ON EXACT GROUP EXTENSIONS

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Abstract. We give conditions for the exactness of $\mathbb{R}^d$-extensions.

§0 Introduction

A fibred system $(X,\mathcal{B},m,T,\alpha)$ is a nonsingular transformation $(X,\mathcal{B},m,T)$ of a standard probability space equipped with a countable, measurable partition $\alpha \subset \mathcal{B}$, generating $\mathcal{B}$ (in the sense that $\sigma(\{T^{-n}a: a \in \alpha, n \geq 0\}) = \mathcal{B}$) such that $T : a \rightarrow Ta$ is invertible, nonsingular for $a \in \alpha$. A fibred system $(X,\mathcal{B},m,T,\alpha)$ is called a Markov map (or Markov fibred system) if $Ta \in \sigma(\alpha)$ mod $m \quad \forall \ a \in \alpha$.

Write $\alpha = \{a_s : s \in S\}$ and endow $S^\infty$ with its canonical (Polish) product topology. Let $\Sigma = \{s = (s_1,s_2,\ldots) \in S^\infty : m(\bigcap_{k=1}^{\infty} T^{-k}a_{s_k}) > 0 \ \forall \ n \geq 1\}$, then $\Sigma$ is a closed, shift invariant subset of $S^\infty$, and there is a measurable map $\phi : \Sigma \rightarrow X$ defined by $\phi(s_1,s_2,\ldots) := \bigcap_{k=1}^{\infty} T^{-(k-1)}a_{s_k}$.

The closed support of the probability $m' = m \circ \phi^{-1}$ is $\Sigma$, and $\phi$ is a conjugacy of $(X,\mathcal{B},m,T)$ with $(\Sigma,\mathcal{B}(\Sigma),m',\text{shift})$. Thus we may, and sometimes do, assume that $X = \Sigma$, $T$ is the shift, and $\alpha = \{[s] : s \in S\}$.

For $n \geq 1$, there are $m$-nonsingular inverse branches of $T$ denoted $v_a : T^n a \rightarrow a$ and defined by $v_a(x) := (a,x)$ ($a \in a_0^{n-1}$) with Radon Nikodym derivatives denoted

$$v'_a := \frac{dm \circ v_a}{dm}.$$ 

Let $(X,\mathcal{B},m,R)$ be a nonsingular transformation of a standard probability space.

The Frobenius-Perron operators $P_{R^n} = P_{R^n,m} : L^1(m) \rightarrow L^1(m)$ are defined by

$$\int_X P_{R^n} f \cdot gdm = \int_X f \cdot g \circ R^n dm$$

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and for the fibred system \((X, \mathcal{B}, m, T, \alpha)\) (as above) have the form

\[ P_{T^n}f = \sum_{\alpha \in \alpha_0^{-1}} 1_{T^n \alpha} v'_{\alpha} \cdot f \circ v_{\alpha}. \]

A fibred system \((X, \mathcal{B}, m, T, \alpha)\) has:

- the Renyi property if \(\exists C > 1\) such that \(\forall n \geq 1, a \in \alpha_0^{-1}, m(a) > 0: |v'_{\alpha}(x)/v'_{\alpha}(y)| \leq C \) for \(m \times m\)-a.e. \((x, y) \in T^n a \times T^n a.\)

It is well known (a proof is recalled in [?]) that any topologically mixing probability preserving Markov map with the Renyi property is exact in the sense that \(\cap_{n \geq 1} T^{-n} \mathcal{B} = \{\emptyset, X\} \mod m.\)

Examples include:

- topological Markov shifts equipped with Gibbs measures ([?],[?]) and
- uniformly expanding, piecewise onto \(C^2\) interval maps \(T : [0, 1] \rightarrow [0, 1]\) satisfying Adler’s condition \(\sup_{x \in [0, 1]} |T^n(x)|/T^n(x)^2 < \infty\) ([?]);

or, generalising the above two examples:

- Gibbs-Markov maps as in [?],

the Markov map \((X, \mathcal{B}, m, T, \alpha)\) being called Gibbs-Markov if it has the Gibbs property that \(\exists C > 1, 0 < r < 1\) such that \(\forall n \geq 1, a \in \alpha_0^{-1}, m(a) > 0: |v'_{\alpha}(x)/v'_{\alpha}(y)| - 1 \leq C r^{t(x,y)}\) for \(m \times m\)-a.e. \((x, y) \in T^n a \times T^n a,\) (see §4.6, §4.7 of [?]);

and the big image property that \(\inf_{\alpha \in \alpha_0} m(Ta) > 0.\)

Now let \(\phi : X \rightarrow \mathbb{R}^d\) be measurable and consider the skew product \(T_\phi : X \times \mathbb{R}^d \rightarrow X \times \mathbb{R}^d\) defined by \(T_\phi(x, y) := (Tx, y + \phi(x))\) with respect to the (invariant) product measure \(m \times m_{\mathbb{R}^d}\) where \(m_{\mathbb{R}^d}\) denotes Lebesgue measure.

We say that \(\phi\) is aperiodic if \(\gamma(\phi) = \int h \circ T\) has no nontrivial solution in \(\gamma \in \mathbb{R}^d, z \in S^1\) and \(h : X \rightarrow S^1\) measurable.

It is not hard to show that if \(T_\phi\) is ergodic, and \(T\) is weakly mixing, then \(T_\phi\) is weakly mixing iff \(\phi\) is aperiodic.

We’re interested in the exactness of \(T_\phi.\)

We establish two (partial) results in this direction.

**Theorem 1**

Suppose that \((X, \mathcal{B}, m, T, \alpha)\) is a probability preserving Markov map with the Renyi property. Let \(N \geq 1\) and \(\phi : X \rightarrow \mathbb{R}^d\) be \(\alpha_0^{-1}\)-measurable (i.e. \(\phi(x) = \phi(\alpha_0^{-1}(x))\)) where \(x \in \alpha_0^{-1}(x) \in \alpha_0^{-1}(x)\).

If \(T_\phi\) is topologically mixing, then \(T_\phi\) is exact.
For the other result, we assume that \((X, B, m, T, \alpha)\) is an exact probability preserving fibred system with the property that there is a Banach space \((L, \| \cdot \|_L)\) of functions with \(\| \cdot \|_2 \leq \| \cdot \|_L\), such that \(P_T : L \to L\) and \(\exists H > 0, 0 < r < 1, N \geq 1\) such that
\[
\| P_T^n f \|_L \leq r \| f \|_L + H \| f \|_1 \quad \forall \ f \in L.
\]
In this case (see \([\text{?}, \text{?}]\)) \(\exists M > 0, \theta \in (0, 1)\) such that
\[
\| P_T^n f - \int_X f dm \|_L \leq M \theta^n \| f \|_L \quad \forall \ f \in L.
\]

Given \(\phi : X \to \mathbb{R}^d\) measurable, we define the characteristic function operators \(P_t(f) = P_T(e^{i(t, \phi)} f) \quad (t \in \mathbb{R}^d)\).
We assume also that \(P_t : L \to L \quad (t \in \mathbb{R}^d)\) and that \(t \mapsto P_t\) is continuous \(\quad (\mathbb{R}^d \to \text{Hom}(L, L)).\)

It is shown in \([\text{?}]\) (see also theorem 4.1 of \([\text{?}]\)) that
\((i)\) there are constants \(\epsilon > 0, K > 0\) and \(\theta \in (0, 1)\); and continuous functions \(\lambda : B(0, \epsilon) \to B_C(0, 1), g : B(0, \epsilon) \to L\) such that
\[
\| P_t^n h - \lambda(t)^n g(t) \int_X h dm \|_L \leq K \theta^n \| h \|_L \quad \forall \ |t| < \epsilon, n \geq 1, h \in L;
\]
\((ii)\) if \(\gamma(\phi) = z \tilde{h} h \circ T\) where \(\gamma \in \mathbb{R}^d, z \in S^1\) and \(h : X \to S^1\) measurable,
then \(h \in L;\)
and
\((ii)\) in case \(\phi\) is aperiodic, then \(\forall \ 0 < \delta < M < \infty, \exists K > 0, 0 < \rho < 1\) such that
\[
\| P_t^n h \|_L \leq K \rho^n \quad \forall \ h \in L, n \geq 1, \delta \leq |\gamma| \leq M.
\]

Examples include:
- (see \([\text{?}],\) \((X, B, m, T, \alpha)\) a Gibbs-Markov maps and \(\phi : X \to \mathbb{R}^d\) uniformly Hölder continuous on partition sets. Here \(L\) is a space of Hölder continuous functions \(f : X \to \mathbb{C}.\)
- (see \([\text{?}],\) \([\text{?}]\), \(X = [0, 1], m\) Lebesgue measure, \(\alpha\) a partition of \(X\) mod \(m\) into open intervals, and \(T : a \to Ta\) an invertible, \(m\)-nonsingular homeomorphism for each \(a \in \alpha\) with \(\inf |T'| > 1\) and \(1/T'\) of bounded variation on \(X;\) and \(\phi : X \to \mathbb{R}^d\) either: of bounded variation on \(X;\) or constant on each \(a \in \alpha.\) Here \(L\) is the space of functions \(f : X \to \mathbb{C}\) of bounded variation on \(X.\)

Set \(\phi_n = \phi + \phi \circ T \ldots + \phi \circ T^{n-1}.\)

**Theorem 2**

Suppose that
\[(o) \quad \forall \lambda > 1 \exists n_k \to \infty \text{ such that } \frac{\phi_{n_k}}{\lambda^{n_k}} \to 0 \text{ a.e. as } k \to \infty.
\]
and that $\phi$ is aperiodic; then $T_\phi$ is exact.

Remarks

1) Theorem 2 generalises the corresponding theorem on page 443 in [?].
2) The condition $(\diamond)$ is satisfied if $m\text{-dist}(\phi)$ is in the domain of attraction of a stable law.
3) The condition $(\diamond)$ is not satisfied iff $\exists \lambda > 1$ and $\epsilon > 0$ such that $m(\{\phi_n > \lambda^n\}) \geq \epsilon \forall n \geq 1$ and there are independent processes like this.
4) For examples satisfying the assumptions of the theorems, let $X = [0,1]$, $Tx = \{1\ x\}$, then $T$ is a piecewise onto $C^2$ interval map with Markov partition $\alpha = \{I_n = (\frac{1}{n+1}, \frac{1}{n}) : n \geq 1\}$. The invariant probability is Gauss’ measure $dm(x) := \frac{1}{\log 2} \frac{dx}{1+x}$. Since $T^2$ is uniformly expanding and satisfies Adler’s condition, we have (passing to the Polish product topology induced by $\alpha$) that $(X, B, m, T, \alpha)$ has the Gibbs property, whence $(T$ is piecewise onto) the Renyi property and is Gibbs-Markov.

It is not hard to show that if $\phi : X \to \mathbb{R}$ is constant on each $I_n$, takes the value 0 and the semigroup generated by the values of $\phi$ is dense in $\mathbb{R}$, then $T_\phi$ is topologically mixing and therefore exact by theorem 1. Such functions $\phi : X \to \mathbb{R}$ are aperiodic by corollary 3.2 of [A-D], and so the exactness of $T_\phi$ is also established by theorem 2. On the other hand, if $\phi(x) = \log \frac{1}{x}$ then $T_\phi$ is not topologically mixing (since $\phi \geq 0$). Nevertheless, $\phi$ is aperiodic by corollary 3.2 of [A-D], and so $T_\phi$ is exact by theorem 2 (but totally dissipative).

§1 Frobenius-Perron operators, exactness and relative exactness

Let $(X, B, m, R)$ be a nonsingular transformation of a standard probability space. The tail $\sigma$-algebra of $(X, B, m, R)$ is $\mathcal{T}(R) := \bigcap_{n=1}^{\infty} R^{-n}B$ and the nonsingular transformation $R$ is called exact if $\{\emptyset, X\} \mod m$.

Theorem 1.1 [?]

$$\|P_{R^n} f\|_1 \to \|E(f|\mathcal{T}(R))\|_1$$ as $n \to \infty \forall f \in L^1(m)$.

In particular (see [?]), $R$ is exact iff $\|P_{R^n} f\|_1 \to 0 \forall f \in L^1(m), \int_X f dm = 0$.

Proof

First note that $|P_{R^n} f| \leq P_{R^n} |f|$ whence $\|P_{R^n} f\|_1 \downarrow$ and $\exists \lim_{n \to \infty} \|P_{R^n} f\|_1$. Next, $\forall n \geq 1 \exists g_n \in L^\infty(B)$ with $\int_X (P_{R^n} f) g_n dm = \|P_{R^n} f\|_1$, whence
\[ \|P_{R^n} f\|_1 = \int_X f g_n \circ R^n \, dm. \]

By weak * compactness, \(\exists n_k \to \infty\) and \(g \in L^\infty(\mathcal{B})\) such that \(g_{n_k} \circ R^{n_k} \to g\) weak * in \(L^\infty(\mathcal{B})\).

It follows that \(g \in L^\infty(\mathcal{T}(R))\), \(\|g\|_\infty \leq 1\) and \(\lim_{n \to \infty} \|P_{R^n} f\|_1 = \int_X f g \, dm\). Thus

\[ \lim_{n \to \infty} \|P_{R^n} f\|_1 \leq \sup \{ \int_X f h \, dm : h \in L^\infty(\mathcal{T}(R)), \|h\|_\infty \leq 1 \} = \|E(f|\mathcal{T}(R))\|_1. \]

To show the converse inequality, note that \(\exists g \in L^\infty(\mathcal{T}(R))\), \(\|g\|_\infty = 1\) such that

\[ \|E(f|\mathcal{T}(R))\|_1 = \int_X E(f|\mathcal{T}(R)) g \, dm = \int_X f g \, dm \]

whence \(\forall n \geq 1, \exists g_n \in L^\infty(\mathcal{B}), g = g_n \circ R^n\) and

\[ \|E(f|\mathcal{T}(R))\|_1 = \int_X f g \, dm = \int_X f g_n \circ R^n \, dm = \int_X (P_{R^n} f) g_n \, dm \leq \|P_{R^n} f\|_1. \]

[\square]

Let \((X, \mathcal{B}, m, R)\) and \((Y, \mathcal{C}, \mu, S)\) be nonsingular transformations of standard probability spaces. A factor map is a function \(\pi : X \to Y\) satisfying \(\pi^{-1} \mathcal{C} \subseteq \mathcal{B}\), \(\pi \circ T = S \circ \pi\), \(m \circ \pi^{-1} = \mu\).

The fibre expectation of the factor map \(\pi : X \to Y\) is an operator

\[ f \mapsto E(f|\pi), \quad L^1(X, \mathcal{B}, m) \to L^1(Y, \mathcal{C}, \mu) \]

defined by \(\int_Y E(f|\pi) g \, d\mu = \int_X f g \circ \pi \, dm\).

The factor map \(\pi : X \to Y\) is called relatively exact if

\[ f \in L^1(\mathcal{B}), \quad E(f|\pi) = 0 \text{ a.e.} \quad \implies \quad \|P_{R^n} f\|_1 \to 0. \]

The corollary below appears in [1]. For the convenience of the reader, we supply a (possibly different) proof.

**Proposition 1.2** Suppose that \(\pi : X \to Y\) is relatively exact, then \(\mathcal{T}(R) = \pi^{-1} \mathcal{T}(S) \mod m\).

**Proof**

Evidently, \(\pi^{-1} \mathcal{T}(S) \subseteq \mathcal{T}(R)\). We show that \(\pi^{-1} \mathcal{T}(S) \supseteq \mathcal{T}(R)\).

By relative exactness and theorem 1.1, if \(f \in L^1(\mathcal{B})\) and \(E(f|\pi) = 0\) a.e., then \(\int_X f g \, dm = 0 \quad \forall g \in L^\infty(\mathcal{T}(R))\).

Thus if \(f \in L^2(\mathcal{B}) \ominus L^2(\pi^{-1} \mathcal{C})\), then \(E(f|\pi) = 0\) a.e. and so

\[ \int_X f g \, dm = 0 \quad \forall g \in L^\infty(\mathcal{T}(R)), \quad \implies \quad f \perp L^2(\mathcal{T}(R)). \]
Thus $L^2(B) \otimes L^2(\pi^{-1}C) \subset L^2(B) \otimes L^2(T(R))$ whence $L^2(T(R)) \subset L^2(\pi^{-1}C)$ and $T(R) \subset \pi^{-1}C \mod m$.

To see that in fact $T(R) \subset \pi^{-1}T(S) \mod m$, fix $N \geq 1$, then

$$T(R) = \bigcap_{n \geq 1} R^{-n}B = \bigcap_{n \geq N+1} R^{-n}B$$

$$= R^{-N}T(R) \subset R^{-N}\pi^{-1}C = \pi^{-1}S^{-N}C.$$ Taking the intersection over $N$ shows the claim. \hfill \Box

**Corollary 1.3** ([?], proposition 1)

*If $S$ is exact and $\pi : X \to Y$ is relatively exact, then $T$ is exact.*

§2 Proof of theorem 1

For a nonsingular transformation $(X, B, m, R)$, define the *tail relation* of $R$:

$$\Sigma(R) := \{((x, y) \in X \times X : \exists \; n \geq 0, R^n x = R^n y\}.$$

Evidently $\Sigma(R)$ is an equivalence relation and if $(X, B, m)$ is standard, then $\Sigma(R) \in B(X \times X)$.

If $R$ is locally invertible, then $\Sigma(R)$ has countable equivalence classes and is nonsingular in the sense that $m(\Sigma(R)(A)) = 0 \forall \; A \in B, \; m(A) = 0$ where $\Sigma(R)(A) := \{y \in X : \exists \; x \in A \; (x, y) \in \Sigma(R)\}$.

A set $A \in B(X)$ is *invariant* under the equivalence relation $\Sigma \in B(X \times X)$ if $\Sigma(A) = A$ and the equivalence relation $\Sigma$ is called *ergodic* if $\Sigma$-invariant sets have either zero, or full measure.

The collection of invariant sets under $\Sigma(R)$ is the tail $\sigma$-algebra $T(R)$ (whence the name "tail relation").

In order to prove theorem 1, it suffices to show that $\Sigma(T_\phi)$ is ergodic.

The tail relation of $T_\phi$ is given by

$$\Sigma(T_\phi)$$

$$= \{((x, s), (y, t)) \in (X \times G)^2 : \exists \; n \geq 0, T^n x = T^n y, \; s - t = \phi_n(y) - \phi_n(x)\}$$

$$= \{((x, s), (y, t)) \in (X \times G)^2 : (x, y) \in \Sigma(T), \; \tilde{\phi}(x, y) = s - t\}$$

where $\tilde{\phi} : \Sigma(T) \to \mathbb{R}^d$ is defined by $\tilde{\phi}(x, y) := \sum_{n=0}^{\infty} (\phi(T_n y) - \phi(T_n x))$.

We prove that $\Sigma(T_\phi)$ is ergodic by the method of Schmidt (explained in [?]), by showing that $\forall \; t \in \mathbb{R}^d, \; U$ a neighbourhood of $t$ and $A \in B \; m(A) > 0, \; \exists \; B \in B \; B \subset A$ and $\tau : B \to B$ nonsingular such that $(x, \tau(x)) \in \Sigma(T)$ and $\tilde{\phi}(x, \tau(x)) \in U \forall \; x \in B$. 
This boils down to showing that
\[ \forall A \in B_+ \text{ } g_0 \in \mathbb{R}^d \text{ } \eta > 0, \exists B \in B, B \subset A, \text{ and } T^n \circ \phi_t \equiv T^n \text{ and } \| \phi_n \circ T^n - g_0 \| < \eta \text{ on } B. \]

The proof of \((\ddagger)\) will be written as a sequence of minor claims, \(\ddagger_0, \ddagger_1, \ldots\).

\(\ddagger_0\) We first claim that there is no loss in generality in assuming that \(N = 1\) (i.e. that \(\phi : X \to \mathbb{R}^d\) is \(\alpha\)-measurable). This is because \((X,B,m,T,\alpha_0^N)\) is also a probability preserving Markov map with the Renyi property and inducing the same (shift) topology on \(X\).

\(\ddagger_1\) \(\forall s, t \in S, \exists \kappa = \kappa_{s,t} \geq 1 \text{ and } a = a_{s,t} = [a_1, \ldots, a_\kappa], b = b_{s,t} = [b_1, \ldots, b_\kappa] \in \alpha_0^{-1}, a_1 = b_1 = s a_\kappa = b_\kappa = t \text{ such that } \| \phi_n(b) - \phi_n(a) - g_0 \| < \eta.\)

This follows from topological mixing of \(T_\phi\).

By the Renyi property, \(\exists M > 1\) such that
\[ M^{-1} m(u) m(v) \leq m(u \cap T^{-k} v) \leq M m(u) m(v) \forall u \in \alpha_0^{k-1}, v \in \alpha_0^{k-1}, [v_1] \subset T[u_k]. \]

Given \(u = [u_1, \ldots, u_n] \in \alpha_0^{-1}\) with \(u_n = t\), define \(\tau = \tau_u : u \cap T^{-n} a \to u \cap T^{-n} b\) by
\[ \tau(u_1, \ldots, u_n, a_1, \ldots, a_\kappa, y) := \tau(u_1, \ldots, u_n, b_1, \ldots, b_\kappa, y). \]

\(\ddagger_2\) \(\tau = \tau_u : u \cap T^{-n} a \to u \cap T^{-n} b\) is invertible nonsingular and \(\frac{dm \circ \tau}{dm} = \frac{M^{-1} m(b)}{m(a)}.\)

**Proof**
\[ \int_{u \cap T^{-n} a \cap c} \frac{dm \circ \tau}{dm} dm = m(u \cap T^{-n} b \cap c) \]
\[ = M^{+2} \frac{m(b)}{m(a)} m(u) m(b) m(c) \]
\[ = M^{+4} \frac{m(b)}{m(a)} m(u \cap T^{-n} a \cap c). \]

\(\ddagger_3\) **Proof of \(\ddagger_2\)**
Fix \(0 < \epsilon < M^{-1} \min \{ m(a_{s,t}), m(b_{s,t}) \}\), then
\[ m(u \cap T^{-n} a_{s,t}), m(u \cap T^{-n} b_{s,t}) \geq \epsilon m(u) \forall u \in \alpha_0^{n-1}, [s] \subset T[u_n]. \]

Let \(\delta > 0\) be so small that \(\delta < \frac{m(b)(\epsilon - \delta)}{M^2 m(a)}.\)

\(\exists n \geq 1\) and \(u \in \alpha_0^{n-1}\) such that \(m(A \cap u) \geq (1 - \delta) m(u)\) and \([s] \subset T[u_n].\)
Consider \( \tau_u : u \cap T^{-n}a \to u \cap T^{-n}b \) as in \( \mathfrak{Q}2 \). Evidently \( T^{n+\kappa} \circ \tau \equiv T^{n+\kappa} \) and \( \| \phi_{n+\kappa} \circ \tau - \phi_{n+\kappa} - g_0 \| < \eta \) on \( u \cap T^{-n}a \).

To complete the proof we claim that \( \exists B \in \mathcal{B}_+ \ B \subset A \cap u \cap T^{-n}a \) such that \( \tau B \subset A \).

To see this, note that
\[
m(u \cap T^{-n}a \cap A) \geq m(u \cap T^{-n}a) - m(u \setminus A) \geq (\epsilon - \delta)m(u),
\]
whence using \( \mathfrak{Q}2 \),
\[
m(\tau(u \cap T^{-n}a \cap A)) \geq \frac{m(b)}{M^4 m(a)} m(u \cap T^{-n}a \cap A) \geq \frac{m(b)(\epsilon - \delta)}{M^4 m(a)} m(u).
\]
Since \( \tau(u \cap T^{-n}a \cap A) \subset u \), the condition on \( \delta > 0 \) ensures that \( m(\tau(u \cap T^{-n}a \cap A) \cap A) > 0 \) whence \( m(B) > 0 \) where \( B := \tau^{-1}\left( \tau(u \cap T^{-n}a \cap A) \cap A \right) \subset A \).

\[\Box\]

\section{3 Proof of theorem 2}

We prove theorem 2 via corollary 1.3. To do this, we must consider \( T_\phi \) as a nonsingular transformation with respect to some probability \( P \sim m \times m_{\mathbb{R}^d} \).

Let \( p : \mathbb{R}^d \to \mathbb{R}_+ \) be continuous with \( \int_{\mathbb{R}^d} p(y)dy = 1 \) and define a probability \( P \) on \( X \times \mathbb{R}^d \) by \( dP(x,y) := p(y)d\nu(x)dy \); then \( (X \times \mathbb{R}^d, \mathcal{B}(X \times \mathbb{R}^d), P, T_\phi) \) is a nonsingular transformation with Frobenius-Perron operators given by
\[
P_{T_\phi^n}(f \cdot 1 \otimes p)(x,y) = \frac{1}{p(y)} P_{T_\phi^n}(f)(x,y)
\]
where \( P_{T_\phi^n} := P_{T_\phi^n m \times m_{\mathbb{R}^d}} \).

Consider the map \( \pi : X \times \mathbb{R}^d \to X \) defined by \( \pi(x,y) = x \). This is a factor map as it satisfies \( \pi^{-1} \mathcal{B}(X) \subset \mathcal{B}(X \times \mathbb{R}^d) \), \( \pi \circ T_\phi = T \circ \pi \), \( P \circ \pi^{-1} = m \).

The fibre expectation of \( \pi \) is given by
\[
E(f|\pi)(x) = \int_{\mathbb{R}^d} f(x,y)p(y)dy \quad (f \in L^1(X \times \mathbb{R}^d, \mathcal{B}(X \times \mathbb{R}^d), P)).
\]

By corollary 1.3 and exactness of \( T \), it suffices to show that \( \pi \) is relatively exact.
To do this, we show that
\[
\int_{\mathbb{R}^d} f(x, y) p(y) dy = 0 \text{ a.e.} \quad \implies \quad \int_{X \times \mathbb{R}^d} |P_{\phi, \lambda}^n(f \cdot 1 \otimes p)| dP = \int_{X \times \mathbb{R}^d} |P_{\phi, \lambda}^n(f \cdot 1)| d(m \times m_{\mathbb{R}^d}) \to 0
\]
as \(n \to \infty\); equivalently (taking \(F(x, y) := f(x, y)p(y)\)),
\[
(*) \quad \int_{\mathbb{R}^d} F(x, y) dy = 0 \text{ a.e.} \quad \implies \quad \int_{X \times \mathbb{R}^d} |P_{\phi, \lambda}^n F| d(m \times m_{\mathbb{R}^d}) \to 0
\]
as \(n \to \infty\).

To prove \((*)\), we first claim that
\[
\mathbb{I}_{1} \text{ for } \lambda > 1, \ h \in L^1(m) \text{ and } f \in L^1(\mathbb{R}^d),
\]
\[
\|P_{\phi, \lambda}^n(h \otimes f)\|_1 \leq C \lambda^{\frac{d}{2}} \|P_{\phi, \lambda}^n(h \otimes f)\|_2 + o(1)
\]
as \(k \to \infty\) where \(C = 2^{\frac{d}{2}} m(B(0, 1))\) and \(\frac{\phi_{n_k}}{x_{n_k}} \to 0 \text{ a.e.}\).

PROOF As can be checked,
\[
P_{\phi, \lambda}^n(h \otimes f)(x, y) = P_{\phi, \lambda}^n(h(\cdot) f(y - \phi_n(\cdot))) (x) \quad (h \in L^1(m), \ f \in L^1(\mathbb{R}^d)).
\]

Denoting \(E(H) := \int_X H dm\) for \(H \in L^1(m)\), we have
\[
\|P_{\phi, \lambda}^n(h \otimes f)\|_1 = \int_{\mathbb{R}^d} |E(P_{\phi, \lambda}^n(h(\cdot) f(y - \phi_n(\cdot))))| dy \leq \int_{|y| \leq 2\lambda_k} + \int_{|y| > 2\lambda_k}.
\]

By the Cauchy-Schwartz inequality,
\[
\int_{|y| \leq 2\lambda_k} \leq \sqrt{m_{\mathbb{R}^d}(B(0, 2\lambda_k))} \|P_{\phi, \lambda}^n(h \otimes f)\|_2 = C \lambda^{\frac{d}{2}} \|P_{\phi, \lambda}^n(h \otimes f)\|_2
\]

whereas
\[
\int_{|y| > 2\lambda_k} \leq \int_{|y| > 2\lambda_k} |E(P_{\phi, \lambda}^n(h(\cdot) f(y - \phi_n(\cdot)) 1_{[\|\phi_n(\cdot)\| \leq \lambda_k]}))| dy
\]
\[
+ \int_{|y| > 2\lambda_k} |E(P_{\phi, \lambda}^n(h(\cdot) f(y - \phi_n(\cdot)) 1_{[\|\phi_n(\cdot)\| > \lambda_k]}))| dy = I + II.
\]

Here as \(k \to \infty\):
\[
II \leq \|f\|_1 E(|h| 1_{[\|\phi_n(\cdot)\| > \lambda_k]}) \to 0
\]
since $\frac{\phi_{nk}}{\lambda^nk} \to 0$ a.e.; and
\[
I \leq \int_{|y|>2\lambda^nk} E(|h||f(y-\phi_{nk})|1_{[0,\lambda^nk]}(y)) \, dy
= E\left(|h|1_{[0,\lambda^nk]} \int_{|y|>2\lambda^nk} |f(y-\phi_{nk})| \, dy\right)
\leq E(|h|) \int_{|y|>2\lambda^nk} |f(y)| \, dy \to 0,
\]
(5)
Substituting (3),(4) and (5) into (2) proves $\square$. \hfill \qed

To complete the proof of $(\ast)$, let $F \in L^1(m \times m_{\mathbb{R}^d})$ satisfy $\int_{\mathbb{R}^d} F(x,y) \, dy = 0$ for $m$-a.e. $x \in X$ and fix $\epsilon > 0$. We show that
\[
\limsup_{n \to \infty} \int_{X \times \mathbb{R}^d} |P_{T^n\phi} F| \, d(m \times m_{\mathbb{R}^d}) < \epsilon.
\]
Standard approximation techniques show that $\forall \epsilon > 0, \exists N \in \mathbb{N}, h_1,\ldots,h_N \in L, g_1,\ldots,g_N \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g_k(y) \, dy = 0$ $(1 \leq k \leq N)$ and
\[
\|F - \sum_{k=1}^N h_k \otimes g_k\|_{L^1(m \times m_{\mathbb{R}^d})} < \frac{\epsilon}{2}.
\]
Next, it follows from theorems 1.6.3 and 1.6.4 in [?] that
\[
\exists f_1,\ldots,f_N \in L^1 \cap L^2
\text{ such that } [f_k \neq 0] \text{ is compact and bounded away from 0 } (1 \leq k \leq N); \quad \text{and}
\]
\[
\|f_k - g_k\|_{L^1(m_{\mathbb{R}^d})} < \frac{\epsilon}{2N\|h_k\|_{L^1(m)}} \quad (1 \leq k \leq N), \text{ whence}
\]
\[
\|\sum_{k=1}^N h_k \otimes f_k - \sum_{k=1}^N h_k \otimes g_k\|_{L^1(m \times m_{\mathbb{R}^d})} \leq \sum_{k=1}^N \|h_k\|_{L^1(m)} \cdot \|f_k - g_k\|_{L^1(\mathbb{R}^d)} < \epsilon,
\]
\[
\|F - \sum_{k=1}^N h_k \otimes f_k\|_{L^1(m \times m_{\mathbb{R}^d})} < \epsilon.
\]
We claim $\square$ If $h \in L$ and $f \in L^1 \cap L^2$ is such that $[\mathcal{F} \neq 0]$ is compact and bounded away from 0, then $\exists 0 < \rho < 1$ such that
\[
\|P_{T^n\phi}(h \otimes f)\|_2 = O(\rho^n) \text{ as } n \to \infty.
\]
(6)
\textbf{Proof}
Let $[\mathcal{F} \neq 0] \subset B(0,M) \setminus B(0,\delta)$. By $(ii)$ (above), $\exists K > 0, \ 0 < \rho < 1$ such that
\[
|P_{T^n\phi} h(x)| \leq K \rho^n \ \forall \ x \in X, \ n \geq 1, \ \delta \leq |\gamma| \leq M,
\]
whence using the fact that the Fourier transform of $y \mapsto P_{T^n\phi}(h \otimes f)(x,y)$ is $\gamma \mapsto \mathcal{F}(\gamma) P_{T^n\phi} h(x)$ and Plancherel’s formula, we have
\[
\| P_{T^n}(h \otimes f) \|_2^2 = \int_X \left( \int_{\mathbb{R}^d} |P_{T^n}(h \otimes f)(x, y)|^2 dy \right) dm(x) \\
= \int_X \left( \int_{\mathbb{R}^d} |\tilde{f}(\gamma)|^2 |P_{T^n} h(x)|^2 d\gamma \right) dm(x) \\
= \int_{\mathbb{R}^d} |\tilde{f}(\gamma)|^2 \| P_{T^n} h \|_2^2 d\gamma \leq K^2 \rho^{2n} \int_{\mathbb{R}^d} |\tilde{f}(\gamma)|^2 d\gamma
\]
proving \( \Box 2. \)

To finish the proof of theorem 2, we claim \( \Box 3 \) if (6) holds for \( h \in L \) and \( f \in L^1 \cap L^2 \), then
\[
\| P_{T^n}(h \otimes f) \|_1 \to 0.
\] (7)

**Proof**

Fix \( \lambda > 1 \) such that \( \lambda^d \rho < 1 \). Suppose that \( \phi_{nk}/x^{n_k} \to 0 \) a.e.. Using (6), we have by \( \Box 1 \),
\[
\| P_{T^{n_k}} (h \otimes f) \|_1 \leq C \lambda^{\frac{n_k d}{2}} \| P_{T^{n_k}} (h \otimes f) \|_2 + o(1) = O(\lambda^{\frac{n_k d}{2}} \rho^{n_k}) + o(1) \to 0
\]
as \( k \to \infty \); establishing (7) since \( \| P_{T^n} (h \otimes f) \|_1 \downarrow \).

This completes the proof of theorem 2.

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