Tableaux with Four Signs as a Unified Framework

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Abstract. We show that the use of tableaux with four types of signed formulas (the signs intuitively corresponding to positive/negative information concerning truth/falsity) provides a framework in which a diversity of logics can be handled in a uniform way. The logics for which we provide sound and complete tableaux systems of this type are classical logic, the most important three-valued logics, the four-valued logic of logical bilattices (an extension of Belnap's four-valued logic), Nelson's logics for constructive negation ($\mathbf{N}^-$ and $\mathbf{N}$), and da Costa's paraconsistent logic C_ω (together with some extensions of which). For the latter we provide new, simple semantics for which our tableaux systems are sound and complete.

1 Introduction

There are two main variants of tableaux systems for classical logic. One employs two sorts of signed formulas: $\mathbf{T}\varphi$ and $\mathbf{F}\varphi$ (intuitively meaning "φ is true" and "φ is false", respectively). The other employs ordinary formulas, replacing $\mathbf{T}\varphi$ simply by φ and using $\neg\varphi$ as a substitute for $\mathbf{F}\varphi$ (where $\neg$ is the negation connective of the language). This alternative for the use of signs works well for classical logic, but a combination of the two methods is frequently needed for handling weaker logics, since usually the refutation of $\mathbf{F}\varphi$ is not equivalent to the validity of φ. Our goal here is to present what we believe to be a better approach, one which allows for a unified treatment of negation (and other standard connectives!) in a diversity of logics. The idea is to use *four* sorts of signed formulas: $\mathbf{T}^+\varphi$, $\mathbf{T}^-\varphi$, $\mathbf{F}^+\varphi$, and $\mathbf{F}^-\varphi$. The intuitive meaning of these signs can best be explained in terms of positive and negative information (see e.g. [Wan93]). $\mathbf{T}^+\varphi$ intuitively means that there is a positive information for the truth of φ, $\mathbf{T}^-\varphi$ means that there is a negative information for the truth of φ, $\mathbf{F}^+\varphi$ means that there is a positive information for the falsity of φ, $\mathbf{F}^-\varphi$ means that there is a negative information for the falsity of φ. [1] In the rest of this paper we demonstrate the usefulness of the four-signs framework by providing within it sound and complete tableaux systems for several well known logics. In the next section we consider logics in which negation is added to positive classical logic. This includes classical logic itself, the most important three-valued logics, and the four-valued logic of logical bilattices (which is an extension of Belnap's famous four-valued logic). In the last section we treat logics in which (true) negation is conservatively added to positive intuitionistic logic. The systems we consider there are Nelson's two logics for constructive negation ($\mathbf{N}^-$ and $\mathbf{N}$), and da Costa's paraconsistent logic C_ω (together with some extensions of which). For the latter we provide new, simple semantics for which our tableaux system(s) are sound and complete.

[1] A similar idea has motivated the introduction and use of bilattices. See [Gin87,Gin88,Fit90b,Fit90a,Fit91,Fit94].

One point should be noted before we proceed. The use of tableaux with more than two signs has already been used in the framework of many-valued logics (see the survey papers [Häh99,BFS00] for the idea and for a extensive list of references). The signs which are used there correspond however to the truth-values of the logic in question (so tableaux systems with exactly n signs are used for any n-valued logic). Here, in contrast, the four signs do *not* correspond to truth-values and we use them even for logics which do not have finite characteristic matrix. On the contrary, our goal is to provide a general framework which (as far as possible) is not essentially connected to any specific type of semantics.

2 Many-valued Extensions of Positive Classical Logic

2.1 Four-valued Logics

In [Bel77b,Bel77a] Belnap suggested the use of logics based on the four truth-values $t, f, \top$, and $\bot$, where t and f are the classical values, $\top$ ("both true and false") represents the truth-value of formulas about which there is inconsistent data, while $\bot$ ("neither true nor false") is the truth-value of formulas on which no data is available. Belnap's structure is nowadays known also as the basic (distributive) *bilattice*, and its logic — as the basic logic of (distributive) bilattices (see [Gin87,Gin88,Fit90b,Fit90a,Fit91,Fit94,AA96,AA98]). The following is an extension (from [AA96]) of Belnap's logic with an appropriate implication connective:

Definition 1. *The matrix* $\mathcal{M}_4 = \langle M_4, D_4, O_4\rangle$*:* [2]

- $M_4 = \{t, f, \top, \bot\}$
- $D_4 = \{t, \top\}$
- *The operations in* O_4 *are defined by:*
  1. $\neg t = f, \ \ \neg f = t, \ \ \neg\top = \top, \ \ \neg\bot = \bot$
  2. $a \vee b = sup_{\leq_t}(a, b), \ \ a \wedge b = inf_{\leq_t}(a, b)$, *where the partial order* $\leq_t$ *is defined by:* $f \leq_t \top, \bot \leq_t t$.
  3. $a \supset b \ = \begin{cases} b & \text{if } a \in D_4 \\ t & \text{if } a \notin D_4 \end{cases}$

As usual, A function v from the set $\mathcal{F}$ of formulas of $\{\neg, \vee, \wedge, \supset\}$ into M_4 is called a valuation in $\mathcal{M}_4$ if it respects the operations in O_4. A valuation v is an $\mathcal{M}_4$-model of a formula φ of $\mathcal{F}$ if $v(\varphi) \in D_4$. v is an $\mathcal{M}_4$-model of a set T of formulas if it is an $\mathcal{M}_4$-model of each element of T. A formula φ follows in $\mathcal{M}_4$ from T ($T \vdash_{\mathcal{M}_4} \varphi$) if every $\mathcal{M}_4$-model of T is also a $\mathcal{M}_4$-model of φ.

The concept of an $\mathcal{M}_4$-model can be extended to signed formulas as follows:

- v is an $\mathcal{M}_4$-model of $\mathbf{T}^+\varphi$ if $v(\varphi) \in \{t, \top\}$
- v is an $\mathcal{M}_4$-model of $\mathbf{T}^-\varphi$ if $v(\varphi) \in \{f, \top\}$
- v is an $\mathcal{M}_4$-model of $\mathbf{F}^+\varphi$ if $v(\varphi) \in \{f, \bot\}$
- v is an $\mathcal{M}_4$-model of $\mathbf{F}^-\varphi$ if $v(\varphi) \in \{t, \bot\}$

[2] The names of the various matrices discussed in this section are taken from [Avr03].

Note that $T \vdash_{\mathcal{M}_4} \varphi$ iff the set $\{\mathbf{T}^+\psi \mid \psi \in T\} \cup \{\mathbf{F}^+\varphi\}$ is not satisfiable.

Note. It is well known from the literature on bilattices (see e.g. [Gin88,Fit94]) that $\mathcal{M}_4$ can be identified with $\{t, f\} \times \{t, f\}$ (so that $\top$ represents (t, t), $\bot$ represents (f, f), t represents (t, f) and f represents (f, t)). This allows an alternative presentation of the semantics, using *two* valuations in $\{t, f\}$, representing independent information concerning truth and falsity of formulas (see [Wan93]). We shall explain more about this approach in the next section.

We present now a tableaux system which is sound and complete with respect to $\vdash_{\mathcal{M}_4}$:

Definition 2. *The tableaux system $Tab(\mathcal{M}_4)$:* [3]

Expansion Rules:

$$
\begin{array}{llll}
(\mathbf{T}^+\neg) & \dfrac{\mathbf{T}^+\neg\varphi}{\mathbf{T}^-\varphi} & \dfrac{\mathbf{T}^-\neg\varphi}{\mathbf{T}^+\varphi} & (\mathbf{T}^-\neg) \\[2ex]
(\mathbf{F}^+\neg) & \dfrac{\mathbf{F}^+\neg\varphi}{\mathbf{F}^-\varphi} & \dfrac{\mathbf{F}^-\neg\varphi}{\mathbf{F}^+\varphi} & (\mathbf{F}^-\neg) \\[2ex]
(\mathbf{T}^+\wedge) & \dfrac{\mathbf{T}^+\varphi \wedge \psi}{\mathbf{T}^+\varphi, \mathbf{T}^+\psi} & \dfrac{\mathbf{T}^-\varphi \wedge \psi}{\mathbf{T}^-\varphi \mid \mathbf{T}^-\psi} & (\mathbf{T}^-\wedge) \\[2ex]
(\mathbf{F}^+\wedge) & \dfrac{\mathbf{F}^+\varphi \wedge \psi}{\mathbf{F}^+\varphi \mid \mathbf{F}^+\psi} & \dfrac{\mathbf{F}^-\varphi \wedge \psi}{\mathbf{F}^-\varphi, \mathbf{F}^-\psi} & (\mathbf{F}^-\wedge) \\[2ex]
(\mathbf{T}^+\vee) & \dfrac{\mathbf{T}^+\varphi \vee \psi}{\mathbf{T}^+\varphi \mid \mathbf{T}^+\psi} & \dfrac{\mathbf{T}^-\varphi \vee \psi}{\mathbf{T}^-\varphi, \mathbf{T}^-\psi} & (\mathbf{T}^-\vee) \\[2ex]
(\mathbf{F}^+\vee) & \dfrac{\mathbf{F}^+\varphi \vee \psi}{\mathbf{F}^+\varphi, \mathbf{F}^+\psi} & \dfrac{\mathbf{F}^-\varphi \vee \psi}{\mathbf{F}^-\varphi \mid \mathbf{F}^-\psi} & (\mathbf{F}^-\vee) \\[2ex]
(\mathbf{T}^+\supset) & \dfrac{\mathbf{T}^+\varphi \supset \psi}{\mathbf{F}^+\varphi \mid \mathbf{T}^+\psi} & \dfrac{\mathbf{T}^-\varphi \supset \psi}{\mathbf{T}^+\varphi, \mathbf{T}^-\psi} & (\mathbf{T}^-\supset) \\[2ex]
(\mathbf{F}^+\supset) & \dfrac{\mathbf{F}^+\varphi \supset \psi}{\mathbf{T}^+\varphi, \mathbf{F}^+\psi} & \dfrac{\mathbf{F}^-\varphi \supset \psi}{\mathbf{F}^+\varphi \mid \mathbf{F}^-\psi} & (\mathbf{F}^-\supset)
\end{array}
$$

Closure Conditions: *A branch is closed iff for some formula φ it contains either $\{\mathbf{T}^+\varphi, \mathbf{F}^+\varphi\}$ or $\{\mathbf{T}^-\varphi, \mathbf{F}^-\varphi\}$.*

Theorem 1. *A set of signed formulas is unsatisfiable in $\mathcal{M}_4$ iff it has a closed tableaux in $Tab(\mathcal{M}_4)$.*

Proof: It is straightforward to check that a set T of formulas for which one of the two closure conditions obtains is unsatisfiable, and that for every expansion rule R, if T has a model in $\mathcal{M}_4$ then so does at least

[3] This system is closely related to the Gentzen-type system for this logic that was presented in [AA96,Avr03], and its completeness can be derived from the completeness of that system. It is however more illuminating to prove it directly.

one of the sets which are obtained from T by R. This shows soundness (i.e.: if T has a closed tableaux then it is unsatisfiable). For the converse it is suffices to prove that the set of formulas Γ of any fully expanded open branch (in a complete tableaux for T) has a model v defined as follows:

$$v(p) = \begin{cases} f & \mathbf{T}^+p \notin \Gamma, \mathbf{F}^-p \notin \Gamma \\ \bot & \mathbf{F}^+p \in \Gamma, \mathbf{F}^-p \in \Gamma \\ \top & \mathbf{T}^+p \in \Gamma, \mathbf{T}^-p \in \Gamma \\ t & otherwise \end{cases}$$

Now the fact that Γ is open (i.e. contains no set of the form $\{\mathbf{T}^+\varphi, \mathbf{F}^+\varphi\}$ or $\{\mathbf{T}^-\varphi, \mathbf{F}^-\varphi\}$) implies that v is well defined, and that if $Sp \in \Gamma$, where S is one of the four signs and p is atomic, then v is a model of Sp. Using induction on the structure of formulas and the fact that Γ is fully expanded it is not difficult to show that v is a model of any formula in Γ.

Corollary 1. *Let φ and ψ be formulas in the language of $\{\neg, \vee, \wedge\}$. Then $\varphi \to \psi$ is a valid first degree entailment of the relevance logic R ([AB75,AB92,Dun86]) iff $\{\mathbf{T}^+\varphi, \mathbf{F}^+\psi\}$ has a closed tableaux in $Tab(\mathcal{M}_4)$.*

Note. Although we use here 4 signs, these signs do *not* correspond to the 4 truth values of the semantics. Indeed, the same signs will be used below with very similar systems for 3-valued logics and even for the classical, two-valued logic.

2.2 Three-valued Logics and Classical Logic

We consider next two basic three-valued logics, whose matrices are submatrices of $\mathcal{M}_4$.

Definition 3. *The matrix $\mathcal{M}_3^{\{t\}} = \langle M_3^{\{t\}}, D_3^{\{t\}}, O_3^{\{t\}} \rangle$:*

- $M_3^{\{t\}} = \{t, f, \bot\}$
- $D_4 = \{t\}$
- *The operations in $O_3^{\{t\}}$ are defined by:*
  1. $\neg t = f, \quad \neg f = t, \quad \neg\bot = \bot$
  2. $a \vee b = sup_{\leq_t}(a, b), \quad a \wedge b = inf_{\leq_t}(a, b),$
  3. $a \supset b \; = \begin{cases} b & \text{if } a \in D_3^{\{t\}} \\ t & \text{if } a \notin D_3^{\{t\}} \end{cases}$

Note. The implication connective of $\mathcal{M}_3^{\{t\}}$ was originally introduced by Słupecki in [Słu36]. It was independently reintroduced in [Mon67,Woj84,Sch86] and [Avr91] (see also [Bus96]). The language of $\mathcal{M}_3^{\{t\}}$ is equivalent ([Avr91]) to that used in the logic LPF of the VDM project ([Jon86]), as well as to the language of Łukasiewicz 3-valued logic Ł$_3$ ([Łuk67]). It is in fact the language of all 3-valued operations which are *classically closed*. It can be shown that by adding one propositional constant to it (corresponding to the truth value $\bot$) we get a functionally complete set of 3-valued connectives (See [Avr99] for further details and references).

Definition 4. *The matrix* $\mathcal{M}_3^{\{t,\top\}} = \langle M_3^{\{t,\top\}}, D_3^{\{t,\top\}}, O_3^{\{t,\top\}} \rangle$*:*

- $M_3^{\{t,\top\}} = \{t, f, \top\}$
- $D_3^{\{t,\top\}} = \{t, \top\}$
- *The operations in* $O_3^{\{t,\top\}}$ *are defined by:*
  1. $\neg t = f, \;\; \neg f = t, \;\; \neg\top = \top$
  2. $a \vee b = sup_{\leq_t}(a,b), \;\; a \wedge b = inf_{\leq_t}(a,b),$
  3. $a \supset b \; = \begin{cases} b & \text{if } a \in D_3^{\{t,\top\}} \\ t & \text{if } a \notin D_3^{\{t,\top\}} \end{cases}$

Note. The implication connective of $\mathcal{M}_3^{\{t,\top\}}$ was first introduced in [DdC70,dC74]. It was independently introduced also in [Avr86]. The language $\{\neg, \vee, \wedge, \supset_{\{t,I\}}\}$ is equivalent to that used in the standard 3-valued paraconsistent logic J_3 ([D'o85,Avr86,Roz89,Eps90]. In [Avr91] it is called *Pac*), as well as to that used in the semi-relevant system RM_3 ([AB75,AB92,Dun86]. See also [Avr86,Avr91]). It is the language of all 3-valued operations which are classically closed and *free* ([Avr99]).

The concepts of $\mathcal{M}_3^{\{t\}}$-model and of $\mathcal{M}_3^{\{t,\top\}}$-model are defined now exactly as in the case of $\mathcal{M}_4$ (but of course only the available truth-values are relevant. Thus practically a valuation v in $\mathcal{M}_3^{\{t\}}$ is an $\mathcal{M}_3^{\{t\}}$-model of $\mathbf{T}^+\varphi$ if $v(\varphi) = t$).

The tableaux system $Tab(\mathcal{M}_3^{\{t\}})$**:** This is the system obtained from $Tab(\mathcal{M}_4)$ by adding to it the following extra closure condition: a branch is closed also if for some formula φ it contains $\{\mathbf{T}^+\varphi, \mathbf{T}^-\varphi\}$.

The tableaux system $Tab(\mathcal{M}_3^{\{t,\top\}})$**:** This is the system obtained from $Tab(\mathcal{M}_4)$ by adding to it the following extra closure condition: a branch is closed also if for some formula φ it contains $\{\mathbf{F}^+\varphi, \mathbf{F}^-\varphi\}$.

Theorem 2.

1. *A set of signed formulas is unsatisfiable in* $\mathcal{M}_3^{\{t\}}$ *iff it has a closed tableaux in* $Tab(\mathcal{M}_3^{\{t\}})$.
2. *A set of signed formulas is unsatisfiable in* $\mathcal{M}_3^{\{t,\top\}}$ *iff it has a closed tableaux in* $Tab(\mathcal{M}_3^{\{t,\top\}})$.

Proof: Similar to that of Theorem 1.

Note. Because of the strong expressive power of the languages of $\mathcal{M}_3^{\{t\}}$ and $Tab(\mathcal{M}_3^{\{t\}})$ (see [Avr99,Avr03]), their tableaux systems can be used as bases for all other 3-valued logics. For example, $\varphi \to \psi$, where $\to$ is Łukasiewicz 3-valued implication, is equivalent in $\mathcal{M}_3^{\{t\}}$ to $(\varphi \supset \psi) \wedge (\neg\psi \supset \neg\varphi)$. This leads to the following 4 rules for it:

$$(\mathbf{T}^+ \to)\ \frac{\mathbf{T}^+\varphi \to \psi}{\mathbf{T}^+\psi \mid \mathbf{T}^-\varphi \mid \mathbf{F}^+\varphi, \mathbf{F}^-\psi} \qquad\qquad \frac{\mathbf{T}^-\varphi \to \psi}{\mathbf{T}^+\varphi, \mathbf{T}^-\psi}\ (\mathbf{T}^- \to)$$

$$(\mathbf{F}^+ \to)\ \frac{\mathbf{F}^+\varphi \to \psi}{\mathbf{T}^+\varphi, \mathbf{F}^+\psi \mid \mathbf{T}^-\psi, \mathbf{F}^-\varphi} \qquad\qquad \frac{\mathbf{F}^-\varphi \to \psi}{\mathbf{F}^+\varphi \mid \mathbf{F}^-\psi}\ (\mathbf{F}^- \to)$$

We end this section with a characterization of classical logic itself:

Theorem 3. *Let $Tab(\mathcal{M}_2)$ be the system obtained from $Tab(\mathcal{M}_4)$ by adding to it both of the new closure conditions of $Tab(\mathcal{M}_3^{\{t\}})$ and $Tab(\mathcal{M}_3^{\{t,\top\}})$. Then a set of signed formulas is unsatisfiable in classical logic iff it has a closed tableaux in $Tab(\mathcal{M}_2)$.*

We leave the proof of this Theorem to the reader.

3 Conservative Extensions of Positive Intuitionistic Logic

3.1 Nelson's Logics for Negation

The logics $\mathbf{N}^-$ and $\mathbf{N}$ are conservative extensions of positive intuitionistic logic which were independently introduced by Nelson (see [AN84]) and Kutschera ([vK69]). The motivation for their introduction has been the wish to provide an adequate treatment of negative information within the framework of constructive logic. See [Wan93] for further details and references.

The standard semantics of $\mathbf{N}^-$ is based on Kripke frames $\mathcal{I} = \langle I, \leq, v^+, v^- \rangle$ in which v^+ and v^- are valuations from $I \times \mathcal{F}$ into $\{t, f\}$ (where $\mathcal{F}$ is the set of formulas) which satisfy the following basic conditions:

(H^+) If $a \leq b$ and $v^+(a,\varphi) = t$ then $v^+(b,\varphi) = t$

(H^-) If $a \leq b$ and $v^-(a,\varphi) = t$ then $v^-(b,\varphi) = t$

v^+ and v^- should further satisfy also the following conditions:

$$\begin{array}{ll}
v^+(a,\varphi \wedge \psi) = t & \text{iff } v^+(a,\varphi) = t \text{ and } v^+(a,\psi) = t \\
v^-(a,\varphi \wedge \psi) = t & \text{iff } v^-(a,\varphi) = t \text{ or } v^-(a,\psi) = t \\
v^+(a,\varphi \vee \psi) = t & \text{iff } v^+(a,\varphi) = t \text{ or } v^+(a,\psi) = t \\
v^-(a,\varphi \vee \psi) = t & \text{iff } v^-(a,\varphi) = t \text{ and } v^-(a,\psi) = t \\
v^+(a,\varphi \supset \psi) = t & \text{iff for all } b \geq a, \text{ either } v^+(b,\varphi) = f \text{ or } v^+(b,\psi) = t \\
v^-(a,\varphi \supset \psi) = t & \text{iff } v^+(a,\varphi) = t \text{ and } v^-(a,\psi) = t \\
v^+(a,\neg\varphi) = t & \text{iff } v^-(a,\varphi) = t \\
v^-(a,\neg\varphi) = t & \text{iff } v^+(a,\varphi) = t
\end{array}$$

Call a frame $\mathcal{I} = \langle I, \leq, v^+, v^- \rangle$ satisfying the above conditions a $\mathbf{N}^-$-frame. A $\mathbf{N}$-frame is defined similarly, with one extra condition: that $v^+(a,\varphi)$ and $v^-(a,\varphi)$ cannot both be t at the same time.

Note. It can be shown that it suffices to demand the H(erditary) conditions (H^+) and (H^-) only for atomic formulas. The other conditions impose them then on all the formulas.

The semantics of formulas and of signed formulas is defined now as follows. Let $\mathcal{I} = \langle I, \leq, v^+, v^- \rangle$ be a $\mathbf{N}^-$-frame, and let $a \in I$. Define:

- $(\mathcal{I}, a)$ is a $\mathbf{N}^-$-model of $\mathbf{T}^+\varphi$ if $v^+(\varphi) = t$
- $(\mathcal{I}, a)$ is a $\mathbf{N}^-$-model of $\mathbf{T}^-\varphi$ if $v^-(\varphi) = t$
- $(\mathcal{I}, a)$ is a $\mathbf{N}^-$-model of $\mathbf{F}^+\varphi$ if $v^+(\varphi) = f$
- $(\mathcal{I}, a)$ is a $\mathbf{N}^-$-model of $\mathbf{F}^-\varphi$ if $v^-(\varphi) = f$

Let now $(\mathcal{I}, a)$ be a $\mathbf{N}^-$-model of an ordinary formula φ iff it is a $\mathbf{N}^-$-model of $\mathbf{T}^+\varphi$ (iff it is not a $\mathbf{N}^-$-model of $\mathbf{F}^+\varphi$). Define the concept of $\mathbf{N}$-model of signed formulas and of ordinary formulas in a similar way, using $\mathbf{N}$-frames instead of $\mathbf{N}^-$-frames.

Note. If we allow only one element in I then what we get is equivalent to the four-valued logic of $\mathcal{M}_4$. As we have already noted above, it is indeed quite common to use two valuations v^+ and v^- from $\mathcal{F}$ to $\{t, f\}$ for an equivalent representation of the semantics of this logic ([Wan93]). The conditions concerning v^+ and v^- are practically identical to those in the case of $\mathbf{N}^-$, with only one exception: instead of the above condition concerning $v^+(a, \varphi \supset \psi) = t$ we have in that logic the simpler condition:

$$v^+(\varphi \supset \psi) = t \text{ iff } v^+(\varphi) = f \text{ or } v^+(\psi) = t$$

It is possible then to define the meanings of the signed formulas in this logic in a way which is completely analogous to the way this was done above in for $\mathbf{N}^-$.

We present now tableaux systems which are sound and complete with respect to $\mathbf{N}^-$ and $\mathbf{N}$.

Definition 5. *The tableaux systems $Tab(\mathbf{N}^-)$ and $Tab(\mathbf{N})$ are obtained from $Tab(\mathcal{M}_4)$ and $Tab(\mathcal{M}_3^{\{t\}})$ (respectively) by replacing their $(\mathbf{F}^+ \supset)$ rule with the following pair of rules:*

$$(\mathbf{F}^+ \supset)^w \qquad \frac{\mathbf{F}^+\varphi \supset \psi}{\mathbf{F}^+\psi}$$

$$(\mathbf{F}^+ \supset)^I \qquad \frac{\mathcal{S}, \mathbf{F}^+\varphi \supset \psi}{T(\mathcal{S}), \mathbf{T}^+\varphi, \mathbf{F}^+\psi}$$

Here $(\mathbf{F}^+ \supset)^I$ is a variant of the usual special intuitionistic rule for refuting implication: if $\mathcal{S}$ is the set of signed formulas on some branch then $T(\mathcal{S})$ is the set of all the elements in $\mathcal{S}$ whose sign is either $\mathbf{T}^+$ or $\mathbf{T}^-$, and an expansion of a branch by this rule requires the creation of a new tableau for the set $T(\mathcal{S}) \cup \{\mathbf{T}^+\varphi, \mathbf{F}^+\psi\}$.

Theorem 4. *A set of signed formulas is unsatisfiable in $\mathbf{N}^-$ iff it has a closed tableaux in $Tab(\mathbf{N}^-)$.*

Theorem 5. *A set of signed formulas is unsatisfiable in $\mathbf{N}$ iff it has a closed tableaux in $Tab(\mathbf{N})$.*

The proofs of these theorems is similar to the proofs of the soundness and completeness of the standard tableaux system for propositional intuitionistic logic, or of the soundness and completeness of the usual Gentzen-type systems for $\mathbf{N}^-$ and $\mathbf{N}$ (as presented e.g. in [Wan93])[4]. Details will be given in the full paper.

3.2 Extensions with Excluded Middle

It is well known that it is impossible to conservatively add to intuitionistic positive logic a negation which is both explosive (i.e.: $\neg\varphi, \varphi \vdash \psi$ for all φ, ψ) and for which LEM (the Law of Excluded Middle: $\neg\varphi \vee \varphi$) is valid. With such an addition we get classical logic. In $\mathbf{N}$ (following the tradition of intuitionistic logic) the

[4] The tableaux systems we present here are of course strongly related to these Gentzen-type systems.

choice was on explosiveness. In the paraconsistent logics of da Costa's school ([dC74,CM02]) explosiveness is rejected, while LEM is accepted. Thus da Costa's basic system C_ω is a conservative extension of positive intuitionistic logic which is obtained from any standard Hilbert-type formulation of this logic by adding as axioms $\neg\varphi \vee \varphi$ and $\neg\neg\varphi \supset \varphi$. We present now a Kripke-style semantics for C_ω which is similar to that we have presented above for $\mathbf{N}$ [5].

Definition 6. *A C_ω-frame is a structure $\mathcal{I} = \langle I, \leq, v^+, v^- \rangle$ in which v^+ and v^- are valuations from $I \times \mathcal{F}$ into $\{t, f\}$ such that:*

1. *There are no $a \in I$ and φ for which both $v^+(a, \varphi) = f$ and $v^-(a, \varphi) = f$.*
2. *The basic conditions (H^+) and (H^-) above are satisfied*
3. *v^+ and v^- satisfy also the following conditions:*

$$\begin{array}{lll}
v^+(a, \varphi \wedge \psi) = t & \textit{iff} & v^+(a, \varphi) = t \textit{ and } v^+(a, \psi) = t \\
v^+(a, \varphi \vee \psi) = t & \textit{iff} & v^+(a, \varphi) = t \textit{ or } v^+(a, \psi) = t \\
v^+(a, \varphi \supset \psi) = t & \textit{iff} & \textit{for all } b \geq a, \textit{ either } v^+(b, \varphi) = f \textit{ or } v^+(b, \psi) = t \\
v^+(a, \neg\varphi) = t & \textit{iff} & v^-(a, \varphi) = t \\
v^-(a, \neg\varphi) = f & \textit{if} & v^+(a, \varphi) = f
\end{array}$$

Thus the conditions concerning v^+ are identical to those in the case of $\mathbf{N}^-$, and are fully deterministic (given v^-). The values assigned to v^-, in contrast, are in general not determined by the values assigned by v^+ and v^- to its subformulas, and they are only subjected to two *constraints* (this implies, among other things, that it does not suffice to assume conditions (H^+) and (H^-) only for atomic formulas, since this does not enforce them for arbitrary formulas).

The concept of a model of a signed formula, and the associated consequence relation are defined now exactly as in the case of $\mathbf{N}^-$ and $\mathbf{N}$. We present now a corresponding tableaux system: [6]

Definition 7. *The tableaux system $Tab(C_\omega)$ has the following rules and closure conditions:*

Closure Conditions: Like in the case of $Tab(\mathcal{M}_3^{\{t,\top\}})$, a branch is closed iff for some formula φ it contains either $\{\mathbf{T}^+\varphi, \mathbf{F}^+\varphi\}$, or $\{\mathbf{T}^-\varphi, \mathbf{F}^-\varphi\}$, or $\{\mathbf{F}^+\varphi, \mathbf{F}^-\varphi\}$.

Expansion Rules: The rules $(\mathbf{T}^+\neg)$, $(\mathbf{F}^+\neg)$, $(\mathbf{T}^+\wedge)$, $(\mathbf{F}^+\wedge)$, $(\mathbf{T}^+\vee)$, $(\mathbf{F}^+\vee)$, $(\mathbf{T}^+\supset)$, and $(\mathbf{T}^-\neg)$ of $Tab(\mathcal{M}_4)$, as well as the rules $(\mathbf{F}^+\supset)^w$ and $(\mathbf{F}^+\supset)^I$ of $Tab(\mathbf{N}^-)$.

Analytic Cuts:

$$\frac{S\varphi}{\mathbf{T}^+\psi \mid \mathbf{F}^+\psi} \qquad\qquad \frac{S\varphi}{\mathbf{T}^-\psi \mid \mathbf{F}^-\psi}$$

Where $S \in \{\mathbf{T}^+, \mathbf{F}^+, \mathbf{T}^-, \mathbf{F}^-\}$ and ψ is a subformula of φ

[5] Though similar, we believe that our new semantics is simpler and more intuitive than the one given in [Baa86].
[6] This system is closely related, but not identical, to the Gentzen-type system given for C_ω in [Rag68].

Theorem 6. *A set of signed formulas is unsatisfiable in* C_ω *iff it has a closed tableaux in* $Tab(C_\omega)$.

Proof: We give an outline, leaving details for the full paper. Call a finite set Γ of signed formulas *saturated* if it satisfies the following conditions:

1. Γ has no closed tableau in $Tab(C_\omega)$.
2. If $S\varphi \in \Gamma$ them for every subformula ψ of φ, either $\mathbf{T}^+\psi \in \Gamma$ or $\mathbf{F}^+\psi \in \Gamma$, and either $\mathbf{T}^-\psi \in \Gamma$ or $\mathbf{F}^-\psi \in \Gamma$.
3. With the exception of $(\mathbf{F}^+ \supset)^I$, Γ respects all the expansion rules of $Tab(C_\omega)$ (e.g.: if $\mathbf{T}^+\varphi \wedge \psi \in \Gamma$ then both $\mathbf{T}^+\varphi \in \Gamma$ and $\mathbf{T}^+\psi \in \Gamma$, while if $\mathbf{F}^+\varphi \wedge \psi \in \Gamma$ then either $\mathbf{F}^+\varphi \in \Gamma$ or $\mathbf{F}^+\psi \in \Gamma$).

Because of the presence of the analytic cuts, it is easy to see that if Δ does not have a closed tableau in $Tab(C_\omega)$ then it can be extended to a finite saturated set Δ^*, so that every (ordinary) formula which occurs in Δ^* is a subformula of some formula in Δ. Let I be the set of all saturated sets which have this property. Obviously $\Delta^* \in I$. Define next v^+ and v^- for $\Gamma \in I$ and $\varphi \in \mathcal{F}$ recursively as follows:

- If φ is atomic, then $v^+(\Gamma, \varphi) = f$ iff $\mathbf{F}^+\varphi \in \Gamma$, and $v^-(\Gamma, \varphi) = f$ iff $\mathbf{F}^-\varphi \in \Gamma$.
- If $\varphi = \psi_1 \wedge \psi_2$ then $v^+(\Gamma, \varphi) = f$ iff either $v^+(\Gamma, \psi_1) = f$ or $v^+(\Gamma, \psi_2) = f$, while $v^-(\Gamma, \varphi) = f$ iff $\mathbf{F}^-\varphi \in \Gamma$.
- If $\varphi = \psi_1 \vee \psi_2$ then $v^+(\Gamma, \varphi) = f$ iff $v^+(\Gamma, \psi_1) = f$ and $v^+(\Gamma, \psi_2) = f$, while $v^-(\Gamma, \varphi) = f$ iff $\mathbf{F}^-\varphi \in \Gamma$.
- If $\varphi = \psi_1 \supset \psi_2$ then $v^+(\Gamma, \varphi) = f$ iff there exists $\Gamma^* \supseteq \Gamma$ in I such that $v^+(\Gamma^*, \psi_1) = t$ and $v^-(\Gamma^*, \psi_2) = f$, while $v^-(\Gamma, \varphi) = f$ iff $\mathbf{F}^-\varphi \in \Gamma$.
- If $\varphi = \neg\psi$ then $v^+(\Gamma, \varphi) = f$ iff $v^-(\Gamma, \psi) = f$, while $v^-(\Gamma, \varphi) = f$ iff either $\mathbf{F}^-\varphi \in \Gamma$ or $v^+(\Gamma, \psi) = f$.

We proceed next to show that $\langle I, \subseteq, v^+, v^- \rangle$ is a C_ω-frame, and that each $\Gamma \in I$ is a model of all the signed formulas of Γ. In particular: Δ^* is a model of all the signed formulas in Δ.

Note. With the exception of $(\mathbf{F}^+ \supset)$ and $(\mathbf{F}^- \supset)$, it is possible to add to $Tab(C_\omega)$ all the other expansion rules of $Tab(\mathcal{M}_4)$, and still get a conservative extension of positive intuitionistic logic. It is possible also to modify the semantics in an appropriate way to get soundness and completeness for the resulting system. On the other hand by adding $(\mathbf{F}^- \supset)$ to $Tab(C_\omega)$ we get classical logic.

Note. The crucial step in the proof of the last theorem is to show that the resulting $\langle I, \subseteq, v^+, v^- \rangle$ is indeed a frame. It is at this point where the addition of $(\mathbf{F}^- \supset)$ causes the argument to fail.

One final remark. It is possible to conservatively add a propsitional constant $\mathbf{f}$ to all the systems we have discussed above, together with the extra closure condition that a branch which contains $\mathbf{T}^+\mathbf{f}$ (and optionally also $\mathbf{F}^-\mathbf{f}$) is closed. Hence we could have assumed that full propositional intutionistic logic is contained in all these systems. It seems difficult to satisfactorily handle intuitionistic "negation" itself within our framework, but this is not so important anyway, since this negation is best understood in terms of $\supset$ and $\mathbf{f}$.

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