Gentzenizing Schroeder-Heister's Natural extension of natural deduction

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In this paper we provide a Gentzen-type formulation of Schroeder-Heister's system of [1]. This system is important from both philosophical and practical points of view: Its philisophical importance is due to the characterization which it provides for the intuitionistic connectives, while the practical one is due to the fact that its notion of higher-order rules and its method of treating the elimination rules were incorporated into the Edinburgh LF (A general logical framework for implementing logical formalisms on a computer, which was developed in the computer science department of the university of Edinburgh. See [4],[5]). We shall show that the notions of S.H. that are the most difficult to handle (discharge functions and subrules) become redundant in the Gentzen-type version. The complex normalization proof of S.H. in [2] can be replaced therefore by a standard cut-elimination proof. Moreover, the unusual form of some of the elimination rules of S.H. corresponds to natural, standard form of antecedent rules in sequential calculi. We believe also that the sequential presentation sheds new light on the connection between S.H.'s higher-order rules and the intuitionistic implication and on S.H.'s characterization of the intuitionistic connectives.

We assume in what follows an acquaintance with at least the introduction and the first two sections of [1].

1 The system GSH

1.1 The language

As customary while trying to get rid of discharge functions, we start by introducing a new *formal* symbol $\vdash$ into the language (in [1] this symbol is used only in the meta-language):

Formulas:

$$A$$

Rules:

$$R ::= A \mid (R_1, \ldots, R_n \Rightarrow A)$$

Sequents:

$$S ::= \vdash A \mid R_1, \ldots, R_n \vdash A$$

We shall use A, B as syntactic variables for formulas, R for rules, Γ, Δ for finite sets of rules. $\Gamma \Rightarrow A$ just means A in case Γ is empty.

1.2 The pure system:

Logical axioms:

$$A \vdash A$$

Weakening:

$$\frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}$$

($\Rightarrow\vdash$):

$$\frac{\Delta_1, \Gamma_1 \vdash A_1 \quad \ldots \quad \Delta_n, \Gamma_n \vdash A_n}{\Delta_1, \Delta_2, \ldots, \Delta_n, ((\Gamma_1 \Rightarrow A_1), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow A) \vdash A}$$

1.3 General GSH systems

S.H. permits in [1] the addition of various basic "rules" to the basic system. In this way we really get a *family* of systems. Now, for each system in this family we can construct a corresponding ***general GSH-system*** by adding to pure GSH local inference rules as follows: Whenever $(\Gamma_1 \Rightarrow A_1), (\Gamma_2 \Rightarrow A_2), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow A$ is a "basic rule" add to pure GSH the non-logical rule:

$$\frac{\Delta_1, \Gamma_1 \vdash A_1 \quad \ldots \quad \Delta_n, \Gamma_n \vdash A_n}{\Delta_1, \ldots, \Delta_n \vdash A}$$

Note that since we are using sequents rather than formulas in our calculi, we do not need the concept of "discharge functions" which was used by S.H.!

Notation: Let R be the rule $(R_1, \ldots, R_n \Rightarrow A)$. We shall use $\Delta \vdash R$ as an abbreviation for $\Delta, R_1, \ldots R_n \vdash A$ and $\Delta \vdash_{\mathrm{G}} R$ as an abreviation for "$\Delta \vdash R$ is provable".

Note: The notation $\Delta \vdash R$ was used already by S.H..

2 The relations between GSH and S.H. system

A natural translation of a proof-tree in S.H. formalism into a tree of sequents can be obtained, as usual, by replacing each formula A in the original tree by the sequent $\Gamma \vdash A$,where Γ is the set of assumptions (in the original proof) on which A depends. It is immediately seen that any ***direct*** applications of an assumption rule is transformed in this way into an application of ($\Rightarrow\vdash$), while every direct application of a basic rule is transformed into an application of the corresponding non-logical inference rule. In general, however, the resulting tree of sequents is ***not*** a proof tree in GSH, since S.H. permits in his formalism also ***indirect*** applications of a rule, through a direct application of one of its subrules (see definition below). Nevertheless, we shall show that the resulting tree of sequents can be converted into a GSH- proof-tree of the root sequent.

Definition:

1. R is a subrule of itself.
2. If every element of Γ'_i is a subrule of some element of Γ_i $(1 \leq i \leq n)$ then
$$(\Gamma'_1 \Rightarrow B_1), \ldots, (\Gamma'_n \Rightarrow B_n) \Rightarrow A$$
is a subrule of
$$(\Gamma_1 \Rightarrow B_1), \ldots, (\Gamma_n \Rightarrow B_n) \Rightarrow A.$$

At the rest of this section "$\vdash_G$" will refer to some fix general GSH.

Proposition 1: If $\Gamma \vdash_G R$ and every rule in Γ is a subrule of some rule in Δ then $\Delta \vdash_G R$.

Proof: By induction on the length of the proof of the given sequent. The base case requires the use of weakening. The other cases follow easily from the induction hypothesis.

Corollary: If R' is a subrule of R then $R \vdash_G R'$.

Proof: By induction on the complexity of R it is easy to prove that $R \vdash_G R$. The corollary is then proved by applying the proposition to this sequent.

Theorem 1: $\Gamma \vdash$ A in a S.H.-system iff $\Gamma \vdash_G$ A in the corresponding GSH.

(Note that "$\Gamma \vdash$ A" is a metaproposition for S.H. systems, while it is a formal assertion in the corresponding GSH. "$\Gamma \vdash_G$ A" is again a metaproposition for GSH).

Proof: The implication from right to left is proved by induction. The proof uses the fact that the basic rules of a general GSH exactly correspond to the way in which formal "higher-order rules" are applied according to S.H.'s definition of his system (but where $\Gamma \vdash A$ has the original meaning of a proposition in the meta-language of S.H.'s formalism).

For the converse we prove by induction on length of proofs the following general claim: if A is derivable in S.H.'s system from the assumptions Γ, and every rule in Γ is a subrule of some rule in Δ, then $\Delta \vdash_G R$. Details are similar to the proof of proposition 1, and are left to the reader.

Theorem 2:("cut elimination") If $\Gamma \vdash_G R$ and $\Delta, R \vdash_G R'$ then $\Gamma, \Delta \vdash_G R'$.
Proof: By double induction on the complexity of R and on the sum of the lengths of the proofs of the two given sequents. The treatment of $\Rightarrow$ is similar to that of $\supset$ in intuitionsitic sequential calculus, and so we omit the standard details.

Corollary 1: Let Δ be a finite set of rules. Then $\Gamma \vdash A$ is derivable in the corresponding general GSH (see section 1.3) iff $\Delta, \Gamma \vdash A$ is derivable in the pure system.

Proof: Using the corollary of proposition 1 it is easy to show that $\vdash R$ is provable in the corresponding general GSH for every R in Δ. It follows therefore from theorem 2 (applied to the general GSH) that if $\Delta, \Gamma \vdash A$ is derivable in the pure system then $\Gamma \vdash A$ is derivable in the generalized one. The converse can be proved by a direct induction on the length of proofs in the general GSH.

Corollary 2: Let Δ be a finite set of rules. Then $\Gamma \vdash A$ is derivable in the corresponding general GSH iff it is derivable in the system which is obtained from the pure one by adding $\vdash R$ as an axiom for every R in Δ and taking cut as a primitive rule.

Proof: This follows easily from the characterization given in the previous corollary and the cut elimination theorem for the pure system.

Discussion: It might be useful to make a digression here for discussing the role of cut-elimination in Gentzen-type systems. As we emphasize in [6], what characterizes a logic is usually not only its set of theorems, but the ***consequence relation*** (C.R.) defined by it. Now , given a Gentzen-type formal system, there are two basic ways of associating C.R.s with it (compare [7]):

The external C.R. $\vdash_E$**:** $A_1, \ldots, A_n \vdash_E B$ iff the sequent $\vdash B$ is derivable if we add $\vdash A_1, \ldots, \vdash A_n$ as axioms (i.e., basic sequents) to the system.

The internal C.R. $\vdash_I$**:** $A_1, \ldots, A_n \vdash_I B$ iff the sequent $A_1, \ldots, A_n \vdash B$ is provable (in the given formalism).

The cut rule should be taken as primitive in order for $\vdash_E$ to actually be a C.R.. It is then immediate that it is at least as strong as $\vdash_I$. In the case of $\vdash_I$, on the other hand, Cut should be taken as primitive ***or else be shown admissible***. If the Gentzen type system satisfies some natural conditions (see [6]) then in either case $\vdash_I$ will be equivalent to $\vdash_E$. The real meaning of cut elimination then is that $A_1, \ldots, A_n \vdash_E B$ iff there exists a cut-free proof of $A_1, \ldots, A_n \vdash B$. If the given formalism is well-behaved then this might imply the sub-formula property and/or decidability for $\vdash_E$. From this point of view Theorem 1 presents a true cut-elimination theorem in the case of the pure calculus. [1]

The two charaterizations of provability in a general GSH which were given in the last two corrolaries have a clear correspondence with the two C.R.s described above. They also describe two natural ways of defining a C.R. using a natural deduction formalism (usually, but not always, these two ways are trivially equivalent). The official definition which was given in 1.3 corresponds to a third way which was available in the present case and was chosen by S.H.. The main property of the systems which are obtained by this definition is that their non-logical rules can never introduce in an antecedent something that has not been already present at one of the antecedents of the premises. The fact that cut elimination obtains for these general GSH systems is due to this proprty. However, decidability and the subformula property are not gauranteed in the general case by this cut elimination, and Theorem 1 may have therefore little significance in case there is an infinite number of non-logical rules. This is the case, e.g., when the set of basic rules is defined using schemes—recall that S.H.'s notion of a "rule" is a local one and is identical to what usually is taken only as ***an instance*** of a rule of inference! In section 4 we shall have the opportunity to charaterize an important class of cases in which a ***significant*** version of cut-elimination again obtains.

The limitation of the language to sequents of the form $\Gamma \vdash$ A is, to our opinion, artificial. Notationally, in fact, we have already abandoned it (and so did S.H. himself!). The next theorem "officially" removes this limitation:

[1] It would have been better to call it the "sub-rule" property here. Unfortunately, this name has already another meaning!

Theorem 3: $\Gamma \vdash_G R$ iff $\Gamma \vdash R$ is provable in the system obtained by generalizing the concept of a sequent to allow ***rules*** on the right side of the $\vdash$ and by adding to GSH the inference-rule:

$$\frac{\Gamma, R_1, \ldots, R_n \vdash \mathrm{A}}{\Gamma \vdash (R_1, \ldots, R_n \Rightarrow \mathrm{A})}$$

moreover: Theorems 1 and 2 are true for the extended system.

we leave the proof to the reader.

3 GSH and intuitionistic implicational calculus

Theorem 3 above indicates that S.H.'s system is just a new (somewhat strange) formulation of the pure implicational intuitionistic calculus. We claim that there is no real difference between S.H.'s $\Rightarrow$ and the intuitionistic $\supset$. The following definition and proposition make this claim precise:

Definition: Let RU be the set of S.H. rules, IMP— the set of sentences of the pure implicational calculus, $\boldsymbol{Int}_{\rightarrow}$—the pure intuitionistic implicational calculus. Define $\boldsymbol{v}$:RU$\rightarrow$IMP, $\boldsymbol{u}$:IMP$\rightarrow$RU as follows:

1. $\boldsymbol{v}(\mathrm{A}) = \mathrm{A}$
2. $\boldsymbol{v}(R_1, \ldots, R_n \Rightarrow \mathrm{A}) = \boldsymbol{v}(R_1) \supset (\boldsymbol{v}(R_2) \supset (\cdots \supset (\boldsymbol{v}(R_n) \supset \mathrm{A}))\ldots)$
3. $\boldsymbol{u}(\boldsymbol{p}) = \boldsymbol{p}$ (p atomic)
4. $\boldsymbol{u}(A_1 \supset (A_2 \supset (\cdots (A_n \supset \boldsymbol{p}))\ldots)) = \boldsymbol{u}(A_1), \ldots, \boldsymbol{u}(A_n) \Rightarrow \boldsymbol{p}$ ($\boldsymbol{p}$ atomic)

Note that since$\{R_1, \ldots, R_n\}$ is a set, $\boldsymbol{v}$ is multiple-valued!

Theorem 4:

1. $\boldsymbol{u}(\boldsymbol{v}(R)) = R$
2. $\vdash_{Int_{\rightarrow}} \boldsymbol{v}(\boldsymbol{u}(A)) \equiv A$
3. If $R_1, \ldots, R_n \vdash_{S.H.} R$ then $\boldsymbol{v}(R_1), \ldots, \boldsymbol{v}(R_n) \vdash_{Int_{\rightarrow}} \boldsymbol{v}(R)$.
4. If $A_1, \ldots, A_n \vdash_{Int_{\rightarrow}} B$ then $\boldsymbol{u}(A_1), \ldots, \boldsymbol{u}(A_n) \vdash_{S.H.} \boldsymbol{u}(B)$.

proof: Easy.

Theorem 4 and the formulation of GSH suggest a new Gentzen-type formulation of intuitionistic logic, in which the usual ($\supset\vdash$) rule is replaced by:

$$\frac{\Gamma_1 \vdash A_1 \cdots \Gamma_n \vdash A_n}{\Gamma_1, \ldots, \Gamma_n, (A_1 \supset (A_2 \supset \ldots \supset B)\ldots) \vdash B}$$

It is not difficult to show that this formulation is correct and that it admits cut-elimination. It might even seem more intuitive than the ordinary one. The trouble with it is that the new rule is not exactly a rule in the ordinary sense, not even a rule schema: It includes an infinite number of rule-schemes (for each n there is a corresponding rule with exactly n premises). A natural question to ask is therefore: What should be done in order to replace this infinite number by a finite number of rules with a fix number of premisesΓ Well, the answer should allow us to derive every n-instance of the new "rule". This sould naturally be done by induction. For the base case we need the rule:

$$\frac{\Gamma \vdash A}{\Gamma, A \supset B \vdash B}$$

For the induction step we need a rule which will permit us to pass from:

$$\Gamma, A_n \supset (A_{n-1} \supset \cdots \supset (A_1 \supset B)\ldots) \vdash B$$

and from:

$$\Gamma \vdash A_{n+1}$$

to

$$\Gamma, A_{n+1} \supset (A_n \supset \cdots \supset (A_1 \supset B)\ldots) \vdash B$$

If we denote A_{n+1} by A, $A_n \supset (\cdots \supset (A_1 \supset B)\ldots)$ by C we get that the rule needed is:

$$\frac{\Gamma \vdash A \qquad \Gamma, C \vdash B}{\Gamma, A \supset C \vdash B}$$

This is, of course, the ordinary ($\supset\vdash$) rule in Gentzen systems. (Since $B \vdash B$ is an axiom, the rule needed for the base case is also covered by this rule!)

4 On introduction and elimination rules

S.H. presents in his paper the following method for adding new n-ary operators to a language L:

Let $\Phi_1(A_1, \ldots, A_n), \ldots, \Phi_m(A_1, \ldots, A_n)$ $(m \geq 0)$ be a list of lists of rule-*schemes*. A new n-operator S, expressing the "common content" of $\Phi_1, \ldots, \Phi_m$, can then be introduced by the following rule schemes:

Introduction rules:

$$\Phi_i(A_1,\ldots,A_n) \Rightarrow S(A_1,\ldots,A_n) \quad (1 \leq i \leq m)$$

Elimination rule:

$$(\Phi_1 \Rightarrow A),(\Phi_2 \Rightarrow A),\ldots,(\Phi_m \Rightarrow A),(S(A_1,\ldots,A_n)) \Rightarrow A$$

(where $\Phi_i = \Phi_i(A_1,\ldots,A_n)$ $i = 1,\ldots,m$)

S.H. shows then that the validity of these rules is a necessary and sufficient condition for the following to be true:

(*)For all $A_1,\ldots,A_n$ and for every R:

$S(A_1,\ldots,A_n) \vdash R$ iff for all $1 \leq i \leq m$ $\Phi_i(A_1,\ldots,A_n) \vdash R$.

Examples:

Conjunction: Here $m = 1$, $\Phi_1(A,B) = \{A,B\}$. The rules are:

intro. : $A,B \Rightarrow A \wedge B$

elim. : $(A,B \Rightarrow C),(A \wedge B) \Rightarrow C$

Disjunction: Here $m = 2$, $\Phi_1(A,B) = \{A\}$, $\Phi_2(A,B) = \{B\}$.

intro. : $A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$

elim. : $(A \Rightarrow C),(B \Rightarrow C),(A \vee B) \Rightarrow C$

Implication: Here $m = 1$, $\Phi_1(A,B) = \{A \Rightarrow B\}$.

intro. : $(A \Rightarrow B) \Rightarrow A \supset B$

elim. : $((A \Rightarrow B) \Rightarrow C),(A \supset B) \Rightarrow C$

Turning now to our Gentzen-type version, suppose $\Phi_i = \{R_{i,1},\ldots,R_{i,m_i}\}$ $(i = 1,\ldots,m)$. Then for each i, the i-introduction rule of S.H. is translated according to 1.3 into the basic rule:

$$\frac{\Delta_1 \vdash R_{i,1}(A_1,\ldots,A_n) \quad \Delta_2 \vdash R_{i,2}(A_1,\ldots,A_n) \quad \cdots \quad \Delta_{m_i} \vdash R_{i,m_i}(A_1,\ldots,A_n)}{\Delta_1,\Delta_2,\ldots,\Delta_{m_i} \vdash S(A_1,\ldots,A_n)}$$

The elimination rule, on the other hand, becomes:

$$\frac{\Gamma_1,\Phi_1 \vdash A \quad \Gamma_2,\Phi_2 \vdash A \quad \cdots \quad \Gamma_m,\Phi_m \vdash A \quad \Delta \vdash S(A_1,\ldots,A_n)}{\Delta,\Gamma_1,\Gamma_2,\ldots,\Gamma_m \vdash A}$$

In the context of Gentzen-type systems it is, however, much more natural to replace the natural-deduction-style "elimination" rules by ***introduction*** rules in the antecedent. This can be achieved by substituting $\{S(A_1,\ldots,A_n)\}$ for Δ at the above version of the elimination rules. Using the axioms we then obtain:

$$\frac{\Gamma_1,\Phi_1\vdash A\ \ \Gamma_2,\Phi_2\vdash A\ \cdots\ \Gamma_m,\Phi_m\vdash A}{S(A_1,\ldots,A_n),\Gamma_1,\Gamma_2,\ldots,\Gamma_m\vdash A}$$

Using cuts it is not difficult to show that the two formulations above are in fact equivalent.

Examples:

disjunction: The rules we get are:

$$\frac{\Gamma\vdash A}{\Gamma\vdash A\vee B}\quad\frac{\Gamma\vdash B}{\Gamma\vdash A\vee B}\qquad\qquad\frac{\Gamma_1,A\vdash C\qquad\Gamma_2,B\vdash C}{\Gamma_1,\Gamma_2,A\vee B\vdash C}$$

conjunction: We get:

$$\frac{\Gamma_1\vdash A\qquad\Gamma_2\vdash B}{\Gamma_1,\Gamma_2\vdash A\wedge B}\qquad\qquad\frac{\Gamma,A,B\vdash C}{\Gamma,A\wedge B\vdash C}$$

implication: We get:

$$\frac{\Gamma\vdash A\Rightarrow B}{\Gamma\vdash A\supset B}\quad(\equiv\quad\frac{\Gamma,A\vdash B}{\Gamma\vdash A\supset B})\qquad\qquad\frac{\Gamma,A\Rightarrow B\vdash C}{\Gamma,A\supset B\vdash C}$$

Note that the rules we got for conjunction and disjunction are the usual Gentzen -type rules for them. This is true even for conjunction, the elimination rule for which looks somewhat unusual in S.H.'s formulation. For $\supset$ we just get the identity between $\Rightarrow$ and $\supset$ [2]. Again the corresponding rules are more intuitive than in the setting of S.H., but the fact that S.H. treats $\Rightarrow$ as basic forces us in this case to derive the usual rules for $\supset$ by the detour through $\Rightarrow$ which we made in the previous section.

Using standard methods, it is not difficult to show that any system which can be defined by gradually introducing new operators, using the above two kinds of introduction rules, admits cut-elimination [3]. From this we can easily deduce S.H.'s result concerning the conservative character of his introduction

[2] This is another evidence that there is no real difference between the two

[3] This is the important case we promise at the discussion in section 2!

and elimination rules (theorem 4.8 of [1]). S.H. himself used for this a normalization theorem which he has proved in [2]. Normalization is, of course, the natural-deduction counterpart of cut-elimination. However, at least in the present case cut-elimination is easier to show and to use, since we are free here from the complications caused by the notions of subrules and discharge functions.

Finally, a remark about S.H.'s characterization of the intuitionistic connectives and their definability power. As we show above, there is no real difference between $\Rightarrow$ and the intuitionistic $\supset$. It is almost trivial, therefore, that the "common content" of $\Phi_1, \ldots, \Phi_m$ is given by

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{m_i} v(R_{i,j}) \quad (\text{where } \Phi_i = \{R_{i,1}, \ldots, R_{i,m_i}\})$$

This is the real content of S.H.'s theorem about the definability within the intuitionistic propositional calculus of all the connectives which can gradually be defined using his method of intro. and elim. rules ($\perp$ corresponds to the case $m = 0$) [4]. The philosophical significance of this characterization depends on the degree of priority one is willing to attach to S.H.'s "higher-order rules" over the corresponding implicational sentences. Because of the local character of S.H.'s rules (as opposed to what we ***usually*** call "rules of inference") the two notions seem to me, at least, just to be variants of each other. Another reservation which I have concerning this characterization is that it seems to me to force us to regard negation as a derived rather than a primitive connective: I see no way of directly defining it by intro. and elim. rules without introducing $\perp$ first!

References

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