APERIODICITY OF COCYCLES AND CONDITIONAL LOCAL LIMIT THEOREMS

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Abstract. We establish conditions for aperiodicity of cocycles (in the sense of [**GH**]), obtaining, via a study of perturbations of transfer operators, conditional local limit theorems and exactness of skew–products. Our results apply to a large class of Markov and non–Markov interval maps, including beta transformations.

1. Introduction

Let $(X, \mathcal{B}, m, T)$ be a non–singular transformation, and $\phi : X \to \mathbb{G}$ be a measurable function taking values in a locally compact Abelian polish (LCAP) group $\mathbb{G}$. We say that ϕ is *aperiodic* [**GH**] if the only solutions for $\gamma \circ \phi = \lambda g / g \circ T$ *a.e.* with $\gamma \in \widehat{\mathbb{G}}$, $|\lambda| = 1$ and a measurable *transfer function* $g : X \to \mathbb{S}^1$ are $\gamma \equiv 1$, $\lambda = 1$ and g constant almost everywhere. This condition is crucial for establishing a local limit theorem (LLT) for the m–distributions of $\{\sum_{i=0}^{n-1} \phi \circ T^i\}_{n \geq 1}$, and exactness for the skew–product $T_\phi(x, t) = \big(Tx, t + \phi(x)\big)$ (see [**G**], [**GH**], [**AD2**]).

We focus on fibred systems. A *fibred system* is a quintuple $(X, \mathcal{B}, m, T, \alpha)$ where $(X, \mathcal{B}, m, T)$ is a non–singular transformation on a σ-finite measure space and $\alpha \subset \mathcal{B}$ is a finite or countable partition $\mod m$ such that:

1. $\bigvee_{i=0}^{\infty} T^{-i}\alpha$ generates $\mathcal{B}$;
2. every $A \in \alpha$ has positive measure;
3. for every $A \in \alpha$, $T|_A : A \to TA$ is bimeasurable invertible with non–singular inverse.

The first aim of the paper is to find sufficient conditions for the aperiodicity of α–measurable $\phi : X \to \mathbb{G}$ where $\mathbb{G}$ is a LCAP group.

The reader is invited to prove this when $(X, \mathcal{B}, m, T, \alpha)$ is *independent* in the sense that $m(\bigcap_{j=0}^{n} T^{-j}A_j) = \prod_{j=0}^{n} m(A_j)$ for all $n \geq 1$, $A_1, \dots, A_n \in \alpha$, and $\phi : X \to \mathbb{G}$ (α–measurable) does not take values in a non-trivial, closed coset of $\mathbb{G}$ (see also §2).

In case $(X, \mathcal{B}, m, T, \alpha)$ is *Markov*, i.e. TA is α–measurable for all $A \in \alpha$, and if also α is finite and m is an equilibrium measure (see [**Ke1**]), one can use the work of Livsic [**L**] to obtain periodic point conditions for aperiodicity. Simpler conditions have been established in [**AD1**] for the α–measurable case, using a technique of Kowalski [**Ko1**].

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The non-Markov case is not so well–understood. Morita has a condition for aperiodicity for a certain class of non-Markov Lasota-Yorke maps ([**M**], proof of theorem 5.2), but this class does not include the β–transformation (see below). Kowalski also has a related result ([**Ko2**], theorem 9). We also mention Nicol and Scott [**NS**] who provide rigidity results for the equation $\phi = h - h \circ T$ with T the β-transformation and ϕ Lipschitz or Hölder on $[0,1]$ (see also Pollicott and Yuri [**PY**]).

We give a brief account of our results on aperiodicity. We consider fibred systems which are *skew-product rigid* (§2, Definition 1), a property shared, for example, by many piecewise monotonic interval maps. For such systems, we identify a collection of sets $\mathcal{M}_{rec}$ (§3, Definition 2) for which we prove (theorem 2): if $\gamma \circ \phi = \lambda g/g \circ T$ a.e., then g has a version which is constant on every element of $\mathcal{M}_{rec}$.

We then study the collection $\mathcal{M}_{rec}$, seeking conditions for it to cover X with overlaps, so that every function which is constant on every element of $\mathcal{M}_{rec}$ is necessarily constant everywhere. We call systems of this type *almost onto*, in analogy with the Markov case which was discussed in [**AD1**]. We give examples of skew-product rigid almost onto systems in §4.

For these systems, if $\gamma \circ \phi = \lambda g/g \circ T$, then g is constant, and the dynamical aperiodicity condition reduces to aperiodicity of the distribution of ϕ, i.e. the non-existence of non-trivial $\gamma \in \widehat{\mathbb{G}}, \lambda \in \mathbb{S}^1$ such that $\gamma \circ \phi = \lambda$ a.e. This is equivalent to $\{\phi(x) - \phi(y) : x, y \in X\}$ generating a dense subgroup of $\mathbb{G}$.

Our tests for aperiodicity in non-Markov situations are complemented by a corresponding study of perturbations of transfer operators. In §5 we prove continuity of perturbations for a large class of expanding interval maps, which leads to sufficient conditions for the exactness of skew products and to conditional local limit theorems.

As an illustration, consider the β–transformation $T : [0,1] \to [0,1]$, $T(x) = \beta x \bmod 1$ for $\beta > 1$, together with its absolutely continuous invariant probability measure $d\mathbb{P} = q(x)dx$ (see Parry [**P**]). Define for $x \in [0,1]$, $X_n(x) := [\beta T^{n-1}x]$. The sequence $\{X_n(x)\}_{n\geq 1}$ is called the (greedy) *β–expansion* of x, because

$$x = \sum_{n=1}^{\infty} \frac{1}{\beta^n} X_n(x).$$

We apply our results to the study of the stochastic behaviour of $\{X_n\}_{n\geq 1}$. If β is an integer, then X_n are i.i.d's. We prove that the following stochastic properties, well–known for i.i.d's (see Feller [**F**], §IV.6 and §VII.4), persist for non–integer β (when $\{X_n\}_{n\geq 1}$ may not be Markov):

(1) *de Moivre's approximation:* If $S_n := \sum_{k=1}^{n} X_k$, then

$$\sigma\sqrt{n}\mathbb{P}(S_n = k_n) \to \tfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ as } n \to \infty,\ k_n \in \mathbb{Z},\ \tfrac{k_n - nE(X_1)}{\sigma\sqrt{n}} \to x$$

uniformly as $x \in K$ for all $K \subset \mathbb{R}$ compact.

(2) *Asymptotics of random walks on $\mathbb{R}$ driven by "β-jumps"*: Suppose that $\psi : [0,1] \to \mathbb{R}$ satisfies $E(\psi) = 0$ and $\psi(x) = a_{[\beta x]}$ where $\{a_i - a_j : 0 \leq i, j \leq [\beta]\}$ are rationally independent, then T_ψ is conservative, exact and pointwise dual ergodic with $a_n(T_\psi) \propto \sqrt{n}$ (as defined in e.g. [**A**]).

(3) *The Hewitt-Savage zero–one law:* Call $x, y \in [0,1]$ β*-exchangeable* if their β–expansions differ by a finite permutation. If a Borel set E satisfies

$$x \in E,\ y \text{ and } x \text{ are } \beta\text{-exchangeable} \implies y \in E,$$

then $\mathbb{P}(E)$ is equal to zero or one.

De Moivre's approximation follows from aperiodicity of $\phi : [0,1] \to \mathbb{Z}$, $\phi(x) := [\beta x]$ (see [**RE**]), and the Hewitt–Savage zero–one law (for non-integer β) follows from the aperiodicity of $F^{\#} : [0,1] \to \mathbb{Z}^{[\beta]}$, $F^{\#}(x) = (\delta_{\phi(x),1}, \ldots, \delta_{\phi(x),[\beta]})$ (see [**G**]). Details are given in §6.

2. Fibred Systems, Skew-Products and skew-product rigidity

Let $(X, \mathcal{B}, m, T, \alpha)$ be a fibred system. Elements of $\alpha_n := \bigvee_{i=0}^{n-1} T^{-i}\alpha$ are called *cylinders of length* n. We agree to call X (the) cylinder of length zero. We denote the cylinder of length n which contains $x \in X$ by $\alpha_n(x)$. We say that a set $E \subseteq X$ is *almost open* $\bmod m$, if for almost every $x \in E$, there exists an n such that $\alpha_n(x) \subseteq E \bmod m$. Ergodic sums of $\phi : X \to \mathbb{G}$ are denoted by $\phi_n := \phi + \phi\circ T + \ldots + \phi \circ T^{n-1}$.

Throughout m_λ will denote Lebesgue measure. A *piecewise monotonic (resp. increasing) map of the interval* is a triple (X, T, α) where X is an interval, α is a finite or countable generating partition $(\bmod m_\lambda)$ of X into open intervals, and $T : X \to X$ is a map such that $T|_A$ is continuous and strictly monotonic (resp. increasing) for each $A \in \alpha$. For piecewise monotonic maps of the interval equipped with a non-atomic measure, all cylinders are intervals, and therefore a set is almost open iff it is equal to an open set $\bmod m_\lambda$.

Recall that the *Frobenius–Perron operator* or *transfer operator* of a non-singular transformation $(X, \mathcal{B}, m, T)$ is the (unique) operator $P_T : L^1(m) \to L^1(m)$ which satisfies

$$\forall g \in L^\infty, f \in L^1 \int g \cdot P_T f dm = \int g \circ T \cdot f dm.$$

If $(X, \mathcal{B}, m, T, \alpha)$ is a fibred system, then $T : A \to TA$ has a non singular inverse $v_A : TA \to A$ for each $A \in \alpha$, and the Frobenius-Perron operator of T is

$$P_T f = \sum_{A \in \alpha} 1_{TA} v_A' \cdot f \circ v_A, \text{ where } v_A' := \frac{dm \circ v_A}{dm}.$$

We are interested in the collection of all skew-products of the form

$$\tau_S : X \times Y \to X \times Y\ ,\ \tau_S(x,y) = \big(Tx, S(\alpha(x))(y)\big)$$

where $(Y, \mathcal{F}, \mu)$ is a Lebesgue probability space, $Aut(Y)$ is the collection of its automorphisms (invertible bi-measurable measure–preserving transformations), and $S : \alpha \to Aut(Y)$ is arbitrary. We call these transformations *skew–products over* α. We note for future reference that $\tau_S^n(x,y) = \big(T^n x, S(\alpha_n(x))(y)\big)$, where for every cylinder $C = [A_0, \ldots, A_{n-1}]$, $S(C) := S(A_{n-1}) \circ \ldots \circ S(A_0)$, and that the transfer operator of τ_S is

$$(P_{\tau_S} f)(x,y) = \sum_{A \in \alpha} 1_{TA}(x) v_A'(x) f\big(v_A(x), S\big(A\big)^{-1} y\big). \tag{1}$$

Definition 1. *A fibred system* $(X, \mathcal{B}, m, T, \alpha)$ *is called* skew-product rigid *if a.e.* $x \in X$ *is included in a cylinder of finite measure, and if for every invariant density* $h(x,y)$ *(not necessarily integrable) of an arbitrary skew–product over* α, $[h(\cdot, y) > 0]$ *is almost open for a.e.* $y \in Y$.

The following proposition shows a stronger property for independent fibred systems.

Proposition 1 ([**M1**]). *Let $(X, \mathcal{B}, m, T, \alpha)$ be an independent fibred system and suppose that $S : \alpha \to Aut(Y)$. If $h \in L^1(m \times \mu)$ satisfies $P_{\tau_S} h = \lambda h$ for some $\lambda \in \mathbb{S}^1$, then h is $X \times \mathcal{F}$-measurable.*

Proof. A calculation shows that

$$P^n_{\tau_S} h(x, y) = P^n_T(h(\cdot, S(\alpha_n(\cdot))^{-1}(y))(x).$$

To see that h is $X \times \mathcal{F}$-measurable, let h_n be $\alpha_n \times \mathcal{F}$-measurable so that $\|h - h_n\|_{L^1(m\times\mu)} \to 0$. Evidently

$$\begin{aligned} P^n_{\tau_S} h_n(x, y) = P^n_T(h_n(\cdot, S(\alpha_n(\cdot))^{-1}(y))(x) = \\ = E(h_n(\cdot, S(\alpha_n(\cdot))^{-1}(y)) = E(P^n_{\tau_S} h_n | X \times \mathcal{F}). \end{aligned}$$

This allows us to bound $\|h - E(h|X \times \mathcal{F})\|_1$ by

$$\begin{aligned} &\|h - P^n_{\tau_S} h\|_1 + \|P^n_{\tau_S} h - P^n_{\tau_S} h_n\|_1 + \\ &+ \|E(P^n_{\tau_S} h_n | X \times \mathcal{F}) - E(P^n_{\tau_S} h | X \times \mathcal{F})\|_1 + \|E(P^n_{\tau_S} h | X \times \mathcal{F}) - E(h|X \times \mathcal{F})\|_1 \le \\ &\le 2\|h - P^n_{\tau_S} h\|_1 + 2\|h - h_n\|_1 = 2|1 - \lambda^n| \cdot \|h\|_1 + 2\|h - h_n\|_1. \end{aligned}$$

The limit inferior of this estimate is zero, so $h = E(h|X \times \mathcal{F})$ almost everywhere. □

Corollary 1. *If $(X, \mathcal{B}, m, T, \alpha)$ is independent, $\mathbb{G}$ is a LCAP group, $\phi : X \to \mathbb{G}$ is α–measurable and does not take values in a non-trivial, closed coset of $\mathbb{G}$, then ϕ is aperiodic.*

Proof. Suppose that $\gamma \circ \phi = \frac{\lambda g \circ T}{g}$ where $\gamma \in \widehat{\mathbb{G}}$, $\lambda \in \mathbb{S}^1$ and $g : X \to \mathbb{S}^1$ is measurable. Setting $Y = \mathbb{S}^1$, $\mu =$ Lebesgue measure and $S(a)(y) := \gamma \circ \phi(a) y$ we see that $P_{\tau_S} h = \lambda h$ where $h(x, y) := g(x)y$. By the previous proposition $h(x, y) = h(y)$, whence $\gamma \circ \phi \equiv \lambda$. It follows that $\gamma \equiv 1 = \lambda$. □

We discuss some other examples. Consider the following properties for a piecewise monotonic map of the interval (X, T, α):

- (A) *Adler's condition*: for all $A \in \alpha$, $T|_A$ extends to a C^2 map on $\overline{A}$ and $T''/(T')^2$ is bounded on X.
- (F) *Finite images*: $\{TA : A \in \alpha\}$ is finite.
- (U) *Uniform expansion*: $\inf |T'| > 1$.
- (N) *Non-uniform expansion*: there is a *finite* set of partition sets $\zeta \subseteq \alpha$ such that every $Z \in \zeta$ has an indifferent fixed point $x_Z \in \partial Z$ with *Thaler's assumptions*:
  - (a) $Tx \xrightarrow[x \to x_Z, x \in Z]{} x_Z$ and $T'x \xrightarrow[x \to x_Z, x \in Z]{} 1$.
  - (b) x_Z is a *one-sided regular source*: T' decreases on $(-\infty, x_Z) \cap Z$ and increases on $(x_Z, \infty) \cap Z$ (one of these conditions is empty).
  - (c) for every $\epsilon > 0$ there exists $\rho(\epsilon) > 1$ such that $|T'| \ge \rho(\epsilon)$ on

$$X_\epsilon := X \setminus \bigcup_{Z \in \zeta} Z \cap (x_Z - \epsilon, x_Z + \epsilon).$$

Piecewise monotonic maps (X, T, α) of the interval with properties (A),(F),(U) (respectively (A),(F),(N)) will be called *AFU maps* (respectively *AFN maps*). They admit at least one finite (respectively σ–finite) absolutely continuous invariant measure m, cf. [**Z1**]. In this context, $\mathcal{B}$ will always denotes the Borel σ–algebra.

Suppose that (X, T, α) is an AFU map, and let $h(x, y)$ be some invariant density of an arbitrary skew-product over α, then:

(1) If α is a Markov partition, then it can be shown that for almost every y, $h(\cdot, y) : I \to I$ has a piecewise Hölder version (see [**Ko1**] and proposition 3.6 in [**AD1**]).
(2) When α is not necessarily a Markov partition, it can be shown that for almost every y, $h(\cdot, y) : I \to \mathbb{R}$ has a version with bounded variation (see lemma 4 in [**Ko2**]).

Thus, AFU maps (with or without the Markov property) are skew-product rigid. Actually the same is true for AFN maps, which shows that this property of fibred systems does not depend on the existence of an absolutely continuous invariant probability measure.

Theorem 1. *AFN maps are skew-product rigid.*

We begin with an account of the basic structure of AFN maps (cf. [**Z1**, **Z2**]): Every AFN map has an absolutely continuous, invariant measure (a.c.i.m.) $m \ll m_\lambda$ with the following decomposition:

$$X = \biguplus_{i=1}^{N} \biguplus_{j=0}^{N_i - 1} T^j X_i \bmod m \ , \ T^{N_i} X_i = X_i \bmod m_\lambda$$

$$\text{and} \quad X = \bigcup_{n=1}^{\infty} T^{-n} \left(\bigcup_{i=1}^{N} \biguplus_{j=0}^{N_i - 1} T^j X_i \right) \bmod m_\lambda.$$

Each X_i is a finite union of intervals and $T^{N_i} : X_i \to X_i$ is conservative exact. Moreover, $m(X_i) = \infty$ iff X_i contains a (possibly one–sided punctured) neighbourhood of x_Z for some $Z \in \zeta$, and in this case $N_i = 1$.

The restriction of an AFN (resp. AFU) map to one of its ergodic components $\bigcup_{j=0}^{N_i - 1} T^j X_i$'s is called a *basic* AFN (resp. AFU) map.

The proof of theorem 1 is based on an inducing procedure which we now describe. Let $(X, \mathcal{B}, m, T, \alpha)$ be a conservative ergodic measure–preserving fibred system. Fix some α–measurable set A with an α–measurable partition η (for interval maps this will be the partition into connected components), and write $A = \biguplus_{i \in \Lambda} A_i$ with $A_i \in \alpha$. The *induced system on A* is the fibred system $(A, \mathcal{B}_A, m_A, T_A, \alpha_A)$ where $\mathcal{B}_A := \{E \in \mathcal{B} : E \subseteq A\}$, $m_A = m|_{\mathcal{B}_A}$, $T_A = T^\varphi$ where

$$\varphi(x) = 1_A(x) \inf\{n \geq 1 : T^n(x) \in A\}$$

and $\alpha_A = \alpha_A(\eta) = \big\{[A_i, B_1, \ldots, B_n, C] : i \in \Lambda, n \geq 0, B_j \in \alpha \setminus \{A_k\}_{k \in \Lambda}, C \in \eta\big\}$.

Lemma 1. *Let $(X, \mathcal{B}, m, T, \alpha)$ be a conservative ergodic measure–preserving fibred system. Suppose that*

$$(2) \qquad \forall n, k \forall C \in \alpha_k,\ T^n(C) \textit{ is almost open mod } m.$$

If there is some α–measurable set A such that $(A, \mathcal{B}_A, m_A, T_A, \alpha_A)$ is skew-product rigid, then so is $(X, \mathcal{B}, m, T, \alpha)$.

Proof. Fix some skew–product over α, $\tau = \tau_S : X \times Y \to X \times Y$ where $(Y, \mathcal{F}, \mu)$ is some standard probability space, and suppose $h(x, y) \geq 0$ is an invariant density for τ . We show that $[h(\cdot, y) > 0]$ is almost open mod m for a.e. y.

We check that τ is conservative. Indeed, for every $B \in \alpha$

$$\sum_{n=1}^{\infty} 1_{B\times Y}\circ\tau^n \equiv \sum_{n=1}^{\infty} 1_B\circ T^n = \infty \quad m\times\mu\text{–almost everywhere in } B\times Y.$$

so $B\times Y$ is in the conservative part of τ for all $B\in\alpha$. It follows that we can induce τ on $A\times Y$. The result is a skew–product over α_A, $\tau_{S_A}: A\times Y\to A\times Y$ where $S_A:\alpha_A\to Aut(Y)$ is

$$S_A([A_i,B_1,\ldots,B_{n-1},A_j]) := S(B_{n-1})\circ\ldots\circ S(B_1)\circ S(A_i).$$

The set $A\times Y$ is a sweep–out set for τ, because $\bigcup_{k=1}^{\infty}\tau^{-k}(A\times Y)=\bigcup_{k=1}^{\infty}T^{-k}A\times Y$ and T is conservative ergodic. We can therefore apply Kac's formula. Writing $\widetilde{h}=h\cdot 1_{A\times Y}$ and recalling the definition of the Frobenius–Perron operator of τ, P_τ, we get for all $f\in L^\infty(X\times Y)$:

$$\begin{aligned}
\int_{X\times Y} fhd(m\times\mu) &= \int_{A\times Y}\sum_{i=0}^{\varphi-1} f\circ\tau^i hd(m\times\mu)\\
&= \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_{X\times Y} 1_{[\varphi=n]\times Y}\widetilde{h}f\circ\tau^i d(m\times\mu)\\
&= \sum_{i=0}^{\infty}\sum_{n=i+1}^{\infty}\int_{X\times Y} P_\tau^i\left(1_{[\varphi=n]\times Y}\widetilde{h}\right) fd(m\times\mu).
\end{aligned}$$

It follows that

$$h=\sum_{n=0}^{\infty}P_\tau^n\left(\widetilde{h}1_{[\varphi>n]\times Y}\right).$$

If $\widetilde{\tau}=(\tau)_{A\times Y}=\tau_{S_A}$, then $\widetilde{h}d(m\times\mu)$ is $\widetilde{\tau}$–invariant. By assumption, the system $(A,\mathcal{B}_A,m_A,T_A,\alpha_A)$ is skew-product rigid, and it is easy to use this to check that

$$[\widetilde{h}(\cdot,y)>0] \text{ is almost open mod } m, \text{ for } \mu\text{–a.e. } y\in Y. \tag{3}$$

Now, if $H^y:=[h(\cdot,y)>0]$, then (1) gives, $\bmod m$,

$$\begin{aligned}
H^y &= \Big\{x\in X:\sum_{n=1}^{\infty}P_\tau^n\left(\widetilde{h}1_{[\varphi>n]\times Y}\right)>0\Big\}\\
&= \bigcup_{n=1}^{\infty}\Big\{x\in X: P_\tau^n\left(\widetilde{h}1_{[\varphi>n]\times Y}\right)>0\Big\}\\
&= \bigcup_{n=1}^{\infty}\Big\{x\in X:\sum_{C\in\alpha_n}1_{T^nC}(x)v_C'(x)\widetilde{h}\big(v_C(x),S(C)^{-1}(y)\big)1_{[\varphi>n]}(v_C(x))>0\Big\}\\
&= \bigcup_{n=1}^{\infty}\bigcup_{C\in\alpha_n}T^n(C)\cap T^n([\varphi>n]\cap C)\cap\{x\in X:\widetilde{h}\big(v_C(x),S(C)^{-1}(y)\big)>0\}\\
&= \bigcup_{n=1}^{\infty}\bigcup_{C\in\alpha_n}T^n\big([\varphi>n]\cap C\big)\cap\big\{x\in X:\widetilde{h}\big(v_C(x),S(C)^{-1}(y)\big)>0\big\}.
\end{aligned}$$

$T^n\big([\varphi>n]\cap C\big)$ is a union of images of cylinders, so it is almost open mod m by (2). We claim that $G(y,n,C):=\big\{x\in X:\widetilde{h}\big(v_C(x),S(C)^{-1}(y)\big)>0\big\}$ is almost open mod m for μ–almost all $y\in Y$.

By (3), and since α_A-cylinders are α-cylinders, there exists some $Y_1 \subseteq Y$ such that

$$\mu(Y \setminus Y_1) = 0 \text{ and } \forall y \in Y_1 \; A \cap [\widetilde{h}(\cdot, y) > 0] \text{ is } m\text{-almost open.}$$

Set $Y' := \bigcap\{S(C')^{-1}(Y_1) : C' \text{ is a cylinder }\}$. Since for every C', $S(C') \in Aut(Y)$,

$$\mu(Y \setminus Y') = 0 \text{ and } \forall y \in Y' \forall C' \in \alpha_n,\ C' \cap [\widetilde{h}(\cdot, S(C')^{-1}y) > 0] \text{ is } m\text{-almost open.}$$

Now fix $y \in Y'$. Since $m \circ v_C^{-1} \sim m$, for almost every $x \in [\widetilde{h}(v_C(x), S(C)^{-1}y) > 0]$ there is a cylinder B such that

$$v_C(x) \in B \subseteq C \cap [\widetilde{h}(\cdot, S(C)^{-1}y) > 0].$$

Choose, using (2), a cylinder $D \subseteq \alpha_n(x) \cap T^n(B)$ which contains x. If $x' \in D$, then $v_C(x') \in B$ and so $h(v_C(x'), S(C)^{-1}y) > 0$. It follows that $x \in D \subseteq G(y, n, C)$. This shows that $G(y, n, C)$ is almost open for all $y \in Y_2$, $n \in \mathbb{N}$, and $C \in \alpha_n$. Since, again by (2), $T^n([\varphi > n] \cap C)$ is almost open mod m, we have that $H^y = [h(\cdot, y) > 0]$ is almost open mod m for μ–almost every y, completing the proof. $\square$

Proof of theorem 1. We can assume without loss of generality that $(X, \mathcal{B}, m, T)$ is conservative and ergodic (otherwise decompose T to its basic components as explained in the the beginning of the section, and treat each component separately).

Lemma 8 of [**Z2**] shows that every conservative ergodic AFN–map has an α_2–measurable sweep out set $A \subseteq X$ with a finite partition η into connected components such that the induced system on A is AFU, and hence skew-product rigid. It follows from lemma 1 that $(X, \mathcal{B}, m, T, \alpha_2)$ also has this property . (AFN maps are piecewise monotonic, so (2) holds, because cylinders are intervals, and images of intervals are almost open.) It remains to observe that $(X, \mathcal{B}, m, T, \alpha)$ is skew-product rigid as soon as $(X, \mathcal{B}, m, T, \alpha_2)$ is. $\square$

3. Aperiodicity

Let $(X, \mathcal{B}, m, T, \alpha)$ be a fibred system. Elements of

$$\mathcal{M} := \{T^n \alpha_n(x) : n \geq 1, x \in X\} \cup \{X\}$$

are called *image sets*. We will be mainly interested in fibred systems for which every image set is almost open. This is the case for piecewise monotonic maps of the interval, for example.

Definition 2. *A cylinder C of length n_0 is called a* cylinder of full returns*, if for almost all $x \in C$ there exist $n_k \uparrow \infty$ such that $T^{n_k} \alpha_{n_k + n_0}(x) = C$. In this case we say that $T^{n_0}(C)$ is a* recurrent image set*, and write*

$$\mathcal{M}_{rec} := \{J : J \textit{ is a recurrent image set}\}.$$

Here, we agree to call X is a cylinder of length zero.

A measurable map $f : X \to S$ (S some arbitrary set) is called a *colouring* of a collection $\mathcal{C} \subset \mathcal{B}$, if $f|_C$ is almost everywhere equal to a constant for every $C \in \mathcal{C}$. The constant colourings are called *trivial colourings.*

Definition 3. *A fibred system is called* almost onto *if all the colourings of $\mathcal{M}_{rec}$ are trivial (in particular, $X = \bigcup \mathcal{M}_{rec} \bmod m$). A map for which $X \in \mathcal{M}_{rec}$ is called* quasi–beta.

The beta transformation is quasi–beta (see §4.2). Other examples of almost-onto maps (quasi-beta and not quasi–beta) are given in §4.

Theorem 2 ($\mathcal{M}_{rec}$-measurability of the transfer function g)**.** *Let $(X, \mathcal{B}, m, T, \alpha)$ be a skew-product rigid measure–preserving fibred system whose image sets are almost open. Let $\mathbb{G}$ be a LCAP group. If $\gamma \circ \phi = \lambda g / g \circ T$ a.e. where $\phi : X \to \mathbb{G}$ is α-measurable, $\gamma \in \widehat{\mathbb{G}}$, and $\lambda \in \mathbb{S}^1$, then g is constant on every recurrent image set.*

Corollary 2. *If in addition $(X, \mathcal{B}, m, T, \alpha)$ is almost onto, then ϕ is aperiodic iff the group generated by $\{\phi(x) - \phi(y) : x, y \in X\}$ is dense in $\mathbb{G}$.*

Proof. Suppose $\gamma \circ \phi = \lambda g / g \circ T$. By theorem 2, g is a colouring of $\mathcal{M}_{rec}$, whence constant. It follows that $\gamma \circ \phi = \lambda$ and the corollary easily follows. $\square$

Remark 1. *If α is a Markov partition and T is conservative, then $\mathcal{M}_{rec} = \{TA : A \in \alpha\}$ and the theorem reduces to theorem 3.1 in* [**AD1**].

Proof. In this case, every cylinder of positive measure is a cylinder of full returns, and for every cylinder $C = [A_0, \ldots, A_{n_0-1}]$, $T^{n_0}(C) = T(A_{n_0-1})$. Therefore $\mathcal{M}_{rec} = \{T(A) : A \in \alpha\}$. The map is almost onto iff the only colouring of $\{T(A) : A \in \alpha, m(A) > 0\}$ is trivial, and this is equivalent to the almost onto condition mentioned in [**AD1**]: $\forall A, A' \in \alpha, \exists B_1, \ldots, B_n \in \alpha$ such that

$$m(TA \cap TB_1), m(TB_1 \cap TB_2), \ldots, m(TB_n \cap TA') > 0.$$

This reduces theorem 2 to theorem 3.1 in [**AD1**]. $\square$

Remark 2. *If $(X, \mathcal{B}, m, T, \alpha)$ is Markov and skew product rigid, then it is almost onto iff $F^\sharp$ is aperiodic, where fixing $a_0 \in \alpha$, $F^\sharp : X \to \mathbb{Z}^{\alpha \setminus \{a_0\}}$ is defined by $F^\sharp(x)_a := \delta_{a, \alpha(x)}$ $(a \in \alpha \setminus \{a_0\})$. Thus the almost onto condition in Corollary 1 cannot be omitted.*

Proof. Almost onto implies $F^\sharp$ aperiodic by Corollary 1.

To see the converse, it suffices to show that if T is not almost onto, then there exists an α measurable $\phi : X \to \mathbb{Z}$ which is not aperiodic, even though $\{\phi(x) - \phi(y) : x, y \in X\}$ generates $\mathbb{Z}$.

If the system is not almost onto, then there exists some α-measurable two-set partition $\widetilde{\alpha} = \{A_-, A_+\}$ of X such that each TA, $A \in \alpha$, is contained in A_- or in A_+. (Let α_* be the finest partition with the property that each TA is contained in some atom of α_*. By assumption, α_* is nontrivial. Fix any $A_- \in \alpha_*$ and let $A_+ := X \setminus A_-$.) Define

$$\phi(x) := \begin{cases} 0 & \text{if } x \in X_0 := \{x : \widetilde{\alpha}(x) = \widetilde{\alpha}(Tx)\} \\ 1 & \text{if } x \in X_1 := \{x : \widetilde{\alpha}(x) \neq \widetilde{\alpha}(Tx)\} \end{cases}$$

which is measurable α since the sets X_0, X_1 are. By transitivity, the X_i are nonempty. Letting

$$g(x) := \begin{cases} 1 & \text{if } x \in A_+ \\ -1 & \text{if } x \in A_- \end{cases}$$

we have

$$e^{i\pi\phi} = \frac{g}{g \circ T} = \begin{cases} 1 & \text{on } X_0 \\ -1 & \text{on } X_1. \end{cases}$$

This shows that $\phi : X \to \mathbb{Z}$ is not aperiodic, even though $\{\phi(x) - \phi(y) : x, y \in X\}$ generates $\mathbb{Z}$. $\square$

Proof of theorem 2. The proof is based on the following statement:

$(*)$ g is constant on every cylinder of full returns.

Given $(*)$ the proof of the theorem is as follows. Fix $J \in \mathcal{M}_{rec}$, and choose some cylinder of full returns $C \in \alpha_{n_0}$ such that $J = T^{n_0}(C)$. Let $g(C)$ be the value of g on C, and define $v_C : J \to C$ to be the inverse of $T^{n_0} : C \to J$. Then $\gamma \circ \phi_{n_0} = \lambda^{n_0} g / g \circ T^{n_0}$, whence $\gamma \circ \phi_{n_0} \circ v_C = \lambda^{n_0} g \circ v_C / g$. Therefore, if $\phi_{n_0}(C)$ is the value of ϕ_{n_0} on C, then

$$g(x) = \frac{\lambda^{n_0} g(C)}{\gamma(\phi_{n_0}(C))} \quad (x \in J)$$

which proves that g is constant on J. This proves the theorem.

We prove $(*)$ first under the additional assumption that T is quasi–beta, and then in the general case. We use the following concept, essentially due to to Kowalski [**Ko1**, **Ko2**]:

Definition 4. *Let $(X, \mathcal{B}, m, T, \alpha)$ be a fibred system.*

(1) *A skew–product over α is called* simple *if each of its invariant densities $h(x,y)$ satisfies $[h > 0] \in \alpha \otimes \mathcal{F}$.*[1]
(2) *$(X, \mathcal{B}, m, T, \alpha)$ is* weak quasi–Markov (wqM)*, if all skew–products over α are simple.*

Remark 3. 1) These definitions can be made with $Aut(Y)$ replaced by the collections of the null–preserving transformations of $(Y, \mathcal{F}, \mu)$, or the non–singular transformations of $(Y, \mathcal{F}, \mu)$. The corresponding properties are then called *strong quasi–Markov* (sqM) and *quasi–Markov* (qM). Note that [**Ko1**] states $qM \Rightarrow \alpha$ is a Markov partition, but only proves $sqM \Rightarrow \alpha$ is a Markov partition.

2) It is not hard to show that a probability preserving fibred system $(X, \mathcal{B}, m, T, \alpha)$ is wqM iff for every $S : \alpha \to \mathrm{Aut}(Y)$ every τ_S–invariant set is $\alpha \otimes \mathcal{F}$–measurable.

3) Using 2) it is not hard to show that if $(X, \mathcal{B}, m, T, \alpha)$ is an almost onto, wqM probability preserving fibred system, and $S : \alpha \to \mathrm{Aut}(Y)$, then the joint ergodicity of $\{S(a) : \ a \in \alpha\}$ implies the ergodicity of τ_S, and indeed, $\lambda \in \mathbb{S}^1$ is an eigenvalue for τ_S iff there is an $h : Y \to \mathbb{C}$ satisfying $h \circ S(a) = \lambda h$ for all $a \in \alpha$.

Returning to the proof of theorem 2, we show that if $(X, \mathcal{B}, m, T, \alpha)$ is skew-product rigid and is quasi-beta, then it is weakly quasi–Markov. We then show that the weak quasi–Markov property implies $(*)$, thus proving the theorem in the case of quasi-beta systems.

Step 1. *A skew-product rigid fibred system which is quasi–beta is weak quasi–Markov.*

Proof. Let $(X, \mathcal{B}, m, T, \alpha)$ be a skew-product rigid quasi–beta fibred system. Fix some standard probability space $(Y, \mathcal{F}, \mu)$ and let $\tau_S : X \times Y \to X \times Y$ be some skew–product over α. We must show that every non-negative measurable solution of $P_{\tau_S} h = h$ satisfies $[h > 0] \in \alpha \otimes \mathcal{F}$. Fix such an h and set $E := [h > 0]$.

Recall that the y–section of a set E is $E^y := \{x \in X : (x,y) \in E\}$. For every $B \in \mathcal{B}$ set $F_E(B) := \{y \in Y : B \subseteq E^y \bmod m\}$. This is $\mathcal{F}$–measurable, because

[1] Here and throughout $\mathcal{F}_1 \otimes \mathcal{F}_2$ denotes the *completion* of the product σ–algebra, and invariant densities are not required to be integrable.

$E \in \mathcal{B} \otimes \mathcal{F}$.[2] We show that $E = E_1 \bmod m \times \mu$, where

$$E_1 := \{(x, y) \in E : \alpha_1(x) \times \{y\} \subseteq E\} \equiv \bigcup_{A \in \alpha} A \times F_E(A)$$

thereby proving the proposition.

We claim that

$$E = \bigcup_{C \text{ cylinder}} C \times F_E(C) \bmod m \times \mu,$$

(1) $RHS \subseteq LHS$: Enough to see that for every cylinder C, $(m \times \mu)\big([C \times F_E(C)] \setminus E\big) = 0$. This is because $[C \times F_E(C)] \setminus E = \bigcup_{y \in F_E(C)} \{x \in C : x \in C \setminus E^y\} \times \{y\}$ and every y–section of this set has measure zero.

(2) To see the other inclusion, fix y and suppose $x \in (LHS \setminus RHS)^y := \{x : (x, y) \in LHS \setminus RHS\}$. Then $x \in [h(\cdot, y) > 0]$ and there is no n such that $\alpha_n(x) \subseteq [h(\cdot, y) > 0]$. The system being skew-product rigid, we find that

$$m\left((LHS \setminus RHS)^y\right) = 0 \;\text{ for } \mu\text{-a.e } y \in Y.$$

It follows from Fubini's theorem that $LHS \subseteq RHS \bmod m \times \mu$.

Therefore, if $E \neq E_1 \bmod m \times \mu$, then there is a cylinder of positive measure $C = [A_0, \dots, A_{n-1}]$ such that $\mu(F_E(C) \setminus F_E(A_0)) > 0$ (otherwise $C \times F_E(C) \subseteq E_1$ for all cylinders C, and this implies $E_1 \supseteq E$). If $F := F_E(C) \setminus F_E(A_0)$, then $E \setminus E_1 \supseteq C \times F \bmod m \times \mu$. This shows that if $E \neq E_1 \bmod m \times \mu$, then there is a cylinder C and an $\mathcal{F}$–measurable F such that

$$E \setminus E_1 \supseteq C \times F \text{ and } (m \times \mu)(C \times F) > 0.$$

We show that $C \supseteq \widetilde{C}$ where $\widetilde{C} = [A_0, \dots A_{N-1}]$ is a cylinder of length N such that if $\widetilde{S} = S(\widetilde{C})$, then

$$m(\widetilde{C}) > 0\ ,\ T^N(\widetilde{C}) = X\ ,\ \mu\big(F \cap \widetilde{S}(F)\big) > 0. \tag{4}$$

The quasi-beta property is that for a.e. $x \in C$, $T^n \alpha_n(x) = X$ infinitely often. It follows that $C \supseteq C' = [A_0, \dots, A_{m-1}]$ where $A_i \in \alpha$, $m(C') > 0$, and $T^m(C') = X$. Set $\overline{S} := S_{A_{m-1}} \circ \cdots \circ S_{A_0}$. This is an automorphism of $(Y, \mathcal{F}, \mu)$, so there exists some $k \geq 1$ such that $\mu\big(F \cap \overline{S}^k(F)\big) > 0$. If $\widetilde{C} := \bigcap_{i=0}^{k-1} T^{-im} C' \in \alpha_{km}$, then

$$\begin{aligned}
T^{mk}(\widetilde{C}) &= T^{mk}(C' \cap T^{-m}C' \cap \ldots \cap T^{-(k-1)m}C') \\
&= T^{mk}\big((T^m|_{C'})^{-1}(C' \cap T^{-m}C' \cap \ldots \cap T^{-m(k-2)}C')\big) \\
&= T^{m(k-1)}\big(C' \cap T^{-m}C' \cap \ldots \cap T^{-m(k-2)}C'\big) = \ldots = T^m(C') = X.
\end{aligned}$$

Finally, note that the local invertibility property of $(X, \mathcal{B}, m, T, \alpha)$ and $T^N \widetilde{C} = X$ imply that $m(\widetilde{C}) > 0$, so (4) is satisfied with $N = mk$ and $\widetilde{C}$.

We can now derive the contradiction which proves that $E \neq E_1 \bmod m \times \mu$ is impossible. Set $\widetilde{F} = F \cap \widetilde{S}(F)$, and consider $\widetilde{C} \times \widetilde{F}$. By construction,

$$A_0 \times \widetilde{F} \subseteq T^N \widetilde{C} \times \big(F \cap \widetilde{S}(F)\big) \subseteq \tau_S^N(\widetilde{C} \times F) \subseteq \tau_S^N(C \times F) \subseteq \tau_S^N(E) \subseteq E$$

[2] To see this define $\psi(y) := \int_B 1_E(x, y)dm(x)$. Then $y \in F_E(B)$ iff $\psi(y) = m(B)$. The measurability of $F_E(B)$ now follows from Fubini's theorem, which says that ψ is $\mathcal{F}$–measurable.

because $E = [h > 0]$ and h is an invariant density, so $\tau_S(E) \subseteq E$.[3]

It follows that $A_0 \times \widetilde{F} \subseteq E_1$, whence $\widetilde{C} \times \widetilde{F} \subseteq E_1$. But this is impossible, since $\widetilde{C} \times \widetilde{F} \subset C \times F \subseteq E \setminus E_1 \bmod m \times \mu$ and $(m \times \mu)(\widetilde{C} \times \widetilde{F}) > 0$.

Step 2. *The weak quasi–Markov property implies that g is α–measurable for measure–preserving fibred systems.*

Proof ([Kow], [AD]). Set $\psi := \gamma \circ \phi : X \to \mathbb{S}^1$. Let $Y = \mathbb{S}^1$ equipped with Lebesgue measure μ, and consider $S : \alpha \to Aut(Y)$ given by $S(A)(y) = \overline{\lambda} y \cdot \psi(A)$ where $\psi(A)$ is the value of ψ on A. The corresponding skew–product, $\tau_S : X \times Y \to X \times Y$, is $\tau_S(x, y) = (Tx, \overline{\lambda} y \cdot \psi(x))$.

A calculation shows that $h(x, y) := g(x) \cdot y$ satisfies $h \circ \tau_S = h$. Thus, every level set of h, $A_t = [h < t]$ is τ_S–invariant, and since T preserves m, 1_{A_t} is an invariant density for τ_S. By the weak quasi-Markov property, $A_t \in \alpha \otimes \mathcal{B}(\mathbb{S}^1)$, and it follows that h is $\alpha \otimes \mathcal{B}(\mathbb{S}^1)$–measurable. This can only happen if g is α–measurable.

This proves $(*)$ and the theorem in the case when T is quasi–beta.

We now consider the general case. First, we note that we may assume without loss of generality that $\lambda = 1$. Indeed, suppose $\lambda = e^{i\theta}$, and define $\widetilde{\mathbb{G}} := \mathbb{G} \times \mathbb{R}$, $\widetilde{\phi}(x) = (\phi(x), -\theta)$, and $\widetilde{\gamma}(x, t) = e^{it}\gamma(x)$. Then $\widetilde{\gamma} \circ \widetilde{\phi} = g/g \circ T$, and $\widetilde{\phi}$ is α–measurable, so we in the situation of the theorem but with $\lambda = 1$. Henceforth, assume that $\lambda = 1$.

Next fix some cylinder $C \in \alpha_{n_0}$ of full returns. If $n_0 = 0$, T is quasi-beta and we are done, so assume that $n_0 > 0$. Next define

$$\varphi_C(x) := 1_C(x) \inf\{n \geq 1 : T^n(x) \in C\}, \alpha_C := \{\alpha_{\varphi_C(x)+n_0}(x) : x \in X\}, T_C := T^{\varphi_C}$$

and let m_C and $\mathcal{B}_C$ be the restrictions of m and $\mathcal{B}$ to C. Then $(C, \mathcal{B}_C, m_C, T_C, \alpha_C)$ is a fibred map with almost open image sets (w.r.t (T_c, α_C)).

We claim that this system is quasi–beta. Indeed, C has full returns, so for almost every $x \in C$ there are $n_k \uparrow \infty$ with $T^{n_k}\alpha_{n_k+n_0}(x) = C$. Since n_k is a time of return, there exists some m_k such that

$$n_k = \varphi_C(x) + \varphi_C(T_C x) + \ldots \varphi_C(T_C^{m_k-1}x).$$

By the definition of α_C,

$$(\alpha_C)_{m_k}(x) = \alpha_{n_k+n_0}(x)$$

whence $(T_C)^{m_k}(\alpha_C)_{m_k}(x) = T^{n_k}\alpha_{n_k+n_0}(x) = C$.

If we set $\phi_C := \phi + \phi \circ T + \ldots \phi \circ T^{\varphi_C - 1}$, we get for almost every $x \in C$, $\gamma \circ \phi_C = g/g \circ T_C$. Since ϕ_C is α_C–measurable and T_C is quasi–beta, we have by the first part of the proof that g is constant on C, whence $(*)$. $\square$

To conclude this section we mention another aspect of cylinders of full returns:

Remark 4 (Relation to Iterated Function Systems)**.** We can also characterize cylinders of full returns in terms of a suitable *iterated function system (IFS)*. Let $(X, \mathcal{B}, m, T, \alpha)$ be a fibred system, and let $\widetilde{\alpha}_+ := \{A \in \bigcup_{n \geq 1} \alpha_n : m(A) > 0\}$. Given $C \in \widetilde{\alpha}_+$ we let $\mathcal{W}_C := \{W \in \widetilde{\alpha}_+ : T^{|W|}W \supseteq C\}$ and notice that $[W_0, \ldots, W_{k-1}] \in \mathcal{W}_C$ implies $[W_i, \ldots, W_{k-1}] \in \mathcal{W}_C$ for $i < k$.

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[3]Proof: for every $f \geq 0$ such that $fh \in L^1(m \times \mu)$, $\int\int 1_E fh = \int\int 1_E \circ \tau_S f \circ \tau_S h \leq \int\int f \circ \tau_S h = \int\int fh = \int\int 1_E fh$, so that $\leq$ is actually $=$. Since f was arbitrary, $1_E \circ \tau_S = 1$ a.e. on $E = [h > 0]$ so that $\tau_S(E) \subseteq E \bmod m$.

Consider the IFS $\mathfrak{X}_C := \{v = (T^{|W|}\,|_W)^{-1}\,|_C\colon W \in \mathcal{W}_C\}$ consisting of maps $v : C \to X$. Observe that if $v_i = (T^{|W_i|}\,|_{W_i})^{-1}\,|_C \in \mathfrak{X}_C$, $i \in \{1,2\}$, and $v_2(C) \cap C \neq \varnothing$, then $v_2(C) \subseteq C$ (since $v_2(C) = [W_2, C] \in \xi_{|W_2|+|C|}$) and $v_1 \circ v_2 \in \mathfrak{X}_C$ with $v_1 \circ v_2(C) = [W_1, W_2, C]$.

Therefore, if $x = v_1 \circ \ldots \circ v_n(x_n)$ for some $x_n \in C$ and $v_i \in \mathfrak{X}_C$, then $x \in [W, C]$ for some $W \in \mathcal{W}_C$ with $|W| \geq n$. Hence if $\mathfrak{X}^* \subseteq \mathfrak{X}_C$ and $X^* \subseteq X$ are such that $X^* = \bigcup_{v \in \mathfrak{X}^*} v(C \cap X^*)$, then any $x \in X^*$ belongs to infinitely many $[W, C]$, $W \in \mathcal{W}_C$. By lemma 2, if X^* has positive measure, then C is a cylinder of full returns.

Specifically, if we let $\mathcal{W}_C^* := \{W \in \mathcal{W}_C : \nexists W_1 \in C \cap \mathcal{W}_C$ such that $W = [W_0, W_1]$ with $\mid W_i \mid > 0\}$ and define $\mathfrak{X}_C^*$ like $\mathfrak{X}_C$ with $\mathcal{W}_C$ replaced by $\mathcal{W}_C^*$, lemma 2 also shows necessity of X being covered (mod m) by the images of $v \in \mathfrak{X}_C^*$, so that

$$C \text{ is a cylinder of full returns iff } X = \bigcup_{v \in \mathfrak{X}_C^*} v(C \cap X) \pmod{m}.$$

4. Examples II: Almost Onto Systems

4.1. **Finding recurrent image sets.** The following result contains the information we need on the structure of $\mathcal{M}_{rec}$:

Theorem 3 (The family of recurrent image sets)**.** *Let T be a basic AFU map with partition α and absolutely continuous invariant measure m. Then:*

1. *J is a recurrent image set iff $[T^n\alpha_n(x) = J$ infinitely often$]$ has positive measure, and in this case this set is of full measure.*
2. *If $\inf |(T^N)'| > 2$, then at least one of the elements of α_N is a cylinder of full returns.*
3. *If J is a recurrent image set for T, and $J \supseteq C$ where C is a cylinder, then $T^{|C|}(C)$ is again a recurrent image set.*
4. *X is covered (up to finitely many points) by some finite $\mathcal{M}'_{rec} \subseteq \mathcal{M}_{rec}$.*

In what follows y is called a *fixed point* in a cylinder A if

1. T is orientation preserving in A, $y \in \overline{A}$ and $T(x) \to y$ as $x \to y$ in A, or
2. T is orientation reversing in A, $y \in int(A)$, and $T(x) \to y$ as $x \to y$ in A.

(This is intended to prevent ambiguity when y is a discontinuity point.)

Theorem 4 (Recurrent image sets at fixed points)**.** *Let (X, T, α) be a basic AFU-map, and suppose y is a fixed point in $A \in \alpha$. If $T(A) \supseteq A$, then each of the images $T(I_1), T(I_2)$ of the components I_1, I_2 of $A \setminus \{y\}$ is covered by a recurrent image set.*

In particular, if T has a full branch, then there are two recurrent image sets $J, J' \in \mathcal{M}_{rec}$ such that $X = J \cup J'$ up to end points of J, J' (and if $y \in \partial A$, then $J = J'$).

The theorem suggests the following test for the almost onto property: Define the *full-image transition graph* $\mathcal{J} = (T\alpha, \rightsquigarrow)$ by requiring that $I \rightsquigarrow J$ iff I covers some $C \in \alpha$ with $TC = J$. Then:

Corollary 3. *Let T be a basic AFU-map on the interval X with $\bigcup T\alpha$ connected, and suppose that $\inf_X |T'| > 2$ or that there is an orientation preserving fixed point at ∂A for some $A \in \alpha$. Then $T\alpha \cap \mathcal{M}_{rec} \neq \varnothing$, and if $\mathcal{J}$ is irreducible, then T is almost onto.*

Proof. Part 2 of theorem 3 and theorem 4 show that under the assumptions of the corollary, $T\alpha \cap \mathcal{M}_{rec} \neq \varnothing$. It is enough to show that $T\alpha \subseteq \mathcal{M}_{rec}$ (since $\bigcup \mathcal{M}_{rec}$, too, is connected in this case). Theorem 3 says that there is some $J \in T\alpha \cap \mathcal{M}_{rec}$, and that if $J = J_0 \rightsquigarrow J_1 \rightsquigarrow \ldots \rightsquigarrow J_{k-1}$, then each J_i is in $T\alpha \cap \mathcal{M}_{rec}$. □

Example 1. *Set* $Z_i = [\frac{i}{4}, \frac{i+1}{4})$, $(i = 0, \ldots, 3)$ *and fix* $\theta \in (0, \frac{1}{2})$*. Let* $T_\theta : [0,1) \to [0,1)$ *be the map given by* $T|_{Z_i}$ *maps* Z_i *affinely onto* B_i*, where* $B_0 = B_2 = [0, 1-\theta)$ *and* $B_1 = B_3 = [\theta, 1)$*, together with* $\alpha = \{Z_0, \ldots, Z_3\}$*. Then* T_θ *is almost onto but not quasi–beta.*

Proof. $T\alpha = \{[0, 1-\theta), [\theta, 1)\}$ and $[0, 1-\theta) \rightsquigarrow [\theta, 1) \rightsquigarrow [0, 1-\theta)$ so that the conditions of the lemma are satisfied, whence T is almost onto. T is not quasi–beta because there are no cylinders with image equal to the whole interval. □

The remainder of this section is dedicated to the proof of theorem 3. We need the following lemma:

Lemma 2. *Let* $(X, \mathcal{B}, m, T, \alpha)$ *be a conservative ergodic fibred system. The following are equivalent:*

1. $C \in \alpha_{n_0}$ *is a cylinder of full returns,*
2. $\varphi^C(x) := \min\{n \geq 1 : T^n \alpha_{n+|C|}(x) = C\}$ *is finite for* m*-a.e.* $x \in C$.
3. *There exists* M *of positive measure such that for almost every* $x \in M$*, there are* $n_k \uparrow \infty$ *with* $T^{n_k}\alpha_{n_k+n_0}(x) = C$.
4. *For almost every* $x \in X$*, there are* $n_k \uparrow \infty$ *such that* $T^{n_k}\alpha_{n_k+n_0}(x) = C$.

In particular, if the system is conservative ergodic and $J \in \mathcal{M}_{rec}$*, then for a.e.* $x \in X$*,* $T^n \alpha_n(x) = J$ *for infinitely many* $n \in \mathbb{N}$.

Proof. (1) ⇒ (2) is trivial.

In order to prove that (2) ⇒ (3), it is enough to prove that (2) ⇒ (1) because (1) ⇒ (3) (take $M = C$). Consider the *full return map* T^C defined a.e. on C by $T^C x := T^{\varphi^C(x)} x \in C$, whose natural partition α^C is given by $\alpha^C(x) = \alpha_{\varphi^C(x)+|C|}(x)$ (mod m). Then $m\,|_C \circ (T^C)^{-1} \ll m\,|_C$, so that $\varphi^C \circ T^C$ is defined a.e. on C, and (by induction) so are all powers $(T^C)^n$, $n \geq 1$, proving (1).

We prove (3) ⇒ (4). For this purpose define $\mathcal{F}_\ell : X \to 2^{\alpha_\ell}$, $\ell \geq 1$ by

$$\mathcal{F}_\ell(x) := \{C \in \alpha_\ell : T^n \alpha_{n+\ell}(x) = C \text{ for infinitely many } n \in \mathbb{N}\}.$$

Observe that for every x, $T^n \alpha_{n+\ell}(x) = T^{n-1}\alpha_{n+\ell-1}(Tx)$: $\subseteq$ is a set-theoretic identity and this forces $=$ because both sets are ℓ–cylinders. It follows that

$$\mathcal{F}_\ell \circ T \supseteq \mathcal{F}_\ell.$$

Our system is assumed to be conservative ergodic. It is not difficult to deduce from this that $\mathcal{F}_\ell$ is constant a.e. on X, so that if $C \in \mathcal{F}_\ell(x)$ for a.e. $x \in M$ and $m(M) > 0$, then $C \in \mathcal{F}_\ell(x)$ for a.e. $x \in X$, whence (3) ⇒ (4).

The last implication (4) ⇒ (1) is trivial, so the lemma is proved. □

Proof of theorem 3. The proof uses *Canonical Markov Extensions (C.M.E.)*, which we now turn to describe. Let $\mathcal{M}$ be the collection of image sets of T, and define for every $J \in \mathcal{M}$, $\widehat{J} := J \times \{J\}$. Let $\widehat{\mathcal{M}} := \{\widehat{J} : J \in \mathcal{M}\}$, and define

$$\widehat{X} := \bigcup \widehat{\mathcal{M}}\ ,\ \widehat{T} : \widehat{X} \to \widehat{X}\ ,\ \widehat{T}(x, J) := \big(Tx, T(\alpha(x) \cap J)\big).$$

We equip $\widehat{X}$ with the natural Borel structure induced by $\widehat{J} \cong J$. Note that $\pi : \widehat{X} \to X$, $\pi(x, J) = x$ is a factor map, and that $\widehat{\alpha} := \widehat{\mathcal{M}} \vee \pi^{-1}\alpha$ is a Markov partition for $\widehat{T}$. $(\widehat{X}, \widehat{T}, \widehat{\alpha})$ is called the *Canonical Markov Extension* of (X, T, α) (Hofbauer [**H**], Keller [**Ke2**]).

We define the *levels* of the extension as follows: Level zero is $X \times \{X\}$, and Level n for $n \geq 1$, is $\bigcup\big(\widehat{\mathcal{M}}_n \setminus \widehat{\mathcal{M}}_{n-1}\big)$, where

$$\widehat{\mathcal{M}}_n = \{\widehat{J} : J \in T^k\alpha_k, k \leq n\} \text{ and } \widehat{\mathcal{M}}_0 = \{X \times \{X\}\}.$$

The *height* $\Lambda(\widehat{x})$ of $\widehat{x} \in \widehat{X}$ is the index of the level set which contains $\widehat{x}$. Some basic properties of $\widehat{T}$ (see [**Ke2**] for proofs):

(1) The collection of image sets of $\widehat{T}$ is $\widehat{\mathcal{M}} = \{\widehat{J} : J \in \mathcal{M}\}$, and this collection is pairwise disjoint.
(2) $\widehat{T}^n(x, J) = \big(T^n x, T^n(\alpha_n(x) \cap J)\big)$. Since α_n shrinks to points, for every $x \in int(J)$ there is $n_0 = n_0(x)$ such that for all $n \geq n_0$, $\widehat{T}^n(x, J) = \big(T^n x, T^n\alpha_n(x)\big)$.
(3) $\widehat{\alpha}_n(x, J) = \big(\alpha_n(x) \cap J\big) \times \{J\}$, and for every $x \in int(J)$ there is $n_0 = n_0(x)$ such that for $n \geq n_0$, $\widehat{\alpha}_n(x, J) = \alpha_n(x) \times \{J\}$.
(4) $\widehat{T}^n\widehat{\alpha}_n(x, J) = T^n(\alpha_n(x) \cap J) \times \{T^n(\alpha_n(x) \cap J)\}$, and for every $x \in int(J)$ there is $n_0 = n_0(x)$ such that for every $n \geq n_0$, $\widehat{T}^n\widehat{\alpha}_n(x, J) = T^n\alpha_n(x) \times \{T^n\alpha_n(x)\}$.
(5) if $|T\alpha| < \infty$, then $\pi^{-1}\{x\} \cap [\Lambda = n]$ is finite for all $n \geq 0, x \in X$.

We will also need the following strong lifting result for basic AFN maps [**Z2**]: Let $dm = hdm_\lambda$ be the a.c.i.m of T. There exists a $\widehat{T}$–invariant conservative ergodic Borel measure $\widehat{m}$ such that $\widehat{m} \circ \pi^{-1} = m$, for which the following is true:

$$(\dagger) \qquad \forall \widehat{J} \in \widehat{\mathcal{M}}, \text{ if } \widehat{m}(\widehat{J}) > 0, \text{ then } \widehat{m}|_{\widehat{J}} \sim m \circ \pi|_J.$$

This is not stated explicitly in [**Z2**] but can be derived from results there as follows: Define the *regularity* of a $\widehat{u} : \widehat{X} \to \mathbb{R}_+$ which is differentiable on every $\widehat{J}$ to be the supremum over all $\widehat{J}$ of

$$R_{\widehat{J}}(\widehat{u}) := \begin{cases} \sup_{\widehat{J}} |\widehat{u}'/\widehat{u}| & \widehat{u} > 0 \text{ on } \widehat{J} \\ 0 & \widehat{u} \equiv 0 \text{ on } \widehat{J} \\ \infty & \text{otherwise.} \end{cases}$$

The proof of proposition 1 in [**Z2**] implies that that $d\widehat{m} = \widehat{h}d\widehat{m_\lambda}$ where $\widehat{m_\lambda}$ is the sum of the Lebesgue measures on $\widehat{J} \cong J$, and the regularity of $\widehat{h}$ is finite. This implies (†).

Let $\widehat{\mathcal{M}}_{rec}$ be the collection of recurrent image sets for $\widehat{T}$ (w.r.t. $\widehat{m}$), and denote by $\widehat{\mathcal{M}}_+$ the collection of image sets of $\widehat{T}$ with positive $\widehat{m}$ measure. We derive the theorem from the following characterization of $\mathcal{M}_{rec}$:

$$(\ddagger) \quad \mathcal{M}_{rec} = \big\{T^{|C|}(C) : C \text{ is a cylinder such that } C \subseteq J \in \pi(\widehat{\mathcal{M}}_+)\big\} = \pi(\widehat{\mathcal{M}}_+).$$

We explain how this implies the theorem:

Proof of part 1. Lemma 1 says that if J is a recurrent image set, then $[T^n\alpha_n(x) = J$ infinitely often$]$ has positive (in fact full) measure. We show the other direction. Suppose $m[T^n\alpha_n(x) = J \text{ i.o.}] > 0$. Then $\widehat{m}[T^n\alpha_n\big(\pi(\widehat{x})\big) = J \text{ i.o.}] > 0$, because

$\widehat{m} \circ \pi^{-1} = m$. But for almost every $\widehat{x}$ if n is large enough, then $\widehat{T}^n \widehat{\alpha}_n(\widehat{x}) = T^n \alpha_n(\pi(\widehat{x})) \times \{T^n \alpha_n(\pi(\widehat{x}))\}$, so that in fact $\widehat{m}[\widehat{T}^n \widehat{a}_n(\pi(\widehat{x})) = \widehat{J}$ i.o.$] > 0$. This means that $\widehat{m}[\widehat{T}^n(\widehat{x}) \in \widehat{J}$ i.o.$] > 0$, whence (trivially) $\widehat{m}[\exists n$ such that $\widehat{T}^n(\widehat{x}) \in \widehat{J}] > 0$. The invariance of $\widehat{m}$ now implies that $\widehat{m}(\widehat{J}) > 0$, so that $J = \pi(\widehat{J}) \in \pi(\widehat{\mathcal{M}}_+)$. Therefore, by (‡), J is a recurrent image set.

Proof of part 2. Without loss of generality $N = 1$ (else work with $(X, \mathcal{B}, m, T^N, \alpha_N)$). The first telescope lemma of [**Z1**] says that for almost every x there are $n_k \uparrow \infty$ such that $T^{n_k} \alpha_{n_k}(x) = T\alpha(T^{n_k - 1}x)$. There are only finitely many possibilities for $T\alpha(T^{n_k-1}x)$, because of (F). Therefore, there is a $J \in T\alpha$ such that

$$m\{x : T^n \alpha_n(x) = J \text{ infinitely often}\} > 0.$$

This implies, by part one, that J is a recurrent image set.

Proof of part 3. Suppose $J \in \mathcal{M}_{rec}$ and $C \subseteq J$ is a cylinder. By (‡), $J \in \pi(\widehat{\mathcal{M}}_+)$. But this means that $C \subseteq J \in \pi(\widehat{\mathcal{M}}_+)$, so (‡) gives $T^{|C|}(C) \in \mathcal{M}_{rec}$ and part 3 follows.

Proof of part 4. By lemma 6 of [**Z2**] and the proof of proposition 1 there, we know that $h(x) = \sum_{\pi(\widehat{x})=x} \widehat{h}(\widehat{x})$ outside some countable set E, and that for $\varepsilon := \inf_X h/2 > 0$ there is some $\eta \in \mathbb{N}$ such that $\sum_{\pi(\widehat{x})=x, \Lambda(\widehat{x})>\eta} \widehat{h}(\widehat{x}) < \varepsilon$ for all $x \in X$. Consequently, $\sum_{\pi(\widehat{x})=x, \Lambda(\widehat{x})\leq\eta} \widehat{h}(\widehat{x}) > 0$ outside E, showing that each $x \in X \setminus E$ is contained in some member J of the finite collection

$$\mathcal{M}'_{rec} := \{J : \widehat{J} \in \widehat{\mathcal{M}}_+ \cap \{\Lambda \leq \eta\}\}.$$

By (‡), $\mathcal{M}'_{rec} \subseteq \mathcal{M}_{rec}$. But since X only has a finite number of components, we conclude that $\mathcal{M}'_{rec}$ covers X up to finitely many points. Part 4 follows.

This shows that the theorem follows from (‡), which we now prove, using the following steps:

(A) $\mathcal{M}_{rec} \supseteq \{T^{|C|}(C) : C \text{ is a cylinder such that } C \subseteq J \in \pi(\widehat{\mathcal{M}}_+)\}$
(B) $\{T^{|C|}(C) : C \text{ is a cylinder such that } C \subseteq J \in \pi(\widehat{\mathcal{M}}_+)\} \supseteq \pi(\widehat{\mathcal{M}}_{rec})$
(C) $\pi(\widehat{\mathcal{M}}_{rec}) \supseteq \pi(\widehat{\mathcal{M}}_+)$
(D) $\pi(\widehat{\mathcal{M}}_+) \supseteq \mathcal{M}_{rec}$

Proof of (A): Suppose $C \subseteq J \in \pi(\widehat{\mathcal{M}}_{rec})$. Then $\widehat{C} := \widehat{J} \cap \pi^{-1}C \in \widehat{\alpha}_{|C|}$ is a full lift of C, i.e. $\pi\widehat{C} = C$, so that each $x \in C$ has a unique lift $\widehat{x} \in \widehat{C}$. For any $n \in \mathbb{N}$ and $x \in C$, $\alpha_{n+|C|}(x) = \pi(\widehat{\alpha}_{n+|C|}(\widehat{x})) \cap J = \pi(\widehat{\alpha}_{n+|C|}(\widehat{x}))$. It follows that $T^n \alpha_{n+|C|}(x) = \pi(\widehat{T}^n \widehat{\alpha}_{n+|C|}(\widehat{x}))$, showing that the full return times $\tau^C(x)$ and $\widehat{\tau}^{\widehat{C}}(\widehat{x})$ agree for all $x \in C$ (cf. lemma 2). It is therefore enough to prove that that $\widehat{C}$ is a cylinder of full returns for $\widehat{T}$ w.r.t. $\widehat{m}$.

By assumption, $\widehat{m}(\widehat{J}) > 0$. (‡) implies that $\widehat{m}(\widehat{C}) > 0$, because $\pi|_{\widehat{J}}(\widehat{C}) = C$ and $m(C) > 0$. It follows that $\widehat{C}$ is a $\widehat{T}$–cylinder of positive $\widehat{m}$–measure, and this means that it is a cylinder of full returns, because $\widehat{T}$ is Markov and $\widehat{m}$ is conservative, and in the Markov case, every cylinder with positive measure is a cylinder of full returns.

Proof of (B): Suppose $J \in \pi(\widehat{\mathcal{M}}_{rec})$. Then $\widehat{J} \equiv J \times \{J\}$ is a recurrent image set, so by lemma 1 for $\widehat{m}$ a.e. $\widehat{x} \in \widehat{X}$, $\widehat{T}^n \widehat{\alpha}_n(\widehat{x}) = \widehat{J}$ infinitely often. Thus

$$\text{for } \widehat{m} \text{ a.e. } \widehat{x} \in \widehat{X},\ J = \pi(\widehat{J}) = \pi\big(\widehat{T}^n \widehat{\alpha}_n(\widehat{x})\big) = T^n \alpha_n\big(\pi(\widehat{x})\big) \text{ infinitely often.}$$

If n is large enough $\alpha_n\big(\pi(\widehat{x})\big) = \pi\big(\widehat{\alpha}_n(\widehat{x})\big)$, and this is contained in some element of $\pi\big(\widehat{\mathcal{M}}_+\big)$ for a.e. $\widehat{x}$. It follows that $J \in \{T^{|C|}(C) : C \subseteq J \in \pi(\mathcal{M}_+)\}$.

Proof of (C): Suppose $J \in \pi(\mathcal{M}_+)$. Then there is a $\widehat{C} \in \widehat{\alpha}$ such that $\widehat{m}(\widehat{C} \cap \widehat{T}^{-1}\widehat{J}) > 0$. For this partition element, $\widehat{m}[\widehat{J} \cap \widehat{T}(\widehat{C})] = \widehat{m}[\widehat{T}^{-1}\widehat{J} \cap \widehat{T}^{-1}\widehat{T}(\widehat{C})] > 0$. But for the canonical Markov extension, $\widehat{\mathcal{M}}$ is a partition, so that if $\widehat{T}(\widehat{C})$ intersects $\widehat{J}$, it is equal to it. Thus $\widehat{J} = \widehat{T}(\widehat{C})$ where $\widehat{C} \in \widehat{\alpha}$ has positive measure. But $\widehat{T}$ is Markov, and for conservative Markov maps every cylinder of positive measure is a cylinder of full returns. It follows that $\widehat{C}$ is a cylinder of full returns, and consequently $\widehat{J} = \widehat{T}(\widehat{C})$ is a recurrent image set for $\widehat{T}$, whence $J \in \pi(\widehat{\mathcal{M}}_{rec})$.

Proof of (D): Suppose $J \in \mathcal{M}_{rec}$. Lemma 1 says that for almost every x, $T^n\alpha_n(x) = J$ infinitely often. Since for $\widehat{m}$–a.e. $\widehat{x}$, $\widehat{T}^n \widehat{\alpha}_n(\widehat{x}) = \alpha_n(\pi\widehat{x}) \times \{T^n \alpha_n(\pi\widehat{x})\}$, we have that with full $\widehat{m}$–probability, $\widehat{T}^n(\widehat{x}) \in \widehat{J}$. This is the same as saying that $\widehat{X} = \bigcup_{n \geq 1} \widehat{T}^{-n}\widehat{J}$ so we must have $\widehat{m}(\widehat{J}) > 0$, whence $J = \pi(\widehat{J}) \in \pi(\mathcal{M}_+)$. □

Proof of theorem 4. We distinguish two cases, the first one being that there is some neighbourhood U of y in X such that $U \cap A$ is contained in some $J \in \mathcal{M}_{rec}$. The cylinders $A_n := [A, \ldots, A] \in \alpha_n$ are (possibly one-sided punctured) neighbourhoods of y shrinking toward this point, so that $A_n \subseteq J$ for $n \geq n_0$. For such n therefore $TA = T^n A_n = T^{|A_n|} A_n \in \mathcal{M}_{rec}$ by part 3 of theorem 3.

As for the second case (where necessarily $y \in int(A)$), theorem 3 part 4 says that there is some finite $\mathcal{M}'_{rec} \subseteq \mathcal{M}_{rec}$ covering X up to finitely many points, so that y is the common end point of two adjacent members J, J' of $\mathcal{M}'_{rec}$. Define A_n as before, then $A_n \setminus \{y\} \subseteq J \cup J'$ for $n \geq n_1$, so that $A_n \cap J, A_n \cap J'$ lift to cylinders $\widehat{A}_n \subseteq \widehat{M}$ and $\widehat{A}'_n \subseteq \widehat{J}'$ from $\widehat{\alpha}_n$ which have positive measure. Hence $T^n(A_n \cap J), T^n(A_n \cap J') \in \mathcal{M}_{rec}$, but as y is a fixed point, these are just the images TI_i of the components I of $A \setminus \{y\}$. □

4.2. **Quasi-Beta Maps.** The strongest form of the almost–onto property is the quasi–beta property: X is a recurrent image set. This section collects examples of such maps. (Examples of almost onto maps which are not quasi–beta are given in the previous section.) Note that the β–transformation is a particular case.

Theorem 5 (Quasi-beta maps I)**.** *Let (X, T, α) be a basic AFU-map. Each of the following conditions implies the quasi–beta property:*

1. *There exists an $A \in \alpha$ such that $T(A) = X$ and such that one of the end points of A is a fixed point in A, or*
2. *There exist $A_1, A_2 \in \alpha$ different such that $T(A_1) = T(A_2) = X$.*

Proof. This follows from theorem 4. (Notice that $\overline{A}$ contains some fixed point y, which is inside A if it is orientation-reversing. Apply theorem 4 to see that each component of $X \setminus \{y\}$ is contained in some recurrent image set. But A° is contained in one of these components, so that by part 3 of theorem 3, $TA = X$ is a recurrent image set.) □

Stronger assumptions on the image structure of the map enable a more direct proof which does not depend on lifting results for Markov extensions (and also allows indifferent fixed points):

Theorem 6 (Quasi-beta maps II)**.** *Let $(I, \mathcal{B}, m, T, \alpha)$ be a piecewise increasing AFN map of the unit interval together with its a.c.i.m. m. Suppose that for every $A \in \alpha$, $0 \in \overline{Ta}$, and that $TA_0 = I$ for some $A_0 \in \alpha$ with the property that $0 \in \overline{A_0}$. Then $(I, \mathcal{B}, m, T, \alpha)$ is quasi-beta.*

Proof. We need the following property:

$$\forall n \in \mathbb{N}\ \forall A \in \alpha_n\ \exists p_A \in \overline{A} \text{ such that } T^n p_A = 0. \tag{5}$$

We use induction on n. For $n = 1$ this is part of our assumptions. Assume this holds for n and choose some $A \in \alpha_{n+1}$. Write $A = B \cap T^{-1}C$ with $B \in \alpha, C \in \alpha_n$. Since T is piecewise increasing, $B = (p_B, q_B)$ and $C = (p_C, q_C)$ where $Tp_B = T^n p_C = 0$, and $(T|_A)^{-1}C = \big((T|_A)^{-1}p_C, (T|_A)^{-1}q_C\big)$. If $A = B \cap T^{-1}C \neq \varnothing$, either $p_B \in \overline{A}$ or $(T|_A)^{-1}p_C \in \overline{A}$. Both map to 0 under T^{n+1} (because 0 must be a fixed point), so $0 \in T^{n+1}\overline{A}$.

Standard arguments (compare [**T**], [**A**], [**Z2**]) show that AFN maps have a *Schweiger collection* ([**Sc**], [**A**]), i.e. a collection of cylinders $\mathfrak{r} \subseteq \bigcup_{n \geq 0} \alpha_n$ with the following properties:

(1) for all n and every $A \in \alpha_n$, $A = \bigcup_{B \in \mathfrak{r}, B \subseteq A} B \bmod m$
(2) there exists some positive constant M such that $\frac{v_B'(x)}{v_B'(y)} \leq M$ for every $B \in \mathfrak{r}$ and $m \times m$-almost all $(x, y) \in T^{|B|}B \times T^{|B|}B$.
(3) $A \in \alpha_n, B \in \mathfrak{r}, A \cap T^{-n}B \neq \varnothing$ implies $A \cap T^{-n}B \in \mathfrak{r}$.

To prove the quasi-beta property, let $A_0 := (0, c_0)$ and define $c_n \in A_0$ by $T^n c_n = c_0$. Fix some $A \in \alpha_N$ and some $B \in \mathfrak{r}$ such that $B \subseteq A \bmod m$, and set $B = (p_B, q_B)$. Let $|B|$ denote the length of B as a cylinder (so $B \in \alpha_{|B|}$). $T^{|B|}$ is increasing on B, so (5) implies that $T^{|B|}B = \big(0, T^{|B|}(q_B)\big)$. Since $c_n \downarrow 0$, there is some n_0 such that $c_{n_0} < T^{|B|}(q_B) \leq c_{n_0 - 1}$ (where $c_{-1} := 1$). By the mean value theorem and the Schweiger property, if $B' := B \cap T^{-|B|}[0, c_{n_0}]$, then

$$\begin{aligned}\frac{m(B')}{m(B)} \equiv \frac{m(B \cap T^{-|B|}[0, c_{n_0}])}{m(B)} &= \left|\frac{v_B(0) - v_B(c_{n_0})}{v_B(0) - v_B(T^{|B|}q_B)}\right| \geq \frac{1}{M} \cdot \frac{c_{n_0}}{T^{|B|}q_B} \\ &\geq \frac{1}{M} \cdot \frac{c_{n_0}}{c_{n_0 - 1}} = \frac{1}{M} \frac{c_n - 0}{T(c_{n_0}) - T(0)} \geq \epsilon_0\end{aligned}$$

where $\epsilon_0 := M^{-1}(\inf\{T'(x) : x \in \overline{A_0}\})^{-1}$. Furthermore,

$$B' := B \cap T^{-|B|}[0, c_{n_0}] \equiv B \cap T^{-|B|}(A_0 \cap T^{-1}A_0 \cap \cdots \cap T^{-n_0}A_0) \in \alpha_{|B| + n_0 + 1}$$

and (since $T^{|B|}B = [0, T^{|B|}q_B] \supseteq [0, c_{n_0}]$)

$$\begin{aligned}T^{|B'|}B' &= T^{n_0+1}T^{|B|}(B \cap T^{-|B|}[0, c_{n_0}]) \\ &= T^{n_0+1}[0, c_{n_0+1}] = T[0, c_0] = TA_0 = [0, 1] \bmod m.\end{aligned}$$

Hence, for every $B \in \mathfrak{r} \cap A$ there exists $B' \in \alpha_{|B'|}$ with $B' \subseteq B$, $T^{|B'|}B' = [0, 1]$ and $m(B') \geq \epsilon_0 m(B)$. Since $A = \bigcup_{B \in \mathfrak{r}, B \subseteq A} B$, this shows that

$$m\{x \in A : \exists n \geq |A| \text{ such that } T^n \alpha_n(x) = [0, 1] \bmod m\} \geq \epsilon_0 m(a).$$

It now follows by the method of exhaustion that for almost all $x \in A$ there is $n \geq |A|$ such that $T^n \alpha_n(x) = [0,1]$. Since $A \in \alpha_N$ and $N \in \mathbb{N}$ were arbitrary, the quasi-beta property follows. $\square$

4.3. **An instructive counterexample.** Fix any $k \in \mathbb{N}$, $k > 1$, let α be the partition (mod m_λ) of $X = [0,1]$ into subintervals with end points $i/2k$, $i \in \{0,1,\dots,2k\} \setminus \{k\}$, and define T to be the piecewise affinely increasing map, symmetric under $x \longmapsto 1-x$, which maps each $(i/2k,(i+1)/2k)$, $i < k-1$, onto $(1/2,1)$, while $T((k-1)/2k,(k+1)/2k) = (0,1)$.

Proposition 2. *(X,T,α) is a basic exact AFU-map which preserves Lebesgue measure m_λ, $|T'| \equiv k$,and has a full branch. However, the system is not almost onto.*

Proof. To see this it is enough to notice that its C.M.E. $(\widehat{X},\widehat{T},\widehat{\alpha})$ only has two levels, the higher of which, $\widehat{X} \cap \{\Lambda = 1\}$, is forward invariant under $\widehat{T}$, so that the base $\widehat{X} \cap \{\Lambda = 0\}$ is dissipative and the only recurrent image sets are the intervals $(0,1/2)$ and $(1/2,1)$, the upper level is made of (see (‡) in the proof of theorem 3).

(Let us also remark that for every non degenerate interval I there is some $j \in \mathbb{N}$ such that $T^j I = (0,1/2) \cup (1/2,1)$, but the fixed point $x = 1/2$ is not contained in any recurrent image set.) $\square$

5. Perturbation Theory and Conditional Local Limit Theorems for Interval Maps

Let $(X,\mathcal{B},m,T,\alpha)$ be a fibred system on a probability space. For $\omega : X \to \mathbb{S}^1$ measurable, define

$$P_\omega f := P_T(\omega f) \quad (f \in L^1(m)),$$

and for $\phi : X \to \mathbb{R}^d$, $\phi = (\phi^{(1)},\dots,\phi^{(d)})$, measurable, and $t \in \mathbb{R}^d$ set $P_t := P_{\chi_t(\phi)}$ where $\chi_t(y) := e^{i\langle t,y\rangle}$. In the *independent case* where ϕ is α-measurable and $\alpha, T^{-1}\alpha, \dots$ are independent,

$$P_t 1 = E(e^{i\langle t,\phi\rangle}),$$

which is why the P_t are sometimes called *characteristic function operators.* The characteristic function operators can be used to study the local and distributional limit behaviour of ϕ_n in the same way as the characteristic function is used in the independent case. In this section we study these operators for certain piecewise monotonic maps of the interval, and establish the properties needed for proving local and distributional limit theorems.

This requires some tools in operator theory which we now explain. Recall that a linear operator P on a Banach space $\mathcal{L}$ is *quasi compact* (on $\mathcal{L}$) if for some $N \geq 1$, $\theta \in (0,1)$, $E_1,\dots,E_N$ projections with finite dimensional images, and $\lambda_1,\dots,\lambda_N \in \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$:

$$\text{(QC)} \qquad \|P^n f - \sum_{k=1}^{N} \lambda_k^n E_k f\|_{\mathcal{L}} \leq M\theta^n \|f\|_{\mathcal{L}} \ \forall\ f \in \mathcal{L}.$$

There are situations when the restriction of the Frobenius-Perron operator to a suitable Banach space is quasi-compact. This can be sometime proved using the following concept:

Definition 5. *Let $\mathcal{C}$, $\mathcal{L}$ be Banach spaces such that $\mathcal{C} \supset \mathcal{L}$ and $\|\cdot\|_{\mathcal{C}} \leq \|\cdot\|_{\mathcal{L}}$.*

(1) *We call the pair* $(\mathcal{C}, \mathcal{L})$ adapted *if* $\mathcal{L}$*-bounded sets are precompact in* $\mathcal{C}$.
(2) *Let* $(\mathcal{C}, \mathcal{L})$ *be an adapted pair of Banach spaces. A linear operator* $P : \mathcal{C} \to \mathcal{C}$ *is said to be a* D-F operator on $(\mathcal{C}, \mathcal{L})$ *if there are* $\theta \in (0, 1)$, $M > 0$, $n \in \mathbb{N}$ *such that*

$$\|P^n f\|_{\mathcal{L}} \leq \theta \|f\|_{\mathcal{L}} + M \|f\|_{\mathcal{C}} \quad \forall\ f \in \mathcal{L}.$$

We will call this latter inequality a D-F inequality.

The terminologies 'D-F inequality' and 'D-F operator' are in honour of W. Doeblin and R. Fortet who first considered such operators (in [**DF**]) for the case when $\mathcal{C}$ is the space continuous functions on a compact metric space X, and $\mathcal{L}$ is the space of Lipschitz continuous functions on X. It was established in [**DF**] that a D-F operator on $(C(X), L(X))$ is quasi compact on $L(X)$ and this was generalized in [**ITM**] to show that a D-F operator on an adapted pair $(\mathcal{C}, \mathcal{L})$ is quasi compact on $\mathcal{L}$. The proof of this uses *inter alia* a kind of rigidity of D-F operators: if P is a D-F operator on $(\mathcal{C}, \mathcal{L})$, then

- if $f \in \mathcal{C}$ and $\lambda \in \mathbb{S}^1$ satisfy $Pf = \lambda f$, then $f \in \mathcal{L}$
- if $\mathfrak{r}(P)$ denotes the spectral radius of $P : \mathcal{L} \to \mathcal{L}$, then $\mathfrak{r}(P) \leq 1$ with equality iff there are some $f \in \mathcal{L}$ and $\lambda \in \mathbb{S}^1$ satisfying $Pf = \lambda f$.

We apply this theory in the context of piecewise monotonic maps with countably many branches. Let $X \subseteq \mathbb{R}$ be an interval. For every measurable f on X taking values in $\mathbb{R}^d$ or $\mathbb{C}$ define $\mathrm{var}_X(f) := \sup \sum_i \|f(x_i) - f(x_{i-1})\|$ where the supremum ranges over all $x_1 < x_2 < \cdots < x_n$ in X. For every $f \in L^1(m_\lambda)$ set

$$\|f\|_{BV} := \|f\|_\infty + \bigvee_X f, \text{ where } \bigvee_X f = \inf\{\mathrm{var}_X(f^*) : f^* = f\ m_\lambda - \text{ a.e. }\}.$$

Finally, let $BV := \{f \in L^1(m_\lambda) : \|f\|_{BV} < \infty\}$. It follows from Helly's theorem that the pair $\big(L^1(m_\lambda), BV\big)$ is adapted.

Let (X, T, α) be a non–singular piecewise monotonic map of the unit interval. Below v'_A will always denote a version of this L_1-function which minimizes variation. We say that T satisfies *Rychlik's condition* [**R**] if

$$\text{(R)} \qquad \sum_{A \in \alpha} \|1_{TA} v'_A\|_{BV} =: \mathcal{R} < \infty.$$

Corollary 1 of [**Z1**] says that every AFU map satisfies Rychlik's condition. The following result is due to M. Rychlik ([**R**]):

Proposition 3 (Ergodic properties of Rychlik maps)**.** *Suppose that* (X, T, α) *is a piecewise monotonic interval map satisfying (R) and (U), then so does* (X, T^n, α_n). T *has a finite ergodic decomposition into products of finite rotations and mixing maps satisfying (R) and (U). If* T *is weakly mixing, then its unique invariant probability density* h *belongs to* BV*, and there are constants* $K > 0$, $\theta \in (0, 1)$ *such that*

$$\|P^n f - (\int_X f dm_\lambda) h\|_{BV} \leq K \theta^n \|f\|_{BV}.$$

We will need the following generalization of Proposition 1 of [**R**].

Proposition 4 (D-F inequality for perturbed P)**.** *Suppose that* (X, T, α) *satisfies (R) and (U) and that* $\omega : X \to \mathbb{S}^1$ *satisfies* $C = C_{\omega,\alpha} := \sup_{A \in \alpha} \bigvee_A \omega < \infty$. *Then there exist* $\theta \in (0, 1)$ *and* $K_0, M_0 > 0$ *such that*

$$\|P_\omega^n f\|_{BV} \leq K_0 \theta^n \|f\|_{BV} + M_0 \|f\|_1.$$

Proof. Let $\omega_n := \prod_{k=0}^{n-1} \omega \circ T^k$. We claim that $C_{\omega_n,\alpha_n} \le nC_{\omega,\alpha} \quad (n \ge 1)$. To see this fix $n \ge 2$, $A \in \alpha_n$, then

$$\bigvee_A \omega_n = \bigvee_A (\omega\, \omega_{n-1} \circ T) \le \bigvee_A \omega + \bigvee_A \omega_{n-1} \circ T \le \bigvee_A \omega + \bigvee_{TA} \omega_{n-1} \le C_{\omega,\alpha} + C_{\omega_{n-1},\alpha_{n-1}}.$$

We let $c := (\sup_{A\in\alpha} \|v_A'\|_\infty)^{-1} > 1$ and fix $n \ge 1$, $\epsilon > 0$ so that $\theta := \frac{2(4+nC)}{c^n} + 2\epsilon < 1$. By Rychlik's condition, there is $\beta \subset \alpha_n$ finite so that $\sum_{A\in\alpha_n\setminus\beta} \|1_{T^nA} v_A'\|_{BV} < \frac{1}{c^n}$.

Now fix $f \in BV$. Note that for every $A \in \alpha_n$, v_A' is nonnegative and also $\|v_A'\|_\infty \le \frac{1}{c^n}$. Thus for $A \in \beta$ there exists a finite partition γ_A of X into intervals whose endpoints are points of continuity for $1_{T^nA} v_A'$, $1_{T^nA} \omega_n \circ v_A$, $1_{T^nA} f \circ v_A$, so that $\sup_{g\in\gamma_A} \bigvee_g (1_{T^nA} v_A') < \frac{1}{c^n} + \epsilon$.

Therefore, since $P_\omega^n f = \sum_{A\in\alpha_n} 1_{T^nA} v_A'\, \omega_n \circ v_A\, f \circ v_A$,

$$\begin{aligned}
\bigvee (P_\omega^n f) &\le \sum_{A\in\alpha_n} \bigvee (1_{T^nA} v_A'\, \omega_n \circ v_A\, f \circ v_A) = \sum_{A\in\alpha_n} \bigvee (P_T^n(1_A f \omega_n)) \\
&= \sum_{A\in\beta} \bigvee (P_T^n(1_A f \omega_n)) + \sum_{A\in\alpha_n\setminus\beta} \bigvee (P_T^n(1_A f \omega_n))
\end{aligned}$$

For $A \in \beta$,

$$\begin{aligned}
\bigvee (P_T^n(1_A f \omega_n)) &= \sum_{g\in\gamma_A} \bigvee_g \big(1_{T^nA} v_A'\, \omega_n \circ v_A\, f \circ v_A\big) \\
&\le \sum_{g\in\gamma_A} \Big(\|v_A'\|_\infty \bigvee_g (1_{T^nA}\, \omega_n \circ v_A\, f \circ v_A) \\
&\qquad\qquad + \bigvee_g (1_{T^nA} v_A') \|1_{g\cap T^nA}\, f \circ v_A\|_\infty \Big) \\
&= \sum_{g\in\gamma_A} \Big(\|v_A'\|_\infty \bigvee_g (1_{T^nA}\, \omega_n \circ v_A\, f \circ v_A) \\
&\qquad\qquad + \bigvee_g (1_{T^nA} v_A') \|1_{v_A(g)} f\|_\infty \Big).
\end{aligned}$$

Now $\|1_{v_A(g)} f\|_\infty \le \frac{1}{m_\lambda(v_A(g))} \|1_{v_A(g)} f\|_1 + \bigvee_{v_A(g)} f$ and

$$\begin{aligned}
\bigvee_g (1_{T^nA}\, \omega_n \circ v_A\, f \circ v_A) &\le 2\|1_{v_A(g)} f\|_\infty + \bigvee_{v_A(g)} \omega_n f \\
&\le (2+nC)\|1_{v_A(g)} f\|_\infty + \bigvee_{v_A(g)} f \\
&\le \tfrac{2+nC}{m_\lambda(v_A(g))} \|1_{v_A(g)} f\|_1 + (3+nC) \bigvee_{v_A(g)} f
\end{aligned}$$

so $\bigvee(P_T^n(1_A f\omega_n))$ is

$$
\begin{aligned}
&\leq \sum_{g\in\gamma_A}(\|v_A'\|_\infty \bigvee_g(1_{T^nA}\,\omega_n\circ v_A\, f\circ v_A)+\bigvee_g(1_{T^nA}v_A')\|1_{v_A(g)}f\|_\infty)\\
&\leq \sum_{g\in\gamma_A}\Bigg(\|v_A'\|_\infty[\tfrac{2+nC}{m_\lambda(v_A(g))}\|1_{v_A(g)}f\|_1+(3+nC)\bigvee_{v_A(g)}f]\\
&\qquad\qquad +\bigvee_g(1_{T^nA}v_A')[\tfrac{1}{m_\lambda(v_A(g))}\|1_{v_A(g)}f\|_1+\bigvee_{v_A(g)}f]\Bigg)\\
&\leq \sum_{g\in\gamma_A}\Bigg([(3+nC)\|v_A'\|_\infty+\bigvee_g(1_{T^nA}v_A')]\bigvee_{v_A(g)}f\\
&\qquad\qquad +[(2+nC)\|v_A'\|_\infty+\bigvee_g(1_{T^nA}v_A')]\tfrac{1}{m_\lambda(v_A(g))}\|1_{v_A(g)}f\|_1\Bigg)\\
&\leq \tfrac{\theta}{2}\bigvee_A f+\max_{A\in\beta,\ g\in\gamma_A}\tfrac{1}{m_\lambda(v_A(g))}\|1_A f\|_1.
\end{aligned}
$$

For $A\in\alpha_n\setminus\beta$,

$$
\begin{aligned}
\bigvee(P_T^n(1_A f\omega_n)) &\leq \|f\|_\infty\bigvee_X(1_{T^nA}v_A')+\|v_A'\|_\infty\bigvee_X(1_{T^nA}\,\omega_n\circ v_A\, f\circ v_A)\\
&\leq \|1_{T^nA}v_A'\|_{BV}\Bigg(\|f\|_\infty+\bigvee_X(1_{T^nA}\,\omega_n\circ v_A\, f\circ v_A)\Bigg).
\end{aligned}
$$

Now

$$\|f\|_\infty\leq\|f\|_1+\bigvee_X f$$

and

$$
\begin{aligned}
\bigvee_X(1_{T^nA}\,\omega_n\circ v_A\, f\circ v_A) &\leq 2\|f\|_\infty+\bigvee_A(\omega_n f)\\
&\leq (2+nC)\|f\|_\infty+\bigvee_A f\\
&\leq (2+nC)\|f\|_1+(3+nC)\bigvee_I f.
\end{aligned}
$$

Thus,

$$\|f\|_\infty+\bigvee_X(1_{T^nA}\,\omega_n\circ v_A\, f\circ v_A)\leq(3+nC)\|f\|_1+(4+nC)\bigvee_X f.$$

Putting things together:

$$\begin{aligned}
\bigvee_X (P_\omega^n f) &\leq \sum_{A\in\beta} \bigvee (P_T^n(1_A f\omega_n)) + \sum_{A\in\alpha_n\setminus\beta} \bigvee (P_T^n(1_A f\omega_n)) \leq \\
&\sum_{A\in\beta}\left(\frac{\theta}{2}\bigvee_A f + \max_{A\in\beta,\ g\in\gamma_A} \frac{1}{m_\lambda(v_A(g))}\|f1_A\|_1\right) + \\
&+ \sum_{A\in\alpha_n\setminus\beta} \|1_{T^nA}v_A'\|_{BV}\left((3+nC)\|f\|_1 + (4+nC)\bigvee_X f\right) \\
&= \left(\frac{\theta}{2} + (4+nC)\sum_{A\in\alpha_n\setminus\beta}\|1_{T^nA}v_A'\|_{BV}\right)\bigvee_X f + M\|f\|_1. \\
&\leq \theta\bigvee_X f + M\|f\|_1.
\end{aligned}$$

The estimate for $\|P_{T^n}(\omega_N f)\|_\infty$ follows from the last statement of the previous proposition. ☐

Proposition 5 (Continuity of the perturbation)**.** *Suppose that (X, T, α) satisfies (U) and (R); and that $\phi : X \to \mathbb{R}^d$ satisfies $C = C_{\phi,\alpha} := \sup_{A\in\alpha}\bigvee_A \phi < \infty$, then, $s \mapsto P_s$ is continuous ($\mathbb{R}^d \to \mathrm{Hom}\,(BV, BV)$), moreover*

$$\|P_s - P_t\|_{BV} \leq 2\sum_{A\in\alpha}\|1_{TA}v_A'\|_{BV}\big((2 + C\|s\|)\sup_A|\chi_s(\phi) - \chi_t(\phi)| + \|s-t\|C\big).$$

If, in addition, (X, T, α) satisfies AFU, then there exists $K > 0$ such that

$$\|P_s - P_t\|_{BV} \leq 2K(2 + C\|s\|)\int_X |1 - \chi_{t-s}(\phi)|dm_\lambda + 2CK(3 + 2C\|s\|)\|s-t\|.$$

Proof. For $g \in BV$ and $t \in \mathbb{R}^d$ we have

$$P_t g = P(e^{i\langle t,\phi\rangle}g) = \sum_{A\in\alpha}\chi_t(\phi\circ v_A)1_{TA}v_A' \cdot g\circ v_A,$$

whence

$$(P_t - P_s)g = \sum_{A\in\alpha} 1_{TA}\chi_t(\phi\circ v_A)(1 - \chi_{s-t}(\phi\circ v_A))v_A' \cdot g\circ v_A.$$

Noting that $|1 - \chi_{s-t}(\phi\circ v_A)| \leq |1 - \chi_{s-t}(\phi(x_A))| + C\|s-t\|$ for any $x_A \in A$, we see that

$$\begin{aligned}
\|(P_t - P_s)g\|_\infty &\leq \sum_{A\in\alpha}\|1 - \chi_{s-t}(\phi\circ v_A)\|_\infty\|1_{TA}v_A' \cdot g\circ v_A\|_\infty \\
&\leq \sum_{A\in\alpha}(|1 - \chi_{s-t}(\phi(x_A))| + C\|s-t\|)\|v_A'\|_\infty\|g\|_\infty;
\end{aligned}$$

and

$$\bigvee((P_t - P_s)g \leq \sum_{A\in\alpha}\bigvee(\chi_s(\phi\circ v_A) - \chi_t(\phi\circ v_A))1_{TA}v_A' \cdot g\circ v_A).$$

We recall the chain rule for BV functions [**AM**]: Let A be an interval and let $\varphi : A \to \mathbb{R}^d$ be a function of bounded variation. Set $\varphi(x^\pm) := \lim_{t\to x,\ \pm(t-x)>0}\varphi(t)$,

$J_\varphi := \{x \in A : \varphi(x^+) \neq \varphi(x^-)\}$, and let μ_φ be the $\mathbb{R}^d$-valued measure determined by $\mu_\varphi\big((a,b]\big) := \varphi(b^+) - \varphi(a^+)$. For every continuously differentiable function $F : \mathrm{Conv}[\varphi(A)] \to \mathbb{C}$, we have

$$d\mu_{F\circ\varphi} = \langle \nabla F \circ \varphi, d\mu_\varphi\rangle + \sum_{x\in J_\varphi} \big[F\big(\varphi(x^+)\big) - F\big(\varphi(x^-)\big)\big] d\delta_x.$$

Passing to total variations, we see that

$$\bigvee_A F\circ\varphi = \int_A \|\nabla F\circ\varphi\|\, d\|\mu_\varphi\| + \sum_{x\in J_\varphi} \big|F\big(\varphi(x^+)\big) - F\big(\varphi(x^-)\big)\big| \leq 2 \sup_{\mathrm{Conv}[\varphi(A)]} \|\nabla F\| \bigvee_A \varphi.$$

Applying this for $F(x) := e^{i\langle t,x\rangle} - e^{i\langle s,x\rangle}$ and $\varphi := \phi$ gives

$$\bigvee_A [\chi_s(\phi) - \chi_t(\phi)] \leq 2C(\|t-s\| + \|s\| \sup_A |\chi_s(\phi) - \chi_t(\phi)|). \tag{6}$$

Using this, we see that for fixed $A \in \alpha$,

$$\begin{aligned}
&\bigvee [(\chi_s(\phi\circ v_A) - \chi_t(\phi\circ v_A))1_{TA}v_A' \cdot g\circ v_A] \leq \\
\leq\ & \|1_{TA}v_A'\|_\infty \bigvee (1_{TA}(\chi_s(\phi\circ v_A) - \chi_t(\phi\circ v_A))g\circ v_A) + \bigvee (1_{TA}v_A')\|g\|_\infty \sup_A |\chi_s(\phi) - \chi_t(\phi)| \\
\leq\ & \|1_{TA}v_A'\|_{BV} \left(3\|g\|_\infty \sup_A |\chi_s(\phi) - \chi_t(\phi)| + \bigvee_A ((\chi_s(\phi) - \chi_t(\phi))g)\right) \\
\leq\ & \|1_{TA}v_A'\|_{BV} \left(4\|g\|_{BV} \sup_A |\chi_s(\phi) - \chi_t(\phi)| + \|g\|_\infty \bigvee_A (\chi_s(\phi) - \chi_t(\phi))\right) \\
\leq\ & 2\|1_{TA}v_A'\|_{BV}\|g\|_{BV} \left((2 + C\|s\|) \sup_A |\chi_s(\phi) - \chi_t(\phi)| + C\|t-s\|\right).
\end{aligned}$$

This proves the first inequality. To verify the second inequality, note first that under AFU there exists $K > 0$ with

$$\|1_{TA}v_A'\|_{BV} \leq K m_\lambda(A) \quad (A \in \alpha).$$

Also

$$\begin{aligned}
\sup_A |\chi_s(\phi) - \chi_t(\phi)| = \sup_A |1 - \chi_{t-s}(\phi)| &\leq \tfrac{1}{m_\lambda(A)} \int_A |1 - \chi_{t-s}(\phi)| dm_\lambda + \bigvee_A \chi_{t-s}(\phi) \\
&\leq \tfrac{1}{m_\lambda(A)} \int_A |1 - \chi_{t-s}(\phi)| dm_\lambda + C\|s-t\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|P_s - P_t\|_{BV} &\leq 2\sum_{A\in\alpha} \|1_{TA}v_A'\|_{BV} \left((2 + C\|s\|) \sup_A |\chi_s(\phi) - \chi_t(\phi)| + C\|t-s\|\right) \\
&\leq 2K \sum_{A\in\alpha} m_\lambda(A) \left((2 + C\|s\|) \sup_A |\chi_s(\phi) - \chi_t(\phi)| + C\|t-s\|\right) \\
&= 2K(2 + C\|s\|) \int_X |1 - \chi_{t-s}(\phi)| dm_\lambda + 2CK(3 + 2C\|s\|)\|s-t\|.
\end{aligned}$$

□

Proposition 6. *Suppose that (X, T, α) a weakly mixing piecewise monotonic map of the interval which satisfies (U), (R) with invariant density h. If $\phi : X \to \mathbb{R}^d$ satisfies $C_{\phi,\alpha} < \infty$, then*

(1) *there are constants $\epsilon > 0$, $K > 0$ and $\theta \in (0,1)$, and continuous functions $\lambda : B(0,\epsilon) \to B_{\mathbb{C}}(0,1)$, $g : B(0,\epsilon) \to \mathrm{Hom}\,(BV, BV)$ such that for $t \in B(0,\epsilon)$: $g(t)$ is a projection, $\dim Im[g(t)] = 1$, $P_t g(t) = \lambda(t)g(t)$,$\lambda(0) = 1$, $g(0)f = (\int_X f dm_\lambda)h$ and*

$$\|P_t^n f - \lambda(t)^n g(t) f\|_{\mathrm{BV}} \le K\theta^n \|f\|_{\mathrm{BV}} \quad \forall\ |t| < \epsilon,\ n \ge 1,\ f \in BV;$$

(2) *if $\gamma(\phi) = z\overline{f} f \circ T$ where $\gamma \in \hat{\mathbb{R}}^d$, $z \in \mathbb{S}^1$ and $f : X \to \mathbb{S}^1$ measurable, then $f \in BV$;*

(3) *in case ϕ is aperiodic, then for all $0 < \delta < M < \infty$ there exist $K > 0$, $0 < \rho < 1$ such that*

$$\|P_\gamma^n f\|_{\mathrm{BV}} \le K\rho^n \|f\|_{BV}\ \forall\ f \in BV,\ n \ge 1,\ \delta \le |\gamma| \le M.$$

Proof. As shown above, P is a D-F operator. By the weak mixing of T and [**ITM**], there exist $M > 0$, $\theta \in (0,1)$ such that

$$\|P^n f - (\int f dm_\lambda)h\|_{\mathrm{BV}} \le M\theta^n \|f\|_{\mathrm{BV}}\ \forall\ f \in \mathrm{BV}.$$

The result now follows as in [**N**] (see also [**DS**], [**RE**] and §4 of [**AD1**]). □

We can now turn to the results mentioned in the introduction.

Theorem 7 (Exactness of skew products)**.** *Let (X, T, α) satisfy (U) and (R), and be weakly mixing with absolutely continuous invariant measure m. Suppose that $\phi : X \to \mathbb{R}$ satisfies $C_{\phi,\alpha} < \infty$. Let $T_\phi : X \times \mathbb{R} \to X \times \mathbb{R}$ be the skew product*

$$T_\phi(x, y) = \big(Tx, y + \phi(x)\big)$$

equipped with the product measure $m \times m_\lambda$. If ϕ is aperiodic and for every $\lambda > 1$ there exist $n_k \to \infty$ such that $\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ a.e. as $k \to \infty$, then T_ϕ is exact.

Proof. This follows from proposition 6 as in the proof of theorem 2 of [**AD2**]. □

The following theorem summarizes the information we need on the characteristic function operators in order to derive conditional LLTs.

Theorem 8 (Expansion of the eigenvalue)**.** *Let (X, T, α) be a weakly mixing AFU map, assume $\phi : X \to \mathbb{R}$ satisfies $C_{\phi,\alpha} < \infty$, and let λ is as in the previous proposition.*

(1) *If $E(\phi^2) < \infty$ and $\frac{1}{n}Var(\phi_n) \to \sigma^2 > 0$, then*

$$\lambda(t) = 1 + itE(\phi) - \frac{t^2\sigma^2}{2}(1 + o(1)) \text{ as } t \to 0.$$

(2) *If the distribution of ϕ is in the domain of attraction of a stable law of index $p \in (0,2)$, then*

$$|\log \lambda(t) - \log E(e^{it\phi})| = o\bigg(|t|^p L(1/|t|)\bigg) \text{ as } t \to 0.$$

Proof. For the first part, check that $t \mapsto P_t$ is C^2 $(\mathbb{R} \to \mathrm{Hom}(BV, BV))$ with $\frac{dP_t}{dt} f = P(i\phi e^{it\phi} f)$ and $\frac{d^2 P_t}{dt^2} f = -P(\phi^2 e^{it\phi} f)$. This implies that $t \mapsto \lambda(t)$ is C^2. The Taylor expansion of λ is then calculated as in [**RE**].

The proof of the second part is as in §5 of [**AD1**], with propositions 5 and 6 replacing theorems 2.4 and 4.1 there. □

We now obtain the advertised conditional distributional and local limit theorems.

Theorem 9 (Conditional central and local limit theorems)**.** *Let (X, T, α) be a weakly mixing AFU map, and suppose that $\phi : X \to \mathbb{R}$ satisfies $C_{\phi,\alpha} < \infty$.*

(1) *If $E(\phi^2) < \infty$ and $\frac{1}{n} Var(\phi_n) \to \sigma^2 > 0$, then*

$$P_{n,x}\left(\frac{\phi_n - E(\phi)}{\sigma\sqrt{n}} \in (a, b)\right) \to \tfrac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2}}\, dt$$

as $n \to \infty$, uniformly in $x \in X$, where $P_{n,x}(A) := P_T^n 1_A(x)$.

(2) *If in addition $\phi : X \to \mathbb{Z}$ is aperiodic, then*

$$\sigma\sqrt{n} P_{n,x}(\phi_n = k_n) \to \tfrac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \ \text{as } n \to \infty,\ k_n \in \mathbb{Z},\ \tfrac{k_n - nE(\phi)}{\sigma\sqrt{n}} \to t$$

uniformly in $x \in X$ and $t \in K$ for all $K \subset \mathbb{R}$ compact.

(3) *If in addition $\phi : X \to \mathbb{R}$ is aperiodic and $I \subset \mathbb{R}$ is a finite interval, then*

$$\sigma\sqrt{n} P_{n,x}(\phi_n \in k_n + I) \to \tfrac{1}{|I|\sqrt{2\pi}} e^{-\frac{t^2}{2}} \ \text{as } n \to \infty,\ k_n \in \mathbb{Z},\ \tfrac{k_n - nE(\phi)}{\sigma\sqrt{n}} \to t$$

uniformly in $x \in X$ and $t \in K$ for all $K \subset \mathbb{R}$ compact.

Theorem 9 follows from (1) of theorem 8 as in [**RE**] (see also [**AD1**]). From (2) of theorem 8, we obtain the analogous stable distributional and local limit theorems. Indeed, it is now routine to check that all the the results in §6 and §7 of [**AD1**] are valid for $(X, \mathcal{B}, m, T, \alpha)$ a mixing, probability preserving AFU map and $\phi : X \to \mathbb{R}^d$ satisfying $C_{\phi,\alpha} < \infty$.

6. Application to β–expansions

Fix $\beta > 1$ and consider $T : [0, 1] \to [0, 1]$ defined by $Tx := \beta x \mod 1$. Let $q, d\mathbb{P} := q(x)dx$, and X_n be as in the introduction.

Proof of De Moivre's theorem for β-expansions

By theorem 1, T is skew-product rigid and almost onto by theorem 5. Since $1 \in \{[\beta x] - [\beta y] :\ x, y \in [0, 1]\}$, by theorem 2, $\phi : X \to \mathbb{Z}$ given by $\phi(x) = [\beta x]$ is aperiodic. De Moivre's theorem now follows from (2) of theorem 9. □

Asymptotics of random walks on $\mathbb{R}$ driven by "β-jumps": Suppose that $\psi : [0, 1] \to \mathbb{R}$ satisfies $E(\psi) = 0$ and $\psi(x) = a_{[\beta x]}$ where $\{a_i - a_j :\ 0 \le i, \le [\beta]\}$ are rationally independent, then by the anologue of theorem 7.1 in [**AD1**] for $(X, \mathcal{B}, m, T, \alpha)$ a mixing, probability preserving AFU map ψ satisfying $C_{\psi,\alpha} < \infty$, T_ψ is conservative, exact and pointwise dual ergodic with $a_n(T_\psi) \propto \sqrt{n}$.

Proof of the Hewitt–Savage zero–one law for β-expansions. Let $\mathcal{R} \in \mathcal{B}(X \times X)$ be an equivalence relation with countable equivalence classes. A Borel isomorphism ψ defined on some $A \in \mathcal{B}$ with image $B \in \mathcal{B}$ is a *holonomy* for $\mathcal{R}$ if $(x, \psi(x)) \in \mathcal{R}$ for any $x \in A$.

A measure μ is *invariant* (non-singular) for $\mathcal{R}$, if it is invariant (non-singular) under all holonomies of $\mathcal{R}$. A set $A \in \mathcal{B}$ is $\mathcal{R}$*-saturated* if $x \in A,\ (x,y) \in \mathcal{R} \ \Rightarrow\ y \in \mathcal{R}$. The measure μ is *ergodic* for $\mathcal{R}$ if every $\mathcal{R}$- saturated set is trivial $\bmod\ \mu$.

Now consider $X = [0,1]$ and the Borel equivalence relations

$$\begin{aligned} \mathfrak{T}(T_\beta) &:= \{(x,y) \in [0,1]^2 :\ \exists\, N \geq 1 \ \text{ such that } T^N x = T^N y\} \\ \mathcal{E}_\beta &:= \{(x,y) \in [0,1]^2 :\ x \text{ and } y \text{ are } \beta\text{-exchangeable}\}. \end{aligned}$$

We are asked to show the $\mathcal{E}_\beta$–ergodicity of Lebesgue's measure.

Lebesgue's measure is $\mathfrak{T}(T_\beta)$-invariant as $\mathfrak{T}(T_\beta)$ is generated by holonomies of form $x \mapsto x + \frac{i}{\beta^n}$. Since $\mathfrak{T}(T_\beta) \supset \mathcal{E}_\beta$, Lebesgue measure is also $\mathcal{E}_\beta$-invariant.

To see that $\mathcal{E}_\beta$- saturated sets are trivial (mod Lebesgue measure), recall from [**ANSS**], $F^\# : [0,1] \to \mathbb{Z}^{[\beta]}$ defined by $F^\#(x)_i = \delta_{[\beta x],i}$ and that

$$\mathcal{E}_\beta = \mathfrak{T}((T_\beta)_{F^\#}) \cap (X \times \{0\})^2$$

where

$$\mathfrak{T}((T_\beta)_{F^\#}) := \{((x,n),(x',n')) \in ([0,1]\mathbb{Z}^{[\beta]})^2 :\ \exists\, N \geq 1,\ T^N_{F^\#}(x,n) = T^N_{F^\#}(x',n')\}.$$

The group generated by $\{F^\#(x) - F^\#(y) :\ \ x,y \in [0,1]\}$ is $\mathbb{Z}^{[\beta]}$ whence by the aperiodicity theorem, $F^\#$ is aperiodic. By theorem 7, $T_{F^\#}$ is exact (with respect to $m \times m_{\mathbb{Z}^{[\beta]}}$). This implies the ergodicity of $\mathcal{E}_\beta$. $\square$

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