\Gamma \vdash \Delta$ then there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ s.t. $\Gamma' \vdash \Delta'$.

When the language is propositional, it is usually required that a monotonic consequence relation for the language will be closed under uniform substitutions:

Definition 2.2 A uniform substitution is a function $\sigma : \mathcal{A} \to \mathcal{W}$ that is extended to all $\mathcal{W}$ by: $\sigma(\diamond(\psi_1, \ldots, \psi_n)) = \diamond(\sigma(\psi_1), \ldots, \sigma(\psi_n))$ for any n-ary connective $\diamond$. $\vdash$ is *uniform* if for every uniform substitution σ and every Γ and Δ, if $\Gamma \vdash \Delta$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$.

We shall use a general definition of semantic structures that does not assume anything about its models, except that there is some satisfaction relation that indicates which formulas are satisfied by each of the models.

Definition 2.3 [1] A *semantic structure* for $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathcal{M}_\mathcal{S}, \models^\mathcal{S} \rangle$, where $\models^\mathcal{S} \subseteq \mathcal{M}_\mathcal{S} \times \mathcal{W}$. $\mathcal{M}_\mathcal{S}$ is a set of *models* and $\models^\mathcal{S}$ is called a *satisfaction relation*. A model $m \in \mathcal{M}_\mathcal{S}$ *satisfies* a formula ψ if $m \models^\mathcal{S} \psi$. m is a *model of* Γ ($m \models^\mathcal{S} \Gamma$) if it satisfies every formula in Γ. The set of the models of Γ is denoted by $mod(\Gamma, \mathcal{S})$. Δ is a *consequence* of Γ in $\mathcal{S}$ ($\Gamma \vdash^\mathcal{S} \Delta$) if for every $m \in mod(\Gamma, \mathcal{S})$, $m \models^\mathcal{S} \phi$ for some $\phi \in \Delta$. $\vdash^\mathcal{S}$ is called the *consequence relation induced by* $\mathcal{S}$. We say that $\mathcal{S}$ is *finitary* if $\vdash^\mathcal{S}$ is finitary.

It is easy to verify that every semantic structure induces an scr.

2.2 Non-deterministic matrices

A common type of semantic structures for propositional logics is the class of multi-valued matrices. These structures employ the classical principle of assigning truth values to formulas, i.e. the value that a valuation assigns to a complex formula is uniquely determined by the values that it assigns to its subformulas. However, an agent acting in the real world often has only incomplete or imprecise knowledge to guide its decisions. This knowledge may even be inconsistent. When this is the case the classical approach becomes useless, and an alternative approach is needed.

One possible such alternative is to borrow the idea of *non-deterministic* computations from automata and computability theory, and apply it for assigning truth-values to complex formulas. This approach has indeed (implicitly) been used in [Batens, 1998] for handling inconsistent data. This was done, however, in an ad-hoc way. Here we introduce a natural generalization of the logical concept of a matrix. In this generalization the value that a valuation assigns to a complex formula can be chosen non-deterministically from a certain nonempty set of options. We therefore call these structures *non-deterministic matrices*:

Definition 2.4 [Avron and Lev, 2000] A *non-deterministic matrix* (*Nmatrix* for short) for a propositional language $\mathcal{L}$ is a tuple $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{T}$ is a non-empty set of *truth values*, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{T}$ (its *designated values*), and for every n-ary connective $\diamond$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{T}^n$ to $2^\mathcal{T} - \{\emptyset\}$. A *valuation* in $\mathcal{S}$ is a function $v : \mathcal{W} \to \mathcal{T}$ that satisfies the condition: if $\diamond$ is an n-ary connective, and $\psi_1, \ldots, \psi_n \in \mathcal{W}$, then $v(\diamond(\psi_1, \ldots, \psi_n)) \in \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$. $\mathcal{V}_\mathcal{S}$ denotes the set of valuations of $\mathcal{S}$. The satisfaction relation $\models^\mathcal{S} \subseteq \mathcal{V}_\mathcal{S} \times \mathcal{W}$ is defined: $v \models^\mathcal{S} \psi$ iff $v(\psi) \in \mathcal{D}$. We identify the Nmatrix $\mathcal{S}$ with the semantic structure $\langle \mathcal{V}_\mathcal{S}, \models^\mathcal{S} \rangle$. $\vdash_\mathcal{S}$ and $\models^\mathcal{S}$ are defined as in Definition 2.3. The same applies for all the other concepts of that definition. We say that $\mathcal{S}$ is (in)finite if $\mathcal{T}$ is (in)finite.

Notes:

1. Every (deterministic) matrix can be identified with an Nmatrix whose functions in $\mathcal{O}$ always return singletons.

[1] See e.g. [Makinson, 1989; Lehmann, 1992].

2. It is easy to verify that if $\mathcal{S}$ is an Nmatrix then $\vdash_{\mathcal{S}}$ is a uniform scr.

In addition to their obvious potential for reasoning under uncertainty and for specification and verification of nondeterministic programs, N-matrices have considerable practical technical applications. It is well known that every propositional logic can be characterized semantically using a multi-valued matrix ([Łos and Suszko, 1958]). However, there are important logics whose characteristic matrices necessarily consist of an infinite number of truth values. Such characteristic matrices are frequently of little help in providing decision procedures for their logics, or in getting real insight into them. Our generalization of the concept of a matrix allows us to replace in many cases an infinite characteristic matrix for a given propositional logic by a characteristic *finite* structure that automatically provides a decision procedure. We provide now examples for such cases. These Nmatrices define monotonic logics that will be the underlying logics of some nonmonotonic logics from the literatures that we shall review in later sections.

Let $\mathcal{L}_{\rm cl}$ be the classical propositional language with the connectives $\{\wedge, \vee, \supset, \neg, f\}$. For a connective $\diamond$ of $\mathcal{L}_{\rm cl}$, $g_\diamond$ denotes the corresponding classical boolean operation. The Nmatrices $\mathcal{S}_p^\top$ and $\mathcal{S}_p^\bot$ for $\mathcal{L}_{\rm cl}$ have the set of truth values $\{t, f\}$ and the designated value t. They interpret the connectives as follows:

$\widetilde{f} = \{f\}$;
if $\diamond \in \{\wedge, \vee, \supset\}$ then $\widetilde{\diamond}(x_1, x_2) = \{g_\diamond(x_1, x_2)\}$;
in $\mathcal{S}_p^\top$, $\widetilde{\neg} f = \{t\}$, $\widetilde{\neg} t = \{t, f\}$;
in $\mathcal{S}_p^\bot$, $\widetilde{\neg} t = \{f\}$, $\widetilde{\neg} f = \{t, f\}$.

We shall use $\vdash_p^\top$ as a shorthand for $\vdash^{\mathcal{S}_p^\top}$, and similarly with other relations.

Note: the consequence relations induced by $\mathcal{S}_p^\top$ and $\mathcal{S}_p^\bot$ cannot be induced by finite matrices (see [Avron and Lev, 2000] for a proof). This result can be generalized as follows:

Theorem 2.5 *Let $\mathcal{S}$ be a two-valued N-matrix which has at least one proper nondeterministic operation. Then $\vdash_{\mathcal{S}}$ has no finite characteristic matrix. If in addition $\mathcal{S}$ includes the classical positive operations, then $\vdash_{\mathcal{S}}$ has no finite weakly-characteristic matrix.*

Definition 2.6 An scr $\vdash$ is paraconsistent w.r.t. $\neg$ when $\psi, \neg\psi \not\vdash \emptyset$ for some ψ, and it is paracompleteness w.r.t. $\neg$ when $\emptyset \not\vdash \psi, \neg\psi$ for some ψ.[2]

Let $\vdash_{\rm pos}$ be the uniform closure of positive classical logic in $\mathcal{L}_{\rm cl}$ (this practically means that every negated formula is treated as an atomic formula, while the semantics of the positive connectives is the classical one). The essential property of $\vdash_p^\top$ is that it is the *minimal* logic in $\mathcal{L}_{\rm cl}$ that contains $\vdash_{\rm pos}$ and in which $\emptyset \vdash_p^\top \psi, \neg\psi$ for all ψ. This means that $\vdash_{\rm pos}$ is paraconsistent and not paracomplete. $\vdash_p^\bot$ is the minimal logic in $\mathcal{L}_{\rm cl}$ that contains $\vdash_{\rm pos}$ and in which $\psi, \neg\psi \vdash_p^\bot \emptyset$ for all ψ, i.e. it is paracomplete and not paraconsistent.

[2] The name 'paraconsistent' was coined by Quesada at the 3rd Latin American Conference on Mathematical Logic in 1976, and the name 'paracomplete' is from [Batens *et al.*, 1999].

To complete the picture, we mention that $\vdash_p^\top$ and $\vdash_p^\bot$ can be characterized using Gentzen-type and Hilbert-type calculi. The Gentzen-type calculus $\mathbf{G}_p^\top$ is obtained from Gentzen's original calculus (in [Gentzen, 1969]) for classical logic (including cut) by omitting the rule $[\neg \Rightarrow]$ for introducing negation on the left (any other version of the classical calculus would do here just as well, as long as its rules for negation are the two standard ones). Similarly, $\mathbf{G}_p^\bot$ is obtained from this calculus by omitting $[\Rightarrow \neg]$. It can be shown[3] that $\vdash_{\mathbf{G}_p^\top} = \vdash_p^\top$ and $\vdash_{\mathbf{G}_p^\bot} = \vdash_p^\bot$. (The standard scr $\vdash_{\mathbf{G}}$ which is associated with a given Gentzen-type system $\mathbf{G}$ is defined by: $\Gamma \vdash_{\mathbf{G}} \Delta$ iff there exist finite $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ such that $\Gamma' \Rightarrow \Delta'$ is provable in $\mathbf{G}$.)

These results can be generalized as follows:

Theorem 2.7 *let $\mathbf{G}_c$ be the standard calculus for classical logic in $\mathcal{L}_{\rm cl}$ in which each connective has exactly two corresponding logical rules. Then every system which is obtained from $\mathbf{G}_c$ by omitting some of its logical rules is decidable, admits cut-elimination, and has a characteristic two-valued N-matrix.*

An even stronger generalization (which will take us too far away from our purpose here) can be found in [Avron and Lev, 2001].

To conclude the discussion on these logics, $\vdash_p^\top$ and $\vdash_p^\bot$ are respectively the same as **CLuN** and **CLaN** from [Batens *et al.*, 1999]. $\vdash_p^\bot$ is also identical to the logic $K/2$ of [Béziau, 1999], where cut-elimination for $\mathbf{G}_p^\bot$ and its completeness have already been claimed. Two-valued N-matrices induce in fact a constructive subclass of the class of bivaluations used in [Béziau, 1999].

We now define some further Nmatrices that can actually be defined as ordinary matrices. $\mathcal{S}_4$ is the matrix for $\mathcal{L}_{\rm cl}$ that has $\{t, f, \top, -\}$ as the truth values and $\{t, \top\}$ as the designated values. $\leq_t$ is a partial ordering of the truth values defined by: $f <_t (\top, -) <_t t$. The interpretation of the connectives is:

$\widetilde{f} = \{f\}$;
$x_1 \widetilde{\wedge} x_2 = \{\inf_{\leq_t}\{x_1, x_2\}\}$; $\quad x_1 \widetilde{\vee} x_2 = \{\sup_{\leq_t}\{x_1, x_2\}\}$;

$$x_1 \widetilde{\supset} x_2 = \begin{cases} \{x_2\} & \text{if } x_1 \in \{t, \top\} \\ \{t\} & \text{otherwise;} \end{cases}$$

$\widetilde{\neg} t = \{f\}$, $\widetilde{\neg} f = \{t\}$, $\widetilde{\neg} \top = \{\top\}$, $\widetilde{\neg} - = \{-\}$.

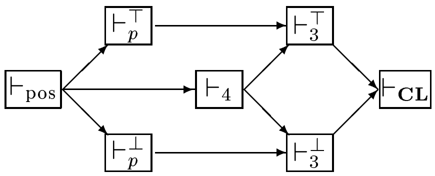
$\widetilde{\supset}$ is defined so that $\supset$ will be an internal implication w.r.t. $\vdash_4$, i.e. $\Gamma, \psi \vdash_4 \phi, \Delta$ iff $\Gamma \vdash_4 \psi \supset \phi, \Delta$. The values t and f behave like the classical values w.r.t. the negation $\neg$. The value $\top$ represents "inconsistency", because a valuation v satisfies both a formula ψ and its negation $\neg\psi$ iff $v(\psi) = \top$. The value $-$ represents "incompleteness", because a valuation satisfies neither a formula ψ nor its negation $\neg\psi$ iff $v(\psi) = -$.
$\mathcal{S}_3^\top$ is the submatrix of $\mathcal{S}_4$ with the truth values $\{t, f, \top\}$.
$\mathcal{S}_3^\bot$ is the submatrix of $\mathcal{S}_4$ with the truth values $\{t, f, -\}$. [4]

[3] See [Avron and Lev, 2000].

[4] These three matrices have been widely investigated in the literature. See e.g. [Kleene, 1950; Belnap, 1977a; 1977b; D'ottaviano, 1985; Avron, 1986; 1991; Ginsberg, 1988; Rozoner, 1989; Epstein, 1990; Fitting, 1990; 1991; Priest, 1989; 1991; Arieli and Avron, 1996; 1998].

The essential property of $\vdash_4$ (respectively, $\vdash_3^\top$, $\vdash_3^\perp$) is that it is the *maximal* logic in $\mathcal{L}_{\rm cl}$ that contains $\vdash_{\rm pos}$ and that is paraconsistent and paracomplete w.r.t. $\neg$ ($\vdash_3^\top$ – paraconsistent (but not paracomplete); $\vdash_3^\perp$ – paracomplete (but not paraconsistent)).

Figure 1 presents the relations between the logics.



$\vdash_x \rightarrow \vdash_y$ means $\vdash_x \subset \vdash_y$

Figure 1: Some basic consequence relations

We conclude this section with a result that will be important for our framework in later sections:

Theorem 2.8 [Avron and Lev, 2000]. *Every finite Nmatrix is finitary.*

3 Nonmonotonic Consequence Relations

3.1 Plausible consequence relations

Monotonic consequence relations are not suitable for many applications in AI, and hence many systems that exhibit nonmonotonic behavior have been developed and studied. [Gabbay, 1985] began a theoretical investigation of conditions that nonmonotonic consequence relations should satisfy. It was suggested that such relations $\mid\!\sim$ should satisfy at least three basic conditions:

Definition 3.1 A *cautious consequence relation* is a binary relation $\mid\!\sim$ between sets of formulas and formulas that satisfies the following conditions:

reflexivity:	$\Gamma \mid\!\sim \psi$ if $\psi \in \Gamma$
weak monotonicity:	if $\Gamma \mid\!\sim \psi$ and $\Gamma \mid\!\sim \phi$ then $\Gamma, \phi \mid\!\sim \psi$
cut:	if $\Gamma \mid\!\sim \psi$ and $\Gamma, \psi \mid\!\sim \phi$ then $\Gamma \mid\!\sim \phi$

Weak monotonicity replaces the usual monotonicity condition (if $\Gamma \mid\!\sim \psi$ then $\Gamma, \phi \mid\!\sim \psi$). The idea is that by adding to Γ one of its conclusions under $\mid\!\sim$, one does not change its set of conclusions, but for an arbitrary formula added to Γ, this is not guaranteed. See [Kraus *et al.*, 1990] for a discussion of why a nonmonotonic relation is expected to satisfy these conditions.

These conditions led to a wide study of general patterns for nonmonotonic reasoning. The basic idea behind most of the works is to classify nonmonotonic formalisms and to recognize logical properties that they should satisfy. Some works continued to study the properties of nonmonotonic relations as independent relations, e.g. [Makinson, 1994] and [Lehmann, 1992]. The latter suggested the concept of a *plausibility logic*. Other works based the nonmonotonic consequence relations $\mid\!\sim$ on underlying monotonic ones $\vdash$. At first ([Kraus *et al.*, 1990]), the nonmonotonic relations were in the classical propositional language and were based on the underlying classical logic (an example for such a connection is the property *right weakening*: if $\vdash_{\mathbf{CL}} \psi \supset \phi$ and $\tau \mid\!\sim \psi$ then $\tau \mid\!\sim \phi$). Later, $\vdash$ was taken as any monotonic logic ([Freund and Lehmann, 1993]), and in any language ([Arieli and Avron, 2000b]) (for other related works see also [Makinson, 1989; Gabbay, 1991; Freund *et al.*, 1991; Lehmann and Magidor, 1992; Schlechta, 1996; Lehmann, 1998].)

In this paper we shall use the following notion:

Definition 3.2 Let $\vdash$ be an scr. A binary relation $\mid\!\sim$ between sets of formulas and sets of formulas is called $\vdash$*-plausible* if it satisfies the following conditions:

Ext $\vdash$*-extension*:
for every $\Gamma, \Delta \neq \emptyset$, if $\Gamma \vdash \Delta$ then $\Gamma \mid\!\sim \Delta$.
RM *right monotonicity*:
if $\Gamma \mid\!\sim \Delta$ and $\Delta \subseteq \Delta'$ then $\Gamma \mid\!\sim \Delta'$.
LCM *left cautious monotonicity*:
if $\Gamma \mid\!\sim \psi$ for every $\psi \in \Gamma'$, and $\Gamma \mid\!\sim \Delta$ then $\Gamma, \Gamma' \mid\!\sim \Delta$.
LCC *left cautious cut*:
if $\Gamma \mid\!\sim \psi, \Delta$ for every $\psi \in \Sigma$ and $\Gamma, \Sigma \mid\!\sim \Delta$ then $\Gamma \mid\!\sim \Delta$.
RCC *right cautious cut*:
if $\Gamma, \psi \mid\!\sim \Delta$ for every $\psi \in \Sigma$ and $\Gamma \mid\!\sim \Sigma, \Delta$ then $\Gamma \mid\!\sim \Delta$.

[Lehmann, 1992; Arieli and Avron, 2000b] use LCC with a finite Σ, and do not use RCC. In the nonmonotonic consequence relations that will interest us, both rules will be valid. Of course, LCC and RCC are also valid in scrs that are induces by semantic structures. What is "cautious" about them is that only Γ is used, in contrast to the Γ, Γ' of the rule C in Definition 2.1 ($\mid\!\sim$ is not "cautious" on its r.h.s. in view of RM).

3.2 Preferential systems

In parallel to the research on syntactic properties of nonmonotonic consequence relations, semantical methods for nonmonotonic reasoning were suggested. Shoham [Shoham, 1987; 1988] proposed the concept of *preferential models* as a generalization of McCarthy's circumscription [McCarthy, 1980]. The main idea is that instead of using all the models of a given theory for checking which conclusions follow from it, the models are ordered by a preference relation, and only the most preferred models are used as relevant for making inferences from the theory.

[Makinson, 1989; Kraus *et al.*, 1990; Lehmann, 1992] use preferential systems to provide semantics for the nonmonotonic relations that they investigate. The nonmonotonic relations that are induced by preferential systems satisfy two of the three basic conditions of Definition 3.1, namely reflexivity and cut, but not necessarily weak monotonicity. In order to ensure this condition, these works identify a condition that the preferential system should satisfy, which is called *smoothness* in the first and *stopperedness* in the other two: for every model of a given theory there should be some most-preferred model of the theory that is comparable with it. The rationale is that the elimination of one of the models should be justified by retaining another model that is preferred over it (and which is a most-preferred model). Indeed, if some model is part of an infinitely-descending chain of models under the preference relation, and there does not exist a most-preferred model that bounds this chain, it is difficult to conceptualize what the preference relation between the models ought to mean. In any case, as [Arieli and Avron, 2000b] has shown, a preferential system that satisfies the stopperedness condition induces a consequence relation that fulfills not only the conditions of Definition 3.1 but is already a plausible consequence relation.

Notation 3.3 If A is a set with a pre-order $\preceq$, $x \prec y$ denotes $x \preceq y$ and $y \not\preceq x$. $\mathrm{Min}_{\preceq}(A) = \{x \in A \mid \forall y \in A.\ y \not\prec x\}$.

Definition 3.4 [5] Let $\mathcal{S}$ be a semantic structure.

1. A *preferential system* in $\mathcal{S}$ is a pair $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$, where $\preceq$ is a pre-order[6] on $\mathcal{M}_{\mathcal{S}}$.
2. A model $m \in mod(\Gamma, \mathcal{S})$ is a $\mathcal{P}$*-preferential model* of Γ if $m \in pmod(\Gamma, \mathcal{P}) = \mathrm{Min}_{\preceq}(mod(\Gamma, \mathcal{S}))$.
3. A set of formulas Γ $\mathcal{P}$*-preferentially entails* a set of formulas Δ (notation: $\Gamma \vdash^{\mathcal{P}} \Delta$) if for every $m \in pmod(\Gamma, \mathcal{P})$ there is a $\phi \in \Delta$ s.t. $m \models^{\mathcal{S}} \phi$.[7] $\vdash^{\mathcal{P}}$ is called the *consequence relation induced by* $\mathcal{P}$.

The term "consequence relation" here is more general than in Definition 2.1. In particular, we do not assume monotonicity (it is possible that $\Gamma \vdash^{\mathcal{P}} \Delta$ but $\Gamma, \psi \not\vdash^{\mathcal{P}} \Delta$ if $\preceq$ is defined in such a way that $pmod(\Gamma \cup \{\psi\}, \mathcal{P}) \not\subseteq pmod(\Gamma, \mathcal{P})$).

Definition 3.5 Let A be a set with a pre-order $\preceq$. A is *well-founded under* $\preceq$ if it does not have an infinitely descending chain under $\prec$. A is *stoppered under* $\preceq$ if every $x \in A$ has $x' \in \mathrm{Min}_{\preceq}(A)$ s.t. $x' \preceq x$.

Note that if A is well-founded under $\preceq$ then it is stoppered under $\preceq$ (the converse does not necessarily hold).

Definition 3.6 [8] A preferential system $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ is *stoppered* if for all Γ, $mod(\Gamma, \mathcal{S})$ is stoppered under $\preceq$.

Theorem 3.7 [9] *If $\mathcal{P}$ is a stoppered preferential system in $\mathcal{S}$ then $\vdash^{\mathcal{P}}$ is $\vdash^{\mathcal{S}}$-plausible.*

Note: The stopperedness condition is introduced because some preferential systems which are not stoppered do not satisfy the condition LCM of Definition 3.2 (the other conditions are always fulfilled by all preferential systems).

As noted in [Kraus *et al.*, 1990; Makinson, 1994], it is usually not easy to check whether a preferential system is stoppered. Preferential systems were originally developed as a framework for providing semantics for nonmonotonic inference relations. They were also used, apparently independently at first, for constructing systems for reasoning with inconsistencies (and other abnormalities) in a way which is on the one hand non-trivial and on the other hand not as weak as monotonic substructural logics (see e.g. [Batens, 1986; Priest, 1991; Kifer and Lozinskii, 1992; Arieli and Avron, 1996]). Interestingly, these ideas, which were developed from motivations different from stopperedness, will provide us with methods for constructing stoppered preferential systems.

[5] Following [Makinson, 1994; Lehmann, 1992].

[6] For the purpose of showing the results in section 3.2 (but not sections 4 and 5), $\preceq$ can be any binary relation.

[7] Note that we do *not* require that $m \in pmod(\{\phi\}, \mathcal{P})$, or that $m \in pmod(\Gamma \cup \{\phi\}, \mathcal{P})$.

[8] Following [Makinson, 1994].

[9] A Generalization of a result in [Arieli and Avron, 2000b].

4 Formula-Preferential Systems

This section provides a natural source of stoppered preferential systems. The idea is to select a subset of the formulas in the language, and to base the comparison between the models in the preferential system on what formulas from this set they satisfy. This idea is a generalization of a method for constructing "adaptive logics" in [Batens, 1998], in which the selected formulas express some kind of abnormality w.r.t. a desired logic. We shall first explain this idea and then show that under a simple condition, formula-preferential systems are stoppered and hence induce plausible relations.

4.1 Definition

The idea behind formula-preferential systems is a generalization of the "minimal-abnormality strategy" from [Batens, 1998]. That paper uses a specific selection of models from $\mathcal{S}_p^{\top}$.[10] Denoting $K(v) = \{\psi \in \mathcal{W}_{\mathrm{cl}} \mid v(\psi \wedge \neg\psi) = t\}$, a model v of Γ is selected iff there is no other model v' of Γ s.t. $K(v') \subset K(v)$. In this way the minimal-abnormality strategy minimizes the abnormalities (here – inconsistencies) in the models of a theory (by "abnormality" we mean a formula that leads to triviality w.r.t. a desired logic, here – classical logic). Other papers consider other kinds of abnormalities (see section 4.3).

Formula-preferential systems form a generalization of this idea. They also select those models of a theory that minimize the satisfaction of formulas from a certain set G of formulas, but it can now be done with respect to any set G. In addition, this type of preferential systems is defined in any semantic structure, since what is important for the preference relation between the models is what formulas they satisfy, and not their inner structure.

Notation 4.1 Let $\mathcal{S}$ be a semantic structure and let $G \subseteq \mathcal{W}$. For $m \in \mathcal{M}_{\mathcal{S}}$ denote: $\mathsf{Sat}_{\mathcal{S},G}(m) = \{\psi \in G \mid m \models^{\mathcal{S}} \psi\}$.

Definition 4.2 Let $G \subseteq \mathcal{W}$. A *formula-preferential system based on* G is a preferential system $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ that satisfies: for all $m_1, m_2 \in \mathcal{M}_{\mathcal{S}}$, $m_1 \preceq m_2$ iff $\mathsf{Sat}_{\mathcal{S},G}(m_1) \subseteq \mathsf{Sat}_{\mathcal{S},G}(m_2)$. $\mathcal{P}$ is called in short a "G-preferential system".

4.2 Stoppered formula-preferential systems

We present now the main technical result of this paper.

Theorem 4.3 *If $\mathcal{P}$ is a formula-preferential system in a finitary semantic structure then $\mathcal{P}$ is stoppered.*

Proof: Suppose that $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ where $\mathcal{S}$ is finitary and $\mathcal{P}$ is based on some $G \subseteq \mathcal{W}$. Let Γ be given – we want to show that $mod(\Gamma, \mathcal{S})$ is stoppered under $\preceq$. The proof consists of two parts which refer to the set $\mathrm{I}\mathcal{C}_{\mathcal{S},G}(\Gamma)$:

Definition 4.4

- $\mathcal{C}_{\mathcal{S},G}(\Gamma) = \{\Delta \subseteq G \mid \Gamma \vdash^{\mathcal{S}} \Delta\}$
- $\mathrm{I}\mathcal{C}_{\mathcal{S},G}(\Gamma) = \{T \subseteq G \mid \forall \Delta \in \mathcal{C}_{\mathcal{S},G}(\Gamma).\ T \cap \Delta \neq \emptyset\}$

[10] In our notations, [Batens, 1998] actually uses the first-order level of $\mathcal{S}_p^{\top}$, but here we discuss only the propositional level.

The first part of the proof (Lemma 4.5) shows that $\mathrm{I}\mathcal{C}_{\mathcal{S},G}(\Gamma)$ is stoppered under $\subseteq$. For the second part of the proof, note that for every $m \in mod(\Gamma, \mathcal{S})$, $\mathsf{Sat}_{\mathcal{S},G}(m) \in \mathrm{I}\mathcal{C}_{\mathcal{S},G}(\Gamma)$, but if $T \in \mathrm{I}\mathcal{C}_{\mathcal{S},G}(\Gamma)$, there does not necessarily exist an $m \in mod(\Gamma, \mathcal{S})$ s.t. $T = \mathsf{Sat}_{\mathcal{S},G}(m)$. In contrast, Lemma 4.9 shows that $m \in pmod(\Gamma, \mathcal{S})$ iff $\mathsf{Sat}_{\mathcal{S},G}(m)$ is a *minimal* element of $\mathrm{I}\mathcal{C}_{\mathcal{S},G}(\Gamma)$.

In the rest of this proof we shall omit the subscripts $\mathcal{S}$, G and $\subseteq$, and also shorten $mod(\Gamma, \mathcal{S})$ to $mod(\Gamma)$ and $pmod(\Gamma, \mathcal{P})$ to $pmod(\Gamma)$.

Lemma 4.5 $\mathrm{I}\mathcal{C}(\Gamma)$ *is stoppered under* $\subseteq$.

Proof: Let $T \in \mathrm{I}\mathcal{C}(\Gamma)$. We need to show that there exists $T' \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ s.t. $T' \subseteq T$. Let $Z_T = \{T' \in \mathrm{I}\mathcal{C}(\Gamma) \mid T' \subseteq T\}$ and let $C \subseteq Z_T$ be a chain w.r.t. $\subset$. We shall show that C is bounded below in Z_T, so by Zorn's lemma Z_T has a minimal element, which is the required $\subset$-minimal element. Indeed, let $T^* = \bigcap C$. Obviously T^* bounds C and $T^* \subseteq T \subseteq G$. It remains to show that $T^* \in \mathrm{I}\mathcal{C}(\Gamma)$. Suppose in contradiction that there is some $\Delta \in \mathcal{C}(\Gamma)$ s.t. $T^* \cap \Delta = \emptyset$. Since $\mathcal{S}$ is finitary, there is a finite $\Delta' \subseteq \Delta$ s.t. $\Delta' \in \mathcal{C}(\Gamma)$. Suppose $\Delta' = \{\psi_1, \ldots, \psi_n\}$. Then for all $1 \leq i \leq n$, $\psi_i \notin T^*$, and since $T^* = \bigcap C$ then for all $1 \leq i \leq n$ there is some $T^{\psi_i} \in C$ s.t. $\psi_i \notin T^{\psi_i}$. Let $C' = \{T^{\psi_i} \mid \psi_i \in \Delta'\}$ and let $T^{**} = \bigcap C'$. Then $T^{**} \cap \Delta' = \emptyset$, and so $T^{**} \notin \mathrm{I}\mathcal{C}(\Gamma)$. But since C is a chain and so is $C' \subseteq C$, there is some $1 \leq k \leq n$ s.t. $T^{\psi_k} \subseteq T^{\psi_j}$ for all $1 \leq j \leq n$ and therefore $T^{\psi_k} = T^{**} \notin \mathrm{I}\mathcal{C}(\Gamma)$, in contradiction to $T^{\psi_k} \in C \subseteq \mathrm{I}\mathcal{C}(\Gamma)$. ■

Lemma 4.6 [11] *If* $T \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ *then for all* $\psi \in T$ *there exists* $\Delta \in \mathcal{C}(\Gamma)$ *s.t.* $T \cap \Delta = \{\psi\}$.

Proof: Suppose in contradiction that for some $T \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ and some $\psi \in T$, $T \cap \Delta \neq \{\psi\}$ for all $\Delta \in \mathcal{C}(\Gamma)$. For all such Δ we know that $T \cap \Delta \neq \emptyset$ since $T \in \mathrm{I}\mathcal{C}(\Gamma)$, so $(T - \{\psi\}) \cap \Delta \neq \emptyset$. Hence $(T - \{\psi\}) \in \mathrm{I}\mathcal{C}(\Gamma)$, in contradiction to $T \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ since $T - \{\psi\} \subset T$. ■

Lemma 4.7 [11] *If* $T \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ *and* $\Delta \subseteq G$ *then* $\Gamma, T \vdash^{\mathcal{S}} \Delta$ *iff* $\Delta \cap T \neq \emptyset$.

Proof: If $\Delta \cap T \neq \emptyset$ then obviously $\Gamma, T \vdash^{\mathcal{S}} \Delta$. For the converse, suppose in contradiction that $\Gamma, T \vdash^{\mathcal{S}} \Delta$ but $\Delta \cap T = \emptyset$. By Lemma 4.6, for each $\psi \in T$ there exists $\Delta^{\psi} \subseteq G$ s.t. $\Gamma \vdash^{\mathcal{S}} \psi, \Delta^{\psi}$ and $T \cap \Delta^{\psi} = \emptyset$. Let $\Delta^* = \bigcup\{\Delta^{\psi} \mid \psi \in T\}$. We show that $\Gamma \vdash^{\mathcal{S}} \Delta^*, \Delta$: suppose $m \in mod(\Gamma)$. If $m \models^{\mathcal{S}} \phi$ for some $\phi \in \Delta$ then we are finished. Otherwise, since $\Gamma, T \vdash^{\mathcal{S}} \Delta$ then $m \not\models^{\mathcal{S}} \psi$ for some $\psi \in T$. Since $\Gamma \vdash^{\mathcal{S}} \psi, \Delta^{\psi}$ is true then $m \models^{\mathcal{S}} \phi$ for some $\phi \in \Delta^{\psi} \subseteq \Delta^*$. So $\Gamma \vdash^{\mathcal{S}} \Delta^*, \Delta$, but $(\Delta^* \cup \Delta) \cap T = \emptyset$. This is impossible because $T \in \mathrm{I}\mathcal{C}(\Gamma)$ and $\Delta^* \cup \Delta \in \mathcal{C}(\Gamma)$. ■

Lemma 4.8 *If* $m \in mod(\Gamma)$ *and* $\mathsf{Sat}(m) \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ *then* $m \in pmod(\Gamma)$.

Proof: By definition, $\mathsf{Sat}(n) \in \mathrm{I}\mathcal{C}(\Gamma)$ for every $n \in mod(\Gamma)$. Thus, If $m \in mod(\Gamma)$ and $\mathsf{Sat}(m) \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ then there cannot be $m' \in mod(\Gamma)$ s.t. $m' \prec m$, because then $\mathsf{Sat}(m') \subset \mathsf{Sat}(m)$ (and $\mathsf{Sat}(m') \in \mathrm{I}\mathcal{C}(\Gamma)$), and so $\mathsf{Sat}(m)$ is not minimal in $\mathrm{I}\mathcal{C}(\Gamma)$. ■

[11] Based on [Batens, 1999a].

Lemma 4.9 [12] $\mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma)) = \{\mathsf{Sat}(m) \mid m \in pmod(\Gamma)\}$.

Proof: For one direction, take some $T \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ and suppose in contradiction that there is no $m \in pmod(\Gamma)$ s.t. $\mathsf{Sat}(m) = T$. By Lemma 4.8, there is also no $m \in mod(\Gamma)$ s.t. $\mathsf{Sat}(m) = T$. In particular, this is true for all $m \in mod(\Gamma \cup T)$. By definition, $\mathsf{Sat}(m) \supseteq T$ for all $m \in mod(\Gamma \cup T)$, so for all such m, $\mathsf{Sat}(m) \supset T$. Now let $\Delta = \bigcup\{\mathsf{Sat}(m) - T \mid m \in mod(\Gamma \cup T)\}$. Then $\Gamma, T \vdash^{\mathcal{S}} \Delta$. But $\Delta \cap T = \emptyset$, in contradiction to Lemma 4.7.

For the converse, if $m \in pmod(\Gamma)$ then in particular $m \in mod(\Gamma)$ and so $\mathsf{Sat}(m) \in \mathrm{I}\mathcal{C}(\Gamma)$. By Lemma 4.5 there is some $T \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ s.t. $T \subseteq \mathsf{Sat}(m)$. By the first direction, there is $m' \in pmod(\Gamma)$ s.t. $T = \mathsf{Sat}(m')$. So $\mathsf{Sat}(m') \subseteq \mathsf{Sat}(m)$, $m \in pmod(\Gamma)$, and therefore $\mathsf{Sat}(m) = \mathsf{Sat}(m') = T \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$. ■

End of the proof of Theorem 4.3: Let $m \in mod(\Gamma)$. Then $\mathsf{Sat}(m) \in \mathrm{I}\mathcal{C}(\Gamma)$. By Lemma 4.5 there is $T \in \mathrm{Min}(\mathrm{I}\mathcal{C}(\Gamma))$ s.t. $T \subseteq \mathsf{Sat}(m)$. By Lemma 4.9, there is $m' \in pmod(\Gamma)$ s.t. $\mathsf{Sat}(m') = T$, so $\mathsf{Sat}(m') \subseteq \mathsf{Sat}(m)$, i.e. $m' \preceq m$. ■

Corollary 4.10 *If* $\mathcal{P}$ *is a formula-preferential system in a finitary semantic structure* $\mathcal{S}$ *then* $\vdash^{\mathcal{P}}$ *is* $\vdash^{\mathcal{S}}$*-plausible.*

Proof: Follows from Theorems 3.7 and 4.3. ■

The main practical importance of our result applies to all finite Nmatrices. Since in practice one usually works with finite structures, this means that the following result has great practical significance.

Corollary 4.11 *If* $\mathcal{P}$ *is a formula-preferential system in a finite Nmatrix* $\mathcal{S}$ *then* $\vdash^{\mathcal{P}}$ *is* $\vdash^{\mathcal{S}}$*-plausible.*

Proof: Follows from Theorem 2.8 and Corollary 4.10. ■

A note about finitariness: it can be shown that formula-preferential systems do not in general preserve finitariness. A G-preferential system $\mathcal{P}$ in a finitary semantic structure $\mathcal{S}$ might not induce a finitary consequence relation. Nevertheless, if Γ has only a finite number of minimal G-consequences ($\mathrm{Min}_{\subseteq}(\mathcal{C}_{\mathcal{S},G}(\Gamma))$ is finite) then $\Gamma \vdash^{\mathcal{P}} \Delta$ implies that there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ s.t. $\Gamma' \vdash^{\mathcal{P}} \Delta'$.

4.3 Examples

We show now how known systems from the literature can be constructed using formula-preferential systems. Since all of them are based on finite Nmatrices, then by Corollary 4.11, the induced consequence relations are plausible.

Closed-World Assumption

In the "Closed-World Assumption" method [Reiter, 1978], a propositional variable that cannot be proved to be true is assumed to be false. A corresponding formula-preferential system is $\mathcal{P} = \langle \mathcal{S}_{\mathbf{CL}}, \preceq \rangle$ that is based on $\mathcal{A}_{\mathrm{cl}}$. The obtained consequence relation $\vdash^{\mathcal{P}}$ is nonmonotonic. E.g., if $\psi \not\vdash_{\mathbf{CL}} p$ then $\psi \vdash^{\mathcal{P}} \neg p$ but $\psi, p \not\vdash^{\mathcal{P}} \neg p$ (provided $\{\psi, p\}$ is classically consistent). $\vdash^{\mathcal{P}}$ is, however, not paraconsistent. Since $\mathcal{P}$ is

[12] Following [Batens, 1998], but the proof here (of the first direction) is different and relies on Lemma 4.7.

based on $\mathcal{S}_{\mathbf{CL}}$, a classically inconsistent theory does not have any models and entails any formula under $\vdash_{\mathbf{CL}}$ and $\vdash^{\mathcal{P}}$. This shows that nonmonotonicity and paraconsistency are independent issues. However, many nonmonotonic systems were designed for handling contradictions in an adequate way, as explained next.

Preferential systems for handling contradictions

Since $\mathbf{CL}$ is unsuitable for reasoning from classically inconsistent theories, one solution is to take the paraconsistent logics $\vdash_3^{\top}$ or $\vdash_4$ (thus if $\Gamma = \{r,\ \neg r,\ p,\ p \supset q\}$ we have that $\Gamma \vdash_3^{\top} q$ and $\Gamma \not\vdash_3^{\top} \neg q$). Nevertheless these consequence relations are too weak. In particular, on classically consistent sets they do not entail all the conclusions that classically follow from them (for example, the Disjunctive Syllogism (from ψ, $\neg\psi \vee \phi$ infer ϕ) is not valid in $\vdash_3^{\top}$ and $\vdash_4$).

Let $\uparrow\psi \equiv \psi \wedge \neg\psi$. A consequence relation that is located between the monotonic "lower-limit logic" $\vdash_3^{\top}$ and the "upper-limit logic" $\vdash_{\mathbf{CL}}$ [13] can be obtained by using the formula-preferential system $\mathcal{P} = \langle \mathcal{S}_3^{\top}, \preceq \rangle$ that is based on $G = \{\uparrow p \mid p \in \mathcal{A}_{\rm cl}\}$. $\vdash^{\mathcal{P}}$ is the same as $\mathbf{LPm}$ of [Priest, 1991] (when $\mathcal{S}_3^{\top}$ is without $\supset$) and $\mathbf{ACLuNs2}$ of [Batens, 1998]. $\vdash^{\mathcal{P}}$ is nonmonotonic: if $\Gamma = \{p, \neg p \vee q\}$ then $\Gamma \vdash^{\mathcal{P}} q$ but $\Gamma, \neg p \not\vdash^{\mathcal{P}} q$ (and $\Gamma, \neg q \not\vdash^{\mathcal{P}} q$). $\vdash^{\mathcal{P}}$ is also paraconsistent: $p, \neg p \not\vdash^{\mathcal{P}} q$, and even $(p \vee q), \neg(p \vee q) \not\vdash^{\mathcal{P}} q$. Moreover, If Γ is classically consistent then $\Gamma \vdash^{\mathcal{P}} \Delta$ iff $\Gamma \vdash_{\mathbf{CL}} \Delta$. For this, notice that for a valuation v in $\mathcal{S}_3^{\top}$, $v \models_3^{\top} \uparrow p$ iff $v(\psi) = \top$. All the classical models v of Γ are also valuations in $\mathcal{S}_3^{\top}$, they are $\preceq$-equivalent (since $\mathsf{Sat}_{\mathcal{S}_3^{\top},G}(v) = \emptyset$), and they are $\prec$-preferred over all the models of Γ in $\mathcal{S}_3^{\top}$ that assign $\top$ to some variable. Thus, if Γ is classically consistent then its $\mathcal{P}$-preferential models are its classical models.

Adaptive Logics

[Batens, 1998] presents the idea of *adaptive logics*. These were originally introduces in [Batens, 1986] by dynamic proof systems that are designed to mimic some aspects of human reasoning with inconsistencies, especially the fact that conclusions that are drawn at a certain stage may be rejected at a later stage because of other conclusions, and then even accepted again. The name "adaptive" is due to the fact that these logics adapt their rules to the given set of premises. E.g. the Disjunctive Syllogism is not valid in $\vdash_3^{\top}$. In contrast, if $\Gamma = \{r,\ \neg r,\ \neg r \vee s,\ p,\ \neg p \vee q\}$ then the adaptive logic **ACLuNs2** that is based on $\vdash_3^{\top}$ does not allow to use this rule on Γ only for inferring s (since r behaves inconsistently) but does allow its use for inferring q from $p,\ \neg p \vee q$ (since there is no reason to suppose that p behaves inconsistently).

Different adaptive logics have been developed (see [Batens, 2000] for a survey). Those that are based on the minimal-abnormality strategy are a special case of the formula-preferential systems where the set G is taken as a set of abnormal formulas.[14] For example, **ACLuN2** (note: not **ACLuNs2**) from [Batens, 1998] is induced by the formula-preferential system in $\mathcal{S}_p^{\top}$ that is based on $G = \{\uparrow\psi \mid \psi \in \mathcal{W}_{\rm cl}\}$ (notice the difference from **ACLuNs2** – the two consequence relations are incomparable. For a comparison between the use of $\vdash_3^{\top}$ and $\vdash_p^{\top}$ as the underlying monotonic relations, see [Batens, 2000]).

ACL∅2 from [Batens, 1999b] is induced by the following formula-preferential system (we give here a simplified version). It is defined in the two-valued Nmatrix $\mathcal{S}_0$ in which all the connectives of $\mathcal{L}_{\rm cl}$ are weakened: for an n-ary connective $\diamond \in \{\wedge, \vee, \supset, \neg, f\}$ and any $\bar{x} \in \{t, f\}^n$, $\tilde{\diamond}(\bar{x}) = \{t, f\}$. To still retain expressive power, we extend $\mathcal{L}_{\rm cl}$ to the language $\mathcal{L}_{\rm cl}^{+}$ with the added connectives $\sim$ and $\&$ which function in $\mathcal{S}_0$ as classical negation and conjunction: $\tilde{\sim} x = \{\text{not } x\}$; $x_1 \tilde{\&} x_2 = \{x_1 \text{ and } x_2\}$. $\mathcal{P}_0$ is the formula-preferential system in $\mathcal{S}_0$ that is based on the set G_0: this set includes all formulas which express the fact that a certain formula $\diamond(\psi_1, \ldots, \psi_n)$ and one or more of $\psi_1, \ldots, \psi_n$ are assigned values that are illegal in a classical valuation, e.g. $\psi \ \&\ \neg\psi$, $\sim\psi \ \&\ \sim\neg\psi$, $(\psi \ \&\ \phi) \ \&\ \sim(\psi \wedge \phi)$, $\sim\psi \ \&\ (\psi \wedge \phi)$, $(\psi \ \&\ \sim\phi) \ \&\ (\psi \supset \phi)$, etc. Whereas the underlying monotonic relation $\vdash_0$ is totally weak in $\mathcal{L}_{\rm cl}$ (i.e. for all $\Gamma, \Delta \subseteq \mathcal{W}_{\rm cl}$, $\Gamma \vdash_0 \Delta$ iff $\Gamma \cap \Delta \neq \emptyset$), $\vdash^{\mathcal{P}_0}$ is still as strong as **CL** on classically consistent sets (in $\mathcal{L}_{\rm cl}$). In comparison to **ACLuN2**, **ACL∅2** is "adaptive" on all the connectives in $\mathcal{L}_{\rm cl}$, not only $\neg$.

Other adaptive logics (e.g. [Vanackere, 1997; 1999]) use a formula-preferential system $\mathcal{P}$ in a more complicated way: the definition of the adaptive logic $\mid\!\sim$ is: $\Gamma \mid\!\sim \Delta$ iff $Tr(\Gamma) \vdash^{\mathcal{P}} Tr(\Delta)$, where Tr is some pre-processing of the formulas.

Further examples of formula-preferential systems will be given in section 5.3.

5 Pointwise-preferential systems

[Arieli and Avron, 2000b] suggests another method for constructing preferential systems that are stoppered. The method is based on a type of preferential systems called *pointwise* preferential systems. The underlying idea is to have a preference between the truth values of a multiple-valued structure and to base the preference between the valuations on this preference. We shall see that these systems can be embedded in formula-preferential systems, and that therefore the finitariness of the underlying semantic structure ensures stopperedness.

5.1 Definition

Consider the truth values of the (N)matrix $\mathcal{S}_4$. We might have a preference between the truth values according to their properties in the valuations. E.g. we might prefer the classical values t and f over $\top$ and $-$, since a valuation satisfies exactly one of ψ and $\neg\psi$ iff it assigns a classical value to ψ. If there are two models for a given set of premises and they assign the same values to all atomic formulas except that one assigns t to p and the other $\top$, we might prefer the first. This is the underlying idea of the following definition.

[13] These terms are borrowed from [Batens, 1999b].

[14] The adaptive logics in [Batens, 1998] and similar papers are defined as single-conclusion consequence relations.

Definition 5.1 [15] Let $\mathcal{S}$ be an Nmatrix with a set of truth values $\mathcal{T}$, and let $\leq$ be a pre-order on $\mathcal{T}$. A *pointwise preferential system* (in $\mathcal{S}$) *based on* $\leq$ is a preferential system $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ that satisfies the condition: for all $v_1, v_2 \in \mathcal{V}_\mathcal{S}$, $v_1 \preceq v_2$ iff for every propositional variable p, $v_1(p) \leq v_2(p)$. If $\leq$ is a partial-order, $\mathcal{P}$ is called *strongly pointwise*. $\mathcal{P}$ will be called in short a "$\leq$-preferential system".

Note that $\preceq$ is indeed a pre-order if $\leq$ is a pre-order.

5.2 Embedding pointwise preferential systems in formula-preferential systems

Pointwise preferential systems are in general a different type of systems than formula-preferential systems. Nevertheless, by adding certain connectives to the language, we can construct for each pointwise preferential system a formula-preferential system that induces the same consequence relation and, in a certain sense, has the same preference relation.

Definition 5.2 Let $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be a Nmatrix for a propositional language $\mathcal{L}$, and let $\mathcal{L}'$ be a propositional language with the same variables as $\mathcal{L}$ but with additional logical connectives. An *extension of* $\mathcal{S}$ *to* $\mathcal{L}'$ is a Nmatrix $\mathcal{S}' = \langle \mathcal{T}, \mathcal{D}, \mathcal{O}' \rangle$ for $\mathcal{L}'$ s.t. $\mathcal{O}' \supseteq \mathcal{O}$. A valuation v' in $\mathcal{S}'$ is an *extension* of a valuation v in $\mathcal{S}$ to $\mathcal{L}'$ if v and v' agree on $\mathcal{W}$.

Definition 5.3 Let $\mathcal{S} = \langle \mathcal{T}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$ and let $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ be a $\leq$-preferential system. A formula-preferential system *associated* with $\mathcal{P}$ is $\mathcal{P}' = \langle \mathcal{S}', \preceq' \rangle$ for the language $\mathcal{L}'$, where $\mathcal{L}'$ is like $\mathcal{L}$ but with the added or defined connectives $\{I_x \mid x \in \mathcal{T}\}$, $\mathcal{S}'$ is an extension of $\mathcal{S}$ to $\mathcal{L}'$ with the same truth values s.t. for every $x, y \in \mathcal{T}$, $\widetilde{I_x} y \subseteq \mathcal{D}$ if $y \geq x$ and $\widetilde{I_x} y \subseteq \mathcal{T} - \mathcal{D}$ otherwise, and $\mathcal{P}'$ is based on $G = \{I_x p \mid x \in \mathcal{T},\ p \in \mathcal{A}\}$.

Note: For all valuations v in $\mathcal{S}'$, $v \models^{\mathcal{S}'} I_x \psi$ iff $v(\psi) \geq x$.

Theorem 5.4 *Let $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ be a $\leq$-preferential system and let $\mathcal{P}' = \langle \mathcal{S}', \preceq' \rangle$ be an associated G-preferential system.*

1. *For all $\Gamma, \Delta \subseteq \mathcal{W}$, $\Gamma \vdash^\mathcal{P} \Delta$ iff $\Gamma \vdash^{\mathcal{P}'} \Delta$.*
2. *For all $v_1, v_2 \in \mathcal{V}_\mathcal{S}$, $v_1 \preceq v_2$ iff for each of their (respective) extensions $v_1', v_2' \in \mathcal{V}_{\mathcal{S}'}$ to $\mathcal{L}'$, $v_1' \preceq' v_2'$.*

Proof: First, Let $\mathcal{P}^* = \langle \mathcal{S}', \preceq^* \rangle$ be the $\leq$-preferential system in $\mathcal{S}'$. Pointwise preferential systems compare only the truth values that valuations assign to propositional variables, and hence for all $v_1, v_2 \in \mathcal{V}_\mathcal{S}$, $v_1 \preceq v_2$ iff for each of their (respective) extensions $v_1', v_2' \in \mathcal{V}_{\mathcal{S}'}$ to $\mathcal{L}'$, $v_1' \preceq^* v_2'$. This means that for all $\Gamma, \Delta \subseteq \mathcal{W}$, $\Gamma \vdash^\mathcal{P} \Delta$ iff $\Gamma \vdash^{\mathcal{P}^*} \Delta$. Now we show that $\preceq' = \preceq^*$, and hence that $\vdash^{\mathcal{P}'} = \vdash^{\mathcal{P}^*}$. Indeed, let $v_1, v_2 \in \mathcal{V}_{\mathcal{S}'}$. $v_1 \preceq' v_2$ iff for all $p \in \mathcal{A}$, $v_1(p) \leq v_2(p)$, iff for all $x \in \mathcal{T}$ and all $p \in \mathcal{A}$, if $v_1 \models^{\mathcal{S}'} I_x p$ then $v_2 \models^{\mathcal{S}'} I_x p$, iff $\mathsf{Sat}_{\mathcal{S}',G}(v_1) \subseteq \mathsf{Sat}_{\mathcal{S}',G}(v_2)$, iff $v_1 \preceq^* v_2$. ∎

Note: for each $x \in \mathcal{T}$ that is a least element ($x \leq y$ for all $y \in \mathcal{T}$), defining G without any formula $I_x p$ will give the same result, since such x guarantees that $v \models^{\mathcal{S}'} I_x p$ for all v, and so the presence of these formulas in G does not influence the preference relation.

Corollary 5.5 *If $\mathcal{P}$ is a pointwise preferential system in a finitary Nmatrix $\mathcal{S}$ then $\mathcal{P}$ is stoppered and $\vdash^\mathcal{P}$ is $\vdash^\mathcal{S}$-plausible.*

Proof: Suppose $\mathcal{P} = \langle \mathcal{S}, \preceq \rangle$ and let $\mathcal{P}' = \langle \mathcal{S}', \preceq' \rangle$ be a formula-preferential systen associated with it. Let $\Gamma \subseteq \mathcal{W}$ and let $v \in mod(\Gamma, \mathcal{S})$. Since $\mathcal{S}'$ is an extension of $\mathcal{S}$, there is $v' \in mod(\Gamma, \mathcal{S}')$ that is an extension of v. By Theorem 4.3, there is $u' \in pmod(\Gamma, \mathcal{P}')$ s.t. $u' \preceq' v'$. It follows from Theorem 5.4(2) that the reduction u of u' to $\mathcal{W}$ is s.t. $u \in pmod(\Gamma, \mathcal{P})$ and $u \preceq v$. Hence $\mathcal{P}$ is stoppered. By Theorem 3.7, $\vdash^\mathcal{P}$ is $\vdash^\mathcal{S}$-plausible. ∎

5.3 Examples

The following pointwise preferential systems are based on finite matrices, so by Theorem 2.8 and Corollary 5.5, the induced consequence relations are plausible.

Minimal knowledge

[Arieli and Avron, 1998; 2000a] considers pointwise preferential systems in matrices that are based on logical bilattices. Bilattices[16] consist of two partial orderings of the truth values, where each one induces a complete lattice: $\leq_t$ measures the amount of *truth* of the values and $\leq_k$ measures the amount of *knowledge* of the values. E.g. in $\mathcal{S}_4$, $f <_t (\top, -) <_t t$ and $- <_k (t, f) <_k \top$. $\leq_t$ and $\leq_k$ are related by a negation operation, which is an involution w.r.t. $\leq_t$ and an order preserving w.r.t. $\leq_k$. A logical bilattice[17] is a pair $\langle \mathcal{L}, \mathcal{F} \rangle$, where $\mathcal{L}$ is a bilattice and $\mathcal{F}$ is a set of designated elements that forms a prime *bifilter* in $\mathcal{L}$, i.e. a prime filter w.r.t. $\leq_t$ and $\leq_k$.

A strongly $\leq_k$-preferential system induces a consequence relation that draws conclusions according to models that assume minimal knowledge concerning the premises. The intuition is that one should not assume anything that is not *really* known. E.g. if $\mathcal{P}_k^4$ is the system that is based on $\leq_k$ in $\mathcal{S}_4$, any variable which is not a subformula of Γ is assigned the "lack of knowledge" truth value $-$ by all the preferential models of Γ. [Arieli and Avron, 2000a] proves that the consequence relation induced by a strongly $\leq_k$-preferential system in a matrix based on a logical bilattice is nonmonotonic and paraconsistent. E.g. $\vdash^{\mathcal{P}_k^4}$ is paraconsistent: $p, \neg p \not\vdash^{\mathcal{P}_k^4} q$ (for $p \neq q$), and nonmonotonic: $q \vdash^{\mathcal{P}_k^4} \neg q \supset p$ but $q, \neg q \not\vdash^{\mathcal{P}_k^4} \neg q \supset p$. However, [Arieli and Avron, 1998] proves that if Δ does not contain $\supset$ then $\Gamma \vdash^{\mathcal{P}_k^4} \Delta$ iff $\Gamma \vdash_4 \Delta$, so $\vdash^{\mathcal{P}_k^4}$ is too weak to be used for adequate reasoning. It is more useful in a composition with the $\vdash^{\mathcal{P}_i}$ below.

According to section 5.2, $\mathcal{P}_k^4$ can be embedded in the formula-preferential system (in $\mathcal{S}_4$) that is based on the set that includes $I_\top p = p \wedge \neg p$, $I_t p = p$, $I_f p = \neg p$, $I_\perp p = \phi \supset \phi$, for all $p \in \mathcal{A}_{\text{cl}}$. According to the remark before Corollary 5.5, $I_\perp p$ is redundant. In this particular case $I_\top p$ is also redundant, i.e. it is enough to take $G = \mathcal{A}_{\text{cl}} \cup \{\neg p \mid p \in \mathcal{A}_{\text{cl}}\}$ (proof: $v_1 \preceq_k^4 v_2$ iff $\forall p \in \mathcal{A}_{\text{cl}}\ v_1(p) \leq_k v_2(p)$; iff $\forall p \in \mathcal{A}_{\text{cl}}$

[15] A generalization of 'pointwise preferential systems' from [Arieli and Avron, 2000b], which are in our notations strongly pointwise preferential systems in matrices.

[16] See e.g. [Ginsberg, 1988; Fitting, 1991; Avron, 1996].

[17] See [Arieli and Avron, 1996].

if $v_1 \models_4 p$ then $v_2 \models_4 p$ and if $v_1 \models_4 \neg p$ then $v_2 \models_4 \neg p$; iff $\mathsf{Sat}_{\mathcal{S}_4,G}(v_1) \subseteq \mathsf{Sat}_{\mathcal{S}_4,G}(v_2)$). Note that contrary to the original motivation behind the minimal-abnormality strategy of [Batens, 1998], in this system (as well as in CWA of section 4.3) we do not regard the formulas in G as abnormal (in particular, all the variables are in G), but rather as the formulas whose satisfaction we want to minimize in the models.

Minimal inconsistency

[Arieli and Avron, 1998] considers another family of systems. The idea is to select a subset $\mathcal{I}$ of the truth values $\mathcal{T}$ as representing inconsistent values (an *inconsistency set*), s.t. for every $x \in \mathcal{T}$, $x \in \mathcal{I}$ iff $\neg x \in \mathcal{I}$, and $x \in \mathcal{D} \cap \mathcal{I}$ iff $x, \neg x \in \mathcal{D}$. The values that are not in $\mathcal{I}$ are preferred over those that are in $\mathcal{I}$ by defining the pre-order $\leq_{\mathcal{I}}$ on $\mathcal{T}$: $x_1 \leq_{\mathcal{I}} x_2$ iff $x_1 \in \mathcal{T} - \mathcal{I}$ or $x_2 \in \mathcal{I}$. The obtained $\leq_{\mathcal{I}}$-preferential systems select the models that assume minimal inconsistency (w.r.t. $\mathcal{I}$) of the premises. The intuition is that contradictory data corresponds to inadequate information about the world and should be minimized.

The preferential system from section 4.3 can be defined as the $\leq_{\mathcal{I}}$-preferential system in $\mathcal{S}_3^{\top}$ where $\mathcal{I} = \{\top\}$. If $\mathcal{T} = \{t, f, \top\}$, this is the only inconsistency set. In the general case, there may be other inconsistency sets. For example, in $\mathcal{S}_4$, both $\mathcal{I}_1 = \{\top\}$ and $\mathcal{I}_2 = \{\top, -\}$ are inconsistency sets, and they induce different consequence relations: if $\mathcal{P}_i$ ($i = 1, 2$) is the pointwise $\leq_{\mathcal{I}_i}$-preferential system in $\mathcal{S}_4$ and $\Gamma = \{p \supset \neg p, \neg p \supset p\}$, then $\Gamma \vdash^{\mathcal{P}_1} p \supset q$ while $\Gamma \not\vdash^{\mathcal{P}_2} p \supset q$, and $\vdash^{\mathcal{P}_2} p \vee \neg p$ but $\not\vdash^{\mathcal{P}_1} p \vee \neg p$. [Arieli and Avron, 2000a] proves that $\leq_{\mathcal{I}}$-preferential systems in matrices based on logical bilattices, where $\mathcal{I}$ is an inconsistency set, induce consequence relations that are nonmonotonic and paraconsistent w.r.t. $\neg$. E.g. $\vdash^{\mathcal{P}_2}$ is nonmonotonic, paraconsistent, and identical to $\vdash_{\mathbf{CL}}$ on classically consistent sets, for the same reasons as the $\vdash^{\mathcal{P}}$ for handling contradictions from section 4.3. The difference between that $\vdash^{\mathcal{P}}$ and $\vdash^{\mathcal{P}_2}$ is that the latter is also paracomplete, and can cope not only with contradictions of the form $\{p, \neg p\}$ but also with classical contradictions of the form $\{p \supset f, \neg p \supset f\}$ (i.e. this set does not entail all the formulas under $\vdash^{\mathcal{P}_2}$).

For a comparison between the consequence relation of a $\leq_{\mathcal{I}}$-preferential system and **LPm**, see [Arieli and Avron, 1998]. [Kifer and Lozinskii, 1992] also proposes a similar relation in the framework of annotated logics - for a comparison between that work and $\leq_{\mathcal{I}}$-preferential systems, see [Arieli and Avron, 1996].

According to section 5.2, $\mathcal{P}_1$ can be embedded in the formula-preferential system (in $\mathcal{S}_4$) that is based on the set G that includes $I_{\top}p = p \wedge \neg p$ for all $p \in \mathcal{A}_{\text{cl}}$ and $I_t p = I_f p = I_{\perp} p = \phi \supset \phi$. According to the remark before Corollary 5.5, only the formulas $I_{\top}p$ are necessary. For $\mathcal{P}_2$, the set includes $I_{\top}p = I_{\perp}p = (p \supset \neg p) \wedge (\neg p \supset p)$ for all $p \in \mathcal{A}_{\text{cl}}$ (and $I_t p = I_f p = I_{\perp} p = \phi \supset \phi$ are redundant).

6 Conclusion

Our main goal in this paper was to demonstrate the central role of formula-preferential systems in non-classical reasoning. We have shown how different systems from the literature for reasoning in the face of inconsistencies and other abnormalities, can be constructed in this framework. Moreover, although most of these systems were not originally part of the theoretical research of nonmonotonic consequence relations, the generalization of their preference relations to the idea of formula-preferential systems provides us with a method for ensuring the condition of stopperedness: formula-preferential systems that are based on finitary semantic structures are stoppered, and hence satisfy theoretical desiderata for a plausible nonmonotonic logic. All the examples from the literature that we have given are of this kind since they are based on finite non-deterministic matrices.

We enumerate some open research questions.

1. Can every plausible consequence relation that is based on an underlying monotonic relation be induced by a (stoppered) preferential system? What are the exact sufficient and necessary conditions for stopperedness? Can the preferential system always be defined as a formula-/pointwise preferential system? What happens when the underlying monotonic relation is not finitary?
2. The examples we provided for the preferential systems were at the propositional level. The next natural thing to do is to extend them to the first-order level.
3. Another important goal is to relate more works and practical applications to the framework presented here (e.g. demonstrating how other "adaptive logics" might be incorporated in it). For some of them it might be necessary to extend and generalize the framework further, e.g. by defining preferences not only between models but also between the formulas of the given set of premises.

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