Simple Consequence Relations

Arnon Avron
Department of Computer Science
School of Mathematical Sciences
Tel-Aviv University
Tel-Aviv
Israel

Abstract

We provide a general investigation of Logic in which the notion of a simple consequence relation is taken to be fundamental. Our notion is more general than the usual one since we give up monotonicity and use multisets rather than sets. We use our notion for characterizing several known logics (including Linear Logic and non-monotonic logics) and for a general, semantics-independent classification of standard connectives via equations on consequence relations (these include Girard's "multiplicatives" and "additives"). We next investigate the standard methods for uniformly representing consequence relations: Hilbert type, Natural Deduction and Gentzen type. The advantages and disadvantages of using each system and what should be taken as *good* representations in each case (especially from the implementation point of view) are explained. We end by briefly outlining (with examples) some methods for developing non-uniform, but still efficient, representations of consequence relations.

1 Introduction

This paper has two main purposes. From a very general perspective it aims to help clarifying basic questions concerning logics. For example:

- What is a logic?
- What is a formal inference system?
- What are the differences between the usual kinds of formal systems and what is common to them?
- Are these kinds of systems the only possible useful ones?
- What is so special about the usual standard connectives?

The second purpose, intimately connected with the first, is to suggest new methods for representing logics, that might make search for proofs, proof checking and implementation easier.

The practical importance of the above goals is immediately realized when one is trying to design a general framework for implementing logical systems on a computer. This exactly is the aim of the Edinburgh LF system (see [HHP], [AHM]), which is currently under development at the Laboratory for Foundations of Computer Science of the University of Edinburgh. The present paper is the result of an attempt to solve basic problems which were encountered while working on the LF project. Its new ideas and methods have proved to be rather helpful for developing this specific system. We hope, however, that they will be of help for any other effort at the same direction that might be made in the future. Our point of view below is, accordingly, a *practical* one. We do not claim, therefore, to solve here the deep problems that exist concerning the foundations of Logic. Still, The content of this paper is certainly relevant to these problems as well.

The structure of the paper is as follows:

In section 2. we explain our notion of a simple consequence relation (which is more general than the usual one) and why we take it to be the fundamental concept of Logic.

In section 3 we provide examples of consequence relations with various properties.

In section 4. we investigate two natural groups of connectives that can be classified via equations on consequence relations. We then proceed to characterize several propositional calculi (including classical, intuitionistic, linear and relevant) in terms of these general connectives.

Between the abstract consequence relations which are discussed in sections 2–4 and their actual implementations on a computer there is a necessary intermediate level. This is the level of formal systems (or, in Gabbay's terminology: algorithmic proof systems). Section 5. provides the main general theoretical background concerning formal systems which is needed (so we believe) for the task of efficiently implementing current logical systems. We discuss in it the standard methods for *uniformly* representing consequence relations: Hilbert type systems, Natural Deduction and Gentzen type systems. We define all three in precise terms. According to our definitions Hilbert systems are a special case of Natural Deduction systems. The latter are, in turn, a special kind of Gentzen-type systems. The advantages and disadvantages of using each system and what should be taken as a *good* representation in each case are explained. The notion of a *pure* system is found to be crucial in this context, especially from the implementation point of view. Various ways in which systems may fail to be pure are described, with examples, at the end of this section.

In section 6. we briefly outline (with examples) some methods for developing non-uniform, but still efficient, representations of consequence relations in the case where good uniform representations are not available.

Because of the general nature of the ideas discussed below it is very difficult to trace the origin of each of them or to give credit. Nevertheless, I like to mention at least the name of tarski, the papers [Sc1], [Sc2] of Scott (in which the notion of a consequence relation of the type dealt with here and many other important ideas were first introduced), [Ha], [Be] and above all — ([Gen]). Exactly like Hacking in [Ha], I see my paper just as a collection of footnotes to this brilliant work of Gentzen.

2 The Notion of a Consequence Relation

2.1 Axiomatic Systems

Traditionally a "formal system" is understood to include the following components:

1. A formal language L with several syntactic categories, one of which is the category of "well formed formulae " (wff).
2. An effective set of wffs called "axioms".
3. An effective set of rules (called "inference rules") for deriving theorems from the axioms.

The set of "theorems" is usually taken to be the minimal set of wffs which includes all the axioms and is closed under the rules of inference.

Systems of this sort have many names in the literature. Here we shall call them *Axiomatic systems* (or, sometimes: Hilbert-systems for theoremhood). Undoubtly they constitute the most basic kind of formal systems. One can argue that in fact all other, more complicated deduction formalisms reduce to systems of this sort. This is true, though, for *every* recursively-defined system. Take for example the wffs in the propositional calculus. One can regard them as the "theorems" of the axiomatic system in which the "wffs" are strings of symbols, the "axioms" are the propositional variables and the "inference rules"- the usual formation rules.[1] The concept of theoremhood in systems of the above sort is not sufficient, therefore, to characterize the notion of a *Logic.* It is too broad a concept. On the other hand the notion of theoremhood of wffs is, at the same time, also too narrow to characterize what a logical system concerning these wffs is all about.

Let us make our last point clearer with two very simple examples from the domain of 3-valued logic [2]. Consider Kleene's 3-valued logic. It has 3

[1]In some recent systems of typed constructive mathematics (see, e.g., [ML]) this resemblance is taken rather seriously and both "proposition" and "theorem" are taken as (different) "judgements" so that there is no significant difference between possible proofs of these "judgements"!

[2]A many-valued logic is a logic which is specified by providing i) a set A of "truth-values" (like $\{\mathbf{t},\mathbf{f}\}$ in classical logic), ii) a set of operations which are defined on A (and correspond to the primitive connectives of the language of the logic. In classical logic these

"truth-values": 1,0 and -1, of which 1 is taken as the only designated one. The operations corresponding to the usual connectives are: $\neg a = -a, a \vee b = max(a,b), a \wedge b = min(a,b)$. Suppose that L is the language of propositional calculus where the wffs are defined as usual. It is immediate then that *no* wff is a theorem of this logic (i.e. there is no wff that gets a designated value under all assignments). The notion of theoremhood seems to be vacuous for this logic. One might ask therefore in what sense it is a "logic".

On the other hand, consider the case in which we take both 1 and 0 to be designated. It is easy then to see that a wff is a theorem of the new logic iff it is a classical tautology. From the point of view of theoremhood there is no difference between this logic and the classical, two-valued one. But are they really the same? Obviously not, a major difference is, e.g. , that the "new" 3-valued logic is *paraconsistent*: It is possible for inconsistent theory to be non-trivial in this logic. (it is possible, e.g. , for P and $\neg P$ to be both "true" while Q is "false").

2.2 Consequence Relations

Both examples above show that sets of "logical truths" are not enough for characterizing logics. The second example indicates that what is really important is what wffs follow from what theories. Indeed, in modern treatments of logic another concept, that of a *Consequence Relation* (C.R.) [3] is taken as fundamental. Logic might be defined, in fact, as the science of consequence relations. Unfortunately, the notion of a C.R. has in the literature several (similar, but not identical) meanings. We shall define first the one which we are going to use here (which is rather general) and then discuss some possible reasonable variations.

Definition A *multiset Consequence Relation* (C.R.) on a set Σ of formulas is a binary relation $\vdash$ between finite multisets of formulas s.t:

(I) Reflexivity: $A \vdash A$ for every formula A.

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operations are defined by the usual truth-tables), iii) A subset Tr of A of the "designated" truth-values (in classical two-valued logic this is just $\{\mathbf{t}\}$). A sentence is logically valid in such a logic iff it gets a designated value under every possible assignment in A which respects the operations.

[3] See, e.g. [Sc1],[Ur],[Ga],[Ha].

(II) Transitivity, or "Cut": if $?_1 \vdash \Delta_1, A$ and $A, ?_2 \vdash \Delta_2$, then $?_1, ?_2 \vdash \Delta_1, \Delta_2$.

2.3 Remarks and variations

1. We use above the notion of a "multiset". By this we mean "sets" in which the number of times each element occurs is significant, but not the order of the elements. Thus, for example, $[A, A, B] = [A, B, A] \neq [A, B]$. In this example we use $[\cdot]$ to denote a multiset. We shall also use "," for denoting the operation of multiset-union (so $[A, A, B], [A, B] = [A, A, B, A, B]$), and omit the "[ ]" whenever there is no danger of confusion.

   It is more customary to take a C.R. to be a relation between *sets*, rather than multisets. This is undoubtly more intuitive and so preferable wherever possible. There are, however, logics the full understanding of which requires us to make finer distinctions that only the use of multisets enable us to make. Examples that we consider below are: Girard's Linear logic, Relevance logics and the finite valued logics of Lukasiewicz. It is possible, of course, to go one step further and to take a C.R. to be between *sequences* of formulas (as Gentzen himself implicitly did). This, however, will considerably complicate the transitivity condition (II), and the need for it seems to be very rare indeed.

   From now on (unless otherwise stated) when we refer to a "Consequence Relation" without any further qualifications we shall mean a multiset C.R..

2. In most definitions of a C.R. that one can find in the literature (including Scott's original papers) there is a third condition besides the two formulated above. This is the *weakening* condition, according to which if $? \vdash \Delta$ then also (i) $\Theta, ? \vdash \Delta$ and (ii) $? \vdash \Delta, \Theta$. (Sometimes only condition (i) is demanded, especially when one is interested only in a single-conclusioned C.R., in which $? \vdash \Delta$ only if Δ consists of a single formula). Again, this is a very natural restriction. It fails however for any system for non-monotonic reasoning, as well as for some C.R.'s based on Linear logic and Relevance Logics.

In the sequel, a multiset C.R. which can be taken to be between sets and is also closed under weakening will be called *ordinary*.

3. We define a C.R. to be a relation between *finite* (multi)sets. This means that we are assuming *compactness* for all the logics we consider. This rules out many "model-theoretic logics" (see [BF]). A partial excuse for this elimination is that we are primarily interested in formal systems that can be *computerized*, while many model-theoretic logics do not even have an r.e. set of valid sentences. Still, effective rules with infinite number of premises *are* possible. We believe that future research will allow us to cope with this possibility by an appropriate extension of the present framework.

4. The above definition of a C.R should more accurately be taken as a definition of a *simple* C.R.. In [Ga] there is another requirement: *uniformity*. This means that $\vdash$ should be "closed under substitutions". This condition involves the inner structure of wffs. In order to make it precise we should take a C.R. to be a *ternary* relation ? $\vdash_{\vec{x}} \Delta$ where ? and Δ are as before and $\vec{x}$ is a finite set of variables *of the language* (This makes sense if the corresponding syntactic categories of the language include special subcategories of variables for these categories). For example: $\forall x A \vdash_{A,y} A(y/x)$ would intuitively means that for every formula ϕ and any individual term t which is free for x in ϕ, $\phi(t/x)$ follows from $\forall x\phi$. Such a general notion of a C.R. should indeed be used to get full characterizations of logics. Using it requires, however, extending the cut condition to some version of resolution (i.e.: unification should be incorporated), and it involves delicate problems concerning substitutions. We prefer therefore to postpone these problems to a sequel to this paper, and here to treat only simple C.R's.

3 Some Examples of Abstract Consequence Relations

3.1 First-order Logic

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ be formulas of some first-order language L (i.e. they may contain free variables).

Truth: $A_1, \ldots, A_n \vdash_t B_1, \ldots, B_m$ iff every assignment in a first-order structure for L which makes all the A_i's true does the same to one of the B_i's.

Validity: $A_1, \ldots, A_n \vdash_v B_1, \ldots, B_m$ iff in every first-order structure (for L) in which all the A_i's are *valid* so is at least one of the B_i's (by "valid" in a structure we mean: true relative to all assignments).

The above are examples of two important C.R.'s which are frequently associated with first order logic. It is important to note that they are *not* identical— not even in the case $m = 1$. $\forall x p(x)$ follows, for example, from $p(x)$ according to the second, but not according to the first. On the other hand, the classical deduction theorem holds for the first but not for the second. The two consequence relations *are* identical, though, from the point of view of *theoremhood*: $\vdash_v A$ iff $\vdash_t A$. Moreover: if all formulas of ? are closed then ? $\vdash_t A$ iff ? $\vdash_v A$. Experience in the LF project shows, however, that completely different attitudes are required for the implementation of these logics on a machine (see [AHM]).

The single-conclusioned fragments of both C.R.'s defined above can be extended to multiple-conclusioned C.R.'s in more than one interesting way, but we shall not go into the details.

3.2 Propositional Modal Logic

Truth: $A_1, \ldots, A_n \vdash_t B$ iff given a frame and a valuation in that frame, B is *true* in every *world* (of the frame) in which all the A_i's are true.

Validity: $A_1, \ldots, A_n \vdash_v B$ iff given a frame, every valuation which makes all the A_i's valid (in that frame) does the same to B (by "valid" we mean: true in all the worlds).

The above are two important *single-conclusioned* C.R.'s. The situation concerning them is similar to that in the previous case: $A \vdash_v \Box A$ but $A \not\vdash_t \Box A$. The deduction theorem obtains for $\vdash_t$ but not for $\vdash_v$. Again the two C.R.'s are identical as far as *theorems* are concerned [4].

[4] The distinction between the two C.R.'s was crucial for the efficient implementation of both in the LF. See [AHM] for further details (some hints are included in section 6 below).

3.3 Three-Valued Logic

Assume again a propositional language with the connectives $\neg, \vee, \wedge$. Let corresponding operations on the truth-values {-1,0,1} be defined as in section 2.1. We define now 5 different (Multiset-) C.R.'s based on the resulting structure. In these definitions v denotes an assignment of truth values to formulas which respects the operations, $? = A_1, \ldots, A_m$ and $\Delta = B_1, \ldots, B_n$.

Kl: $? \vdash_{Kl} \Delta$ iff for all v, $v(B_i) = 1$ for some i or $v(A_j) \in \{-1, 0\}$ for some j.

Pac: $? \vdash_{Pac} \Delta$ iff for all v, either $v(B_i) \in \{1, 0\}$ for some i or $v(A_j) = -1$ for some j.

Lt: $? \vdash_{Lt} \Delta$ iff for all v, either $v(B_i) = 1$ for some i, or $v(A_j) = -1$ for some j, or $v(B_i) = v(A_j) = 0$ for some i,j.

Sob: $? \vdash_{Sob} \Delta$ iff for all v, either $v(B_i) = 1$ for some i or $v(A_j) = -1$ for some j or $v(A_i) = v(B_j) = 0$ for *all* i, j.

Luk: $? \vdash_{Luk} \Delta$ iff for all v, either $v(B_i) = 1$ for some i or $v(A_j) = -1$ for some j or at least *two* formulas in $?, \Delta$ get 0 (under v).

Notes:

1. The first three of these C.R.s are ordinary. The last two are not (see below).

2. $\vdash_{Kl}$ corresponds to taking 1 as the only designated value, and so it is the obvious C.R. defined by the 3-valued logic of Kleene's (given above as an example of a logic with no logical theorems). It was originally introduced by Kleene for the study of recursive functions, and today it is extensively used, e.g., in the VDM project (see [BCJ] or [Jo]). The standard interpretation of 0 in it is "undefined".

3. $\vdash_{Pac}$ corresponds to taking both 1 and 0 as designated. As noted above, it has the same set of theorems as classical propositional calculus, but it is *paraconsistent* ($P, \neg P \not\vdash_{Pac} Q$). Moreover: it is a *maximal* paraconsistent ordinary C.R. in its language (see [Av4]). As such it might proved to be important for future use of inconsistent knowledge bases.

4. $A_1, \ldots, A_m \vdash_{Lt} B_1, \ldots, B_n$ iff for every v, $v(A_1 \wedge A_2 \wedge \ldots \wedge A_m) \leq v(B_1 \vee B_2 \vee \ldots \vee B_n)$. This C.R. also has no theorems. In fact, if $? \vdash_{Lt} \Delta$ then both ? and Δ are non-empty.

5. $A_1, \ldots, A_n \vdash_{Sob} B$ iff $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$ is valid when $\rightarrow$ is defined as in Sobociński 3-valued logic [5]. Moreover, $A_1, \ldots A_m \vdash_{Sob} B_1, \ldots B_n$ iff $\vdash_{RM_3} A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_m \rightarrow (B_1 + \ldots + B_n)) \ldots)$, where $A + B \stackrel{\text{def}}{=} \neg A \rightarrow B$ and RM_3 is the strongest in the family of logics created by the Relevantists school [6]. *This C.R. is not monotonic:* weakening fails for it on both sides (this is our first example of this sort!). It is maximally paraconsistent as well.

6. $A_1 \ldots, A_n \vdash_{Luk} B$ iff $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$ is valid in Lukasiewicz three-valued logic [7] (using negation, it is easy to give a corresponding interpretation for *every* sequent). Its main property is that *contraction fails for it* (on both sides). It is therefore completely necessary to understand it as a *multiset* C.R..

   It is intersting to note that Lukasiewicz' 3-valued logic is equivalent, in a certain sense, to the *full* 3-valued logic used in the VDM project (including the non-monotonic operator Δ—see [BCJ] or [Jo]). See [Av5] for more details.

7. The classical C.R. (our first example) can also be characterized in the present framework by : $? \vdash \Delta$ iff for every v either $v(A_i) = -1$ or $v(B_j) = 1$ or at least one formula in $? \cup \Delta$ gets 0 (the proof of this claim uses known proof-theoretical reductions). Note that this C.R. is *not* the union of $\vdash_{Kl}$ and $\vdash_{Pac}$. Take, for example, $? = \{C \vee \neg P, P \vee R\}, \Delta = \{C, S, R \wedge \neg S\}$. Classically $? \vdash \Delta$, but this is not the case if we interpret $\vdash$ as either $\vdash_{Pac}$ or $\vdash_{Kl}$!

[5]This logic was introduced in [Sob]. It has 0 and 1 as designated and $a \rightarrow b$ is defined in it as $\max(b, -a)$ in case $a \leq b$, $\min(b, -a)$ otherwise (see also [AB], pp. 148-9).

[6]It is obtained from Sobociński's logic by adding to it the connectives $\wedge, \vee$ as defined above. See [AB] or [Du] for more details.

[7]In this logic 1 is the only designated value, while $a \rightarrow b$ is defined as 1 in case $a \leq b$ and b otherwise.

All the consequence relations described above were defined in abstract terms, using semantics. It is possible, of course, to define a C.R. directly in syntactical terms. Historically, for example, the two kinds of modal C.R. described above were (implicitly) in use long before their semantic description was known!

4 Classification of C.R. according to their Basic Connectives

This section has two related objects. The first is to classify certain important connectives via equations on the consequence relations in which they are used. The second is to provide a syntactic characterization of some propositional (multiset-) C.R.'s in terms of the connectives which are definable in them [8]. We examine for these purposes two classes of connectives that are common to many logics and see what general rules they should obey. The rules we find are quite familiar, but the fact that they are common to many different logics is quite significant for the task of constructing general systems for implementing logics [9].

4.1 Internal connectives

[10] Given C.R., an internal connective relative to it is one that makes it possible to transform a given sequent to an equivalent one that has a special required form. By "equivalent" here we mean that one sequent obtains iff the other does (but in most important cases it can be interpreted in a much stronger sense).

In what follows assume $\vdash$ to be a fix C.R.. All notions defined are taken to be relative to $\vdash$.

[8] By a "definable" connective we mean either a primitive connective of the language or one that can be defined as a combination of the primitive connectives using schematic variables (see for example the above definition in Relevance logic of $A + B$ as $\neg A \to B$)

[9] This potential application has not yet been implemented in the LF, but it is soon going to be.

[10] The connectives which we describe in this subsection are closely related to what Girard has called "multiplicatives" in [Gi] and to the "intensional" connectives of Relevance logics.

Internal Disjunction: We call a binary connective $+$ an internal disjunction if for all $?, \Delta, A, B$:

$$? \vdash \Delta, A, B \quad \text{iff} \quad ? \vdash \Delta, A + B \ .$$

Internal Conjunction: We call a binary connective $\circ$ an internal conjunction if for all $?, \Delta, A, B$:

$$?, A, B \vdash \Delta \quad \text{iff} \quad ?, A \circ B \vdash \Delta \ .$$

Internal Negation: We call a unary connective $\neg$ a right internal negation if for all $?, \Delta, A$:

$$?, A \vdash \Delta \quad \text{iff} \quad ? \vdash \Delta, \neg A \ .$$

We call a unary connective $\neg$ a left internal negation if for all $?, \Delta, A$:

$$? \vdash \Delta, A \quad \text{iff} \quad ?, \neg A \vdash \Delta \ .$$

Since it can be shown (see the proposition in the next subsection) that $\neg$ is a right internal negation iff it is a left one, we use the term *internal negation* to mean either. It is important to note that such a negation is *necessarily* involutive (i.e., $A \vdash \neg\neg A$ and $\neg\neg A \vdash A$).

Internal Implications: We call a binary connective $\rightarrow$ an internal implication if for all $?, A, B$:

$$?, A \vdash B \quad \text{iff} \quad ? \vdash A \rightarrow B \ .$$

We call $\rightarrow$ a *strong* internal implication if for all $?, \Delta, A, B$:

$$?, A \vdash \Delta, B \quad \text{iff} \quad ? \vdash \Delta, A \rightarrow B \ .$$

Note: We believe that the notion of an implication is intuitively best understood for single-conclusioned sequents (as reflected in deduction theorems). We have defined internal implication accordingly. Strong implication is a natural generalization in the context of multiple-conclusioned C.R.'s [11]. Note, however, that in the presence of an internal negation an internal implication is necessarily a strong one!

[11] The term "strong" reflect the fact that the corresponding condition is strictly stronger than the one we have taken as primary. See below.

Internal Truth: We call a 0-ary connective $\top$ an internal truth if for all $?, \Delta$:

$$? \vdash \Delta \quad \text{iff} \quad \top, ? \vdash \Delta \ .$$

Internal Falsehood: We call a 0-ary connective $-$ an internal falsehood if for all $?, \Delta$:

$$? \vdash \Delta \quad \text{iff} \quad ? \vdash \Delta, - \ .$$

Proposition: The following are immediate consequences of the above definitions:

1. If $\vdash$ has a (primitive or definable) internal disjunction (conjunction) then any sequent $? \vdash \Delta$ such that Δ ($?$) is not empty is equivalent to a sequent of the form: $? \vdash A$ ($A \vdash \Delta$). If $\vdash$ has also an internal falsehood (truth) then this is true for *any* sequent.

2. If $\vdash$ has internal disjunction, implication and falsehood then for all $?, \Delta$ there is A such that $? \vdash \Delta$ is equivalent to $\vdash A$. Problems concerning such consequence relations can therefore be reduced to problems about *theoremhood* of formulas.

3. If $\vdash$ has a right (left) internal negation then any sequent is equivalent to one in which the left (right) hand side is empty. If in addition it has also an internal disjunction (conjunction) then every non-empty sequent is equivalent to one of the form $\vdash A$ ($A \vdash$).

4. If $\vdash$ has a strong implication then every sequent in which the right-hand side is not empty is equivalent to one with the left-hand side empty.

4.2 Characterizing the Internal Connectives by Gentzen Type Rules

In the previous subsection we have introduced several internal connectives. Their definitions can be split into two rules that are the converse of each other. In each case one of the two rules does not have what Gentzen has called “the subformula property”[12]. In this subsection we describe a uniform

[12] A rule in a calculus of sequents has the subformula property if any formula that occurs in one of its premises is a subformula of some formula that occurs in the conclusion of the rule.

method for deriving rules *with* the subformula property which characterize the same connectives. All the rules we shall find are quite standard in Gentzen-type systems. As an example, we treat the case of strong implication in detail. We then list the rules which are obtained by the same method for the other connectives.

The definition above directly entails that $\rightarrow$ is a strong implication iff $\vdash$ is closed under the rules:

$$\frac{?, A \vdash B, \Delta}{? \vdash A \rightarrow B, \Delta} \quad \frac{? \vdash A \rightarrow B, \Delta}{?, A \vdash B, \Delta} \ .$$

The first of these rules already has the subformula property. The second does not. In order to find an appropriate substitute for it we use the reflexivity and transitivity of $\vdash$. The reflexivity condition, *applied to a formula with* $\rightarrow$ *as the principal connective* yields:

$$A \rightarrow B \vdash A \rightarrow B \ .$$

Hence the second condition above implies that:

$$A, A \rightarrow B \vdash B \ .$$

Taking $A \rightarrow B$ to be the principal formula in the last sequent, we proceed next to eliminate the others using cuts. Suppose, accordingly, that $?_1 \vdash \Delta_1, A$ and $B, ?_2 \vdash \Delta_2$. Then two cuts of these two sequents with the last sequent above result with $?_1, ?_2, A \rightarrow B \vdash \Delta_1, \Delta_2$. We obtain, therefore, that in order for $\rightarrow$ to be a strong implication relative to $\vdash$ this relation should be closed under the rule:

$$\frac{?_1 \vdash \Delta_1, A \qquad B, ?_2 \vdash \Delta_2}{?_1, ?_2, A \rightarrow B \vdash \Delta_1, \Delta_2} \ .$$

Conversely, the closure of $\vdash$ under this rule implies the provability of the sequent $A, A \rightarrow B \vdash B$, and so if $? \vdash A \rightarrow B, \Delta$ then also $?, A \vdash B, \Delta$ (using a cut). This proves the first part of the proposition below.

Proposition:

1. $\rightarrow$ is a strong internal implication iff $\vdash$ is closed under the rules:

$$\frac{?, A \vdash B, \Delta}{? \vdash A \rightarrow B, \Delta} \quad \frac{?_1 \vdash \Delta_1, A \qquad B, ?_2 \vdash \Delta_2}{?_1, ?_2, A \rightarrow B \vdash \Delta_1, \Delta_2} \ .$$

Similarly we can show:

2. $+$ is an internal disjunction iff $\vdash$ is closed under the rules:

$$\frac{? \vdash A, B, \Delta}{? \vdash A + B, \Delta} \qquad \frac{A, ?_1 \vdash \Delta_1 \qquad B, ?_2 \vdash \Delta_2}{?_1, ?_2, A + B \vdash \Delta_1, \Delta_2} \; .$$

3. $\circ$ is an internal conjunction iff $\vdash$ is closed under the rules:

$$\frac{?_1 \vdash \Delta_1, A \qquad ?_2 \vdash \Delta_2, B}{?_1, ?_2 \vdash \Delta_1, \Delta_2, A \circ B} \qquad \frac{?, A, B \vdash \Delta}{?, A \circ B \vdash \Delta} \; .$$

4. $\rightarrow$ is an internal implication iff $\vdash$ is closed under the rules:

$$\frac{?, A \vdash B}{? \vdash A \rightarrow B} \qquad \frac{?_1 \vdash \Delta_1, A \qquad B, ?_2 \vdash \Delta_2}{?_1, ?_2, A \rightarrow B \vdash \Delta_1, \Delta_2} \; .$$

5. $-$ is an internal falsehood iff $\vdash$ is closed under the rules:

$$\frac{? \vdash \Delta}{? \vdash \Delta, -} \qquad - \vdash \; .$$

6. $\top$ is an internal truth iff $\vdash$ is closed under the rules:

$$\vdash \top \qquad \frac{? \vdash \Delta}{\top, ? \vdash \Delta} \; .$$

7. The following conditions are equivalent:

   (a) $\neg$ is an internal negation.

   (b) $\neg$ is a right internal negation.

   (c) $\neg$ is a left internal negation.

   (d) $\vdash$ is closed under the rules:

$$\frac{A, ? \vdash \Delta}{? \vdash \Delta, \neg A} \qquad \frac{? \vdash \Delta, A}{\neg A, ? \vdash \Delta} \; .$$

The following observations are immediate from these characterizations. They are familiar facts about the connectives of classical logic. The foregoing discussion and the examples at the end of this section show, however, that they are not peculiar to classical logic or to truth-functional connectives.

- If $\vdash$ has an internal negation $\neg$ and an internal disjunction $+$ then it also has an internal conjunction and a strong internal implication, defined by $\neg(\neg A + \neg B)$ and $\neg A + B$ respectively. Similar results obtain if $\vdash$ has an internal negation and either an internal conjunction or a strong internal implication.

- If $\vdash$ has a *strong* internal implication $\rightarrow$ and an internal falsehood $-$ then it also has an internal negation, defined by $A \rightarrow -$.

- If $\vdash$ is closed under weakenings and has *theorems* (i.e. formulas A such that $\vdash A$), then each of these theorems is an internal truth. (For having theorems it suffices, e.g. that $\vdash$ has an internal implication, since then $A \rightarrow A$ is a theorem for every A.) If such $\vdash$ has also an internal negation then the negation of each theorem is an internal falsehood.

4.3 The Combining Connectives

Among the rules that we have found for the internal connectives there are rules that take *two* sequents and return a single one. In all of these rules, however, the resulting combination is not reversible: the premises cannot always be recovered from the conclusion. Indeed, one cannot expect to be able to combine *any* two sequents into one which contains exactly the same information as included in the original two. It *is* possible, though, in one important case: When the two sequents to be combined are identical in all formulas except (perhaps) one. The two sequents can then be combined through their exceptional formulas by using a new type of connectives which we shall call *combining* connectives. [13] According to the three possible positions that the exceptional formulas might occupy we get the following principal three:

Combining Conjunction: We call a connective $\wedge$ a combining conjunction iff for all $?, \Delta, A, B$:

$$? \vdash \Delta, A \wedge B \quad \text{iff} \quad ? \vdash \Delta, A \ \text{ and } \ ? \vdash \Delta, B \ .$$

[13] They are usually called "extensional" by the relevantists, while Girard calls them "additive" (see [Gi]).

Combining Disjunction: We call a connective $\vee$ a combining disjunction iff for all $?, \Delta, A, B$:

$$A \vee B, ? \vdash \Delta \quad \text{iff} \quad A, ? \vdash \Delta \text{ and } B, ? \vdash \Delta \ .$$

Combining Implication: We call a connective $\supset$ a combining implication iff for all $?, \Delta, A, B$:

$$A \supset B, ? \vdash \Delta \quad \text{iff} \quad ? \vdash \Delta, A \text{ and } B, ? \vdash \Delta \ .$$

Notes:

1. The choice of these three connectives was guided by tradition. Obviously, there are other possibilities. Thus we could characterize the "Sheffer stroke" by:

$$?, A|B \vdash \Delta \quad \text{iff} \quad ? \vdash \Delta, A \text{ and } ? \vdash \Delta, B \ .$$

   It is not difficult to carry the analysis below to this connective as well. We shall satisfy ourselves, though, with the above three.

2. If an internal negation is available then the existence of one combining connective entails the existence of the rest.

By using exactly the same method we have used for investigating the internal connectives we can show:

Proposition:

1. $\vee$ is a combining disjunction iff $\vdash$ is closed under the rules:

$$\frac{? \vdash \Delta, A}{? \vdash \Delta, A \vee B} \quad \frac{? \vdash \Delta, B}{? \vdash \Delta, A \vee B} \qquad\qquad \frac{?, A \vdash \Delta \quad ?, B \vdash \Delta}{?, A \vee B \vdash \Delta} \ .$$

2. $\wedge$ is a combining conjunction iff $\vdash$ is closed under the rules:

$$\frac{?, A \vdash \Delta}{?, A \wedge B \vdash \Delta} \quad \frac{?, B \vdash \Delta}{?, A \wedge B \vdash \Delta} \qquad\qquad \frac{? \vdash \Delta, A \quad ? \vdash \Delta, B}{? \vdash \Delta, A \wedge B} \ .$$

3. $\supset$ is a combining implication iff $\vdash$ is closed under the rules:

$$\frac{?, A \vdash \Delta}{? \vdash \Delta, A \supset B} \quad \frac{? \vdash \Delta, B}{? \vdash \Delta, A \supset B} \qquad\qquad \frac{? \vdash \Delta, A \quad B, ? \vdash \Delta}{?, A \supset B \vdash \Delta} \; .$$

It is easy to see that relative to an *ordinary* C.R. a connective is a combining disjunction (conjunction, implication) iff it is an internal disjunction (conjunction, strong implication). This is not the case in general, though, as the examples of Linear and Relevance Logics below show.

4.4 Characterization of some Known Logics

As the title suggests, we shall try to use the various notions introduced so far for characterizing several logics (some of them well-known, some others should be). We shall examine each logic according to three criteria:

- Being *regular* or not, where by "regular" we shall mean below that the associated C.R. can be taken to be between *sets*.
- Being monotonic or nonmonotonic.
- The internal and combining connectives which are available in it.

"Multiplicative" Linear Logic: [14] This is the logic which corresponds to the *minimal* (multiset) C.R. which includes all the internal connectives discussed above. If we omit the internal truth and falsehood we get the minimal C.R. which includes the others (the full system is conservative with respect to this fragment). In both versions the system is neither regular nor monotonic.

Propositional Linear Logic (without the "exponentials" and the propositional constants T and F): This corresponds to the minimal consequence relation which contains all the connectives introduced above. Again— it is neither regular nor monotonic. It is important to note that its internal connectives behave quite differently from its combining ones!

[14] see [Gi].

$R_{\underset{\rightarrow}{\sim}}$ -the Intensional Fragment of the Relevance Logic R: [15] This corresponds to the minimal C.R. which contains all the internal connectives (the internal truth and falsehood are again optional) and is *closed under contraction*. It is still neither regular nor monotonic [16].

The purely implicational fragment of this logic was first introduced by Church in [Ch]. It is interesting to note that it is known to correspond to the λI-calculus (see [Ba]) through a suitable version of the Curry-Howard isomorphism. This correspondence is completely similar to that which exists between the implicational fragment of intuitionistic (or minimal) logic and the ordinary λ-calculus.

R^t without Distribution: This corresponds to the minimal C.R. which contains all the connectives which were described above and is closed under contraction. $-$ and $\top$ are again optional.

$RMI_{\underset{\rightarrow}{\sim}}$: [17] This corresponds to the minimal *regular* C.R. which contains internal negation, disjunction, conjunction and implication.

$RM^t_{\underset{\rightarrow}{\sim}}$: [18] This corresponds to the minimal regular C.R. which includes *all* internal connectives described above.

In contrast to Linear Logic and $R_{\underset{\rightarrow}{\sim}}$, $RM^t_{\underset{\rightarrow}{\sim}}$ is *not* a conservative extension of $RMI_{\underset{\rightarrow}{\sim}}$. This means that the addition of the internal truth (or falsehood) to $RMI_{\underset{\rightarrow}{\sim}}$ forces new sequents in the language of this system to obtain. The reason is that the internal constants bring with them part of the power of weakening. This happens to be harmless in

[15] See [AB] or [Du].

[16] A word is in order here of what C.R. associated with $R_{\underset{\rightarrow}{\sim}}$ we have in mind, since there is more than one candidate. The answer is: that which the standard Gentzen type formulation of this system defines. An equivalent definition is: $A_1, \ldots, A_n \vdash_{R_{\underset{\rightarrow}{\sim}}} B_1, \ldots, B_m$ iff $A_1 \rightarrow (A_2 \rightarrow \cdots (A_n \rightarrow B_1 + \cdots + B_m) \ldots)$ is a theorem of this system, where $A + B$ is $\sim A \rightarrow B$ and $(A_n \rightarrow B_1 + \cdots + B_m)$ is $\sim A_n$ in case $m = 0$ (The axioms of the system guarantee that this is indeed a C.R.. In particular: the order of the A_i's and the B_i's is not important.). The last interpretation is the one we have in mind also with respect to the other systems in the Relevance/Linear family.

[17] see [AB] pp. 148-9 and [Av1].

[18] This is the intensional fragment of the system RM^t (see [AB] or [Du]). Without the propositional internal constants this is Sobociński 3-valued logic (see **2.4** above), also called $RMI^1_{\underset{\rightarrow}{\sim}}$ in [Av1].

the context of Linear logic and R, but together with the converse of contraction, (which is available, of course in every regular C.R. but can also be taken as a *very* special case of weakening) it causes sequents like $A, B \Rightarrow A, B$ to be provable. This is true in fact for every regular C.R. which has an internal implication: starting with $(\Rightarrow \top, \top)$, $(A \Rightarrow A)$ and $(B \Rightarrow B)$, two applications of $(\rightarrow\Rightarrow)$ yield $\top \rightarrow A, \top \rightarrow B \Rightarrow A, B$. But since $\top, A \Rightarrow A$ is provable, so are $A \Rightarrow \top \rightarrow A$ and $B \Rightarrow \top \rightarrow B$. Hence two cuts yield $A, B \Rightarrow A, B$ (In order to get a cut-free representation of this system it is necessary to add the following "mingle" rule: from $?_1 \Rightarrow \Delta_1$ and $?_2 \Rightarrow \Delta_2$ infer $?_1, ?_2 \Rightarrow \Delta_1, \Delta_2$)[19].

Classical Propositional Logic: This of course corresponds to the minimal ordinary C.R. which has all the above connectives. Needless to say, there is no difference in it between the combining connectives and the corresponding internal ones.

Intuitionistic Logic: It is a common belief that the main difference between classical and intuitionistic logic is that the latter is single conclusioned while the former is essentially multiple-conclusioned. The origin of this belief is the way in which Gentzen has formulated his sequential version of intuitionistic logic. That version differs from his formalism for classical logic only by limiting sequents to have at most one formula in the succedent. Assuming for a moment that this belief is true, it is straightforward to define for the single-conclusioned case the notions of a regular and an ordinary C.R., the combining connectives and the internal conjunction, implication, truth and falsehood (but not internal negation or strong implication!). It is easy then to see that:

> The usual single-conclusioned C.R. associated with intuitionistic logic is the minimal ordinary single-conclusioned C.R. which has internal implication and falsehood and combining disjunction and conjunction.

We proceed now to offer what we believe to be a deeper analysis, in which we need not treat intuitionistic logic differently from the other

[19]For a proof and for more information about this system see [Av2]. Girard, by the way, is using in [Gi] the name "mix" for this rule.

logics we have considered so far. What we seek therefore is a multiple-conclusioned conservative extension of the above single-conclusioned C.R., which has the same types of basic connectives. The unique solution to this problem is easy to find once we recall that for *ordinary* C.R. a combining disjunction is also an internal one. Accordingly, we define:

> $A_1, \ldots, A_n \vdash_{Int} B_1, \ldots, B_m$ iff there is a proof of $B_1 \vee \cdots \vee B_m$ from $A_1, \ldots, A_n$ in one of the usual formalisms for intuitionistic logic.

It is easy now to see that:

> $\vdash_{Int}$ is the minimal ordinary C.R. which has internal disjunction, conjunction, falsehood and implication. [20]

It is illuminating to compare this to the following possible characterization of classical logic:

> The standard C.R. associated with classical logic is the minimal ordinary C.R. which has internal disjunction, conjunction, falsehood and *strong* implication.

According to these two characterizations classical and intuitionistic logic differ mainly with respect to their *implication* connective, while their disjunctions are the same! Indeed, it is easy to show (using the cut-free multiple-conclusioned Gentzen-type formulations of the two systems) that exactly the same sequents which involve only $\vee, \wedge$ and $-$ are valid in both (note that there are no theorems, i.e. sequents of the form $\vdash A$, among these sequents!). There is however another crucial difference between the two logics that is somewhat hidden in these characterizations: *Intuitionistic logic does not contain any internal negation:* There is no sentence $N(p)$ in the $\{\vee, \wedge, \rightarrow, -\}$-language (in which only the atomic formula p occurs) such that $p, N(p) \vdash$ and $\vdash p, N(p)$ are both valid (This is an immediate consequence of the disjunction property of $\vdash_{Int}$: Since $\vdash p$ is obviously not valid, the validity of $\vdash p, N(p)$ would entail the validity of $\vdash N(p)$. This and the validity of $p, N(p) \vdash$ entail

[20] This characterization corresponds to Maehera's multiple-conclusioned, cut-free Gentzen-type formulation of Intuitionistic logic ([Tak], ch. 1.).

the validity of $p \vdash$ where p is an atomic formula!). This fact might explain why the rules for negation leave something to be desired (compared to the rules for the other connectives) in the context of intuitionistic natural deduction [21], and why authors like Prawitz and Schroeder-Heister prefer to take $-$ rather than negation as a primitive connective of intuitionistic logic. [22]

A final remark concerning implications. From results in [Av1] it easily follows that the minimal *regular* C.R. containing a strong internal implication is strictly stronger than the one which contains just internal implication. In fact, there are *sentences* which are logically valid according to the first but not according to the second. It is not difficult to see, on the other hand, that the purely implicational fragment of *Linear* Logic can be characterized indifferently either as the minimal C.R. which has an internal implication or as the minimal C.R. which has the stronger connective (similar observation applies also to the corresponding fragment of R).

4.5 A Digression: On the Meanings of the Propositional Connectives

There is a long tradition, originated already in Gentzen ([Gen]), about the introduction and elimination rules of Natural Deduction as providing the meanings of the propositional connectives. The famous [Pri] has forced the followers of this tradition to be more careful about this issue. Hence today the emphasis is usually put on the *introduction* rules as those which define the meaning of a connective. Concerning the elimination rules the general principle is that one should not be able to get out of a formula more than the introduction rules can put into it. This principle was used, e.g., by Schroeder-Heister in [SH] for developing an explicit method for *deriving* the (unique) elimination rule for a connective from the corresponding set of introduction rules [23]. Unfortunately, this method does not seem to work beyond the realm of intuitionistic logic. A particularly important connective that seems to escape this type of characterization is the negation. The present paper suggests

[21] The introduction rule for $\neg$, e.g., is the only one in which the introduced connective should occur in one of the premises!

[22] See [Pra2] and [SH].

[23] For more explanations about this tradition—see [Sud2] and the extensive literature cited there.

therefore an alternative method of taking rules as defining the meaning of connectives. It had the following two main properties:

- The meaning of a connective is always something which is *relative* to some C.R..
- What defines a connective is not a set of "introduction rules" but a single rule which is *reversible*.

The reversible rule which defines a connective might introduce it in either the succedent or the antecedent of a sequent. In the first case it usually corresponds to an "introduction" rule of N.D., in the second—to an elimination rule. The combining disjunction, for example, is characterized by what is usually taken as the *elimination* rule for disjunction. In the present context there is no priority, therefore, to introduction rules over elimination rules.

5 Uniform Representations of C.R's

The notion of a C.R., as defined in the first section and exemplified in the second is an *abstract* one. We have seen above several ways, semantical as well as syntactical, of defining or characterizing C.R.'s. However, in order to use a certain abstract C.R. in practice one needs a *concrete* way of *representing* it. This is usually done by using a formal system [24]. There are two basic demands that such representations of a C.R. $\vdash$ should meet. These are:

Faithfulness: If the representation can be used to show that $? \vdash \Delta$ then this is actually the case.

Effectiveness: If someone uses the representation to show that $? \vdash \Delta$ then his success in doing so can mechanically be checked. If we accept Church's thesis then this means that the set of of sequents that can be shown to hold by a given formal representation is an r.e. set.

A third property that we would like an adequate representation of $\vdash$ to have, but can in principle be achieved (by Church's thesis) only if the represented C.R. is r.e. is:

[24] In fact, Hodges ([Ho],p.26) defines a formal system (or a "formal proof calculus") to be "a device for proving sequents in a language L."

Completeness: Whenever $? \vdash \Delta$ the representation can be used to show this.

In an adequate formal representation of a C.R. $\vdash$ it should be possible to express every true fact of the form: $? \vdash \Delta$. Obviously, the most direct way of achieving this goal is to have in the formal language a formal symbol, "$\Rightarrow$", so that the system can show that $? \vdash \Delta$ iff the corresponding *formal* sequent "$? \Rightarrow \Delta$" is derivable in it. In order to unify our treatment we assume the existence of such a formal symbol in all the formal systems we consider below. If officially it does not, then this assumption means that we are considering an *extended* language in which it does, and an *extended* formal system in which the connections between the old one and $\vdash$ are made a part of the formal machinery . For example, an explanation in the metalanguage of the form: "$A_1, \ldots, A_n \vdash B$ iff there is a proof of B from $A_1, \ldots, A_n$" will be translated (in the extended system) to: "From a formal proof of B from $A_1, \ldots, A_n$ infer $A_1, \ldots, A_n \Rightarrow B$". Formulas, formal sequents and proofs will all be objects (usualy of different types) of such an extended formal system. This might look a little bit complex, but it is absolutely necessary for a full representation (that can, e.g. be computerized).

5.1 Using Axiomatic Systems for Representations

The oldest way of representing a C.R. $\vdash$ is by using an axiomatic system (see **2.1**) *which has the same language* (i.e. the same well-formed formulas). Such axiomatic systems are designed, however, to prove *theorems*, and so they can only indirectly be used for representing C.R.'s . There are two main methods for doing this:

The Interpretation Method: One defines a correspondence between sequents of L and sets of wffs of L so that $? \vdash \Delta$ iff at least one of the sentences of the corresponding set (or perhaps all of them) is a theorem of the corresponding axiomatic system. Usually, the corresponding set is just a singleton, and so every sequent is translated into some formula of the language. Such a translation is straightforward if $\vdash$ has the required internal connectives. An interpretation using these connectives has the property that the interpretation of the formal sequent $\Rightarrow A$ is just A. This should not always be the case, though. A famous example in this respect is Gödel's interpretation of classical logic within intuitionistic logic. An example in which the interpretation does not use just singletons may be provided by the intuitionistic pure implicational C.R.. Here $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ iff *for some* i, $A_1 \to (A_2 \to \ldots \to (A_n \to B_i) \ldots)$ is a theorem (of intuitionistic logic).

The Extension Method: Here we say that $A_1, \ldots, A_n \vdash B$ iff B is a theorem of the axiomatic system which is obtained by adding $A_1, \ldots, A_n$ to the axioms of the given one. The resulting $\vdash$ is then a single conclusioned, ordinary C.R. (see 2.3). This second method is the origin and the prototype of what is known as Hilbert-type systems. The next subsection treats this type of systems in detail.

5.2 Hilbert Type Representations

If we uses an axiomatic system AS together with the extension method in order to show that $A_1, \ldots A_n \vdash B$, then we should provide a proof of B in a new axiomatic system $AS + \{A_1, \ldots A_n\}$. Of course, we do not really use an infinite number of new axiomatic systems. We just use one which is, each time, temporarily augmented by a finite number of axioms. In terms of formal sequents what we are doing is best described by the following *single* axiomatic system *for sequents:*

Axioms:

1. $A \Rightarrow A$ for every A.
2. $\Rightarrow A$ whenever A is an axiom of AS.

Rules:

1.

$$\frac{? \Rightarrow A}{\Delta, ? \Rightarrow A} \quad \text{(weakening)}$$

2.

$$\frac{?_1 \Rightarrow A_1 \ldots, ?_n \Rightarrow A_n}{?_1, \ldots ?_n \Rightarrow B}$$

where

$$\frac{A_1, \ldots, A_n}{B}$$

is (an instance of) a rule of AS.

The main fact to note concerning this sequential system is that *all the "activity" is made on the right-hand side of the* $\Rightarrow$. This is, in fact, the main thing that is common to all the usual "Hilbert-type formalisms". Accordingly we define:

Definition: *A Hilbert-type system for consequence* in the language L is an axiomatic system such that:

1. The "formulas" are formal sequents of L.
2. The axioms include $A \Rightarrow A$ for all A. All the other axioms are of the form $\Rightarrow \Delta$.
3. With the possible exception of cut and weakening, the *set* of formulas which appear on the left-hand side of a conclusion of a rule is the union of the *sets* of formulas which appear on the left-hand side of the premises. (We shall call this property the *left-hand side property*).

If we take axioms as rules with 0 premises then Hilbert representations can be characterized as those systems which *have besides the basic reflexivity and transitivity rules only structural rules and/or rules with the left-hand side property.*

Our definition generalizes in three main points from the system which was presented above:

- The system above was single-conclusioned. This is not necessarily the case in general [25].
- The above system was ordinary. In the general case we deal, however, with multisets. We have not demanded, though, that the multiset of formulas on the left-hand side of a conclusion should be the *multiset union* of the l.h.s. of the premises. We require these multisets only to be identical as *sets*. This allows for rules like

$$\frac{? \vdash A \quad ? \vdash B}{? \vdash A \wedge B}$$

[25] An example in which a multiple-conclusioned Hilbert type system is used can be found in [Av6].

(which characterizes the combining conjunction and is important in the context of, e.g., linear and relevance logics).

- The rules of the above system have the following property that rules in general might lack: whenever they allow (for some $\Delta, ?_1, \ldots, ?_n$) the inference of $? \Rightarrow \Delta$ from $?_1 \Rightarrow \Delta_1, \ldots ?_n \Rightarrow \Delta_n$ then for *all* $?'_1 \ldots, ?'_n$ they allow to infer $?'_1, \ldots, ?'_n \Rightarrow \Delta$ from $?'_1 \Rightarrow \Delta_1, \ldots, ?'_n \Rightarrow \Delta_n$. We shall call rules with this property (and also systems which have only such rules) *pure* . They can be represented just by the schema:

$$\frac{\Delta_1 \ldots \Delta_n}{\Delta} \; .$$

(In most known cases Δ and Δ_i are, of course, singletons).

Obviously pure systems are the most frequent type of Hilbert-type systems that one finds in practice. Here is an example of an *impure* system. Take some normal modal logic with the associated *truth* C.R. (see 3.2). A standard Hilbert-type representation will have some axioms and the following two general rules of inference:

$$\frac{?_1 \Rightarrow A \qquad ?_2 \Rightarrow A \rightarrow B}{?_1, ?_2 \Rightarrow B}$$

$$\frac{\Rightarrow A}{\Rightarrow \Box A} \; .$$

The first one (M.P) is pure, the other is not: we cannot add, for example, the formula A to the left-hand-side of its premise and conclusion. Hence this Hilbert-type system is impure (The material-implication $\rightarrow$ of S4, say, is an internal strong implication relative to *this* C.R. It is quite common therfore to represent this C.R. by using the interpretation method of section 5.1).

Remark: It is a common belief that Hilbert-type systems have only to do with *provability*, not with deducibility. [26] The reasons for this belief are partially historical: in the past logicians happened to be interested mainly in proving logical theorems and used for this Hilbert-systems for theoremhood, i.e. axiomatic systems. Another possible reason is that there might exist

[26]see, e.g., [sud1] pp. 134-5.

more than one reasonable way to define a C.R. which is compatible with a given axiomatic system (i.e.: has the same set of *theorems*). The pure one (obtained by the extension method) is always a candidate, but there might be others. We have seen the example of the impure Hilbert-type representations of the truth C.R. in normal modal logics (the pure extension, by the way, corresponds to the *validity* C.R.). We shall encounter more below. What makes this phenomenon possible is that in a Hilbert-type system a proof of a sequent $\Rightarrow \Delta$ always consists of sequents of the same form. This allows for a certain degree of freedom while deciding what *other* sequents should be taken as valid!

5.3 Natural-Deduction Representations

The next type of formal systems that we examine is natural-deduction. We start with a very general definition of this type of systems which closely resembles that given above for Hilbert-type formalisms:

Definition: A Natural-deduction system in the language L is an axiomatic system such that:

1. The formulas are formal sequents of L.
2. $A \Rightarrow A$ is an axiom for each formula A.
3. All the other rules (including 0-premises rules!) have the following property: the set of formulas that appear on the left hand side of their conclusion is a *subset* of the union of the left-hand side of the premises.

If we compare this definition to that of Hilbert-type systems we see that the only difference is that in N.D. (Natural-Deduction) systems we allow certain formulas of the left-hand side of a premise of a rule to be *discharged* from the left-hand side of the conclusion.

The notion of a "pure" system can now be generalized to N.D. systems as follows:

Definition: We call a rule of a N.D. system "pure" if whenever $? \Rightarrow \Delta$ is derivable in it from $?_1 \Rightarrow \Delta_1, \ldots, ?_n \Rightarrow \Delta_n$ there exist sub-multisets

$?'_1, \ldots, ?'_n, \Delta'_1, \ldots, \Delta'_n$ and Δ' of $?_1, \ldots, ?_n, \Delta_1, \ldots, \Delta_n$ and Δ (respectively) such that for every $?''_1, \ldots, ?''_n, \Delta''_1, \ldots, \Delta''_n$ we can infer

$$?''_1, \ldots, ?''_n \Rightarrow \Delta', \Delta''_1, \ldots, \Delta''_n$$

from

$$(?''_1, ?'_1 \Rightarrow \Delta'_1, \Delta''_1), \ldots, (?''_n, ?'_n \Rightarrow \Delta'_n, \Delta''_n) \ .$$

A N.D. system is pure if all its rules are pure. In such systems we can take all the inferences to be applications of rules of the form:

$$\frac{\begin{matrix}[?'_1] \\ \Delta'_1\end{matrix} \quad \cdots \quad \begin{matrix}[?'_n] \\ \Delta'_n\end{matrix}}{\Delta'}$$

(where the $?'_1$ etc. are like in the above definition).

Pure N.D. systems are again the most usual kind of N.D. systems. It is relatively easy to implement them in an economical way and to check proofs in them. In the Edinburgh LF, e.g., it is a simple task to internalize any ordinary, pure, single-conclusioned N.D. system, while impure systems are much more difficult to handle. An example of such an impure N.D. system is Prawitz N.D. system for S4 (see [Pra1]). In this system one can infer $? \Rightarrow \Box A$ from $? \Rightarrow A$ only iff *all* the formulas in $?$ begin with $\Box$, and so this rule is impure [27].

There are important differences that should be noted between pure Hilbert systems and pure N.D. systems. In the process of using pure Hilbert systems one can (and does) write down in a proof-tree of a sequent only the r.h.s of the sequents which participate in that proof. One can then quite easily discover at the end which *sequent* was actually proved, by just looking at the root and the leaves of the tree. One need not know for this what intermediate sequents were proved before the final one was derived. Moreover: one can check each part of the proof separately without needing to examine what happened prior to that part. All these facts fail for N.D. proofs—even pure ones. Here one is *forced to keep track at each stage of a proof of what sequent was actually derived in it.* This is the case regardless of whether formal sequents are employed while implementing the system or other devices are used for this goal. Whatever the method is, the fact remains that N.D. systems are *essentially calculi of sequents.*

[27] This example, as well as many others, are dealt with in [AHM].

Since natural-deduction systems enable us to manipulate sequents and not only formulas, they provide us, within the formal machinery, an access to methods of proofs that necessarily belong to the *meta*-theory in the case of Hilbert-type systems. An obvious example of this ability is the deduction theorem. For many Hilbert-type systems this is an important meta-theorem which is extensively used for *indirectly* showing that something is provable—without actually proving it. In natural deduction systems this method of proof is usually incorporated into the system as one of its rules. The ability to do such things is the main source of power of N.D. systems and their crucial advantage over Hilbert-type systems.

At this point a natural objection may be raised against our definition of natural deduction systems: according to our presentation, every Hilbert-type system is a N.D. system, but not vice versa. This seems to render as pointless the frequent problem of finding a natural-deduction presentation for a C.R. for which a Hilbert-type representation is already known. Something essential seems to be missing: the notion of introduction and elimination rules which [Sud1], for example, takes to be the second major component (besides the possibility of discharging assumptions) of natural deduction systems. It seems quite difficult, though, to find a sufficiently general definition of this notion which will not rule out from the class of N.D. formalisms systems that are taken to be such in the literature. In fact the very first N.D. representation of classical logic in [Gen] has included as axiom the excluded middle, which cannot be characterized in terms of introduction and elimination rules. It really seems that almost only intuitionistic logic, together with a careful choice of the basic connectives, admits single-conclusioned N.D. representation consisting only of matching introduction and elimination rules (Actually, some fragments of Linear and Relevance logics admit such as well, but the corresponding C.R. is not ordinary). As for classical logic, the only way to do so is by using *multiple-conclusioned* N.D. systems, but this certainly is not the *standard* procedure! We conclude, therefore, that the demand for using matching introduction and elimination rules belongs to the methodology of constructing *good* N.D. representations, not to their definition.

The last discussion brings us to the question what makes one formal representation of a C.R. better then another. We suggest the following two criterions:

- ease of finding (and also checking) proofs of sequents.

- usefulness for (constructively) showing significant properties of the represented C.R. (An example of such a property in many logics is Craig's interpolation theorem).

Past experience indicates that in the context of N.D. systems the above two goals are best achieved by first formulating such a system using introduction and (matching) elimination rules. Then using these rules for defining a notion of a "normal proof" with nice properties, and finally proving a "normalization theorem" to the effect that every proof can be converted into a normal proof of the same end sequent. This method seems to be successful especially when the connectives involved are internal [28] and when the represented C.R. can be characterized in terms of them— as was the case in the examples of the last section. The *exact* characterization of the C.R.'s to which this method is applicable is an interesting topic which we are not going to pursue here[29].

A final point to discuss is the status of the cut rule in the context of N.D. (including Hilbert-type) systems. It can easily be checked that the definition we gave above does not exclude cut from being one of their rules. For pure systems the possibility to eliminate it is can be shown rather easily. For impure systems, on the other hand, it might be less easy and should not be taken for granted. Here is an example of a system for which cut-elimination is true but not *completely* trivial: Consider the $\{\neg, \Box\}$ fragment of the N.D. system for S4. It is immediate that in this system we have that $\neg\neg\Box A \vdash \Box A$ and that $\Box A \vdash \Box\Box A$. It is not, at first glance, obvious that $\neg\neg\Box A \vdash \Box\Box A$, since because of the side conditions on the $(\Rightarrow \Box)$ rule, the proofs of these two sequents cannot directly be combined to produce a proof of the third (This is strongly related to the fact that normalization fails for this version of the system- see [Pra1]). The system does admit cut-elimination, though, but this requires at least *some* efforts. It is not inconceivable that more

[28]To a lesser degree—also when they are of the combining type. The resulting N.D. system is usually impure, though, when these combining connectives are not also internal— as is the case in relevance and linear logics.

[29]Some authors, especially those with intuitionistic tendencies, believe the method of introduction and elimination rules, as well as the concept of a normal proof, to be of a crucial philosophical importance. They believe, accordingly, that there are deeper reasons for the success of this method than the above description suggests. The interested reader is referred to the enormous literature on the subject like: [Pra1], [Pra2], [Sud2], [SH] and (of course!) the original paper of Gentzen ([Gen]). Section 4.5 above is also relevant.

complicated impure N.D. systems might offer more serious difficulties while proving cut- elimination, or even that it just may fail for them!

5.4 Gentzen-Type Representations

As we argue above, already pure N.D. systems essentially carry us from proofs of formulas to proofs of sequents. The next obvious step is therefore to take full advantage of the use of sequents by allowing the rules of a system to make significant changes also in the antecedent of a sequent:

Definition: A Gentzen-type representation of a given C.R. $\vdash$ in a language L is an axiomatic system such that:

1. Its formulas are formal sequents of L.
2. A formal sequent $? \Rightarrow \Delta$ is a theorem iff $? \vdash \Delta$.

The concept of a *pure* representation can naturally be extended to Gentzen -type representations as follows:

Definition: We call a rule of a Gentzen-type system *pure* if whenever it allows the inference of $?_0 \Rightarrow \Delta_0$ from $?_1 \Rightarrow \Delta_1, \ldots, ?_n \Rightarrow \Delta_n$ then there are subsets $?'_i, \Delta'_i$ of $?_i, \Delta_i$ (respectively) such that for every $?''_i, \Delta''_i$ we can infer $?''_1, \ldots, ?''_n, ?'_0 \Rightarrow \Delta'_0, \Delta''_1, \ldots, \Delta''_n$ from $?''_i, ?'_i \Rightarrow \Delta'_i, \Delta''_i$ $(i = 1, \ldots, n)$. The systems as a whole is pure if all its rules are.

The definitions of a pure N.D. rule and a Gentzen-type rule are similar. Hence the notation that is frequently used for the former can be generalized to the latter as follows:

$$\frac{\begin{matrix}[?'_1] \\ \Delta'_1\end{matrix} \quad \cdots \quad \begin{matrix}[?'_n] \\ \Delta'_n\end{matrix}}{\begin{matrix}[?'_0] \\ \Delta'_0\end{matrix}}$$

(where the $?'_i, \Delta'_i$ $(0 \leq i \leq n)$ are like in the definition above).

Examples:

1. the standard $\vee-$ elimination rule of intuitionistic (single- conclusioned) N.D. system, usually written as

$$\frac{A \vee B \quad \begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{C}$$

will have in Gentzen-type systems the form:

$$\frac{\begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{\begin{matrix}[A \vee B] \\ C\end{matrix}} \; .$$

And in classical logic just the form: (when ϕ is the empty set):

$$\frac{\begin{matrix}[A] \\ \phi\end{matrix} \quad \begin{matrix}[B] \\ \phi\end{matrix}}{\begin{matrix}[A \vee B] \\ \phi\end{matrix}} \; .$$

The introduction rule in this context will be just:

$$\frac{A, B}{A \vee B} \; .$$

2. The usual antecedent rule for the conjunction in the intuitionistic case is [30]:

$$\frac{\begin{matrix}[A, B] \\ C\end{matrix}}{\begin{matrix}[A \wedge B] \\ C\end{matrix}}$$

In classical logic,on the other hand, we can replace C above with the empty set. For both systems the succedent rule for conjunction is just:

$$\frac{A \quad B}{A \wedge B} \; .$$

[30] The form of this rule resembles that which some rules take in Schroeder-Heister's calculus of higher-order rules. More on the connection between this calculus and Gentzen type calculi can be found in [Av3].

It is an intersting question (the discussion of which we leave to another opportunity) why people are used to write the full sequents involved while doing proofs in Gentzen-type system, while more economical methods are used for N.D. systems. Repeating passive formulas again and again is after all what makes the use of Gentzen-systems tedious, especially if it is done by hand. Now the use of computerized systems should free us of this worry anyway, but it might still be desirable to develop methods to avoid such repetitions in the underlying representations. The above observation concerning representations of pure rules might then be helpful. Another promising recent contribution in this direction is Girard's concepts of *proof-nets* (see [Gi]). The use of tableaux systems can also be regarded as a method of this kind.

A major fact about Gentzen-type systems, even pure ones, is that unlike pure Hilbert-type and N.D. systems, the transitivity (= cut) is never obvious. In the previous types of systems it was an easy consequence of the asymmetry between the roles of antecedents and succedents in their rules. In Gentzen-type systems this asymmetry is abolished, and so the cut-rule should either be taken as an explicit rule (as frequently it must) or else be proved admissible. This explains the fact that cut-elimination which is crucial for *every* formal system, was (and is) investigated only in the context of Gentzen-type systems.

Usually, a Gentzen-type representation of a C.R. has the following form: It has as axioms the reflexivity axioms $A \Rightarrow A$ (sometimes it suffices to take only a subset of these axioms and then derive the rest). The rules are then divided into two groups:

- Structural rules, which operate on the multisets, but do not change the formulas in them (though it may add or delete some). Frequent examples are: weakening, contraction, anticontraction, mingle (from $?_1 \Rightarrow \Delta_1$ and $?_2 \Rightarrow \Delta_2$ infer $?_1, ?_2 \Rightarrow \Delta_1, \Delta_2$) and, of course, cut.

- Rules concerning the constants of the language. Usually they are divided into succedent rules, which specify how a logical constant may be introduced in the succedent of a sequent, and antecedent rules, which do the same for the antecedent.

In order to judge how good a given Gentzen-type presentation of a given C.R. is, we should apply exactly the same criterions as we did for N.D. systems. However, since we allow more in Gentzen-type systems we are usually entitled to demand more. Hence we usually should expect it to be easier to demonstrate the validity of a formal sequent in a Gentzen-type formalism than it is in a corresponding N.D. system. Also we can expect nice proof-theoretic properties to hold. Such properties are usually proved by induction, the major step of which is showing that the various rules preserve the property under discussion. Usually the cut rule is the major obstacle to such a proof. A good example for this is the important subformula property. This property is preserved by all the rules we found as characteristic for the various internal and combining connectives. It is preserved also by all the structural rules considered in the last section—except cut. Hence if we delete cut from the set of rules of each of the systems we discuss in the last section we get formal systems with the subformula property. The cut-elimination theorem (which obtains for each of those systems) mean that these new systems are still representations of C.R.'s and that the C.R.'s they represent are identical to the original ones.

5.5 Three Degrees of Impurity

In the previous subsections a great deal of importance was attached to what we call pure systems. Our purpose in this one is to try to identify the most usual forms that *impurity* may take. We assume, accordingly, that we are dealing with a formal system of one of the types described above. Moreover, we assume that the inference rules leading from sequents to sequents are given by some global rule-schemes of the form described after the definition of a pure Gentzen-type system, but that there are side conditions on the applicability of some of these schemes which make them impure. We shall examine three possible ways in which this might happen:

Level 1: At this level the side conditions are related to the *structure* of the multisets of the side-formulas ($?''_i, \Delta''_i$ above). Examples of such side conditions are:

- Demanding that there be no side formulas. In Hilbert-type systems rules with this condition are usually known as *rules of proof*. The best known example is the necessitation rule in traditional formulations of normal modal logics. Another example is the adjunction rule of many relevance logics (from A and B infer $A \wedge B$) which frequently is taken to be only a rule of proof.

- Demanding, in a multiple-conclusioned C.R., the succedents of the conclusion and the hypotheses of a rule to be singletons. Examples are the rule for the $\Box$ in the Gentzen-type system for the minimal normal modal logic K (see above) and the succedent rule for the intuitionistic implication in the multiple-conclusioned Gentzen-type version which we mention in section 4.4.
- Demanding all the hypotheses of a rule to have *exactly* the same side-formulas. Examples are provided by the rules for the combining connectives in section 4.3.. For logics without contraction or weakening these rules are impure (and this explains why Girard has found (in [Gi]) the "multiplicative" fragment of his logic, which *is* pure, to be better understood than the "additive" one, which is not).

Level 2: Here the applicability of a rule might depend also on the structure of the side-formulas. Examples are:

- The introduction rule for the $\Box$ in Prawitz N.D. system for $S4$. This rule permits the inference of $? \Rightarrow \Box A$ from $? \Rightarrow A$ only if all the formulas in $?$ begin with $\Box$.
- The introduction rule for $\forall$ in the N.D. systems for classical and intuitionistic logics. Here one can infer $? \Rightarrow \forall x A$ from $? \Rightarrow A$ only if x does not occur free in $?$ (This is not the whole story, though, since this rule might become pure in the context of non-simple C.R.'s. There are in fact no problems to deal with *this* kind of impurity in the Edinburgh LF!).

Level 3: Here an applicability of a rule might depend not only on its potential premises, but also *on their proofs.* A possible example may be provided by an attempt to base a N.D. system on the following version of the deduction theorem in classical first-order logic: [31] $? \vdash A \rightarrow B$ iff there is a proof of B from $?, A$ in which no inference of $\forall x C$ from C is made in which C depends on A and x is free in A.

The third level of impurity carries us, in fact, beyond the class of *uniform* systems with which we were dealing so far. The main properties of uniform

[31] See, e.g., in [Men]. This theorem is true for the pure Hilbert type system for *validity* which is presented there.

systems are that they treat exactly one C.R., and that once a sequent was derived in them one can completely forget about *how* it was derived while using it for deriving other sequents. For systems of the third level of impurity this is no longer the case. For each single step in them it might not be enough to find out, once and for all, whether it is valid or not. It might also be necessary either to return to it later for other tests or to record in advance what side conditions that might be important elsewhere it violates. Such systems might therefore be inefficient in the time and space required for proof checking. Moreover, they might create difficult problems for general systems which are designed for implementing a large class of different logics (like the Edinburgh LF). It is advisable, therefore, to try to avoid the use of such systems. In the next section we shall suggest some possible methods for doing this when uniform representations are not available or the available ones are inefficient.

6 Non-Uniform Representations

As we have seen in the previous section, all the standard proof systems are examples of uniform representations of consequence relations. However, if we accept that a formal system is essentially a device for deriving correct sequents (of a C.R. in which we happen to be interested) then there is no reason to limit ourselves to uniform formal systems. I believe that this observation might open the door to a new area of investigations with a wealth of promising possibilities. To demonstrate the potential of non-uniform representations we shall describe now two methods of developing such representations together with examples of their applicability. We hope that other efficient methods will be developed in the future.

6.1 Treating Several Consequence Relations Simultaneously

A major feature of uniform representations is that they treat only a single C.R.. In mathematics it is often much more efficient to simultaneously solve *several* related problems. The problems of representing related consequence relations should not be an exception in this respect. A good example for this is provided by the two C.R.s which we have associated with modal logics in 3.3 . Obviously, there are strong connections between them. On the other hand it is difficult to provide a nice pure representation of either, since each lacks some important properties (which the other has). It is reasonable, therefore, to try to represent them together in one formal system. This really can be done (at least for natural modal systems like $S4, T, K4$ and so on). Examples of rules of the resulting system in the case of $S4$ are:

$$\frac{? \vdash_v A}{? \vdash_v \Box A} \quad \frac{A, ? \vdash_t B, \Delta}{? \vdash_t A \to B, \Delta} \quad \frac{?, A \vdash_v B}{? \vdash_v \Box A \to B} \quad \frac{? \vdash_t A}{? \vdash_v A} \quad \frac{\vdash_v A}{\vdash_t A} .$$

In this system $\vdash_t$ is taken as multiple conclusioned while $\vdash_v$ as single conclusioned. All the impurities of the usual representations are eliminated in the combined one (or—depending on the meaning of "pure" in this context—are reduced to impurities of the first degree - see 5.5.). It is worth noting that suitable variations of this system have actually been used in order to efficiently internalize modal logics in the Edinburgh LF system. More information concerning the problems involved in doing that and how this method helps to solve them in a natural way are described in [AHM].

In [Av7] there is another example of a logic with which *two* principal consequence relations are naturally associated: Girard's Linear Logic. One of these relations is the nonmonotonic linear C.R. which was described in section 4.4 (and is induced by the standard Gentzen-type formulation of the logic). The other is the *ordinary* C.R. which corresponds to its "phase" semantics (see [Gi]). Although the first is the more basic one, it is the second which is easy to internalize in the LF in its present stage of development. The first can then be handled *indirectly*, via a translation into the second.

6.2 Higher-Order Sequents

Uniform representations are axiomatic systems in which the wffs are formal sequents. Past experience shows that it has frequently been useful to extend such a system to a multiple-conclusioned one even if the ultimate interest remained in proving theoremhood of wffs. It is natural therefore to try applying the same process in the present case. This leads us to consider "hypersequents" which are sequents of sequents. Can they be useful? The answer is "yes". The use of hypersequents allow us, for example, to develop cut-free Gentzen-type formulations for $\vdash_{Luk}$ and $\vdash_{Sob}$, the two non-ordinary C.R.'s which were described in section 3.3 (See [Av5] for details).

Another example of this type is provided by the logic LC of Dummet. The corresponding hypersequential calculus can be found in [Av6]. Some other examples may also be found there [32].

It is reasonable, of course, to go further and consider more iterations of the above procedure. The interested reader is referred to [SH], [Do] and [Av3] for examples of cases in which this is done. Indeed, in those papers sequents of arbitrary degree of nesting are fruitfully employed!

7 Conclusion

The work above contains a rather general analysis of logics, formal systems and the relations between them. Still, much work remains to be done. Here is a list of future tasks (some of which were already mentioned above):

- A satisfactory treatment of the subjects of quantification theory, variables and substitutions.
- Generalizing the framework to cope with effective C.R. between possibly infinite (multi)sets.
- Identifying important sources of impurities of rules and systems on the one hand and developing general methods for handling them on the other. For example, the use of two consequence relations in the case of modal logics has an obvious semantical interpretation. Still, the success of the method of introducing an extra C.R. does not seem to depend on this fact. It is rather general, and works, e.g., whenever the impurities are all due to the use of rules of proof (and in many other cases as well).
- Using all these ideas to construct a much more general Logical Framework for implementing logics than the present Edinburgh LF is. Besides its practical importance the existence of such a system might enable us (among other things) to uniformly investigate universal problems which are connected with logical systems, e.g.: their associated search space.

[32] Cut-free Hypersequential calculi were first introduced in [Pot] and, independently, in [Av2].

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