A LOCAL LIMIT THEOREM FOR STATIONARY PROCESSES IN THE DOMAIN OF ATTRACTION OF A NORMAL DISTRIBUTION

JON AARONSON AND MANFRED DENKER

ABSTRACT. We prove local limit theorems for Gibbs-Markov processes in the domain of attraction of normal distributions.

§1 Introduction

It is well known that a random variable X belongs to the domain of attraction of a normal distribution DA(2) if its characteristic function satisfies

$$(*) \qquad \log E\exp[itX] = it\gamma - \frac{1}{2}t^2 L(1/|t|)$$

for some slowly varying function $L : \mathbb{R}_+ \to \mathbb{R}_+$ which is bounded below and some constant $\gamma \in \mathbb{R}$ (cf. [IL]).

The normal (or classical) domain of attraction NDA(2) consists of the class L_2, and is characterised by the boundedness of the slowly varying function L in $(*)$. Here we consider the "non-normal" domain of attraction DA(2) \ NDA(2).

The function L is unbounded and is determined (up to asymptotic equivalence) by the tails of the distribution of X which satisfy

$$\begin{aligned} &1 - G(x) = P(X \geq x) \sim c_1 x^{-2} l(x) \\ (1) \qquad &G(-x) = P(X \leq -x) \sim c_2 x^{-2} l(x) \quad x \to \infty.1 \end{aligned}$$

for some constants $c_1, c_2 \geq 0, c_1 + c_2 = 1$ and some slowly varying function l, which in turn determines L by

$$L(x) = \int_{-x}^{x} u^2 dP_X(u). \qquad (1).2$$

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It follows from (1.2) that

$$l(x) = o(L(x)) \tag{1).3}$$

as $x \to \infty$.

Let $X_1, X_2, \dots$ be a stationary process of independent random variables with $X_k \in \mathrm{DA}(p) \quad (0 < p \le 2)$.

The local limit theorem (LLT) for the partial sums $S_n := X_1 + \cdots + X_n$ is well known, that is $\exists$ constants $A_n, B_n \in \mathbb{R}, \quad B_n \to +\infty$ such that $\forall\ \kappa \in \mathbb{R}$ and $I \subset \mathbb{R}$ (an interval),

$$B_n P(S_n - k_n \in I) \to |I| g(\kappa) \quad \text{as} \quad \frac{k_n - A_n}{B_n} \to \kappa$$

where g is a p-stable density on $\mathbb{R}$. Extensions of the LLT to Markov chains are well known (see [AD] for a more detailed discussion).

In [AD], LLT's were established for Gibbs-Markov functionals (definition below) in the non-normal stable case ($p < 2$).

In the normal case ($p = 2$), such extensions are only known when $X_k \in \mathrm{NDA}(2)$ (see [AD], [Rou], [GH], [M]).

Here we prove the LLT for Gibbs-Markov functionals $X_1, X_2, \dots$ in the case $X_1 \in \mathrm{DA}(2) \setminus \mathrm{NDA}(2)$.

§2 Gibbs-Markov processes and functionals

Definition 2.1: A mixing stationary process $\{Z_n : n \in \mathbb{N}\}$ is called *Gibbs-Markov*, if its state space E is at most countable and if

(1) (*Markov property*)

$$P(Z_1 = a, Z_2 = b) > 0 \quad \text{and} \quad P(Z_1 = b, Z_2 = c) > 0$$
$$\implies P(Z_1 = a, Z_2 = b, Z_3 = c) > 0$$

for all $a, b, c \in E$ and

$$\inf\{ \sum_{b \in E; P(Z_1 = a, Z_2 = b) > 0} P(Z_1 = b) : a \in E\} > 0.$$

(2) (*Gibbs property*) There exist constants $M > 0$ and $0 < r < 1$ such that

$$\left| \frac{P(Z_1 = a_1, ..., Z_n = a_n | Z_{n+1} = b_1, ..., Z_{n+k} = b_k)}{P(Z_1 = a_1, ..., Z_n = a_n | Z_{n+1} = c_1, ..., Z_{n+k} = c_k)} - 1 \right| \le M r^{-\min\{l : c_l \ne b_l\}}$$

for all $a_i, b_j, c_j \in E$, $1 \le i \le n$, $1 \le j \le k$ and all $n, k \ge 1$.

Remarks 2.2:
1) Recall that a process $\mathbf{Z} = \{Z_n : n \geq 1\}$ is called mixing, if for all square-integrable functions $f, g \in L_2(\mathbf{Z})$ one has

$$Ef(\mathbf{Z})g(Z_n, Z_{n+1}, ...) \to Ef(\mathbf{Z})Eg(\mathbf{Z}) \qquad \text{as } n \to \infty,$$

where $L_q(\mathbf{Z})$ ($q \in \mathbb{N} \cup \{\infty\}$) is the space of functions $g : E^{\mathbb{N}} \to \mathbb{R}$ which are q-integrable with respect to the distribution of $\mathbf{Z}$.
2) The coordinate process on $E^{\mathbb{N}}$ of a mixing Gibbs-Markov map (as in [AD]) is a Gibbs-Markov process in the sense of definition 2.1. Conversely, every Gibbs-Markov process has a mixing, shift-invariant distribution on $E^{\mathbb{N}}$ under which the shift is a Gibbs-Markov map.

Definition 2.3: A function $f : E^{\mathbb{N}} \to \mathbb{R}$ is uniformly Lipschitz on states ($f \in Lip$), if

$$D(f) := \sup_{a \in E, x, y \in [a]} r^{\min\{l : x_l \neq y_l\}} |f(x) - f(y)| < \infty,$$

where $[a] = \{(x_1, x_2, ...) \in E^{\mathbb{N}} : x_1 = a\}$.

Definition 2.4: A stationary process $\{X_n : n \in \mathbb{N}\}$ is called a *Gibbs-Markov functional*, if there exists a Gibbs-Markov process $\mathbf{Z} = \{Z_n : n \in \mathbb{N}\}$ and a function $f \in Lip$ such that

$$X_n = f(Z_n, Z_{n+1}, ...).$$

The Frobenius-Perron operators $P^n : L_1(\mathbf{Z}) \to L_1(\mathbf{Z})$ are defined by

$$EP^n f(Z_1, Z_2, ...)g(Z_1, Z_2, ...) = Ef(Z_1, Z_2, ...)g(Z_{n+1}, Z_{n+2}, ...), \quad (2).1$$

and the characteristic function operator for the function $\varphi : E^{\mathbb{N}} \to \mathbb{R}$ by

$$P_t f = P(f \exp[it\varphi]). \qquad (2).2$$

In [AD] it has been shown that when $\varphi \in Lip$, P_t acts on $\mathcal{L} := L_\infty(\mathbf{Z}) \cap Lip$ equipped with the norm $\|f\|_{\mathcal{L}} = \|f\|_\infty + D(f)$. As an operator on $\mathcal{L}$, P_t has a unique eigenvalue of maximal modulus $\lambda(t)$ for $|t| < \epsilon$ and a decomposition

$$P_t^n f = \lambda(t)^n g(t) Ef(\mathbf{Z}) + Q_t^n f \qquad (|t| < \epsilon), \qquad (2).3$$

where the spectral radius of Q_t is uniformly bounded by some $\theta < 1$ and where $g(t)$ is the normalized eigenfunction for $\lambda(t)$. P_t is called the characteristic function operator, since

$$P_t^n 1 = P^n e^{itS_n} = \lambda(t)^n g(t) + Q_t^n 1,$$

where $S_n = X_1 + ... + X_n$.

§3 Local limit theorems

In this section, we assume that $\{X_n : n \geq 1\}$ is a Gibbs-Markov functional with $X_1 = f(\mathbf{Z}) \in \mathrm{DA}(2)$, but $EX_1^2 = \infty$. Let the operator $P_t : \mathcal{L} \to \mathcal{L}$, $\lambda(t)$ and $g(t)$ be defined (as in §2, (2.1)–(2.3)) for $|t| < \epsilon$ and for $\phi = f$. Moreover, let G denote the distribution function of X_1 and l and L the associated slowly varying functions as defined in (1.1) and (1.2).

Theorem 3.1:

$$\log \lambda(t) = it\gamma - \frac{1}{2}|t|^2 L(|t|^{-1})(1 + o(1)) \qquad (3).1$$

as $t \to 0$, where the constant $\gamma \in \mathbb{R}$ is defined by

$$\gamma = \int_{-\infty}^{\infty} xG(dx). \qquad (3).2$$

Remark 3.2: Theorem 3.1 may fail in the 'classical' case where $Ef(\mathbf{Z}) = 0$ and $Ef(\mathbf{Z})^2 < \infty$. Indeed, suppose $\phi \in \mathcal{L}$, then also $f := \phi \circ T - \phi \in \mathcal{L}$ (here T denotes the shift on $E^{\mathbb{N}}$). As can be easily checked,

$$P_t(e^{it\phi}) = e^{it\phi},$$

whence $\lambda(t) = 1$ (see [AD]). On the other hand, it is indicated in [AD] how to prove theorem 3.1 in case $f \in Lip$, $Ef(\mathbf{Z}) = 0$, $Ef(\mathbf{Z})^2 < \infty$, and not of form $f = \phi \circ T - \phi$.

Remark 3.3:

As a corollary, we obtain that under the conditions of theorem 3.1

$$|\log \lambda(t) - \log E \exp[itX_1]| = o(|t|^2 L(1/|t|)) \qquad \text{as } t \to 0.$$

Lemma 3.4:

$$E(|1 - e^{itX_1}|) = O(|t|)$$

as $t \to 0$.

Proof. This estimate follows from the expansion of $E \exp[itX_1]$ (see theorem 2.6.5 in [IL].

Proof of theorem 3.1. Let $\tilde{g}_t = g(t)/Eg(t)(\mathbf{Z})$ denote the eigenfunction of P_t with eigenvalue $\lambda(t)$ satisfying $E\tilde{g}_t(\mathbf{Z}) = 1$, then by (2.1)

$$\lambda(t) = \lambda(t)E\tilde{g}_t(\mathbf{Z}) = E\lambda(t)\tilde{g}_t(\mathbf{Z}) = EP[\tilde{g}_t e^{it\phi}](\mathbf{Z}) = E\tilde{g}_t(\mathbf{Z})e^{itX_1}. \tag{3).3}$$

By theorem 4.1 in [AD], and by lemma 3.4,

$$\|\tilde{g}_t - 1\|_\infty = O(|t|) \quad \text{as } t \to 0.$$

Denote by $\mathcal{F}_0$ the σ-algebra generated by X_1 and let $\hat{g}_t \circ X_1 = E(\tilde{g}_t(\mathbf{Z})|\mathcal{F}_0)$, then by (3.3)

$$\lambda(t) = E\hat{g}_t(X_1)\exp[itX_1] = \int_{-\infty}^{\infty} \hat{g}_t(x)\exp[itx]G(dx), \tag{3).4}$$

$$\|\hat{g}_t - 1\|_{L_\infty(G)} \le \|\tilde{g}_t - 1\|_\infty = O(|t|) \quad \text{as } t \to 0, \tag{3).5}$$

and

$$\int_{-\infty}^{\infty} \hat{g}_t(x)\; G(dx) = 1 \quad \forall\; t \in \mathbb{R}.$$

It follows from (3.5) that for $|t|$ small enough, Re $\hat{g}_t \ge 0$. Write

$$\hat{g}_t = g_t^r + ig_t^+ - ig_t^-$$

where $g_t^\pm := \max\{\pm\text{Im }\hat{g}_t, 0\} \ge 0$ and $g_t^r = \text{Re }\hat{g}_t \ge 0$.

For $* = r, +, -$, we fix $g_t = g_t^*$. Then $dG_t := g_t dG$ is a (positive) measure on $\mathbb{R}$. Note that by (3.5)

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}} |g_t(x) - K| = 0$$

where $K = K_* = 1$ for $* = r$ and $K = 0$ otherwise.

Define distribution functions $G^j,\ G_t^j$ $(j = 1, 2)$ on $\mathbb{R}_+$ by

$$G_t^1(x) := G_t(x) - G_t(0),\ G_t^2(x) := G_t(0) - G_t(-x),$$
$$G^1(x) := G(x) - G(0), \text{ and } \quad G^2(x) := G(0) - G(-x).$$

We have that

$$G_t^j(\infty) - G_t^j(x) = \frac{h_j(x)}{x^2} g_j(t,x), \tag{3).6}$$

where

$$h_j(x) := \begin{cases} x^2(1 - G(x)) = (c_1 + o(1))l(x) & \text{if } \; j = 1 \\ x^2 G(-x) = (c_2 + o(1))l(x) & \text{if } \; j = 2 \end{cases}$$

as $x \to \infty$, and

$$g_1(t,x) := \frac{\int_x^\infty g_t(u)\; G(du)}{\int_x^\infty G(du)}, \quad g_2(t,x) := \frac{\int_{-\infty}^{-x} g_t(u)\; G(du)}{\int_{-\infty}^{-x} G(du)}.$$

It follows from (3.5) again that $\sup_{x\in\mathbb{R}} |g_j(t,x) - K| \to 0$ as $t \to 0$.

We need the following calculations. First note that

$$\begin{aligned}
&\int_{\mathbb{R}} (1 + itx - e^{itx})\ G_t(dx) \\
&= \int_0^\infty (1 + itx - e^{itx})\ G_t^1(dx) + \int_0^\infty (1 - itx - e^{-itx})\ G_t^2(dx),
\end{aligned}$$

and secondly that integration by parts (for $j = 1, 2$) yields

$$\begin{aligned}
&\int_0^\infty (1 - (-1)^j itx - \exp[-(-1)^j itx]) G_t^j(dx) \\
&= -[(G_t^j(\infty) - G_t^j(x))(1 - (-1)^j itx - \exp[-(-1)^j itx])]_0^\infty \\
&\quad + \int_0^\infty (G_t^j(\infty) - G_t^j(x))((-1)^j it \exp[-(-1)^j itx] - (-1)^j it) dx \\
&= i(-1)^j t \int_0^\infty \Big(\exp[-i(-1)^j tx] - 1\Big) g_j(t,x) \frac{h_j(x)}{x^2} dx.
\end{aligned}$$

We split the last integral into three parts:

$$\begin{aligned}
&t \int_{|t|^{-1}}^\infty \Big(\exp[-i(-1)^j tx] - 1\Big) g_j(t,x) \frac{h_j(x)}{x^2} dx \\
&+ t \int_0^{|t|^{-1}} \Big(\exp[-i(-1)^j tx] - 1 + i(-1)^j tx\Big) g_j(t,x) \frac{h_j(x)}{x^2} dx \\
&- t \int_0^{|t|^{-1}} i(-1)^j tx g_j(t,x) \frac{h_j(x)}{x^2} dx.
\end{aligned}$$

For the first integral we obtain using (1.3)

$$\begin{aligned}
&t \int_{|t|^{-1}}^\infty \Big(\exp[-i(-1)^j tx] - 1\Big) g_j(t,x) \frac{h_j(x)}{x^2} dx \\
&= \operatorname{sgn}(t) \int_1^\infty \Big(\exp[-i(-1)^j y \operatorname{sgn}(t)] - 1\Big) g_j(t, y/|t|) \frac{h_j(y/|t|)}{(y/|t|)^2} dy \\
&= O\Big(\int_1^\infty \frac{t^2}{y^2} h_j(y/|t|) dy\Big) \\
&= O\Big(\int_1^\infty \frac{t^2}{y^2} l(y/|t|) dy\Big) \\
&= O\Big(t^2 l(1/|t|)\Big) = o\Big(t^2 L(1/|t|)\Big).
\end{aligned}$$

Since l is slowly varying

$$|t|\int_0^{|t|^{-1}} l(x)dx = O\Big(l(|t|^{-1})\Big).$$

From this and (1.3) we obtain for the second integral that

$$\begin{aligned}
&t\int_0^{|t|^{-1}}\Big(\exp[-i(-1)^j tx]-1+i(-1)^j tx\Big)g_j(t,x)\frac{h_j(x)}{x^2}dx\\
&=O\Big(t^3\int_0^{|t|^{-1}} h_j(x)dx\Big)\\
&=O\Big(t^2 l(1/|t|)\Big)=o\Big(t^2 L(1/|t|)\Big).
\end{aligned}$$

The third integral, multiplied by $i(-1)^j$, is equal to

$$\begin{aligned}
&t^2\int_0^{|t|^{-1}} x g_j(t,x)\frac{h_j(x)}{x^2}dx\\
&=t^2\int_0^{|t|^{-1}} x(G_t^j(\infty)-G_t^j(x))dx\\
&=\frac{t^2}{2}[(G_t^j(\infty)-G_t^j(x))x^2]_0^{|t|^{-1}}+\frac{t^2}{2}\int_0^{|t|^{-1}} x^2 G_t^j(dx)\\
&=\frac{t^2}{2}\int_0^{|t|^{-1}} x^2 G_t^j(dx) \;+\; o(t^2L(1/|t|))\\
&=\begin{cases}(K+o(1))\frac{t^2}{2}\int_0^{|t|^{-1}} x^2G(dx) \;+\; o(t^2L(1/|t|)) & j=1\\ (K+o(1))\frac{t^2}{2}\int_{-|t|^{-1}}^0 x^2G(dx) \;+\; o(t^2L(1/|t|)) & j=2\end{cases}\\
&=(K+o(1))\frac{t^2}{2}L(1/|t|),
\end{aligned}$$

where we used (1.1), (3.5) and (3.6). Finally note that by (3.5)

$$\gamma_t := \int_{\mathbb{R}} x\hat{g}_t(x)G(dx) = \gamma + O(|t|) \text{ as } t\to 0,$$

and, since G is not in the normal domain of attraction, we have $t^2 = o(t^2L(1/|t|))$.

The proof of theorem 3.1 is completed by using (3.4) and the previous estimates:

$$\begin{aligned}
&\log\lambda(t) - it\gamma \sim \lambda(t) - 1 - it\gamma \\
&= \int_{-\infty}^{\infty}\Big(e^{itx} - 1 - itx\Big)\hat{g}_t(x)G(dx) + o(t^2 L(1/|t|)) \\
&= \int_{-\infty}^{\infty}\Big(e^{itx} - 1 - itx\Big)(g_t^r(x) + ig_t^+(x) - ig_t^-(x))G(dx) + o(t^2 L(1/|t|)) \\
&= \frac{t^2}{2}\int_{-|t|^{-1}}^{|t|^{-1}} x^2 \hat{g}_t^r(x)G(dx) + o(t^2 L(1/|t|)) \\
&= t^2 L(1/|t|)(1 + o(1)).
\end{aligned}$$

Let

$$nL(B_n) = B_n^2, \quad A_n = \gamma n. \tag{3).7}$$

The following corollaries contain the local and central limit theorems. Their proofs are straightforward using theorem 3.1 (cf. corresponding statements in [AD]). We write, as before,

$$S_n = X_1 + X_2 + ... + X_n,$$

and denote by ϕ the density of the standard normal distribution.

Corollary 3.5: (Conditional lattice local limit theorem) Suppose that X_1 is aperiodic.

Let $A_n,\ B_n$ be as defined in (3.7), and suppose that $k_n \in \mathbb{Z},\ \frac{k_n - A_n}{B_n} \to \kappa \in \mathbb{R}$ as $n \to \infty$, then

$$\|B_n P^n(1_{[S_n = k_n]}) - \phi(\kappa)\|_\infty \to 0 \quad \text{ as } n \to \infty,$$

and, in particular

$$B_n E1_{[S_n = k_n]} \to \phi(\kappa) \quad \text{ as } n \to \infty.$$

Corollary 3.6: (Conditional non-lattice local limit theorem) Suppose that X_1 is aperiodic.

Let $A_n,\ B_n$ be as defined in (3.7), let $I \subset \mathbb{R}$ be an interval, and suppose that $k_n \in \mathbb{Z},\ \frac{k_n - A_n}{B_n} \to \kappa \in \mathbb{R}$ as $n \to \infty$, then

$$B_n P^n(1_{[S_n \in k_n + I]}) \to |I|\phi(\kappa) \quad \text{ as } n \to \infty$$

where $|I|$ is the length of I, and in particular

$$B_n E1_{[S_n \in k_n + I]} \to |I|\phi(\kappa) \quad \text{ as } n \to \infty.$$

Corollary 3.7: (Distributional limit theorem) Let A_n, B_n be as defined in (3.7). Then

$$\frac{S_n - A_n}{B_n}$$

is asymptotically standard normal.

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(Aaronson) School of Math. Sciences, Tel Aviv University, 69978 Tel Aviv, Israel.

Email address: `aaro@tau.ac.il`

(Denker) Institut für Mathematische Stochastik, Universität Göttingen, Lotzestr. 13, 37083 Göttingen, Germany

Email address: `denkermath.uni-goettingen.de`