Tonk- A Full Mathematical Solution

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1 Introduction: the Tonk Problem

There is a long tradition (See e.g. [9, 10]) starting from [12], according to which the meaning of a connective is determined by the introduction and elimination rules which are associated with it. The supporters of this thesis usually have in mind natural deduction systems of a certain ideal type (explained in Section 3 below). Unfortunately, already the handling of classical negation requires rules which are not of that type. This problem can be solved in the framework of multiple-conclusion Gentzen-type systems (also first introduced in [12]), where instead of introduction and elimination rules there are left introduction rules and right introduction rules.

The thesis according to which the meaning of a connective is given by its introduction (and elimination) rules was strongly challenged by Prior in [13]. In that paper he introduced his famous "connective" Tonk (denoted below by T). This "connective" has two rules of the ideal type. The introduction rule allows to infer $\varphi T \psi$ from φ. The elimination rule allows to infer ψ from $\varphi T \psi$. In the presence of Tonk every formula can be derived from any other formula, making trivial the "logic" which is "defined" by any system which includes this "connective".

Prior's paper has made it clear that not every combination of "ideal" introduction and elimination rules can be used for defining a connective. Some constraints should be imposed on the set of rules. Such a constraint was indeed suggested by Belnap in his famous [6]: the rules for a connective $\diamond$ should be *conservative*, in the sense that if $\mathcal{T} \vdash \varphi$ is derivable using them, and $\diamond$ does not occur in $\mathcal{T} \cup \varphi$, then $\mathcal{T} \vdash \varphi$ can also be derived without using the rules for $\diamond$. This solution to the Tonk problem has at least two problematic aspects:

1. Belnap did not provide any effective necessary and sufficient criterion for checking whether a given set of rules is conservative in the above sense. Without such criterion every connective defined by inference rules (without an independent denotational semantics) is suspected of being a Tonk-like connective, and should not be used until a proof is given that it is "innocent".

2. Belnap formulated the condition of conservativity only with respect to the basic deduction framework, in which no connectives are assumed. But nothing in what he wrote excludes the possibility of a system G having two connectives, each of them "defined" by a set of rules which is conservative over the basic system B, while G itself is not conservative over B. If this happens then it will follow from Belnap's thesis that each of the two connectives is well-defined and meaningful, but they cannot exist together. Such a situation is no less paradoxical than that described by Prior. In order to prevent it one should demand a much stronger conservativity condition than that actually given by Belnap, and it might not be clear even how to *formulate* this stronger condition. Demanding conservativity of the rules for $\diamond$ over *any* system not involving $\diamond$ seems too strong, and in any case totally useless: how can one prove, for example, that even the standard rules for conjunction are conservative over *any* system not involving conjunction? (indeed they are *not* always conservative - and there are plenty of examples of this phenomenon in relevance logics and other logics).

Later attempts of solutions of the Tonk problem insisted on closer connections between the introduction and the elimination rules for a given connectives than those implicit in Belnap's condition of conservativity. Usually it is demanded that the introduction and elimination rules should precisely match (see e.g. [10, 9]) in the sense that the elimination rules should be derived from the introduction rules by some syntactic procedure. In the words of Hodges in [9] (following Prawitz and others): "the elimination rules only allow us to infer from a formula what we had to know in order to introduce the formula". Other (e.g. [7]) emphasize *invertible* rules (either introduction rules or elimination rules) as those defining meaningful connectives. The discussion is still very much alive, but what is common to almost all the works on the subject is a purely syntactic approach to the question what sets of rules define the meaning of connectives. Needless to say, such works may eliminate the immediate threat of the Tonk argument, but cannot persuade someone who does not understand in what sense can rules define meaning of connectives.[1]

In this paper we suggest a completely different solution for the Tonk problem which will settle it (so we believe) once and for all. Our approach is to translate the debatable philosophical thesis of Gentzen (which Prior tried to refute) into a precise, unquestionable *mathematical theorem*. Our results can (imprecisely) be summarized as follow: *Every acceptable set of rules of the ideal type*[2] *determines the meanings of the connectives involved, where the meaning is given in terms of effective conventional denotational semantics (that generalizes in a straightforward way the usual semantics of the classical and intuitionistic logics).* To show this we need first to give precise mathematical definitions of the relevant concepts. This is done in sections 2–4. Then we introduce our algebraic semantic framework and formulate the connection between rules and corresponding semantics in terms of general soundness and completeness theorems. The main idea behind our semantics is to allow variable degrees of *non-determinism* in assigning truth-values to formulas. This idea makes it possible to develop denotational semantics for rules in a modular way.

Two comments before we turn to the technical part:

1. We treat here two frameworks for rules: natural deduction systems, and Gentzen-type systems. For the latter we again consider two types: multiple-conclusion (of which the well-known system for classical logic is paradigmatic) and single-conclusion (of which the well-known system for intuitionistic logic is paradigmatic). Now the problem of Tonk is usually considered in the framework of natural deduction, but the framework of single-conclusion Gentzen-type systems is completely equivalent, and its use allows for greater uniformity in treating multiple-conclusion logics and single-conclusion ones. Hence the emphasis below is on Gentzen-type systems, but our results can easily be translated into the natural deduction framework.

2. The results concerning the multiple-conclusion frameworks have already been presented (with full proofs) in [4, 5]. In contrast, the treatment of the single-conclusion framework is new. In this paper we present only the relevant ideas, concepts, and results, leaving the proofs to a future paper (which will include also results about the validity of the cut-elimination theorem for our various systems)[3].

[1] Some recent work on the Tonk problem can be found in [7, 8, 16, 17].

[2] The technical term "canonical rule" is used below for "rule of the ideal type".

[3] Again in the multiple-conclusion case such results have already been proved in [4, 5].)

2 What is a Propositional Logic?

In what follows $\mathcal{L}$ is a *countable* [4] propositional language, $\mathcal{F}$ is its set of wffs, p, q, r denote atomic formulas, ψ, φ, θ denote arbitrary formulas (of $\mathcal{L}$), T, S denote sets of formulas, and $\Gamma, \Delta, \Sigma, \Pi$ denote finite sets of formulas. $Fv(X)$ denotes the set of atomic formulas occurring in X. We assume that the atomic formulas of $\mathcal{L}$ are $p_1, p_2, \ldots$ (in particular: $\{p_1, p_2, \ldots, p_n\}$ are the first n atomic formulas of $\mathcal{L}$).

Definition 1

1. A *Tarskian consequence relation* (*tcr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ and formulas of $\mathcal{L}$ that satisfies the following conditions:

   strong reflexivity: if $\varphi \in T$ then $T \vdash \varphi$.
   monotonicity: if $T \vdash \varphi$ and $T \subseteq T'$ then $T' \vdash \varphi$.
   Transitivity (*cut*): if $T \vdash \psi$ and $T, \psi \vdash \varphi$ then $T \vdash \varphi$.

2. [14, 15] A *Scott consequence relation* (*scr* for short) for $\mathcal{L}$ is a binary relation $\vdash$ between sets of formulas of $\mathcal{L}$ that satisfies the following conditions:

   strong reflexivity: if $T \cap S \neq \emptyset$ then $T \vdash S$.
   monotonicity: if $T \vdash S$ and $T \subseteq T'$, $S \subseteq S'$ then $T' \vdash S'$.
   Transitivity (*cut*): if $T \vdash \psi, S$ and $T, \psi \vdash S$ then $T \vdash S$.

Conventions: Let $\vdash$ be a tcr. Below $\Gamma \vdash \{\varphi\}$ means that $\Gamma \vdash \varphi$. In particular: $\vdash \{\varphi\}$ (or $\vdash \varphi$) means $\emptyset \vdash \varphi$. On the other hand $\Gamma \vdash$ (or $\Gamma \vdash \emptyset$) means that $\Gamma \vdash \varphi$ for every $\varphi \in \mathcal{F}$. Provided that the cardinality of Δ is at most 1, these conventions will allow us to use statements of the form $\Gamma \vdash \Delta$ even in case $\vdash$ is a tcr. As usual, we shall write Γ, φ instead of $\Gamma \cup \{\varphi\}$, and Γ, Γ' instead of $\Gamma \cup \Gamma'$.

Definition 2 An scr or tcr $\vdash$ for $\mathcal{L}$ is *structural* (or substitution-invariant) if for every uniform $\mathcal{L}$-substitution σ and every Γ and Δ, if $\Gamma \vdash \Delta$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$. $\vdash$ is *finitary* if the following condition holds for all $T, S \subseteq \mathcal{F}$: if $T \vdash S$ then there exist finite $\Gamma \subseteq T$ and $\Delta \subseteq S$ such that $\Gamma \vdash \Delta$. $\vdash$ is *consistent* (or *non-trivial*) if there exist non-empty Γ and Δ s.t. $\Gamma \not\vdash \Delta$.

[4] This requirement is not essential, but it is convenient.

The following proposition has been proved in [5]:

Proposition 1 *There are exactly four inconsistent finitary scrs and two inconsistent finitary tcrs in any given language.*

The four inconsistent finitary scrs and the two inconsistent finitary tcrs are trivial. We exclude them therefore from our definition of a *logic*:

Definition 3 A single-conclusion (or Tarskian) propositional *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language and $\vdash$ is a tcr for $\mathcal{L}$ which is structural and consistent. Similarly, a multiple-conclusion (or Scottian) propositional *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language and $\vdash$ is an scr for $\mathcal{L}$ which is structural and consistent. In both cases $\langle \mathcal{L}, \vdash \rangle$ is finitary if $\vdash$ is finitary.

3 What Is a (Canonical) Rule?

An examination of the standard examples of rules used in natural deduction systems or Gentzen-type system shows that an ideal logical rule should be an introduction rule or an elimination rule for one connective (where in the Gentzen-type case by an "elimination rule" we mean an introduction rule in the antecedent), and its formulation should include exactly one occurrence of that connective, while no other occurrences of that connective or any other connective should be mentioned anywhere else in it. Moreover: the rule should be *pure* in the sense of [1] (i.e., there should be no side conditions limiting its application), and its active formulas should be immediate subformulas of its principal formula. The definitions in this section formulate this idea in exact terms.

3.1 The Multiple-conclusion Case

Definition 4

A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas. A sequent is called a *clause* if it consists of atomic formulas only.

Definition 5 ([5])

1. A *canonical rule* is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}/C$, where $m \geq 0$, C is either $\diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ or $\Rightarrow \diamond(p_1, p_2, \ldots, p_n)$

for some connective $\diamond$ of arity n, and for all $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a nonempty clause such that $\Pi_i, \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.[5]

2. An *application* of a canonical rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} \ / \ \diamond(p_1, \ldots, p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^*\}_{1 \leq i \leq m}}{\Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \Delta}$$

   where Π_i^* and Σ_i^* are obtained from Π_i and Σ_i (respectively) by substituting ψ_j for p_j (for all $1 \leq j \leq n$), and Γ, Δ are any finite sets of formulas. An application of a canonical rule with a conclusion of the form $\Rightarrow \diamond(p_1, \ldots, p_n)$ is defined similarly.

Note: While sequents are written in a metalanguage for $\mathcal{L}$ (which includes the extra symbol $\Rightarrow$), a canonical rule is formulated in a meta-metalanguage of $\mathcal{L}$ (which includes another extra symbol: /).

3.2 The Single-conclusion Case

Definition 6 A *positive Horn clause* is a sequent of the form $\Pi \Rightarrow \{q\}$, where q is an atomic formula, and Π is a finite set of atomic formulas. (we shall usually write $\Pi \Rightarrow q$ instead of $\Pi \Rightarrow \{q\}$). A *negative Horn clause* is a sequent of the form $\Pi \Rightarrow$, where Π is a finite set of atomic formulas. A *Horn clause* is either a positive Horn clause or a negative one.

Definition 7 A *single-conclusion sequent* is an expression $\Gamma \Rightarrow \varphi$ where Γ is finite set of formulas, and φ is a formula.

Definition 8

1. A *canonical (logical) introduction 1-rule* is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} / \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$, where $m \geq 0$, $\diamond$ is a connective of arity n, and for all $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a positive Horn clause such that $\Pi_i \cup \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.

2. A *canonical (logical) elimination 1-rule* is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} / \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$, where $m \geq 0$, $\diamond$ is a connective of arity n, and for all $1 \leq i \leq m$, $\Pi_i \Rightarrow \Sigma_i$ is a Horn clause (either positive or negative) such that $\Pi_i \cup \Sigma_i \subseteq \{p_1, p_2, \ldots, p_n\}$.

[5] Recall that $\{p_1, p_2, \ldots, p_n\}$ are the first n atomic formulas.

3. An *application* of the 1-rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ is any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \varphi_i\}_{1\leq i\leq m}}{\Gamma \Rightarrow \diamond(\psi_1, \ldots, \psi_n)}$$

   where Γ is a finite set of formulas, and for all $1 \leq i \leq m$, $\Pi_i^* \Rightarrow \varphi_i$ is obtained from $\Pi_i \Rightarrow \Sigma_i$ by substituting ψ_j for p_j $(1 \leq j \leq n)$.

4. An *application* of the 1-rule $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \varphi_i\}_{1\leq i\leq m}}{\Gamma, \diamond(\psi_1, \ldots, \psi_n) \Rightarrow \theta}$$

   where θ is a formula, Γ is a finite set of formulas, and for all $1 \leq i \leq m$: Π_i^* is obtained from Π_i by substituting ψ_j for p_j $(1 \leq j \leq n)$, and $\varphi_i = \psi_j$ in case $\Sigma_i = \{p_j\}$, $\varphi_i = \theta$ in case Σ_i is empty.

Note. We formulated the definition above in terms of Gentzen-type systems. However, we could have formulated them in terms of natural deduction systems instead. The definition of an application of an introduction rule is defined in this context exactly as above, while an application of an elimination 1-rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \ \diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ is in the context of natural deduction any inference step of the form:

$$\frac{\{\Gamma, \Pi_i^* \Rightarrow \varphi_i\}_{1\leq i\leq m+1}}{\Gamma \Rightarrow \theta}$$

where Γ, θ, Π_i^*, φ_i $(1 \leq i \leq m)$ are as above, while Π_{m+1}^* is empty, and φ_{m+1} is $\diamond(\psi_1, \ldots, \psi_n)$.

3.3 Examples

3.3.1 Canonical Rules for Conjunction

The two usual rules for conjunction are: $\{p_1, p_2 \Rightarrow\ \} \ / \ p_1 \wedge p_2 \Rightarrow$ and $\{\ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\} \ / \ \ \Rightarrow p_1 \wedge p_2$ (denoted by $(\wedge \Rightarrow)$ and $(\Rightarrow \wedge)$, respectively). In the multiple-conclusion context applications of these rules have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

In the single-conclusion context applications of these rules have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta}{\Gamma, \psi \wedge \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \wedge \varphi}$$

In natural deduction systems applications of these rules have the form:

$$\frac{\Gamma, \psi, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \wedge \varphi}{\Gamma \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \wedge \varphi}$$

Note that the above elimination rule can easily be shown to be equivalent to the combination of the two more usual elimination rules for conjunction. Moreover: this is the form obtained when one applies in the case of conjunction the standard general method for deriving the elimination rule for a connective from the introduction rules for that connective.

3.3.2 Canonical Rules for Disjunction

The two usual classical rules for disjunction are: $\{p_1 \Rightarrow, \ p_2 \Rightarrow\} \ / \ p_1 \vee p_2 \Rightarrow$ and $\{\Rightarrow p_1, p_2\} \ / \ \Rightarrow p_1 \vee p_2$ (denoted by $(\vee \Rightarrow)$ and $(\Rightarrow \vee)$, respectively).

In the multiple-conclusion context applications of these rules have the form:

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

In the single-conclusion context applications of the first rule have the form:

$$\frac{\Gamma, \psi \Rightarrow \theta \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \vee \varphi \Rightarrow \theta}$$

In the natural-deduction version applications of the same rule have the form:

$$\frac{\Gamma, \psi \Rightarrow \theta \quad \Gamma, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \vee \varphi}{\Gamma \Rightarrow \theta}$$

In contrast, the second rule is not applicable in the single-conclusion context, since its premise is not a Horn clause. Instead, the following two introduction rules are usually employed in this context: $\{\Rightarrow p_1\} \ / \ \Rightarrow p_1 \vee p_2$ and $\{\Rightarrow p_2\} \ / \ \Rightarrow p_1 \vee p_2$. Applications of these rules have then the form:

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi \vee \varphi} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \vee \varphi}$$

It should be noted that this splitting of the introduction rule for $\vee$ (on the right hand side of a sequent) can be done in the multiple-conclusion case as well. Similarly, the introduction for $\wedge$ on the left hand side of a sequent can be split into two simpler rules by using the same method.

3.3.3 Canonical Rules for Implication

The two usual rules for implication are: $\{\Rightarrow p_1\ ,\ \ p_2 \Rightarrow\}\ /\ p_1 \supset p_2 \Rightarrow$ and $\{p_1 \Rightarrow p_2\}\ /\ \ \Rightarrow p_1 \supset p_2$ (denoted by ($\supset\Rightarrow$) and ($\Rightarrow\supset$), respectively).

In the multiple-conclusion context applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi}$$

In the single-conclusion context applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \supset \varphi \Rightarrow \theta} \qquad \frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \psi \supset \varphi}$$

In the natural-deduction version applications of the first rule have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta \quad \Gamma \Rightarrow \psi \supset \varphi}{\Gamma \Rightarrow \theta}$$

Again this form of the rule is obviously equivalent to the more usual one (from $\Gamma \Rightarrow \psi$ and $\Gamma \Rightarrow \psi \supset \varphi$ infer $\Gamma \Rightarrow \varphi$).

3.3.4 Canonical Rules for Semi-implication

Suppose we introduce a "semi-implication" $\leadsto$ with the following two rules: $\{\Rightarrow p_1\ ,\ \ p_2 \Rightarrow\}\ /\ p_1 \leadsto p_2 \Rightarrow$ and $\{\Rightarrow p_2\}\ /\ \ \Rightarrow p_1 \leadsto p_2$.

In the multiple-conclusion context applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \leadsto \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \leadsto \varphi}$$

In the single-conclusion context applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \leadsto \varphi \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \leadsto \varphi}$$

Again in the natural-deduction version applications of the first rule are equivalent to MP for $\leadsto$ (from $\Gamma \Rightarrow \psi$ and $\Gamma \Rightarrow \psi \leadsto \varphi$ infer $\Gamma \Rightarrow \varphi$).

3.3.5 Canonical Rules for Negation

The two usual classical rules for negation are: $\{\ \Rightarrow p_1\}/\ \neg p_1 \Rightarrow$ and $\{\ p_1 \Rightarrow\ \}\ /\ \ \Rightarrow \neg p_1$ (denoted by $(\neg\Rightarrow)$ and $(\Rightarrow\neg)$, respectively).
In the multiple-conclusion case applications of these rules have the form:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

In the single-conclusion context applications of the first rule have the form:

$$\frac{\Gamma \Rightarrow \psi}{\Gamma, \neg\psi \Rightarrow \theta}$$

In the natural-deduction version applications of the same rule have the form:

$$\frac{\Gamma \Rightarrow \psi \quad \Gamma \Rightarrow \neg\psi}{\Gamma \Rightarrow \theta}$$

In contrast, the second rule is not applicable in the single-conclusion context, since its premise is not a *positive* Horn clause. There is indeed no satisfactory introduction rule for negation in the single-conclusion context. [6]

3.3.6 Canonical Rules for "Tonk"

Prior's two rules for Tonk are: $\{p_2 \Rightarrow\ \}\ /\ p_1 T p_2 \Rightarrow$ and $\{\ \Rightarrow p_1\}\ /\ \ \Rightarrow p_1 T p_2$ (denoted by $(T\Rightarrow)$ and $(\Rightarrow T)$, respectively).

In the multiple-conclusion context applications of these rules have the form:

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi T \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi T \psi}$$

In the single-conclusion context applications of these rules have the form:

$$\frac{\psi, \Gamma \Rightarrow \theta}{\varphi T \psi, \Gamma \Rightarrow \theta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi T \psi}$$

[6] There are two standard solutions to this problem. One is to allow (in the Gentzen-type framework) the use of sequents of the form $\Gamma \Rightarrow$ (in addition to normal single-conclusion sequents of the form $\Gamma \Rightarrow \varphi$). This is somewhat artificial in the single-conclusion context. The other solution is to introduce a propositional constant $\bot$, together with the axioms $\bot, \Gamma \Rightarrow \psi$ for every Γ and ψ, and then to define $\neg\psi$ as $\psi \supset \bot$. Note that the axioms for $\bot$ can be viewed as applications of the canonical 1-elimination rule: $\emptyset/\ \bot\Rightarrow$.

In the natural-deduction version applications of the first rule have the form:

$$\frac{\psi, \Gamma \Rightarrow \theta \quad \Gamma \Rightarrow \varphi T \psi}{\Gamma \Rightarrow \theta}$$

This form of the rule is obviously equivalent to that given by Prior: from $\Gamma \Rightarrow \varphi T \psi$ infer $\Gamma \Rightarrow \psi$.

4 What Sets of Rules Are Acceptable?

We start by defining canonical systems, and the consequence relations associated with them.

Definition 9

1. A multiple-conclusion Gentzen-type system is called *canonical* if its axioms are the sequents of the form $\Gamma \Rightarrow \Delta$ where $\Gamma \cap \Delta \neq \emptyset$, Cut is one of its rules, and each of its other rules is a canonical logical rule.

2. A single-conclusion Gentzen-type system (or an n.d. system) is called *canonical* if its axioms are the sequents of the form $\Gamma \Rightarrow \varphi$ where $\varphi \in \Gamma$, Cut is one of its rules[7], and each of its other rules is either a canonical 1-introduction rule or a canonical 1-elimination rule.

Definition 10 Let **G** be a canonical Gentzen-type system of either sort.

1. The tcr $\vdash^{seq}_{\mathbf{G}}$ between *sequents* is defined by: $S \vdash^{seq}_{\mathbf{G}} s$ (where s is a sequent and S is a set of sequents) if there is a derivation in **G** of s from S.

2. The tcr $\vdash_{\mathbf{G}}$ between *formulas* which is induced by **G** is defined by: $T \vdash_{\mathbf{G}} \varphi$ iff there exists a finite $\Gamma \subseteq T$ such that $\vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \varphi$.

3. If G is multiple-conclusion then the scr $\vdash^{m}_{\mathbf{G}}$ between formulas is defined by: $T \vdash^{m}_{\mathbf{G}} S$ if there exist finite $\Gamma \subseteq T, \Delta \subseteq S$ such that $\vdash^{seq}_{\mathbf{G}} \Gamma \Rightarrow \Delta$.

Proposition 2

1. *If* **G** *is canonical then* $\vdash_{\mathbf{G}}$ *is a structural and finitary tcr. If* G *is also multiple-conclusion then* $\vdash^{m}_{\mathbf{G}}$ *too is structural and finitary.*

2. $T \vdash_{\mathbf{G}} \varphi$ *iff* $\{\Rightarrow \psi \mid \psi \in T\} \vdash^{seq}_{\mathbf{G}} \Rightarrow \varphi$.

[7]This condition is not necessary for natural deduction systems.

The last proposition does not guarantee that every canonical system induces a *logic* (in the sense of Definition 3). For this the system should satisfy one more condition:

Definition 11 A canonical Gentzen-type system **G** (either single-conclusion or multiple-conclusion) is called *coherent* if it satisfies the following condition: If both $S_1/\diamond(p_1, p_2, \ldots, p_n) \Rightarrow$ and $S_2/\ \Rightarrow \diamond(p_1, p_2, \ldots, p_n)$ are rules of **G**, then the set of clauses $S_1 \cup S_2$ is classically inconsistent (and so the empty clause can be derived from it using cuts).

Theorem 1 *Let* **G** *be a canonical Gentzen-type system (either multiple-conclusion or single-conclusion).* $\langle \mathcal{L}, \vdash_{\mathbf{G}} \rangle$ *is a single-conclusion logic (i.e.* $\vdash_{\mathbf{G}}$ *is consistent) iff* **G** *is coherent.*[8]

Note. The last theorem implies that coherence is a minimal demand from any acceptable canonical system. As will be shown in the next section, this minimal condition is in fact sufficient for the existence of an effective denotational semantics. Hence it is not necessary to demand that the introduction and elimination rules of a canonical system should exactly match, or that one of these two sets would be derivable from the other. All is needed is that there will be no conflict between any two rules of the system.

Examples.

- *Any* subset of the set of rules described above for the connectives $\wedge, \vee, \supset, \leadsto$ and $\neg$ induces a coherent canonical system. [9] It follows that each such subset R induces a single-conclusion logic. Note that if R does not include $(\Rightarrow \neg)$ (which can be used only in calculi employing multiple-conclusion sequents) then R induces in fact *two* single-conclusion logics (depending on whether the corresponding canonical system **G** is taken as a calculus of multiple-conclusion sequents or single-conclusion sequents). These two logics are in general different.

[8]Similarly, if **G** is multiple-conclusion then $\langle \mathcal{L}, \vdash^m_{\mathbf{G}} \rangle$ is a multiple-conclusion logic iff **G** is coherent.

[9]For example: for the first two rules for conjunction described in section 3.3.1 we have $S_1 = \{p_1, p_2 \Rightarrow\ \}$, $S_2 = \{\ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$, and $S_1 \cup S_2$ is the classically inconsistent set $\{p_1, p_2 \Rightarrow\ ,\ \ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$ (from which the empty sequent can be derived using two cuts). If the first of these two rules $(\wedge \Rightarrow)$ is split into $\{p_1 \Rightarrow\ \}\ /\ p_1 \wedge p_2 \Rightarrow$ and $\{p_2 \Rightarrow\ \}\ /\ p_1 \wedge p_2 \Rightarrow$, then each of these new rules again forms with $(\Rightarrow \wedge)$ a coherent pair. Thus in the case of the first $S_1 \cup S_2 = \{p_1 \Rightarrow\ ,\ \ \Rightarrow p_1\ ,\ \ \Rightarrow p_2\}$, from which the empty sequent can be derived using a single cut.

Note also that R may include no rule at all for some of the connectives, or only one rule, or just two rules which do *not* perfectly match, like the rules of $\rightsquigarrow$. Still, in all these cases R induces a legitimate logic.

- If the rules of a canonical system **G** include both of the rules for T ("tonk") described in section 3.3.6, then **G** is not coherent: The union of the sets of premises of these two rules is $\{p_2 \Rightarrow\ ,\ \Rightarrow p_1\}$, and this is a classically consistent set of clauses. It follows that such a system **G** *does not define a logic* (and this is "what is wrong with tonk").

5 The Semantics Induced by Canonical Systems

In this section we describe the denotational semantics induced by coherent canonical systems.

5.1 The Multiple-conclusion Case

Definition 12 An $\mathcal{L}$-*valuation* is a function $v : \mathcal{F} \rightarrow \{t, f\}$,

Definition 13 An $\mathcal{L}$-valuation v is a model of a formula φ if $v(\varphi) = t$. It is a model of a theory $\mathcal{T}$ if it is a model of every $\varphi \in \mathcal{T}$.

Definition 12 for itself imposes no restrictions on valuations (in particular: the truth-values a valuation assigns to a complex formula needs not be a function of the truth-values it assigns to atomic formulas). The role of rules is to limit the set of allowed valuations by imposing some constraints on them. The idea is that given a canonical system **G**, only those valuations which respect all the rules of **G** should be taken into account.

Definition 14 Let v be an $\mathcal{L}$-*valuation*, and let σ assigns a formula in $\mathcal{F}$ to every atomic formula. We say that σ satisfies in v a clause $\Pi \Rightarrow \Sigma$ if $v(\sigma(p)) = f$ for some $p \in \Pi$, or $v(\sigma(q)) = t$ for some $q \in \Sigma$.

Definition 15 Let v be an $\mathcal{L}$-*valuation.*

1. v respects a rule of the form $S/\Rightarrow \diamond(p_1, \ldots, p_n)$ if $v(\diamond(\psi_1, \ldots, \psi_n)) = t$ whenever all elements of S are satisfied in v by an assignment σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. [10]

[10] The values of $\sigma(q)$ for $q \notin \{p_1, \ldots, p_n\}$ are immaterial here.

2. v respects a rule of the form $S/\diamond(p_1,\ldots,p_n)\Rightarrow$ if $v(\diamond(\psi_1,\ldots,\psi_n))=f$ whenever all elements of S are satisfied in v by an assignment σ such that $\sigma(p_i)=\psi_i$ $(1\leq i\leq n)$.

3. Let **G** be a canonical Gentzen-type system for $\mathcal{L}$. v is **G**-legal if it respects all the rules of **G**.

Examples.

- A valuation v respects the rule $(\supset\Rightarrow)$ iff $v(\varphi\supset\psi)=f$ whenever $v(\varphi)=t$ and $v(\psi)=f$. It respects the rule $(\Rightarrow\supset)$ iff $v(\varphi\supset\psi)=t$ whenever $v(\varphi)=f$ or $v(\psi)=t$. Note that these two constraints are independent of each other (but do not contradict each other). Together the two rules dictate that v treats $\supset$ according to the classical truth-table for $\supset$.

- A valuation v respects the rule $(\rightsquigarrow\Rightarrow)$ iff $v(\varphi\rightsquigarrow\psi)=f$ whenever $v(\varphi)=t$ and $v(\psi)=f$. It respects the rule $(\Rightarrow\rightsquigarrow)$ iff $v(\varphi\rightsquigarrow\psi)=t$ whenever $v(\psi)=t$. Note that none of these two rules imposes any constraint in case $v(\varphi)=f$ and $v(\psi)=f$. In other words: valuations which respect both rules treat the semi-implication $\rightsquigarrow$ according to the following *non-deterministic matrix* ([5, 2, 3]):

$\rightsquigarrow$	t	f
t	t	f
f	t	$\{t,f\}$

- A valuation v respects $(T\Rightarrow)$ if $v(\varphi T\psi)=f$ whenever $v(\psi)=f$. It respects $(\Rightarrow T)$ if $v(\varphi T\psi)=t$ whenever $v(\varphi)=t$. The two constraints contradict each other in case both $v(\varphi)=t$ and $v(\psi)=f$. This is a semantic explanation why the "connective" T is meaningless.

Theorem 2 ([4, 5]) *Every canonical multiple-conclusion Gentzen-type system* **G** *for $\mathcal{L}$ is strongly sound and complete with respect to the semantics of* **G***-legal valuations. In other words: $\mathcal{T}\vdash_{\mathbf{G}}\varphi$ iff every* **G***-legal model of $\mathcal{T}$ is also a model of φ.*

5.2 The Single-conclusion Case

Definition 16 A generalized *$\mathcal{L}$-frame* is a triple $\mathcal{W}=\langle W,\leq,v\rangle$ such that:

1. $\langle W,\leq\rangle$ is a nonempty partially ordered set.

2. v is a function from $\mathcal{F}$ to the set of persistent functions from W into $\{t, f\}$ (A function $h : W \to \{t, f\}$ is *persistent* if $h(a) = t$ implies that $h(b) = t$ for every $b \in W$ such that $a \leq b$).

Notation: We shall usually write $v(a, \varphi)$ instead of $v(\varphi)(a)$.

Definition 17 A generalized $\mathcal{L}$-frame $\langle W, \leq, v\rangle$ is a model of a formula φ if $v(\varphi) = \lambda a \in W.t$ (i.e.: $v(a, \varphi) = t$ for every $a \in W$). It is a model of a theory $\mathcal{T}$ if it is a model of every $\varphi \in \mathcal{T}$.

Definition 18 Let $\langle W, \leq, v\rangle$ be a generalized $\mathcal{L}$-frame, let $a \in W$, and let σ assigns a formula in $\mathcal{F}$ to every atomic formula. We say that σ satisfies in a a positive Horn clause $\Pi \Rightarrow q$ if for every $b \geq a$, either $v(b, \sigma(p)) = f$ for some $p \in \Pi$, or $v(b, \sigma(q)) = t$. We say that σ satisfies in a a negative Horn clause $\Pi \Rightarrow$ if $v(a, \sigma(p)) = f$ for some $p \in \Pi$.

Note: Because of the persistence condition, σ satisfies in a a positive Horn clause of the form $\Rightarrow q$ iff $v(a, \sigma(q)) = t$.

Definition 19 Let $\mathcal{W} = \langle W, \leq, v\rangle$ be a generalized $\mathcal{L}$-frame.

1. Let r be a 1-introduction rule for the n-ary connective $\diamond$. $\mathcal{W}$ *respects* r if $v(a, \diamond(\psi_1, \ldots, \psi_n)) = t$ whenever all the premises of r are satisfied in a by an assignment σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$. [11] [12]

2. Let r be a 1-elimination rule for the n-ary connective $\diamond$. $\mathcal{W}$ *respects* r if $v(a, \diamond(\psi_1, \ldots, \psi_n)) = f$ whenever there exists $b \geq a$ in which all the premises of r are satisfied by an assignment σ such that $\sigma(p_i) = \psi_i$ $(1 \leq i \leq n)$.

3. Let **G** be a canonical Gentzen-type system for $\mathcal{L}$. $\mathcal{W}$ is **G**-legal if it respects all the rules of **G**.

Examples.

- A generalized $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v\rangle$ respects the rule $(\supset\Rightarrow)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = f$ whenever there exists $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$. $\mathcal{W}$ respects $(\Rightarrow\supset)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = t$ whenever for every $b \geq a$, either $v(b, \varphi) = f$ or $v(b, \psi) = t$. Hence the two rules together impose exactly the well-known Kripke semantics for intuitionistic implication ([11]).

[11] Again the values of $\sigma(q)$ for $q \notin \{p_1, \ldots, p_n\}$ are immaterial here.

[12] Note that since the premises of a 1-introduction rule are all positive, this is equivalent to: for every $b \geq a$, all the premises of R are satisfied in b by σ.

- A generalized $\mathcal{L}$-frame $\mathcal{W} = \langle W, \leq, v \rangle$ respects the rule $(\rightsquigarrow\Rightarrow)$ iff for every $a \in W$, $v(a, \varphi \supset \psi) = f$ whenever there exists $b \geq a$ such that $v(b, \varphi) = t$ and $v(b, \psi) = f$. $\mathcal{W}$ respects $(\Rightarrow\rightsquigarrow)$ iff for every $a \in W$, $v(a, \varphi \rightsquigarrow \psi) = t$ whenever $v(a, \psi) = t$ (recall that this is equivalent to: $v(b, \psi) = t$ for every $b \geq a$). Note that in this case too the two rules for $\rightsquigarrow$ do not always determine the value assigned to $\varphi \rightsquigarrow \psi$: if $v(a, \psi) = f$, and also $v(b, \varphi) = f$ for every $b \geq a$, then $v(a, \varphi \supset \psi)$ is free to be either t or f. So the semantics of this connective is non-deterministic also in the single-conclusion context.
- A valuation v respects $(T \Rightarrow)$ if $v(a, \varphi T \psi) = f$ whenever $v(a, \varphi) = f$. It respects $(\Rightarrow T)$ if $v(a, \varphi T \psi) = t$ whenever $v(a, \psi) = t$. Again the two constraints contradict each other in case both $v(a, \varphi) = f$ and $v(a, \psi) = t$. This is a semantic explanation why the "connective" T is meaningless even in the single-conclusion context.

Theorem 3 *Every canonical single-conclusion system* **G** *is strongly sound and complete with respect to the semantics of* **G***-legal generalized frames (i.e.: $\mathcal{T} \vdash_{\mathbf{G}} \varphi$ iff every* **G***-legal model of $\mathcal{T}$ is also a model of φ).*

6 Applications of the Semantics

The soundness and completeness theorems of the previous section can be used to show the following two theorems (which are valid for multiple-conclusion systems, as well as for single-conclusion systems):

Theorem 4 . *Let* **G** *be a canonical Gentzen type system. Then* **G** *is decidable (and so it is decidable whether $\Gamma \vdash_{\mathbf{G}} \varphi$, where φ is formula and Γ a finite set of formulas).*

Theorem 5 *Let* $\mathbf{G_1}$ *be a coherent canonical Gentzen-type system in a language $\mathcal{L}_1$, and let* $\mathbf{G_2}$ *be a coherent canonical Gentzen-type system in a language $\mathcal{L}_2$. Assume that $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$ by some set of connectives, and that* $\mathbf{G_2}$ *is obtained from* $\mathbf{G_1}$ *by adding to the latter canonical rules for connectives in $\mathcal{L}_2 - \mathcal{L}_1$. Then* $\mathbf{G_2}$ *is a conservative extension of* $\mathbf{G_1}$ *(i.e.: If all formulas in $\mathcal{T} \cup \{\varphi\}$ are in $\mathcal{L}_1$ then $\mathcal{T} \vdash_{\mathbf{G_1}} \varphi$ iff $\mathcal{T} \vdash_{\mathbf{G_2}} \varphi$).*

Note. The last theorem shows that a very strong form of Belnap's conservativity criterion is valid for coherent canonical systems. Hence it provides a full answer to the second objection concerning this criterion which was raised in the Introduction. The first one is met, of course, by our coherence criterion for canonical system, since coherence of a finite set of canonical rules can effectively be checked.

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