A New Approach to Predicative Set Theory

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Abstract

We suggest a new basic framework for the Weyl-Feferman predicativist program by constructing a formal predicative *set theory* PZF which resembles ZF. The basic idea is that the predicatively acceptable instances of the comprehension schema are those which determine the collections they define in an absolute way, independent of the extension of the "surrounding universe". This idea is implemented using syntactic safety relations between formulas and sets of variables. These safety relations generalize both the notion of domain-independence from database theory, and Godel notion of absoluteness from set theory. The language of PZF is type-free, and it reflects real mathematical practice in making an extensive use of statically defined abstract set terms. Another important feature of PZF is that its underlying logic is ancestral logic (i.e. the extension of FOL with a transitive closure operation).

1 Introduction

The predicativist program for the foundations of mathematics, initiated by Poincaré in [35,36] [1], and first seriously developed by Weyl in [50], seeks to establish certainty in mathematics without revolutionizing it (as the intuitionistic program does). The program as is usually conceived nowadays (following Weyl and Feferman) is based on the following two basic principles:

(PRE) Higher order constructs, such as sets or functions, are acceptable only when introduced through definitions. These definitions cannot be circular. Hence in defining a new construct one can only refer to constructs which were introduced by previous definitions.

(NAT) The natural-numbers sequence is a basic well understood mathematical concept, and as a totality it constitutes a set.

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[1] Though its kernel can be found in Richard's discussion of his paradox [38].

The first of these principles, (PRE), was interpreted by Russell according to his philosophical views of logic, [39], [40], and incorporated as the *ramified type theory* (RTT) in *Principia Mathematica* ([51]). In RTT objects are divided into types, and each higher-order type is further divided into levels. However, the use of levels makes it impossible to develop mathematics in RTT, and so Russell had to add a special axiom of reducibility which practically destroyed the predicative nature of his system ([37]). The principle was then taken again by Weyl in [50], but instead of Russell's ramified hierarchy, Weyl adopted the second principle, (NAT), which also goes back to Poincaré. Weyl's predicativist program was later extensively pursued by Feferman, who in a series of papers (see e.g. [15,17,19,20]) developed proof systems for predicative mathematics. Feferman's systems are less complex than RTT, and he has shown that a very large part of classical analysis can be developed within them. He further conjectured that predicative mathematics in fact suffices for developing all the mathematics that is actually indispensable to present-day natural sciences.

Despite this success, Feferman's systems failed to receive in the mathematical community the interest they deserve. Unlike constructive mathematics, they were also almost totally ignored in the computer science community. The main reason for this seems to be the fact that on the one hand Feferman's systems are not "revolutionary" (since they allow the use of *classical* logic), but on the other hand they are still rather complicated in comparison to the impredicative formal set theory ZF, which provides the standard foundations and framework for developing mathematics. In particular: Feferman's systems still use complicated systems of types, and both functions and classes are taken in them as independent primitives. Therefore working within Feferman's systems is not easy for someone used to ZF (or something similar).

The main goal of this paper is to suggest a new framework for the Weyl-Feferman predicativist program by constructing an absolutely (at least in our opinion) reliable predicative *set theory* PZF which is suitable for mechanization, and has the following properties:

(1) Its language is type-free, and it reflects real mathematical practice by making an extensive use of *abstract set terms* (i.e. terms of the form $\{x \mid \varphi\}$). [2].
(2) Like ZF, it is a *pure* set theory, in which everything (including functions) is assumed to be a set. Moreover: from a platonic point of view, the universe V of ZF (whatever this universe is) is a model of it.
(3) ZF itself (or each intuitively true extension of it) is obtainable from it in a straightforward way.

[2] The use of such terms, albeit in a somewhat cumbersome form, more complicated than that actually used in mathematical texts, is also a major feature of the systems developed in [8,9].

2 The Main Ideas

2.1 Interpreting and Implementing Principle (PRE)

According to our approach, a predicative set theory need *not* exclude the possibility that "arbitrary (undefinable) sets of integers", or "real numbers", or even "arbitrary sets of reals", do exist in some sense, and that propositions about them might be meaningful. However, it cannot be committed to the existence of such entities. Accordingly, one may formulate and use in such a theory propositions that refer to all sets. However, only those of them which are true independently of the exact extension of "the true universe V of sets" may be theorems. Therefore classical logic is acceptable, but there should be restrictions on principles that entail the *existence* "in the universe" of certain objects. Now the major existence principle of naive set theory is given by the comprehension scheme, and so it is this principle that should be restricted. We suggest that principle (PRE) means that the predicatively acceptable instances of the comprehension scheme are those which determine the collections they define in an absolute way, independently of any "surrounding universe". In other words: according to our interpretation of (PRE) in the context of set theory, a formula ψ is predicative (with respect to x) if the collection $\{x \mid \psi(x, y_1, \ldots, y_n)\}$ is completely and uniquely determined by the identity of the parameters $y_1, \ldots, y_n$, and the identity of other objects referred to in the formula (all of which should be well-determined beforehand). [3] Next we translate this idea into an exact definition. For simplicity of presentation, we assume in our definition the "platonic" cumulative universe V of ZF.

Notations. We denote by $Fv(exp)$ the set of free variables of exp, and by $\varphi\{t_1/x_1, \ldots, t_n\}$ the result of simultaneously substituting t_i for the free occurrences of x_i in φ $(i = 1, \ldots, n)$.

Definition 1 Let T be a set theory, and let $Fv(\varphi) = \{y_1, \ldots, y_n, x_1, \ldots, x_k\}$. We say that φ is *predicative* in T for $\{x_1, \ldots, x_k\}$ if $\{\langle x_1, \ldots, x_k\rangle \mid \varphi\}$ is a set for all values of the parameters $y_1, \ldots, y_n$, and the following is true (in V) for every transitive model $\mathcal{M}$ of T:

$$\forall y_1 \ldots \forall y_n . y_1 \in \mathcal{M} \wedge \ldots \wedge y_n \in \mathcal{M} \rightarrow [\varphi \leftrightarrow (x_1 \in \mathcal{M} \wedge \ldots \wedge x_k \in \mathcal{M} \wedge \varphi_{\mathcal{M}})]$$

Thus a formula $\varphi(x)$ is predicative (in T) for x if it has the same extensions in all transitive models of T which contains the values of its other parameters.

[3] Our notion of predicativity of formulas seems to be less restrictive than that used by Weyl and Feferman, since it makes the l.u.b. principle valid for predicatively acceptable sets of reals.

Note on the other hand that φ is predicative for $\emptyset$ iff it is absolute in the usual sense of set theory. (see e.g. [33]).

The main problem in formulating a predicative, type-free, set theory is how to syntactically impose this predicativity property on formulas without introducing syntactic types or levels. The solution suggested here to this problem comes from the observation that this is an instance of a more general task, not peculiar only to set Theory. In fact, in [3] and [6] an appropriate purely logical framework that can be used for this task has been introduced. This framework unifies different notions of "safety" of formulas, coming from different areas of mathematics and computer science, like: domain independence in database theory ([1,48]), decidability of arithmetical formulas in computability theory and metamathematics, and absoluteness in set theory. In the next definition we review this Framework.

Notation. Let σ be a first-order signature without function symbols, and let S_1 and S_2 be two structures for σ. $S_1 \subseteq_\sigma S_2$ denotes that the domain of S_1 is a subset of the domain of S_2, and the interpretations in S_1 and S_2 of the *individual constants* of σ are identical.

Definition 2

(1) Let $S_1 \subseteq_\sigma S_2$, and let $Fv(\varphi) = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$, where φ is a formula in σ. φ is d.i. (domain-independent) for S_1 and S_2 with respect to $\{x_1, \ldots, x_n\}$ (notation: $\varphi \succ^{S_1;S_2} \{x_1, \ldots, x_n\}$), if for all $a_1, \ldots, a_n \in S_2$ and $b_1 \ldots, b_m \in S_1$:[4]

$$S_2 \models \varphi(\overrightarrow{a}, \overrightarrow{b}) \;\leftrightarrow\; a_1 \in S_1 \wedge \ldots \wedge a_n \in S_1 \;\wedge S_1 \models \varphi(\overrightarrow{a}, \overrightarrow{b})$$

(2) A *d.i.-signature* is a pair (σ, F), where σ is an ordinary first-order signature with equality and *no function symbols*, and F is a function which assigns to every n-ary predicate symbol from σ (other than equality) a subset of $\mathcal{P}(\{1, \ldots, n\})$.
(3) Let (σ, F) be a d.i.-signature, and let S_1 and S_2 be structures for σ. S_2 is called a $(\sigma, F)-$*extension* of S_1 (and S_1 is a $(\sigma, F)-$substructure of S_2) if $S_1 \subseteq_\sigma S_2$, and $p(x_1, \ldots, x_n) \succ^{S_1;S_2} \{x_{i_1}, \ldots, x_{i_k}\}$ whenever p is an n-ary predicate of σ, $x_1, \ldots, x_n$ are n distinct variables, and $\{i_1, \ldots, i_k\} \in F(p)$.
(4) Let (σ, F) be a d.i.-signature. A formula φ of σ is called $(\sigma, F)-$*d.i. w.r.t.* X (notation: $\varphi \succ_{(\sigma,F)} X$) if $\varphi \succ^{S_1;S_2} X$ whenever S_2 is a $(\sigma, F)-$extension of S_1. φ is called $(\sigma, F)-$*absolute* if $\varphi \succ_{(\sigma,F)} \emptyset$.

[4] Below we use the informal notation $S \models \varphi(a_1, \ldots, a_n)$ (or even just $\varphi(a_1, \ldots, a_n)$, in case S is the "universe of sets") instead of the more precise, but cumbersome, "$S, V \models \varphi$, where $Fv(\varphi) = \{x_1, \ldots, x_n\}$, and V is an assignment in the domain of S such that $V(x_i) = a_i$ $(i = 1, \ldots, n)$". This notation should not be confused with the notation $\varphi\{t_1/x_1, \ldots, t_n/x_n\}$ for substituting terms of a language for variables.

Examples.

- Let $\sigma_{\overrightarrow{P}} = \{P_1, \ldots, P_k\}$. Assume that the arity of P_i is n_i, and define $F_{\overrightarrow{P}}(P_i) = \{\{1, \ldots, n_i\}\}$. Then φ is $(\sigma_{\overrightarrow{P}}, F_{\overrightarrow{P}})$−d.i. w.r.t. $Fv(\varphi)$ iff it is domain-independent in the sense of database theory (see [1,48]).
- Let $\sigma_{\mathcal{N}} = \{0, <, P_+, P_\times\}$, where 0 is a constant, $<$ is binary, and $P_+, P_\times$ are ternary. Define $F_{\mathcal{N}}(<) = \{\{1\}\}$, $F_{\mathcal{N}}(P_+) = F_{\mathcal{N}}(P_\times) = \{\emptyset\}$. Then the standard structure $\mathcal{N}$ for $\sigma_{\mathcal{N}}$ (with the usual interpretations of 0 and $<$, and the (graphs of the) operations $+$ and $\times$ on N as the interpretations of P_+ and $P_\times$, respectively) is a $(\sigma_{\mathcal{N}}, F_{\mathcal{N}})$-extension of a structure S for $\sigma_{\mathcal{N}}$ iff the domain of S is an initial segment of $\mathcal{N}$ (where the interpretations of the relation symbols are the corresponding reductions of the interpretations of those symbols in $\mathcal{N}$). It was shown in [6] that every Δ_0-formula of $\sigma_{\mathcal{N}}$ is $(\sigma_{\mathcal{N}}, F_{\mathcal{N}})$-absolute, that every $(\sigma_{\mathcal{N}}, F_{\mathcal{N}})$-absolute formula defines a decidable relation on the set of natural numbers, and that a relation on the natural numbers is r.e. iff it is definable by a formula of the form $\exists y_1, \ldots, y_n \psi$, where the formula ψ is $(\sigma_{\mathcal{N}}, F_{\mathcal{N}})$-absolute.
- Let $\sigma_{ZF} = \{\in\}$ and let $F_{ZF}(\in) = \{\{1\}\}$. Then S_2 is a (σ_{ZF}, F_{ZF})−extension of S_1 iff $S_1 \subseteq_{\sigma_{ZF}} S_2$, and $x_1 \in x_2 \succ^{S_1;S_2} \{x_1\}$. The latter condition means that S_1 is a transitive substructure of S_2 (In particular, the universe V is a (σ_{ZF}, F_{ZF})−extension of the transitive sets and classes). Therefore $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_k) \succ_{(\sigma_{ZF}, F_{ZF})} \{x_1, \ldots, x_n\}$ iff the following holds whenever S_1 is a transitive substructure of S_2, and $y_1, \ldots, y_k \in S_1$:

$$\{\langle x_1, \ldots, x_n\rangle \mid S_1 \models \varphi\} = \{\langle x_1, \ldots, x_n\rangle \mid S_2 \models \varphi\}$$

  In particular, a formula is (σ_{ZF}, F_{ZF})-absolute iff it is absolute in the usual sense this notion is used in set theory.

Obviously, "domain independence" and "predicativity" in the sense of "universe independence" are very close relatives. Accordingly, a plausible interpretation of principle (PRE) is that φ is predicative with respect to x iff $\varphi \succ_{(\sigma_{ZF}, F_{ZF})} \{x\}$. However, it follows from results in [6] that the relation $\succ_{(\sigma_{ZF}, F_{ZF})}$ is undecidable. Therefore in order to base predicative formal systems on this interpretation of principle (PRE) we should replace the semantic relation of (σ, F)-d.i. by a useful syntactic approximation. Now the most natural way to define a syntactic approximation of a semantic logical relation concerning formulas is by a structural induction. Such an inductive definition should be based on the behavior with respect to the original semantic relation of the atomic formulas and of the logical connectives and quantifiers. The next theorem from [6] lists the most obvious and useful relevant properties that every relation $\succ_{(\sigma,F)}$ has in the first-order framework:

Theorem 1 *$\succ_{(\sigma,F)}$ has the following properties:*

(1) $p(t_1, \ldots, t_n) \succ_{(\sigma,F)} X$ in case p is an n-ary predicate symbol of σ, and

there is $I \in F(p)$ *such that:*

(a) For every $x \in X$ *there is* $i \in I$ *such that* $x = t_i$.

(b) $X \cap Fv(t_j) = \emptyset$ *for every* $j \in \{1, \ldots, n\} - I$.

(2) (a) $\varphi \succ_{(\sigma,F)} \{x\}$ *if* $\varphi \in \{x \neq x, x = t, t = x\}$, *and* $x \notin Fv(t)$.

(b) $t = s \succ_{(\sigma,F)} \emptyset$.

(3) $\neg\varphi \succ_{(\sigma,F)} \emptyset$ *if* $\varphi \succ_{(\sigma,F)} \emptyset$.

(4) $\varphi \vee \psi \succ_{(\sigma,F)} X$ *if* $\varphi \succ_{(\sigma,F)} X$ *and* $\psi \succ_{(\sigma,F)} X$.

(5) $\varphi \wedge \psi \succ_{(\sigma,F)} X \cup Y$ *if* $\varphi \succ_{(\sigma,F)} X$, $\psi \succ_{(\sigma,F)} Y$, *and* $Y \cap Fv(\varphi) = \emptyset$.

(6) $\exists y\varphi \succ_{(\sigma,F)} X - \{y\}$ *if* $y \in X$ *and* $\varphi \succ_{(\sigma,F)} X$.

(7) If $\varphi \succ_{(\sigma,F)} \{x_1, \ldots, x_n\}$, *and* $\psi \succ_{(\sigma,F)} \emptyset$, *then* $\forall x_1 \ldots x_n(\varphi \rightarrow \psi) \succ_{(\sigma,F)} \emptyset$.

By a "safety relation" we shall henceforth mean a relation $\succ$ between formulas of σ_{ZF} and finite sets of variables which satisfies the clauses in Theorem 1 with respect to F_{ZF} [5]. The least safety relation is a plausible syntactic approximation of predicativity. However, a better approximation is obtained if greater power is given to the first two clauses by providing a much more extensive set of terms than that provided by σ_{ZF} (the only terms of which are its variables). This is achieved by allowing $\{x \mid \psi\}$ to be a legal term whenever $\psi \succ \{x\}$. Note that this is in full coherence with our intended meaning of $\succ$. Moreover, this move is still justified by Theorem 1, since its proof remains valid also for languages which include complex terms (not just variables and constants), as long as $x = t \succ_{(\sigma,F)} \{x\}$ whenever $x \notin Fv(t)$.

2.2 Interpreting and Implementing Principle (NAT)

First we note that by "acceptance of the set N of natural numbers" we understand here also acceptance of principles and ideas implicit in the construction of N. This includes proofs by mathematical induction, as well as the idea of iterating (an operation or a relation) an arbitrary (finite) number of times. Hence finitary inductive definitions of sets, relations, and functions are accepted. In particular, the ability to form the transitive closure of a given relation (like forming the notion of an ancestor from the notion of a parent) should be taken as a major ingredient of our logical abilities (even prior to our understanding of the natural numbers). In fact, in [2] it was argued that this concept is the key for understanding finitary inductive definitions and reasoning, and evidence was provided for the thesis that systems which are based on it provide the right framework for the formalization and mechanization of mathematics. This suggestion will be used as our main tool for implementing (NAT). Hence in addition to allowing the use of set terms we shall also go be-

[5] Property 7 is easily derivable from the others. Hence if $\forall$ and $\rightarrow$ are taken as defined in terms of the other logical constants, then the same relation is obtained if we omit property 7 from the list in Theorem 1.

yond FOL (First-Order Logic) by introducing an operation TC for transitive closure [6]. The corresponding language and semantics are defined as follows (see, e.g., [30,29,47,28,13]):

Definition 3 Let σ be a signature for a first-order language with equality. The language $L^1_{TC}(\sigma)$ is defined like the usual first-order language which is based on σ, but with the addition of the following clause: If φ is a formula, x, y are distinct variables, and t, s are terms, then $(TC_{x,y}\varphi)(s,t)$ is a formula (in which all occurrences of x and y in φ are bound). The intended meaning of $(TC_{x,y}\varphi)(s,t)$ is the following "infinite disjunction": (where $w_1, w_2, \ldots,$ are all new variables):

$$\varphi\{s/x, t/y\} \vee \exists w_1(\varphi\{s/x, w_1/y\}) \wedge \varphi\{w_1/x, t/y\}) \vee$$
$$\vee \exists w_1 \exists w_2(\varphi\{s/x, w_1/y\} \wedge \varphi\{w_1/x, w_2/y\} \wedge \varphi\{w_2/x, t/y\}) \vee \ldots$$

The most important relevant facts shown in [2] concerning TC are:

(1) If σ contains a constant 0 and a (symbol for) a pairing function, then all types of finitary inductive definitions of relations and functions (as defined by Feferman in [21]) are available in $L^1_{TC}(\sigma)$. This result, in turn, allows for presenting a simple version of Feferman's framework FS_0, demonstrating that TC-logics provide an excellent framework for mechanizing formal systems.

(2) Let V_0 be the smallest set including 0 and closed under the operation of pairing. Then a subset S of V_0 is recursively enumerable iff there exists a formula $\varphi(x)$ of $\mathcal{P}TC^+$ such that $S = \{x \in V_0 \mid \varphi(x)\}$, where the language $\mathcal{P}TC^+$ is defined as follows:

Terms of $\mathcal{P}TC^+$

(a) The constant 0 is a term.
(b) Every (individual) variable is a term.
(c) If t and s are terms then so is (t, s).

Formulas of $\mathcal{P}TC^+$

(a) If t and s are terms then $t = s$ is a formula.
(b) If φ and ψ are formulas then so are $\varphi \vee \psi$ and $\varphi \wedge \psi$.
(c) If φ is a formula, x, y are two different variables, and t, s are terms, then $(TC_{x,y}\varphi)(t, s)$ is a formula.

[6] It is well known (see [47]) that the language of FOL enriched with TC is equivalent in its expressive power to the language of weak SOL. So taking "transitive closure" as primitive is equivalent to taking "finite set" as primitive (which is the approach of [23], though the system presented there is essentially first-order). We prefer the former as primitive, because it allows a very natural treatment of induction as a logical rule, as well as a neat extension of the safety relation - see below.

(3) By generalizing a particular case which has been used by Gentzen in [26], mathematical induction can be presented as a logical rule of languages with TC. Indeed, Using a Gentzen-type format, a general form of this principle can be formulated as follows:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta, \psi(y/x)}{\Gamma, \psi(s/x), (TC_{x,y}\varphi)(s,t) \Rightarrow \Delta, \psi(t/x)}$$

where x and y are not free in Γ, Δ, and y is not free in ψ.

Now in order to combine the two central ideas described above, a clause concerning TC should be added to the list of clauses in Theorem 1. Such a clause was suggested in [2]. To understand it, let us look at the first three disjuncts in the infinite disjunction θ which corresponds to $(TC_{x,y}\varphi)(x,y)$:

$$\varphi(x,y) \vee \exists w_1(\varphi(x,w_1) \wedge \varphi(w_1,y)) \vee \exists w_1 \exists w_2(\varphi(x,w_1) \wedge \varphi(w_1,w_2) \wedge \varphi(w_2,y))$$

Call this finite disjunction ψ. From the clauses in Theorem 1 concerning $\wedge, \exists$ and $\vee$ it follows that if $\varphi \succ_{(\sigma,F)} X$ and $y \in X$ (or $x \in X$) then $\psi \succ_{(\sigma,F)} X$. This remains true for every finite subdisjunction of θ. Hence every such finite subdisjunction is d.i. with respect to X, and this easily implies that so is the whole disjunction. This observation leads to the following new condition (in which the variables x and y may be elements of X):

- $(TC_{x,y}\varphi)(x,y) \succ_{(\sigma,F)} X$ if either $\varphi \succ_{(\sigma,F)} X \cup \{x\}$ or $\varphi \succ_{(\sigma,F)} X \cup \{y\}$.

3 PZF and Its Formal Counterparts

In this section we use the ideas described in the previous section for introducing a family of systems for predicative set theory. All these systems share the same language and the same axioms. They differ only with respect to the strength of their formal underlying apparatus. We shall denote by PZF the strongest (and non-axiomatizable) system in this family.

3.1 Language

We define the terms and formula of the language $\mathcal{L}_{PZF}$, as well as the safety relation $\succ_{PZF}$ between formulas and finite sets of variables, by simultaneous recursion as follows (where $Fv(exp)$ denotes the set of free variables of exp):

Terms:

- Every variable is a term.

- If x is a variable, and φ is a formula such that $\varphi \succ_{PZF} \{x\}$, then $\{x \mid \varphi\}$ is a term (and $Fv(\{x \mid \varphi\}) = Fv(\varphi) - \{x\}$).

Formulas:

- If t and s are terms than $t = s$ and $t \in s$ are atomic formulas.
- If φ and ψ are formulas, and x is a variable, then $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $\exists x\varphi$ are formulas (where $Fv(\exists x\varphi) = Fv(\varphi) - \{x\}$).
- If φ is a formula, t and s are terms, and x and y are distinct variables then $(TC_{x,y}\varphi)(t, s)$ is a formula, and

$$Fv((TC_{x,y}\varphi)(t, s)) = (Fv(\varphi) - \{x, y\}) \cup Fv(t) \cup Fv(s)$$

The Safety Relation $\succ_{PZF}$:

(1) (a) $\varphi \succ_{PZF} \emptyset$ if φ is atomic.
    (b) $\neg\varphi \succ_{PZF} \emptyset$ if $\varphi \succ_{PZF} \emptyset$.
(2) $\varphi \succ_{PZF} \{x\}$ if $\varphi \in \{x \in x, x = t, t = x, x \in t\}$, and $x \notin Fv(t)$.
(3) $\varphi \vee \psi \succ_{PZF} X$ if $\varphi \succ_{PZF} X$ and $\psi \succ_{PZF} X$.
(4) $\varphi \wedge \psi \succ_{PZF} X \cup Y$ if $\varphi \succ_{PZF} X$, $\psi \succ_{PZF} Y$ and either $Y \cap Fv(\varphi) = \emptyset$ or $X \cap Fv(\psi) = \emptyset$.
(5) $\exists y\varphi \succ_{PZF} X - \{y\}$ if $y \in X$ and $\varphi \succ_{PZF} X$.
(6) $(TC_{x,y}\varphi)(x, y) \succ_{PZF} X$ if $\varphi \succ_{PZF} X \cup \{x\}$, or $\varphi \succ_{PZF} X \cup \{y\}$.

Note 1 The intended *intuitive* meaning of "$\varphi \succ_{PZF} \{y_1, \ldots, y_k\}$", where $Fv(\varphi) = \{x_1, \ldots, x_n, y_1, \ldots, y_k\}$, is that for every "accepted" sets $a_1, \ldots, a_n$, the collection of all tuples $\langle y_1, \ldots, y_k\rangle$ such that $\varphi(a_1, \ldots, a_n, y_1, \ldots, y_k)$ is a *set* which is constructed in an absolute, "universe independent" way from previously "accepted" sets and from (elements in the transitive closure of) $a_1, \ldots, a_n$. Since this is an imprecise explanation, it cannot be proved in the strict sense of the word. However, it is not difficult to convince oneself that $\succ_{PZF}$ indeed has this property. For example, assume that $\theta = \varphi \wedge \psi$, where $Fv(\varphi) = \{x, z\}, Fv(\psi) = \{x, y, z\}, \varphi \succ_{PZF} \{x\}$, and $\psi \succ_{PZF} \{y\}$. Given some absolute set c, by induction hypothesis the collection $Z(c)$ of all x such that $\varphi(x, c)$ is an absolute set. Again by induction hypothesis, for every d in this set the collection $W(c, d)$ of all y such that $\psi(d, y, c)$ is an absolute set. Now the collection of all $\langle x, y\rangle$ such that $\theta(x, y, c)$ is the union for $d \in Z(c)$ of the sets $\{d\} \times W(c, d)$. Hence it is a set containing only previously accepted, absolute collections, and its identity is obviously absolute too. This is exactly what $\theta \succ_{PZF} \{x, y\}$ (which holds in this case by the clause concerning conjunction in the definition of $\succ_{PZF}$) intuitively means.

Note 2 Officially, the language we use does not include the universal quantifier $\forall$ and the implication connective $\rightarrow$. Below they are taken therefore as defined (in the usual way) in terms of the official connectives and $\exists$.

Note 3 It is not difficult to show that $\succ_{PZF}$ has the following properties:

- If $\varphi \succ_{PZF} X$ then $X \subseteq Fv(\varphi)$.
- If $\varphi \succ_{PZF} X$ and $Z \subseteq X$, then $\varphi \succ_{PZF} Z$.
- If $\varphi \succ_{PZF} \{x_1, \ldots, x_n\}$, $v_1, \ldots v_n$ are n distinct variables not occurring in φ, and φ' is obtained from φ by replacing *all* (not only the free) occurrences of x_i by v_i ($i = 1, \ldots, n$), then $\varphi' \succ_{PZF} \{v_1, \ldots, v_n\}$.
- If $x \notin Fv(t)$, and $\varphi \succ_{PZF} \emptyset$, then both $\forall x(x \in t \rightarrow \varphi) \succ_{PZF} \emptyset$, and $\exists x(x \in t \wedge \varphi) \succ_{PZF} \emptyset$. Hence $\varphi \succ_{PZF} \emptyset$ for every Δ_0 formula φ in $\mathcal{L}_{ZF}$.

The following proposition can easily be proved:

Proposition 1 *There is an algorithm which given a string of symbols E determines whether E is a term of $\mathcal{L}_{PZF}$, a formula of $\mathcal{L}_{PZF}$, or neither, and in case E is a formula it returns the set of all X such that $E \succ_{PZF} X$.*

3.2 Axioms

We turn to the axioms of PZF and its formal counterparts. The basic idea here is to use a version of the "ideal calculus" ([14]) for naive set theory, in which the comprehension schema is applicable only to safe formulas. In addition we include also $\in$-induction, which seems to be quite natural within a predicative framework. Here is the resulting list of axioms:

Extensionality:

- $\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$

The Comprehension Schema: [7]

- $\forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi)$

The Regularity Schema ($\in$-induction):

- $(\forall x(\forall y(y \in x \rightarrow \varphi\{y/x\}) \rightarrow \varphi)) \rightarrow \forall x \varphi$

3.3 Logic

The logic which underlies PZF is TC-logic (transitive closure logic, also called ancestral logic): the logic which corresponds to ordinary first-order logic (with equality) augmented with TC, the operator which produces the transitive

[7] This name is justified here because for φ which is predicative with respect to x (i.e. $\varphi \succ_{PZF} \{x\}$) it easily entails the usual formulation: $\exists Z \forall x(x \in Z \leftrightarrow \varphi)$.

closure of a given binary relation. Now the set of valid formulas of this logic is not r.e. (or even arithmetical). Hence no sound and complete *formal* system for it exists. It follows that PZF, our version of predicative set theory, cannot be fully formalized. The problem whether the above set of axioms is sound and complete for predicative set theory should therefore be understood as being relative to this underlying logic. This means that according to our approach, *no single formal system can capture the whole of predicative mathematics*. It also follows that the problem of producing formal systems for actually using PZF (for making formal deductions in predicative mathematics) reduces to finding appropriate formal approximations of this underlying *logic*. Hence what we introduce here together with PZF is really a family of formal systems.

One crucial logical rule that should be available in any such approximation is the general rule of induction formulated in subsection 2.2:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta, \psi(y/x)}{\Gamma, \psi(s/x), (TC_{x,y}\varphi)(s,t) \Rightarrow \Delta, \psi(t/x)}$$

(where x and y are not free in Γ, Δ, and y is not free in ψ). Two other obvious rules introduce TC on the right hand side of sequents: [8]

$$\frac{\Gamma \Rightarrow \Delta, \varphi\{t/x, s/y\}}{\Gamma \Rightarrow \Delta, (TC_{x,y}\varphi)(t,s)}$$

$$\frac{\Gamma \Rightarrow \Delta, (TC_{x,y}\varphi)(r,s) \qquad \Gamma \Rightarrow \Delta, (TC_{x,y}\varphi)(s,t)}{\Gamma \Rightarrow \Delta, (TC_{x,y}\varphi)(r,t)}$$

Henceforth we denote by PZF_0 the formal approximation of PZF in which the underlying formal logic is the extension of first-order logic with these three rules for TC. PZF_0 suffices for everything we do below, and we believe (but this remains to be confirmed) that it should in fact suffice for (most of) applicable mathematics. Now PZF_0 is relatively a week system. Thus it can easily be interpreted in Kripke-Platek set theory KP together with the infinity axiom (see [7,31,11]) [9]. However, it should again be emphasized that PZF as a whole is open-ended, and transcends any given formal system.

Note 4 In addition to having TC (which is the major difference between our underlying logic and FOL), one should also note that the language of PZF provides a class of terms which is much richer than those allowed in orthodox first-order systems. In particular: a variable can be bound in it within a term. The notion of a term being free for substitution should be generalized accordingly (also for substitutions within terms!). As usual this amounts to avoiding the capture of free variables within the scope of an operator which binds them. Otherwise the rules/axioms concerning the quantifiers and terms

[8] The resulting system is equivalent to Myhill's system for ancestral logic in [34].

[9] KP itself includes the Δ_0-collection schema, which is not predicatively justified.

remain unchanged (for example: $\forall x\varphi \to \varphi\{t/x\}$ is valid for *every* term t which is free for x in φ). We also assume α-conversion to be a part of the logic[10].

For simplicity of presentation and understanding, we again assume in the rest of this paper the platonic cumulative universe V (although its exact extension is irrelevant). Predicatively meaningful counterparts of our various claims can be formulated and proved, but we leave this task to another opportunity.

The straightforward proof of the following proposition was practically given in Note 1 (see [5] for a proof of a stronger claim):

Proposition 2 *V is a model of PZF.*

4 The Expressive Power of PZF

4.1 Some Standard Notations for Sets

In $\mathcal{L}_{PZF}$ we can introduce as *abbreviations* most of the standard notations for sets used in mathematics. Note that all these abbreviations can be introduced in a purely static way: unlike in the extension by definition procedure (see [46]), no formal proofs within the system (of corresponding justifying existence and uniqueness propositions) are needed before introducing them.

- $\emptyset =_{Df} \{x \mid x \in x\}$.
- $\{t_1, \ldots, t_n\} =_{Df} \{x \mid x = t_1 \vee \ldots \vee x = t_n\}$ (where x is new).
- $\langle t, s\rangle =_{Df} \{\{t\}, \{t, s\}\}$
- $\{x \in t \mid \varphi\} =_{Df} \{x \mid x \in t \wedge \varphi\}$, provided $\varphi \succ_{PZF} \emptyset$. (where $x \notin Fv(t)$).
- $\{t \mid x \in s\} =_{Df} \{y \mid \exists x.x \in s \wedge y = t\}$ (where y is new, and $x \notin Fv(s)$).
- $s \times t =_{Df} \{x \mid \exists a \exists b.a \in s \wedge b \in t \wedge x = \langle a, b\rangle\}$ (where x, a and b are new).
- $s \cap t =_{Df} \{x \mid x \in s \wedge x \in t\}$ (where x is new).
- $s \cup t =_{Df} \{x \mid x \in s \vee x \in t\}$ (where x is new).
- $S(x) =_{Df} x \cup \{x\}$
- $\bigcup t =_{Df} \{x \mid \exists y.y \in t \wedge x \in y\}$ (where x and y are new).
- $\bigcap t =_{Df} \{x \mid \exists y(y \in t \wedge x \in y) \wedge \forall y(y \in t \to x \in y)\}$ (where x, y are new).
- $\iota x\varphi =_{Df} \bigcap\{x \mid \varphi\}$ (provided $\varphi \succ_{PZF} \{x\}$).
- $P_1(z) =_{Df} \iota x.\exists v \exists y(v \in z \wedge x \in v \wedge y \in v \wedge z = \langle x, y\rangle) \quad (\vdash_{PZF} P_1(\langle t, s\rangle) = t)$.
- $P_2(z) =_{Df} \iota y.\exists v \exists x(v \in z \wedge x \in v \wedge y \in v \wedge z = \langle x, y\rangle) \quad (\vdash_{PZF} P_2(\langle t, s\rangle) = s)$.
- $\omega =_{Df} \{x \mid x = \emptyset \vee \exists y.y = \emptyset \wedge (TC_{x,y}(x = S(y)))(x, y)\}$

[10] Other rules, like substitution of equals for equals within any context (under the usual conditions concerning bound variables) are derivable from the usual first-order axioms for equality by using the axioms of PZF.

- $TH(x) =_{Df} x \cup \{y \mid (TC_{x,y} y \in x)(x, y)\}$ (the transitive hull of x).

Our term above for $\bigcap t$ is valid (and so denotes a set) whenever t is valid. It is easy to see that if t denotes a non-empty set A then $\bigcap t$ indeed denotes the intersection of all the elements of A. On the other hand, if the set denoted by t is empty, then the set denoted by the term $\bigcap t$ is empty as well. With the help of the extensionality axiom this in turn implies that if $\varphi \succ_{PZF} \{x\}$ then the term above for $\iota x\varphi$ denotes $\emptyset$ if there is no set which satisfies φ, and it denotes the intersection of all the sets which satisfy φ otherwise. In particular: if there is exactly one set which satisfy φ then $\iota x\varphi$ denotes this unique set. All these facts are theorems of PZF_0. In particular we have:

Proposition 3 *If* $\varphi \succ_{PZF} \{x\}$ *then* $\vdash_{PZF_0} \exists! x\varphi \rightarrow \forall x(\varphi \leftrightarrow x = \iota x\varphi)$.

From Proposition 3 it follows that if a formula $\varphi(y_1, \ldots, y_n, x)$ implicitly defines in PZF a function f_φ such that for all $y_1, \ldots, y_n$, $f_\varphi(y_1, \ldots, y_n)$ is the unique x such that $\varphi(y_1, \ldots, y_n, x)$, and if $\varphi \succ_{PZF} \{x\}$, then there is a term in PZF which explicitly denotes f_φ, and no extension by definitions of the language is needed for introducing it. Moreover: in PZF we can introduce as abbreviations the terms used in the λ-calculus for handling explicitly defined functions (except that our terms for functions should specify the domains of these functions, which should be explicitly definable sets):

- $\lambda x \in s.t =_{Df} \{\langle x, t\rangle \mid x \in s\}$ (where $x \notin Fv(s)$)
- $f(x) =_{Df} \iota y.\exists z \exists v(z \in f \wedge v \in z \wedge y \in v \wedge z = \langle x, y\rangle)$
- $Dom(f) =_{Df} \{x \mid \exists z \exists v \exists y(z \in f \wedge v \in z \wedge y \in v \wedge x \in v \wedge z = \langle x, y\rangle\}$
- $Rng(f) =_{Df} \{y \mid \exists z \exists v \exists x(z \in f \wedge v \in z \wedge y \in v \wedge x \in v \wedge z = \langle x, y\rangle\}$
- $f \restriction s =_{Df} \{\langle x, f(x)\rangle \mid x \in s\}$ (where x is new).

Identifying $\perp$ from domain theory with $\emptyset$, we can easily check now that rules β and η obtain in PZF:

- $\vdash_{PZF_0} u \in s \rightarrow (\lambda x \in s.t)u = t\{u/x\}$ (if u is free for x in t).
- $\vdash_{PZF_0} u \notin s \rightarrow (\lambda x \in s.t)u = \emptyset$ (if u is free for x in t).
- $\vdash_{PZF_0} \lambda x \in s.tx = t \restriction s$ (in case $x \notin Fv(t)$).

4.2 RST and Rudimentary Functions

Let $\mathcal{L}_{RST}$ and $\succ_{RST}$ be defined like $\mathcal{L}_{PZF}$ and $\succ_{PZF}$ (respectively), but without using the TC operator. Let RST be the first-order system in $\mathcal{L}_{RST}$ which is based on the three axioms of PZF (and with a suitable version of ordinary first-order logic as the underlying logic). It should be noted that with the exception of ω and $TH(x)$, all the constructions above have actually been done in the framework of $\mathcal{L}_{RST}$ (and can be justified in RST). Now HF, the set of hereditarily finite sets, is a model of RST. Hence ω is not definable in $\mathcal{L}_{RST}$, and so TC is indeed necessary for its definition. [11]

Note 5 RST can be shown to be equivalent to Gandy's basic set theory ([25]), and to the system called BST_0 in [43].

The following theorem and its two corollaries determine the expressive power of $\mathcal{L}_{RST}$, and connect it (and $\succ_{RST}$) with the class of rudimentary set functions — a refined version of Gödel basic set functions (from [27]) which was independently introduced by Gandy in [25] and Jensen in [32] (See also [10]).

Theorem 2

(1) If F is an n-ary rudimentary function, then there exists a formula φ_F with the following properties:
(a) $Fv(\varphi_F) \subseteq \{y, x_1, \ldots, x_n\}$
(b) $\varphi_F \succ_{RST} \{y\}$
(c) $F(x_1, \ldots, x_n) = \{y \mid \varphi_F\}$.
(2) If φ is a formula of $\mathcal{L}_{RST}$ such that:
(a) $Fv(\varphi) \subseteq \{y_1, \ldots, y_k, x_1, \ldots, x_n\}$
(b) $\varphi \succ_{RST} \{y_1, \ldots, y_k\}$
then there exists a rudimentary function F_φ such that:

$$F_\varphi(x_1, \ldots, x_n) = \{\langle y_1, \ldots, y_k \rangle \mid \varphi\}$$

(3) If t is a term of $\mathcal{L}_{RST}$ such that $Fv(t) \subseteq \{x_1, \ldots, x_n\}$, then there exists a rudimentary function F_t such that $F_t(x_1, \ldots, x_n) = t$ for every $x_1, \ldots, x_n$.

Proof: We prove part (1) by induction, following the definition of the rudimentary functions given in [10]:

- If $F(x_1, \ldots, x_n) = x_i$ then φ_F is $y \in x_i$. Here $\varphi_F \succ_{RST} \{y\}$ by clause (2) of the definition of $\succ_{RST}$.
- If $F(x_1, \ldots, x_n) = \{x_i, x_j\}$ then φ_F is $y = x_i \vee y = x_j$. Here $\varphi_F \succ_{RST} \{y\}$ by clauses (2) and (3) of the definition of $\succ_{RST}$.
- If $F(x_1, \ldots, x_n) = x_i - x_j$ then φ_F is $y \in x_i \wedge \neg(y \in x_j)$. Here $\varphi_F \succ_{RST} \{y\}$ by clause (2), (1a), (1b), and (4) of the definition of $\succ_{RST}$.
- Suppose $F(x_1, \ldots, x_n) = H(G_1(x_1, \ldots, x_n), \ldots, G_k(x_1, \ldots, x_n))$, where H and $G_1, \ldots, G_k$ are rudimentary. Let $w_1, \ldots, w_k$ be new variables. Then φ_F is $\exists w_1 \ldots w_k(w_1 = \{y \mid \varphi_{G_1}\} \wedge \ldots \wedge w_k = \{y \mid \varphi_{G_k}\} \wedge \varphi_H(y, w_1, \ldots, w_k))$. Here $\varphi_F \succ_{RST} \{y\}$ by clauses (2), (4), and (5) of the definition of $\succ_{RST}$.

[11] It is known (see e.g. [25]) that the property of being a finite ordinal is definable by a Δ_0-formula $\varphi(x)$, but this φ does not satisfy $\varphi \succ_{PZF} \{x\}$ (it only satisfies $\varphi \succ_{RST} \emptyset$, like any other Δ_0-formula). Hence $\{x \mid \varphi\}$ is not a legal term of RST.

- Suppose $F(x_1,\ldots,x_n) = \bigcup_{z\in x_1} G(z,x_2,\ldots,x_n)$, where G is rudimentary. Then φ_F is $\exists z(z \in x_1 \wedge \varphi_G(y,z,x_2,\ldots,x_n))$. Here again $\varphi_F \succ_{RST} \{y\}$ by clauses (2), (4), and (5) of the definition of $\succ_{RST}$.

Next we prove parts (2) and (3) together by induction on the complexity of φ and t.

- If t is x_i then $F_t(x_1,\ldots,x_n) = x_i$.
- If t is $\{y \mid \varphi\}$, where $\varphi \succ_{RST} \{y\}$, then $F_t = F_\varphi$.
- If φ is $t = s$ and $k = 0$ then

$$F_\varphi(x_1,\ldots,x_n) = \begin{cases} \{\emptyset\} & F_t(x_1,\ldots,x_n) = F_s(x_1,\ldots,x_n) \\ \emptyset & F_t(x_1,\ldots,x_n) \neq F_s(x_1,\ldots,x_n) \end{cases}$$

  The case in which φ is $t \in s$ and $k = 0$ is treated similarly.
- If φ is $\neg\psi$ and $k = 0$ then $F_\varphi(x_1,\ldots,x_n) = \{\emptyset\} - F_\psi(x_1,\ldots,x_n)$.
- If φ is $y_1 \neq y_1$ (and $k = 1$), then $F_\varphi(x_1,\ldots,x_n) = \emptyset$.
- If φ is $y_1 = t$ or $t = y_1$, where $y_1 \notin Fv(t)$ (and $k = 1$), then $F_\varphi(x_1,\ldots,x_n) = \{F_t(x_1,\ldots,x_n)\}$.
- If φ is $y_1 \in t$, where $y_1 \notin Fv(t)$ (and $k = 1$), then $F_\varphi(x_1,\ldots,x_n) = F_t(x_1,\ldots,x_n)$.
- If φ is $\psi_1 \vee \psi_2$ then $F_\varphi(x_1,\ldots,x_n) = F_{\psi_1}(x_1,\ldots,x_n) \cup F_{\psi_2}(x_1,\ldots,x_n)$.
- If φ is $\psi \wedge \theta$, where $\psi \succ_{RST} \{y_1,\ldots,y_l\}$ $(l \leq k)$, $\theta \succ_{RST} \{y_{l+1},\ldots,y_k\}$, and $Fv(\psi) \cap \{y_{l+1},\ldots,y_k\} = \emptyset$, then

$$F_\varphi(x_1,\ldots,x_n) = \bigcup_{\langle y_1,\ldots,y_l\rangle \in F_\psi(x_1,\ldots,x_n)} \; \bigcup_{\langle y_{l+1},\ldots,y_k\rangle \in F_\theta(x_1,\ldots,x_n,y_1,\ldots,y_l)} \{\langle y_1,\ldots,y_k\rangle\}$$

- Suppose $\varphi = \exists z\psi$, where $\psi \succ_{RST} \{z,y_1,\ldots,y_k\}$. Then $F_\varphi(x_1,\ldots,x_n) = \bigcup_{\langle z,y_1,\ldots,y_k\rangle \in F_\psi(x_1,\ldots,x_n)} \{\langle y_1,\ldots,y_k\rangle\}$.

It is not difficult to see that all functions defined above are indeed rudimentary.

Corollary 1 *Every term of $\mathcal{L}_{RST}$ with n free variables explicitly defines an n-ary rudimentary function. Conversely, every rudimentary function is defined by some term of $\mathcal{L}_{RST}$.*

Corollary 2 *If $Fv(\varphi) = \{x_1,\ldots,x_n\}$, and $\varphi \succ_{RST} \emptyset$, then φ defines a rudimentary predicate P. Conversely, if P is a rudimentary predicate, then there is a formula φ such that $\varphi \succ_{RST} \emptyset$, and φ defines P.*

4.3 Recursion and Inductive Definitions

The inclusion of the operation TC in $\mathcal{L}_{PZF}$ strongly extends its expressive power. As a simple example of this power we take primitive recursion on ω:

Proposition 4 *Assume g is a function on ω^2 which is definable by a (closed) term of $\mathcal{L}_{PZF}$. Let f be a function on ω defined by $f(0) = a$, $f(n+1) = g(n, f(n))$ (where a is definable in $\mathcal{L}_{PZF}$). Then f is definable (as a set of pairs) by a closed term of $\mathcal{L}_{PZF}$.*

Proof: Assume t_g is a term which defines g in $\mathcal{L}_{PZF}$. Let $\psi_1(z, w)$ be the formula $(TC_{z,w} w = \langle S(P_1(z)), t_g(z)\rangle)(z, w)$ (note that we use here the notation for function application which was introduced in subsection 4.1). Let ψ_2 be the formula $z = \langle 0, a\rangle \wedge \psi_1(z, w) \wedge P_1(w) = n \wedge P_2(w) = x$, and φ the formula $\exists z \exists w \psi_2$. Since $w = \langle S(P_1(z)), t_g(z)\rangle \succ_{PZF} \{w\}$, also $\psi_1 \succ_{PZF} \{w\}$ (by the clause concerning TC in the definition of $\succ_{PZF}$). Hence $\psi_2 \succ_{PZF} \{z, w, n, x\}$ (by the clauses concerning $\wedge$ and $=$ in the definition of $\succ_{PZF}$). It follows that $\varphi \succ_{PZF} \{n, x\}$, and so $\iota x \varphi$ is defined. Since it is easy to prove by induction that $\vdash_{PZF_0} \forall n \in \omega \exists! x \varphi$, Proposition 3 entails that $\lambda n \in \omega . \iota x \varphi$ is a term as required.

Proposition 4 is only a special case of the following much more general theorem, which implies that all types of *finitary inductive* definitions (as characterized in [21]) are available in $\mathcal{L}_{PZF}$. Its proof is similar to the proof of Theorem 15 in [2]:

Theorem 3 *For $1 \leq j \leq p$, let $\varphi_1(y, x_1, \ldots, x_{n_1}), \ldots, \varphi_p(y, x_1, \ldots, x_{n_p})$ be p formulas such that $\varphi_j \succ_{PZF} \{y\}$, and let $k_1(j), \ldots, k_{n_j}(j)$ and $o(j)$ be (not necessarily distinct) natural numbers between 1 and m. Assume that $A_1, \ldots, A_m$ are sets, and that $B_1, \ldots, B_m$ are the least $X_1, \ldots, X_m$ which satisfy the following conditions (for $1 \leq i \leq m$ and $1 \leq j \leq p$):*

(1) $A_i \subseteq X_i$
(2) If $a_1 \in X_{k_1(j)}, \ldots, a_{n_j} \in X_{k_{n_j}(j)}$ and $\varphi_j(b, a_1, \ldots, a_{n_j})$ then $b \in X_{o(j)}$

Then $B_1, \ldots, B_m$ are definable by terms of $\mathcal{L}_{PZF}$ with parameters $A_1, \ldots, A_m$.

Example: The set HF of hereditarily finite sets is the least X such that $\{\emptyset\} \subseteq X$, and $y \in X$ whenever $a \in X$, $b \in X$, and $y = a \cup \{b\}$. Hence HF is defined by a closed term of $\mathcal{L}_{PZF}$.

5 The Predicativity of PZF

The following theorem implies that PZF indeed satisfies condition (PRE):

Theorem 4

(1) If $\varphi \succ_{PZF} X$ then φ is predicative in PZF for X.

(2) *If t is a valid term of PZF then t is predicative in the sense that it satisfies the following condition: If $Fv(t) = \{y_1, \ldots, y_n\}$ then the following is true (in V) for every transitive model $\mathcal{M}$ of PZF:*

$$\forall y_1 \ldots \forall y_n . y_1 \in \mathcal{M} \wedge \ldots \wedge y_n \in \mathcal{M} \to t_{\mathcal{M}} = t$$

where $t_{\mathcal{M}}$ denotes the relativization of t to $\mathcal{M}$.

Proof: By a simultaneous induction on the complexity of t and φ.

Discussion. By Theorem 4, every term t of $\mathcal{L}_{PZF}$ has the same interpretation in all transitive models of PZF which contains the values of its parameters. Thus the identity of the set denoted by t is independent of the exact extension of the assumed universe of sets. This already justifies seeing PZF as predicative. However, we want to argue that the predicativity of PZF intuitively goes deeper than this. The argument will necessarily be less exact (and on a more intuitive level) than that given by Theorem 4.

The problem with Theorem 4 is that it is a theorem of platonistic mathematics, and so it assumes an all-encompassing collection V which includes all potential "sets" and contains all "universes", but is itself a universe too (meaning that classical logic holds within it). This assumption is doubtful from a predicativist point of view [12] . To see how we can do without it, call two universes $\mathcal{M}_1$ and $\mathcal{M}_2$ *compatible* if the following conditions are satisfied:

(1) Suppose a is an object in both $\mathcal{M}_1$ and $\mathcal{M}_2$. Then b is an object in $\mathcal{M}_1$ such that $\mathcal{M}_1 \models b \in a$ iff b is an object in $\mathcal{M}_2$ such that $\mathcal{M}_2 \models b \in a$.
(2) Suppose a and b are objects in $\mathcal{M}_1$ and $\mathcal{M}_2$ (respectively), and that the collections $\{x \in \mathcal{M}_1 \mid \mathcal{M}_1 \models x \in a\}$ and $\{x \in \mathcal{M}_2 \mid \mathcal{M}_2 \models x \in b\}$ are identical. Then a and b are identical. (Here we temporarily use the notation $\{x \mid A\}$ *in the metalanguage* to denote classes of objects.)

It is now not difficult to check that if t is a term of $\mathcal{L}_{PZF}$ then the value of t for the assignment $x_1 := a_1, \ldots, x_n := a_n$ (where $Fv(t) = \{x_1, \ldots, x_n\}$) is the same in any two compatible universes which include $\{a_1, \ldots, a_n\}$. This is what we really had in mind when we talked above about "universe independence" (note that if the platonic universe V exists, then every two transitive subcollections of V are compatible according to the definition above).

Turning next to Principle (NAT), we first of all note again that the set of all natural numbers is available in PZF in the form of ω. This easily implies that PA, the first-order Peano's Arithmetics, has a natural interpretation in PZF_0 (see Proposition 4 for a partial proof). However, the availability of ω alone is

[12] Thus both Sanchis in [42] and Weaver in [49] argue that classical logic is unsuitable for dealing with the whole of V, and intuitionistic logic should be used for it instead.

not sufficient for getting the full power of mathematical induction, since the full separation schema is not available in PZF. Nevertheless, the fact that the underlying logic is the TC-logic implies that the following induction *schema* is available (alternatively, this schema can be derived from the availability of ω with the help of $\in$-induction):

$$\vdash_{PZF_0} \varphi\{\emptyset/x\} \wedge \forall x(\varphi \to \varphi\{S(x)/x\}) \to \forall x.x \in \omega \to \varphi$$

No less crucial than the ability to use induction is the ability to use inductive definitions. Theorem 3 (see also Proposition 4) entails that the most important form of using such definitions is available in $\mathcal{L}_{PZF}$.

Note 6 Unlike in the case of proofs by induction (where $\in$-induction would do), the TC-machinery is essential for the ability to use in PZF inductive definitions. Now in previous systems for predicative mathematics, recursion in ω was obtained using Δ-comprehension (or Δ-collection). The explanation was that a Δ-formula φ is both upward absolute and downward absolute, and so it is absolute. This argument implicitly assumes the platonic universe V, and so it is doubtful in view of the discussion of (PRE) in this section (without V as a maximal universe, or some other doubtful assumptions concerning universes, I do not see why the combination of upward absoluteness and of downward absoluteness entails absoluteness).

6 Relations with the Axioms of ZF

The definability of $\{t, s\}$, $\bigcup t$, and ω means that the axioms of pairing, union, and infinity are provable in PZF. On the other hand $\{x \in t \mid \varphi\}$ is a valid term only if $\varphi \succ_{PZF} \emptyset$. Hence we do not have in PZF the full power of the other comprehension axioms of ZF. Instead we have the following counterparts:

The predicative separation schema: If $\varphi \succ_{PZF} \emptyset$; ψ is equivalent in PZF_0 to φ; x, w, Z are distinct variables and $Z \notin Fv(\psi)$, then:

$$\vdash_{PZF_0} \forall w \exists Z \forall x(x \in Z \leftrightarrow x \in w \wedge \psi)$$

The predicative replacement schema: If $x \notin Fv(t)$ then

$$\vdash_{PZF_0} \forall w \exists Z \forall x(x \in Z \leftrightarrow \exists y.y \in w \wedge x = t)$$

The predicative collection schema: If $\varphi \succ_{PZF} \{x\}$; ψ is equivalent in PZF_0 to φ; x, y, w, Z are distinct variables, and $Z \notin Fv(\psi)$, then:

$$\vdash_{PZF_0} \forall w \exists Z \forall x(x \in Z \leftrightarrow \exists y.y \in w \wedge \psi)$$

The predicative powerset schema: If $\varphi \succ_{PZF} \{x\}$; ψ is equivalent in PZF_0 to φ; x, w, Z are distinct variables, and $Z \notin Fv(\psi)$, then:

$$\vdash_{PZF_0} \forall w \exists Z \forall x (x \in Z \leftrightarrow x \subseteq w \wedge \psi)$$

Thus although $P(\omega)$, the powerset of ω, is not available in PZF (This easily follows from Theorem 4, and the fact that $P(\omega)$ is not absolute), every set of the form $\{x \in P(\omega) \mid \varphi\}$ where $\varphi \succ_{PZF} \{x\}$ is available nevertheless.

At this point it is interesting to note that TZF, a system similar to PZF_0 which is intuitively sound (from a platonistic point of view), and does have the full power of ZF (though not ZFC), can be defined in a way similar to PZF_0, but using another relation $\succ_{TZF}$, instead of $\succ_{PZF}$. $\succ_{TZF}$ is the relation obtained by adding to the definition of $\mathcal{L}_{PZF}$ the following three conditions:

(1) $\varphi \succ_{TZF} \emptyset$ for *every* formula φ.
(2) $x \subseteq t \succ_{TZF} \{x\}$ if $x \notin Fv(t)$.
(3) $\exists y \varphi \wedge \forall y(\varphi \rightarrow \psi) \succ_{TZF} X$ if $\psi \succ_{TZF} X$, and $X \cap Fv(\varphi) = \emptyset$.

In [4,5] it was shown that a first-order system which is equivalent to ZF (but more natural and easier to mechanize than the usual presentation of ZF) is obtained from TZF if the underlying logic is changed to classical first-order logic (in a first-order language enriched with abstract terms), and instead of using TC, a special constant for ω is added to the language, together with Peano's axioms for it. This shows that ZF and PZF are indeed close in spirit.

7 The Minimal Model of PZF

7.1 The Basic Universe

Next we show that in the spirit of (PRE), we may take our universe to be the collection of predicatively definable sets.

Definition 4 PD_0 (for "predicatively Definable") is the set (in V) of all sets (in V) which are defined by closed terms of $\mathcal{L}_{PZF}$.

Lemma 1 *Let s be a term of $\mathcal{L}_{PZF}$.*

(1) If s is free for y in the term t of $\mathcal{L}_{PZF}$, then $t\{s/y\}$ is a term of $\mathcal{L}_{PZF}$.
(2) If s is free for y in the formula φ of $\mathcal{L}_{PZF}$, $\varphi \succ_{PZF} X$, $y \notin X$, and $Fv(s) \cap X = \emptyset$, then $\varphi\{s/y\} \succ_{PZF} X$.

The proof is by a simultaneous induction on the complexity of t and φ.

Notation.

(1) If t is a term of $\mathcal{L}_{PZF}$, and v is an assignment in V, we denote by $\|t\|_v$ the value (in V) that t gets under v. In case t is closed we denote by $\|t\|$ the value of t in V.
(2) Let φ be a formula of $\mathcal{L}_{PZF}$, and let v be an assignment in V. $v \models \varphi$ denotes that v satisfies φ in V.
(3) If φ is a formula of $\mathcal{L}_{PZF}$, $X \subseteq Fv(\varphi)$, and v is an assignment in V, we denote by $\|\varphi\|_v^X$ the class of all $a \in V$ for which there exists an assignment v' such that $a = v'(x)$ for some $x \in X$, $v'(y) = v(y)$ for $y \notin X$, and $v' \models \varphi$.

Lemma 2 *Let $Fv(t) = \{x_1, \ldots, x_n\}$, and let $s_1, \ldots, s_n$ be closed terms of $\mathcal{L}_{PZF}$. Suppose v is an assignment such that $v(x_i) = \|s_i\|$ for $i = 1, \ldots, n$. Then $\|t\|_v = \|t\{s_1/x_1, \ldots, s_n/x_n\}\|$.*

Theorem 5 *PD_0 is transitive (in other words: all elements of a predicatively definable set are themselves predicatively definable).*

Proof: Denote by HPD_0 (for "Hereditarily Predicatively Definable") the set of all sets $a \in V$ such that $TC(\{a\}) \subseteq PD_0$. Obviously, HPD_0 is a transitive subset of PD_0. Hence it suffices to show that $PD_0 \subseteq HPD_0$ (implying that $PD_0 = HPD_0$). For this we prove the following by a simultaneous induction on the complexity of t and φ:

(1) $\|t\|_v \in HPD_0$ if t is a term of $\mathcal{L}_{PZF}$, and v is an assignment in HPD_0.
(2) $\|\varphi\|_v^X \subseteq HPD_0$ in case $\varphi \succ_{PZF} X$, and v is an assignment in HPD_0 (Equivalently: if $\varphi \succ_{PZF} X$, $v \models \varphi$, and $v(x) \in HPD_0$ for $x \notin X$, then $v(x) \in HPD_0$ also for $x \in X$).

- The case where t is a variable is trivial.
- Suppose t is $\{x \mid \varphi\}$. Then $\|t\|_v \in PD_0$ by Lemma 2. Obviously $a \in \|t\|_v$ iff $a \in \|\varphi\|_v^{\{x\}}$. Hence $\|t\|_v \subseteq HPD_0$ by the I.H. for φ. It follows that $\|t\|_v \in HPD_0$.
- The cases where $\varphi \succ_{PZF} \emptyset$ and $X = \emptyset$, or φ is $x \neq x$ and $X = \{x\}$ are trivial.
- Suppose φ is $x \in t$ where $x \notin Fv(t)$, and $X = \{x\}$. Then $\|\varphi\|_v^X = \|t\|_v$. Hence $\|\varphi\|_v^X \subseteq HPD_0$ by the I.H. concerning t and the transitivity of HPD_0.
- Suppose φ is $x = t$ (or $t = x$) where $x \notin Fv(t)$, and $X = \{x\}$. Then $\|\varphi\|_v^X = \{\|t\|_v\}$. Hence $\|\varphi\|_v^X \subseteq HPD_0$ by the I.H. concerning t.
- Suppose φ is $\varphi_1 \vee \varphi_2$, where $\varphi_1 \succ_{PZF} X$ and $\varphi_2 \succ_{PZF} X$. Then $\|\varphi\|_v^X = \|\varphi_1\|_v^X \cup \|\varphi_2\|_v^X$. Hence $\|\varphi\|_v^X \subseteq HPD_0$ by the I.H. concerning φ_1 and φ_2.
- Suppose φ is $\varphi_1 \wedge \varphi_2$, where $\varphi_1 \succ_{PZF} Y$, $\varphi_2 \succ_{PZF} Z$, $X = Y \cup Z$, and $Z \cap Fv(\varphi_1) = \emptyset$. To prove the claim for φ and X, it suffices to show that if $v' \models \varphi$, and $v'(w) \in HPD_0$ in case $w \notin X$, then $v'(x) \in HPD_0$ also for

$x \in X$. So let v' be such an assignment. Then $v' \models \varphi_1$ and $v' \models \varphi_2$. Let v_1 be any assignment such that $v_1(x) = v'(x)$ for $x \notin Z$, and $v_1(x) \in HPD_0$ if $x \in Z$. Since $Z \cap Fv(\varphi_1) = \emptyset$, also $v_1 \models \varphi_1$. By the induction hypothesis concerning φ_1 and Y, this and the fact that $v_1(x) \in HPD_0$ in case $x \notin Y$ together imply that $v_1(x) \in HPD_0$ also in case $x \in Y$. It follows that $v'(x) \in HPD_0$ in case $x \in Y$, and that v_1 is an assignment in HPD_0. Now v' differs from v_1 only for variables in Z. This and the facts that $v' \models \varphi_2$ and $\varphi_2 \succ_{PZF} Z$, together entail that $v'(z) \in \|\varphi_2\|_{v_1}^Z$ for every $z \in Z$. Hence the I.H. for φ_2 implies that $v'(z) \in HPD_0$ in case $z \in Z$. Since we have already shown that $v'(y) \in HPD_0$ in case $y \in Y$, it follows that $v'(x) \in HPD_0$ for every $x \in X$.

- Suppose φ is $\exists z\psi$, where $\psi \succ_{PZF} X \cup \{z\}$. Then $\|\varphi\|_v^X \subseteq \|\psi\|_v^{X\cup\{z\}}$. Hence $\|\varphi\|_v^X \subseteq HPD_0$ by the I.H. concerning ψ.
- Suppose φ is $(TC_{x,y}\psi)(x,y)$, where $\psi \succ_{PZF} X \cup \{y\}$ (say). For $n \geq 0$, let φ_n be $\exists w_1 \ldots \exists w_n.\psi(x, w_1) \wedge \psi(w_1, w_2) \wedge \ldots \wedge \psi(w_{n-1}, w_n) \wedge \psi(w_n, y)$ (where $w_1, \ldots, w_n$ are distinct variables not occurring in φ). Then $\|\varphi\|_v^X = \bigcup_{n\geq 0} \|\varphi_n\|_v^X$. Now it is easy to show by induction on n (using the I.H. for ψ and the cases concerning $\wedge$ and $\exists$ already dealt with above) that$\|\varphi_n\|_v^X$ is a subset of HPD_0 for every $n \geq 0$. Hence $\|\varphi\|_v^X \subseteq HPD_0$.

Let now $a \in PD_0$. Then there is a closed term t of $\mathcal{L}_{PZF}$ such that $a = \|t\|$. Hence $a \in HPD_0$ as a special case of (1), and so $a \subseteq PD_0$.

Next we show that PD_0 is a minimal model of PZF.

Definition 5 Let the language $\mathcal{L}_{PZF}^{\mathcal{M}}$ be defined like $\mathcal{L}_{PZF}$, but with the additional constant $\mathcal{M}$. For every term t and formula φ of $\mathcal{L}_{PZF}$ we define in $\mathcal{L}_{PZF}^{\mathcal{M}}$ the corresponding relativization $t_{\mathcal{M}}$ and $\varphi_{\mathcal{M}}$ (respectively):

- $x_{\mathcal{M}} = \{y \in \mathcal{M} \mid y \in x\}$.
- $\{x \mid \varphi\}_{\mathcal{M}} = \{x \mid x \in \mathcal{M} \wedge \varphi_{\mathcal{M}}\}$
- $(sRt)_{\mathcal{M}} = s_{\mathcal{M}} R t_{\mathcal{M}}$ for R in $\{\in, =\}$.
- $(\neg\varphi)_{\mathcal{M}} = \neg\varphi_{\mathcal{M}}$
- $(\varphi * \psi)_{\mathcal{M}} = \varphi_{\mathcal{M}} * \psi_{\mathcal{M}}$ for $*$ in $\{\vee, \wedge\}$.
- $(\exists x\varphi)_{\mathcal{M}} = \exists x.x \in \mathcal{M} \wedge \varphi_{\mathcal{M}}$.
- $((TC_{x,y}\varphi)(s,t))_{\mathcal{M}} = (TC_{x,y} x \in \mathcal{M} \wedge y \in \mathcal{M} \wedge \varphi_{\mathcal{M}})(s_{\mathcal{M}}, t_{\mathcal{M}})$.

Theorem 6 *Suppose the constant $\mathcal{M}$ is interpreted in V as PD_0.*

(1) If t is term of $\mathcal{L}_{PZF}$ and v is an assignment in PD_0 then $\|t_{\mathcal{M}}\|_v = \|t\|_v$.
(2) Suppose that φ is a formula of $\mathcal{L}_{PZF}$ s.t. $Fv(\varphi) = \{y_1, \ldots, y_n, x_1, \ldots, x_k\}$, and $\varphi \succ_{PZF} \{x_1, \ldots, x_k\}$. Then the following is true in V:

$$\forall y_1 \ldots \forall y_n.y_1 \in \mathcal{M} \wedge \ldots \wedge y_n \in \mathcal{M} \rightarrow [\varphi \leftrightarrow (x_1 \in \mathcal{M} \wedge \ldots \wedge x_k \in \mathcal{M} \wedge \varphi_{\mathcal{M}})]$$

Proof: As usual, the proof is by a simultaneous induction on the complexity of t and φ.

- If t is a variable x then $\|t\|_v = \|t_{\mathcal{M}}\|_v$ follows from Theorem 5, because in this case $\|x_{\mathcal{M}}\|_v = \|x\|_v \cap PD_0$, and $\|x\|_v \in PD_0$.
- If t is $\{x \mid \varphi\}$ then the claim for t follows from the I.H. concerning φ.
- If φ is $s \in t$ or $s = t$ then the claim for φ immediately follows from the I.H. concerning t and s.
- If φ is $x \in t$, where $x \notin Fv(t)$, then the claim for φ follows from Lemma 2, Theorem 5, and the I.H. concerning t.
- If φ is $x = t$ or $t = x$, where $x \notin Fv(t)$, then the claim for φ follows from Lemma 2, and the I.H. concerning t.

The proofs of the other cases are similar to those given in the proof of the predicativity of $\mathcal{L}_{PZF}$, and are left for the reader.

Theorem 7 *PD_O is a minimal model of PZF.*

Proof: That PD_O is a model of PZF easily follows from Theorem 5 and Theorem 6. Minimality is obvious from the fact that every element in PD_O is denoted by some closed term of $\mathcal{L}_{PZF}$ (and the absoluteness of the interpretations of these closed terms).

7.2 Ordinals in PD_0

Theorem 8 *If α is an ordinal and $\alpha < \omega^\omega$ then $\alpha \in PD_O$.*

Proof: We prove that for every $n \in N$ there exists a term t_n of PZF such that $Fv(t_n) = \{a\}$, and for every assignment v in V, if $v(a)$ is an ordinal, then $\|t_n\|_v = v(a) + \omega^n$. Obviously, t_0 is $S(a)$ (see subsection 4.1). Assume that t_n has been constructed, and let t_{n+1} be $\bigcup\{y \mid (TC_{a,y} y = t_n)(a, y)\}$. Given v, from the induction hypothesis concerning t_n it follows that $\|t_{n+1}\|_v$ is $\bigcup_{k \in N} v(a) + \omega^n k$. Hence $\|t_{n+1}\|_v = v(a) + \omega^{n+1}$.

Now let s_n be the closed term obtained from t_n by substituting 0 (i.e. $\emptyset$) for a. From what we have proved it follows that $\|s_n\| = \omega^n$. Hence $\omega^n \in PD_O$ for every $n \in N$. Since for every $\alpha < \omega^\omega$ there exists $n \in N$ such that $\alpha \in \omega^n$, the transitivity of PD_O implies that $\alpha \in PD_O$ for every $\alpha < \omega^\omega$.

Theorem 9 *$\rho(a) < \omega^\omega$ for every $a \in PD_O$ (where $\rho(a)$ is the rank of a).*

Proof: We first show the following two facts:

(1) For every term t of $\mathcal{L}_{PZF}$ there exists $n(t) \in N$ such that the following inequality obtains for every assignment v in V:

$$\rho(v(t)) < max\{\rho(v(y)) \mid y \in Fv(t)\} + \omega^{n(t)}$$

(2) Let φ be a formula of $\mathcal{L}_{PZF}$ such that $Fv(\varphi) = X \uplus Y$, and $\varphi \succ_{PZF} X$. Then there exists $n(\varphi) \in N$ for which the following inequality obtains for every assignment v in V such that $v \models \varphi$:

$$max\{\rho(v(x)) \mid x \in X\} < max\{\rho(v(y)) \mid y \in Y\} + \omega^{n(\varphi)}$$

The proof is by a simultaneous induction on the complexity of t and φ:

- If t is a variable we take $n(t) = 0$.
- Suppose t is $\{x \mid \varphi\}$. By the induction hypothesis concerning φ, we can take $n(t) = n(\varphi) + 1$.
- The cases where $\varphi \succ_{PZF} \emptyset$ and $X = \emptyset$, or φ is $x \neq x$ and $X = \{x\}$ are trivial.
- If φ is $x \in t$ or $x = t$ (and $X = \{x\}$) then we take $n(\varphi) = n(t)$.
- Suppose φ is $\varphi_1 \vee \varphi_2$, where $\varphi_1 \succ_{PZF} X$ and $\varphi_2 \succ_{PZF} X$. Take $n(\varphi) = max\{n(\varphi_1), n(\varphi_2)\}$.
- Suppose φ is $\varphi_1 \wedge \varphi_2$, where $\varphi_1 \succ_{PZF} X_1$, $\varphi_2 \succ_{PZF} X_2$, $X = X_1 \cup X_2$, and $X_2 \cap Fv(\varphi_1) = \emptyset$. By induction hypothesis for φ_1:

  $$max\{\rho(v(x)) \mid x \in X_1\} < max\{\rho(v(y)) \mid y \in Y\} + \omega^{n(\varphi_1)}$$

  While by induction hypothesis for φ_2:

  $$max\{\rho(v(x)) \mid x \in X_2\} < max\{\rho(v(y)) \mid y \in Y \cup X_1\} + \omega^{n(\varphi_2)}$$

  Together these two inequalities imply:

  $$max\{\rho(v(x)) \mid x \in X\} < max\{\rho(v(y)) \mid y \in Y\} + \omega^{n(\varphi_1)} + \omega^{n(\varphi_2)}$$

  It follows that we can take $n(\varphi) = max\{n(\varphi_1), n(\varphi_2)\} + 1$.
- Suppose φ is $\exists z \psi$, where $\psi \succ_{PZF} X \cup \{z\}$. Then obviously we can take $n(\varphi) = n(\psi)$.
- Suppose φ is $(TC_{z,y}\psi)(z, y)$, where $\psi \succ_{PZF} X \cup \{z\}$ (say, where possibly $z \in X$), and suppose $v \models \varphi$. Then for some $k \in N$:

  $$v \models \exists w_1 \dots \exists w_n . \psi(z, w_1) \wedge \varphi(w_1, w_2) \wedge \dots \wedge \varphi(w_{n-1}, w_n) \wedge \varphi(w_n, y)$$

  (where $w_1, \dots, w_n$ are distinct variables not occurring in φ). By induction hypothesis for ψ applied k times, this entails:

  $$max\{\rho(v(x)) \mid x \in X\} < max\{\rho(v(y)) \mid y \in Y\} + \omega^{n(\psi)} \cdot k$$

  It follows that we can take $n(\varphi) = n(\psi) + 1$.

This ends the proof of the two facts. Now in case t is a closed term of $\mathcal{L}_{PZF}$ fact (1) implies that $\rho(\|t\|) < \omega^\omega$. From this the theorem is immediate.

Corollary 3 *$\omega^\omega \notin PD_0$.*

Corollary 4 *ω^ω is the set of ordinals in PD_0.*

Corollary 5 *Ordinal addition (+) is not definable by a term of $\mathcal{L}_{PZF}$*

Proof: Had + been definable, so would have been (using TC) multiplication by ω (since such a multiplication is equivalent to a repeated addition of the same ordinal). Again using TC, this would have made the set $\{\omega^n \mid n \in N\}$ definable, and so its union, ω^ω, would have been definable too, in contradiction to the previous corollary.

Note 7 Let $RST\omega$ be the system obtained from RST by adding the constant HF (for hereditarily finite) to its language, together with its defining axioms. A similar analysis to that given above shows that $\omega \cdot 2$ is the set of ordinals which are definable by some closed term of $RST\omega$.

Theorem 10 *Suppose F is a monotonic set operation definable by some term of $\mathcal{L}_{PZF}$. Define a transfinite sequence of operations $F^{(\alpha)}$ by:*

- $F^{(0)}(a) = a$
- $F^{(\alpha+1)}(a) = F(F^{(\alpha)}(a))$
- $F^{(\alpha)}(a) = \bigcup_{\beta<\alpha} F^{(\beta)}(a)$ *in case α is a limit ordinal.*

Than for every $\alpha < \omega^\omega$, $F^{(\alpha)}$ is definable by some term of $\mathcal{L}_{PZF}$.

Proof: The following two facts can easily be shown:

(1) $F^{(\alpha+\beta)} = F^{(\beta)} \circ F^{(\alpha)}$
(2) $F^{(\alpha\cdot\beta)} = (F^{(\alpha)})^{(\beta)}$

Since every ordinal $\alpha < \omega^\omega$ can be obtained from 0, 1, and ω using addition and multiplication, it follows from these two facts that it suffices to prove that $F^{(\omega)}$ is definable whenever F is. So let t be a term of $\mathcal{L}_{PZF}$ which defines F ($Fv(t) = \{a\}$). Then $F^{(\omega)}$ is defined by the term $a \cup \bigcup\{x \mid (TC_{a,x} x = t)(a, x)\}$.

Corollary 6 *If F is a monotonic set operation definable by some term of $\mathcal{L}_{PZF}$, and $a \in PD_0$, then $F^{(\alpha)}(a) \in PD_0$ for every $\alpha < \omega^\omega$.*

Note 8 Theorem 8 is a special case of Corollary 6 (take $F = S$).

Corollary 7 *$J_\alpha \in PD_0$ for every $\alpha < \omega^\omega$.*

Proof: $J_{\alpha+1}$ is obtained from J_α using a finitary inductive definition (it is the closure of J_α under the 9 operations listed in Lemma 1.11 of Chapter VI of [10]). Hence this monotonic operation is defined by a term of $\mathcal{L}_{PZF}$. The claim follows therefore from Corollary 6.

Theorem 11 $PD_0 = J_{\omega^\omega}$

Proof: From Corollary 7 it follows that $J_{\omega^\omega} \subseteq PD_0$.

For the converse, we first prove the following two facts:

(1) For any term t of $\mathcal{L}_{PZF}$ there exists a natural number $n(t)$ and a term t^* of $\mathcal{L}_{RST}$ such that $Fv(t^*) \subseteq Fv(t) \cup \{w\}$ (where $w \notin Fv(t)$)), and the following holds for every ordinal α and valuation v: If $v(x) \in J_\alpha$ for every $x \in Fv(t)$, and $v(w) = J_\beta$ where $\beta \geq \alpha + \omega^{n(t)}$, then $\|t\|_v = \|t^*\|_v$.

(2) Let $X = \{x_1, \ldots, x_n\}$. For any formula φ of $\mathcal{L}_{PZF}$ such that $\varphi \succ_{PZF} X$ and $w \notin Fv(\varphi)$, there exist a natural number $n(\varphi)$ and a formula φ^* of $\mathcal{L}_{RST}$ such that $Fv(\varphi^*) \subseteq Fv(\varphi) \cup \{w\}$, and for every ordinal α and valuation v, if $v(y) \in J_\alpha$ for every $y \in Fv(\varphi) - X$, and $v(w) = J_\beta$ where $\beta \geq \alpha + \omega^{n(\varphi)}$, then $\|\{\langle x_1, \ldots, x_n\rangle \mid \varphi\}\|_v = \|\{\langle x_1, \ldots, x_n\rangle \in J_\beta \mid \varphi^*\}\|_v$.

As usual, the proof of these two facts is by induction on the structure of t and φ, and is similar to the proof of Theorem 9. The only case which is not straightforward is when φ is $(TC_{y,x}\psi)(y, x)$, where $\psi \succ_{PZF} \{x\}$ (for simplicity, we suppress other variables). In this case $n(\varphi) = n(\psi) + 1$, and φ^* is:

$$\exists f \in w \exists n \in N.F(f) \wedge Dom(f) = n+1 \wedge f(0) = y \wedge f(n) = x \wedge \forall k < n.\psi^*(f(k), f(k+1))$$

where $F(f)$ is the Δ_0 formula which says that F is a function.

Suppose now that $a \in PD_0$. Then $a = \|t\|$ for some closed term t of $\mathcal{L}_{PZF}$. By (1) it follows that $a = \|t^*\|_v$, where v is a valuation such that $v(w) = J_{\omega^{n(t)}}$. Since $J_{\omega^{n(t)}} \in J_{\omega^\omega}$, J_{ω^ω} is closed under rudimentary functions, and t^* is a term of $\mathcal{L}_{RST}$ (and so defines a rudimentary function by Corollary 1), $\|t^*\|_v \in J_{\omega^\omega}$. Hence $a \in J_{\omega^\omega}$. It follows that $PD_0 \subseteq J_{\omega^\omega}$.

8 Directions for Further Research

8.1 Strengthening PZF

PZF is a rich set theory, which is sufficient for the goals described in the introduction. Still, it is far from capturing the potential of predicative set theory. Thus although ω^n is definable in PZF for each n, and there is an effective procedure to derive a definition of ω^{n+1} from a a definition of ω^n, the set $\{\omega^n \mid n \in N\}$ and the function $\lambda n \in N.\omega^n$ are not definable in $\mathcal{L}_{PZF}$, even though their identity is clearly absolute and predicatively acceptable. There are at least five possible directions to remedy this by extending the definability power of PZF:

New Constants and Autonomous Progressions: A system $RST\omega$ where ω is definable can be obtained from RST by adding to $\mathcal{L}_{RST}$ a constant HF that denotes the set of sets which are defined by terms of RST, and by adding to RST appropriate closure axioms concerning this new constant. Similarly, it is not difficult to show that by adding to $\mathcal{L}_{PZF}$ a constant denoting J_{ω^ω} with appropriate closure axioms, we get a system in which it is easy to construct closed terms for $\lambda n \in N.\omega^n$ and for ω^ω, and prove their main properties. Obviously this process can be repeated using transfinite recursion, creating by this a transfinite progression of languages and theories. To do so, we need first of all to precisely define the process of passing from a theory $\mathbf{T}_\alpha$ to $\mathbf{T}_{\alpha+1}$, and of constructing $\mathbf{T}_\alpha$ for limit α. Moreover, like in the systems for predicative analysis of Feferman and Schütte (see [15,44]), the progression should be autonomous, in the sense that only ordinals justified in previous systems may be used. Now instead of using indirect systems of (numerical) notations for ordinals, it would be much more natural to use terms of our systems which provably denote in them *von Neumann's ordinals*. We expect that every ordinal less than Γ_0, the Feferman-Schütte ordinal for predicativity ([15,17,44,45]), should be obtainable in this way.

Decoding: Although $\{\omega^n \mid n \in N\}$ and $\lambda n \in N.\omega^n$ are not definable in PZF, $\{\ulcorner\omega^n\urcorner \mid n \in N\}$ and $\lambda n \in N.\ulcorner\omega^n\urcorner$ *are* definable, where $\ulcorner\omega^n\urcorner$ is some natural Gödel code in HF for the term of $\mathcal{L}_{PZF}$ that defines ω^n. Now there should exist predicatively acceptable methods for passing from, say, $\{\ulcorner\omega^n\urcorner \mid n \in N\}$ to $\{\omega^n \mid n \in N\}$, and the language and proof system of PZF might be extended using these methods.

Dynamic Safety Relations: The safety relations we used in our 3 basic systems are all *static*, and are prior to the proof system. More power can be gained by allowing dynamic connections between safety and provability. Thus Δ-comprehension is equivalent to the following dynamic condition: $\exists y\varphi(y) \succ \emptyset$ in case $\varphi(y) \succ \emptyset, \psi(z) \succ \emptyset$, and $\vdash_{PZF} \exists y\varphi(y) \leftrightarrow \forall z\psi(z)$.

Inductive Definitions: The use of TC makes it possible to provide inductive definitions of relations and functions which are *sets*. In certain cases it also allows for defining global relations (using formulas of the language). However, its use is quite limited for inductively defining global operations Take e.g. the ternary operation $G(n, k, a) = a + \omega^n \cdot (k + 1)$ (where $n, k \in N$). G can be inductively defined as follows: $G(0, 0, a) = a \cup \{a\}$, $G(n+1, 0, a) = \bigcup_{k\in N} G(n, k, a)$, $G(n+1, k+1, a) = G(n+1, 0, G(n+1, k, a))$. Intuitively, G should therefore be a predicatively acceptable operation. However, it is not definable in $\mathcal{L}_{PZF}$ by a term $t(n, k, a)$. Another possible direction for extending the power of $\mathcal{L}_{PZF}$ is therefore to allow stronger methods of inductive definitions over the natural numbers, as well as predicatively accepted transfinite recursion.

Introducing Classes Introducing global operations might be done by allowing terms for classes (of the form $[x \mid \varphi]$ where $\varphi \succ_{PZF} \emptyset$).

8.2 Other Directions

A necessary direction of research is to determine the relations of our framework and systems with previous works concerned with predicative set theory. This includes first of all Feferman's various systems for predicative mathematics, especially his system PS_1E for predicative set theory ([16,18]), and his system W from [20]. Also relevant are the proof-theoretic investigations of systems of Kripke-Platek set theory by Jäger, Pohlers, and Rathjen (a partial list), as well as the works on constructive set theory by Aczel, Beeson, Friedman, Gambino, Rathjen, and many others. Another work that seems closely related is Weaver's recent work (officially unpublished) on predicative mathematics.

Beyond this, a major future project should be to produce concrete formal systems within the framework of PZF (based on valid, sufficiently strong formal systems for TC-logics), to determine their proof-theoretical strength, and to actually developed large portions of classical mathematics in them.

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