ON EXACT GROUP EXTENSIONS

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ABSTRACT. We give conditions for the exactness of $\mathbb{R}^d$-extensions.

§0 INTRODUCTION

A *fibred system* $(X, \mathcal{B}, m, T, \alpha)$ is a nonsingular transformation $(X, \mathcal{B}, m, T)$ of a standard probability space equipped with a countable, measurable partition $\alpha \subset \mathcal{B}$, generating $\mathcal{B}$ (in the sense that $\sigma(\{T^{-n}a : a \in \alpha, n \geq 0\}) = \mathcal{B}$) such that $T : a \to Ta$ is invertible, nonsingular for $a \in \alpha$. A fibred system $(X, \mathcal{B}, m, T, \alpha)$ is called a *Markov map* (or Markov fibred system) if $Ta \in \sigma(\alpha) \mod m \;\; \forall\, a \in \alpha$.

Write $\alpha = \{a_s : s \in S\}$ and endow $S^{\mathbb{N}}$ with its canonical (Polish) product topology. Let

$$\Sigma = \{s = (s_1, s_2, \dots) \in S^{\mathbb{N}} : \; m(\bigcap_{k=1}^{n} T^{-k} a_{s_k}) > 0 \;\; \forall\; n \geq 1\},$$

then Σ is a closed, shift invariant subset of $S^{\mathbb{N}}$, and there is a measurable map $\phi : \Sigma \to X$ defined by $\{\phi(s_1, s_2, \dots)\} := \bigcap_{k=1}^{\infty} T^{-(k-1)} a_{s_k}$.

The closed support of the probability $m' = m \circ \phi^{-1}$ is Σ, and ϕ is a conjugacy of $(X, \mathcal{B}, m, T)$ with $(\Sigma, \mathcal{B}(\Sigma), m', \text{shift})$. Thus we may, and sometimes do, assume that $X = \Sigma$, T is the shift, and $\alpha = \{[s] : s \in S\}$.

For $n \geq 1$, there are m-nonsingular inverse branches of T denoted $v_a : T^n a \to a$ and defined by $v_a(x) := (a, x) \;\; (a \in \alpha_0^{n-1})$ with Radon Nikodym derivatives denoted

$$v_a' := \frac{dm \circ v_a}{dm}.$$

Let $(X, \mathcal{B}, m, R)$ be a nonsingular transformation of a standard probability space.

The *Frobenius-Perron* operators $P_{R^n} = P_{R^n, m} : L^1(m) \to L^1(m)$ are defined by

$$\int_X P_{R^n} f \cdot g dm = \int_X f \cdot g \circ R^n dm$$

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and for the fibred system $(X,\mathcal{B},m,T,\alpha)$ (as above) have the form

$$P_{T^n}f = \sum_{a\in\alpha_0^{n-1}} 1_{T^n a}v_a' \cdot f\circ v_a.$$

A fibred system $(X,\mathcal{B},m,T,\alpha)$ has:

the *Renyi property* if $\exists\ C>1$ such that $\forall\ n\ge 1,\ a\in\alpha_0^{n-1},\ m(a)>0$: $\left|\frac{v_a'(x)}{v_a'(y)}\right| \le C$ for $m\times m$-a.e. $(x,y)\in T^n a\times T^n a$.

It is well known (a proof is recalled in [ADU93]) that any topologically mixing probability preserving Markov map with the Renyi property is *exact* in the sense that $\bigcap_{n\ge 1} T^{-n}\mathcal{B} = \{\varnothing, X\} \mod m$.

Examples include:

• topological Markov shifts equipped with Gibbs measures ([Bow08],[BR75]) and

• uniformly expanding, piecewise onto C^2 interval maps $T:[0,1]\to[0,1]$ satisfying

Adler's condition $\sup_{x\in[0,1]}\frac{|T''(x)|}{T'(x)^2}<\infty$ ([Adl73]);

or, generalising the above two examples:

• Gibbs-Markov maps as in [AD96],

the Markov map $(X,\mathcal{B},m,T,\alpha)$ being called *Gibbs-Markov* if it has the *Gibbs property* that $\exists\ C>1,\ 0<r<1$ such that $\forall\ n\ge 1,\ a\in\alpha_0^{n-1},\ m(a)>0$:

$\left|\frac{v_a'(x)}{v_a'(y)}-1\right| \le Cr^{t(x,y)}$ for $m\times m$-a.e. $(x,y)\in T^n a\times T^n a$, (see §4.6, §4.7 of [Aar97]);

and the *big image property* that $\inf_{a\in\alpha} m(Ta)>0$.

Now let $\phi: X\to\mathbb{R}^d$ be measurable and consider the skew product $T_\phi: X\times\mathbb{R}^d\to X\times\mathbb{R}^d$ defined by $T_\phi(x,y) := (Tx, y+\phi(x))$ with respect to the (invariant) product measure $m\times m_{\mathbb{R}^d}$ where $m_{\mathbb{R}^d}$ denotes Lebesgue measure.

We say that ϕ is *aperiodic* if $\gamma(\phi)=z\overline{h}h\circ T$ has no nontrivial solution in $\gamma\in\widehat{\mathbb{R}^d},\ z\in S^1$ and $h: X\to S^1$ measurable.

It is not hard to show that if T_ϕ is ergodic, and T is weakly mixing, then T_ϕ is weakly mixing iff ϕ is aperiodic.

We're interested in the exactness of T_ϕ.

We establish two (partial) results in this direction.

Theorem 1

Suppose that $(X,\mathcal{B},m,T,\alpha)$ is a probability preserving Markov map with the Renyi property. Let $N\ge 1$ and $\phi: X\to\mathbb{R}^d$ be α_0^{N-1}-measurable (i.e. $\phi(x)=\phi(\alpha_0^{N-1}(x))$ where $x\in\alpha_0^{N-1}(x)\in\alpha_0^{N-1}$).

If T_ϕ is topologically mixing, then T_ϕ is exact.

For the other result, we assume that $(X, \mathcal{B}, m, T, \alpha)$ is an exact probability preserving fibred system with the property that there is a Banach space $(L, \|\cdot\|_L)$ of functions with $\|\cdot\|_2 \le \|\cdot\|_L$, such that $P_T : L \to L$ and $\exists H > 0,\ 0 < r < 1,\ N \ge 1$ such that

$$\|P_{T^N} f\|_L \le r\|f\|_L + H\|f\|_1 \ \forall\ f \in L.$$

In this case (see [DF37], [ITM50]) $\exists M > 0,\ \theta \in (0,1)$ such that

$$\|P_{T^n} f - \int_X f dm\|_L \le M\theta^n \|f\|_L \ \forall\ f \in L.$$

Given $\phi : X \to \mathbb{R}^d$ measurable, we define the *characteristic function operators* $P_t(f) = P_T(e^{i\langle t,\phi\rangle} f) \quad (t \in \mathbb{R}^d)$.

We assume also that $P_t : L \to L \ (t \in \mathbb{R}^d)$ and that $t \mapsto P_t$ is continuous $(\mathbb{R}^d \to \mathrm{Hom}\,(L, L)$.

It is shown in [Nag57] (see also theorem 4.1 of [AD96]) that

(i) there are constants $\epsilon > 0,\ K > 0$ and $\theta \in (0,1)$; and continuous functions $\lambda : B(0,\epsilon) \to B_{\mathbb{C}}(0,1),\ g : B(0,\epsilon) \to L$ such that

$\|P_t^n h - \lambda(t)^n g(t) \int_X h dm\|_L \le K\theta^n \|h\|_L \quad \forall\ |t| < \epsilon,\ n \ge 1,\ h \in L;$

(ii) if $\gamma(\phi) = z\overline{h} h \circ T$ where $\gamma \in \widehat{\mathbb{R}^d},\ z \in S^1$ and $h : X \to S^1$ measurable, then $h \in L$;

and

(ii) in case ϕ is aperiodic, then $\forall\ 0 < \delta < M < \infty,\ \exists\ K > 0,\ 0 < \rho < 1$ such that

$$\|P_\gamma^n h\|_L \le K\rho^n \quad \forall\ h \in L,\ n \ge 1,\ \delta \le |\gamma| \le M.$$

Examples include:

• (see [AD96]), $(X, \mathcal{B}, m, T, \alpha)$ a Gibbs-Markov maps and $\phi : X \to \mathbb{R}^d$ uniformly Hölder continuous on partition sets. Here L is a space of Hölder continuous functions $f : X \to \mathbb{C}$.

• (see [RE83], [Ryc83]), $X = [0,1]$, m Lebesgue measure, α a partition of $X \mod m$ into open intervals, and $T : a \to Ta$ an invertible, m-nonsingular homeomorphism for each $a \in \alpha$ with $\inf |T'| > 1$ and $\frac{1}{T'}$ of bounded variation on X; and $\phi : X \to \mathbb{R}^d$ either: of bounded variation on X; or constant on each $a \in \alpha$. Here L is the space of functions $f : X \to \mathbb{C}$ of bounded variation on X.

Set $\phi_n = \phi + \phi \circ T + \ldots + \phi \circ T^{n-1}$.

Theorem 2

Suppose that

$$(\diamond) \qquad \forall\ \lambda > 1\ \exists\ n_k \to \infty \ \textit{such that}\ \frac{\phi_{n_k}}{\lambda^{n_k}} \to 0 \ \textit{a.e. as}\ k \to \infty$$

and that ϕ is aperiodic;
then T_ϕ is exact.

Remarks

1) Theorem 2 generalises the corresponding theorem on page 443 in [Gui89].

2) The condition ($\diamond$) is satisfied if m-dist$\,(\phi)$ is in the domain of attraction of a stable law.

3) The condition ($\diamond$) is not satisfied iff $\exists\ \lambda > 1$ and $\epsilon > 0$ such that $m([|\phi_n| > \lambda^n]) \ge \epsilon\quad \forall\ n \ge 1$ and there are independent processes like this.

4) For examples satisfying the assumptions of the theorems, let $X = [0,1]$, $Tx = \{\frac{1}{x}\}$, then T is a piecewise onto C^2 interval map with Markov partition $\alpha = \{I_n = (\frac{1}{n+1}, \frac{1}{n}] :\ n \ge 1\}$. The invariant probability is Gauss' measure $dm(x) := \frac{1}{\log 2}\frac{dx}{1+x}$. Since T^2 is uniformly expanding and satisfies Adler's condition, we have (passing to the Polish product topology induced by α) that $(X, \mathcal{B}, m, T, \alpha)$ has the Gibbs property, whence (T is piecewise onto) the Renyi property and is Gibbs-Markov.

It is not hard to show that if $\phi : X \to \mathbb{R}$ is constant on each I_n, takes the value 0 and the semigroup generated by the values of ϕ is dense in $\mathbb{R}$, then T_ϕ is topologically mixing and therefore exact by theorem 1.

Such functions $\phi : X \to \mathbb{R}$ are aperiodic by corollary 3.2 of [A-D], and so the exactness of T_ϕ is also established by theorem 2. On the other hand, if $\phi(x) = \log\frac{1}{x}$ then T_ϕ is not topologically mixing (since $\phi \ge 0$). Nevertheless, ϕ is aperiodic by corollary 3.2 of [A-D], and so T_ϕ is exact by theorem 2 (but totally dissipative).

§1 Frobenius-Perron operators, exactness and relative exactness

Let $(X, \mathcal{B}, m, R)$ be a nonsingular transformation of a standard probability space. The *tail σ-algebra* of $(X, \mathcal{B}, m, R)$ is $\mathcal{T}(R) := \bigcap_{n=1}^\infty R^{-n}\mathcal{B}$ and the nonsingular transformation R is called *exact* if $= \{\varnothing, X\} \mod m$.

Theorem 1.1 [DL84]

$$\|P_{R^n} f\|_1 \to \|E(f|\mathcal{T}(R))\|_1 \ \textit{as } n \to \infty\ \forall\ f \in L^1(m).$$

In particular (see [Lin71]), R is exact iff $\|P_{R^n} f\|_1 \to 0\ \forall\ f \in L^1(m),\ \int_X f dm = 0$.

Proof

First note that $|P_T f| \le P_T|f|$ whence $\|P_{R^n} f\|_1 \downarrow$ and $\exists\ \lim_{n\to\infty} \|P_{R^n} f\|_1$. Next, $\forall\ n \ge 1\ \exists\ g_n \in L^\infty(\mathcal{B})$ with $\int_X (P_{R^n} f) g_n dm = \|P_{R^n} f\|_1$, whence

$$\|P_{R^n} f\|_1 = \int_X f g_n \circ R^n dm.$$

By weak $*$ compactness, $\exists\ n_k \to \infty$ and $g \in L^\infty(\mathcal{B})$ such that $g_{n_k} \circ R^{n_k} \rightharpoonup g$ weak $*$ in $L^\infty(\mathcal{B})$.

It follows that $g \in L^\infty(\mathcal{T}(R))$, $\|g\|_\infty \le 1$ and $\lim_{n\to\infty} \|P_{R^n} f\|_1 = \int_X f g dm$. Thus

$$\lim_{n\to\infty} \|P_{R^n} f\|_1 \le \sup\{\int_X f h dm :\ h \in L^\infty(\mathcal{T}(R)),\ \|h\|_\infty \le 1\} = \|E(f|\mathcal{T}(R))\|_1.$$

To show the converse inequality, note that $\exists\ g \in L^\infty(\mathcal{T}(R))$, $\|g\|_\infty = 1$ such that

$$\|E(f|\mathcal{T}(R))\|_1 = \int_X E(f|\mathcal{T}(R)) g dm = \int_X f g dm$$

whence $\forall\ n \ge 1,\ \exists\ g_n \in L^\infty(\mathcal{B}), g = g_n \circ R^n$ and

$$\|E(f|\mathcal{T}(R))\|_1 = \int_X f g dm = \int_X f g_n \circ R^n dm = \int_X (P_{R^n} f) g_n dm \le \|P_{R^n} f\|_1.$$

□

Let $(X, \mathcal{B}, m, R)$ and $(Y, \mathcal{C}, \mu, S)$ be nonsingular transformations of standard probability spaces. A *factor map* is a function $\pi : X \to Y$ satisfying $\pi^{-1}\mathcal{C} \subset \mathcal{B},\ \pi \circ T = S \circ \pi,\ m \circ \pi^{-1} = \mu$.

The *fibre expectation* of the factor map $\pi : X \to Y$ is an operator

$$f \mapsto E(f|\pi),\ L^1(X, \mathcal{B}, m) \to L^1(Y, \mathcal{C}, \mu)$$

defined by $\int_Y E(f|\pi) g d\mu = \int_X f g \circ \pi dm$.

The factor map $\pi : X \to Y$ is called *relatively exact* if

$$f \in L^1(\mathcal{B}),\ E(f|\pi) = 0 \text{ a.e.} \implies \|P_{R^n} f\|_1 \to 0.$$

The corollary below appears in [Gui89]. For the convenience of the reader, we supply a (possibly different) proof.

Proposition 1.2 *Suppose that $\pi : X \to Y$ is relatively exact, then $\mathcal{T}(R) = \pi^{-1}\mathcal{T}(S)$ mod m.*

Proof

Evidently, $\pi^{-1}\mathcal{T}(S) \subseteq \mathcal{T}(R)$. We show that $\pi^{-1}\mathcal{T}(S) \supseteq \mathcal{T}(R)$.

By relative exactness and theorem 1.1, if $f \in L^1(\mathcal{B})$ and $E(f|\pi) = 0$ a.e., then $\int_X f g dm = 0\ \forall\ g \in L^\infty(\mathcal{T}(R))$.

Thus if $f \in L^2(\mathcal{B}) \ominus L^2(\pi^{-1}\mathcal{C})$, then $E(f|\pi) = 0$ a.e. and so

$$\int_X f g dm = 0\ \forall\ g \in L^\infty(\mathcal{T}(R)), \implies f \perp L^2(\mathcal{T}(R)).$$

Thus $L^2(\mathcal{B})\ominus L^2(\pi^{-1}\mathcal{C}) \subset L^2(\mathcal{B})\ominus L^2(\mathcal{T}(R))$ whence $L^2(\mathcal{T}(R)) \subset L^2(\pi^{-1}\mathcal{C})$ and $\mathcal{T}(R) \subset \pi^{-1}\mathcal{C} \mod m$.

To see that in fact $\mathcal{T}(R) \subseteq \pi^{-1}\mathcal{T}(S) \mod m$, fix $N \geq 1$, then

$$\begin{aligned}\mathcal{T}(R) &= \bigcap_{n\geq 1} R^{-n}\mathcal{B} = \bigcap_{n\geq N+1} R^{-n}\mathcal{B} \\ &= R^{-N}\mathcal{T}(R) \subset R^{-N}\pi^{-1}\mathcal{C} = \pi^{-1}S^{-N}\mathcal{C}.\end{aligned}$$

Taking the intersection over N shows the claim. □

Corollary 1.3 ([Gui89], **proposition 1)**
If S is exact and $\pi : X \to Y$ is relatively exact, then T is exact.

§2 Proof of theorem 1

For a nonsingular transformation $(X, \mathcal{B}, m, R)$, define the *tail relation* of R:

$$\mathfrak{T}(R) := \{(x,y) \in X \times X : \ \exists\ n \geq 0,\ R^n x = R^n y\}.$$

Evidently $\mathfrak{T}(R)$ is an equivalence relation and if $(X, \mathcal{B}, m)$ is standard, then $\mathfrak{T}(R) \in \mathcal{B}(X \times X)$.

If R is locally invertible, then $\mathfrak{T}(R)$ has countable equivalence classes and is nonsingular in the sense that $m(\mathfrak{T}(R)(A)) = 0\ \forall\ A \in \mathcal{B},\ m(A) = 0$ where $\mathfrak{T}(R)(A) := \{y \in X : \ \exists\ x \in A\ (x,y) \in \mathfrak{T}(R)\}$.

A set $A \in \mathcal{B}(X)$ is *invariant* under the equivalence relation $\mathfrak{T} \in \mathcal{B}(X \times X)$ if $\mathfrak{T}(A) = A$ and the equivalence relation $\mathfrak{T}$ is called *ergodic* if $\mathfrak{T}$-invariant sets have either zero, or full measure.

The collection of invariant sets under $\mathfrak{T}(R)$ is the tail σ-algebra $\mathcal{T}(R)$ (whence the name "tail relation").

In order to prove theorem 1, it suffices to show that $\mathfrak{T}(T_\phi)$ is ergodic.

The tail relation of T_ϕ is given by

$$\begin{aligned}&\mathfrak{T}(T_\phi) \\ &= \{((x,s),(y,t)) \in (X \times G)^2 : \ \exists\ n \geq 0,\ T^n x = T^n y,\ s - t = \phi_n(y) - \phi_n(x)\} \\ &= \{((x,s),(y,t)) \in (X \times G)^2 : \ (x,y) \in \mathfrak{T}(T),\ \tilde{\phi}(x,y) = s - t\}\end{aligned}$$

where $\tilde{\phi} : \mathfrak{T}(T) \to \mathbb{R}^d$ is defined by $\tilde{\phi}(x,y) := \sum_{n=0}^{\infty}(\phi(T^n y) - \phi(T^n x))$.

We prove that $\mathfrak{T}(T_\phi)$ is ergodic by the method of Schmidt (explained in [Sto66]), by showing that $\forall\ t \in \mathbb{R}^d$, U a neighbourhood of t and $A \in \mathcal{B}\ m(A) > 0,\ \exists\ B \in \mathcal{B}\ B \subset A$ and $\tau : B \to B$ nonsingular such that $(x, \tau(x)) \in \mathfrak{T}(T)$ and $\tilde{\phi}(x, \tau(x)) \in U\ \forall\ x \in B$.

This boils down to showing that

$$\forall\ A \in \mathcal{B}_+\ g_0 \in \mathbb{R}^d\ \eta > 0,\ \exists\ B \in \mathcal{B}_+\ B \subset A,\ n \geq 1$$
$$\text{and } \tau : B \to \tau B \subset A \text{ nonsingular such that}$$
$$T^n \circ \tau \equiv T^n \text{ and } \|\phi_n \circ \tau - \phi_n - g_0\| < \eta \text{ on } B. \tag{‡}$$

The proof of (‡) will be written as a sequence of minor claims, ¶0,¶1,....

¶0 We first claim that there is no loss in generality in assuming that $N = 1$ (i.e. that $\phi : X \to \mathbb{R}^d$ is α-measurable). This is because $(X, \mathcal{B}, m, T, \alpha_0^{N-1})$ is also a probability preserving Markov map with the Renyi property and inducing the same (shift) topology on X as $(X, \mathcal{B}, m, T, \alpha)$.

¶1 $\forall\ s, t \in S,\ \exists\ \kappa = \kappa_{s,t} \geq 1$ and $a = a_{s,t} = [a_1, \ldots a_\kappa],\ b = b_{s,t} = [b_1, \ldots b_\kappa] \in \alpha_0^{\kappa-1},\ a_1 = b_1 = s\ a_\kappa = b_\kappa = t$ such that $\|\phi_\kappa(b) - \phi_\kappa(a) - g_0\| < \eta$.

This follows from topological mixing of T_ϕ.

By the Renyi property, $\exists\ M > 1$ such that

$$M^{-1}m(u)m(v) \leq m(u \cap T^{-k}v) \leq Mm(u)m(v)\ \ \forall\ u \in \alpha_0^{k-1},\ v \in \alpha_0^{\ell-1},\ [v_1] \subset T[u_k].$$

Given $u = [u_1, \ldots, u_n] \in \alpha_0^{n-1}$ with $u_n = t$, define $\tau = \tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ by

$$\tau(u_1, \ldots, u_n, a_1, \ldots a_\kappa, y) := \tau(u_1, \ldots, u_n, b_1, \ldots b_\kappa, y).$$

¶2 $\tau = \tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ is invertible nonsingular and $\frac{dm\circ\tau}{dm} = M^{\pm 4}\frac{m(b)}{m(a)}$.

Proof

$$\begin{aligned}\int_{u \cap T^{-n}a \cap c} \frac{dm \circ \tau}{dm} dm &= m(u \cap T^{-n}b \cap c) \\ &= M^{\pm 2}\frac{m(b)}{m(a)} m(u)m(b)m(c) \\ &= M^{\pm 4}\frac{m(b)}{m(a)} m(u \cap T^{-n}a \cap c).\end{aligned}$$

□

¶3 Proof of ‡

Fix $0 < \epsilon < M^{-1}\min\{m(a_{s,t}),\ m(b_{s,t})\}$, then

$$m(u \cap T^{-n}a_{s,t}),\ m(u \cap T^{-n}b_{s,t}) \geq \epsilon m(u)\ \ \forall\ u \in \alpha_0^{n-1},\ [s] \subset T[u_n].$$

Let $\delta > 0$ be so small that $\delta < \frac{m(b)(\epsilon-\delta)}{M^4 m(a)}$.

$\exists\ n \geq 1$ and $u \in \alpha_0^{n-1}$ such that $m(A \cap u) \geq (1-\delta)m(u)$ and $[s] \subset T[u_n]$.

Consider $\tau_u : u \cap T^{-n}a \to u \cap T^{-n}b$ as in ¶2. Evidently $T^{n+\kappa} \circ \tau \equiv T^{n+\kappa}$ and $\|\phi_{n+\kappa} \circ \tau - \phi_{n+\kappa} - g_0\| < \eta$ on $u \cap T^{-n}a$.

To complete the proof we claim that $\exists\ B \in \mathcal{B}_+\ B \subset A \cap u \cap T^{-n}a$ such that $\tau B \subset A$.

To see this, note that

$$m(u \cap T^{-n}a \cap A) \geq m(u \cap T^{-n}a) - m(u \setminus A) \geq (\epsilon - \delta)m(u),$$

whence using ¶2,

$$m(\tau(u \cap T^{-n}a \cap A)) \geq \frac{m(b)}{M^4 m(a)} m(u \cap T^{-n}a \cap A) \geq \frac{m(b)(\epsilon - \delta)}{M^4 m(a)} m(u).$$

Since $\tau(u \cap T^{-n}a \cap A) \subset u$, the condition on $\delta > 0$ ensures that $m(\tau(u \cap T^{-n}a \cap A) \cap A) > 0$ whence $m(B) > 0$ where $B := \tau^{-1}\Bigg(\tau(u \cap T^{-n}a \cap A) \cap A\Bigg) \subset A$. $\square$

§3 Proof of theorem 2

We prove theorem 2 via corollary 1.3. To do this, we must consider T_ϕ as a nonsingular transformation with respect to some probability $P \sim m \times m_{\mathbb{R}^d}$.

Let $p : \mathbb{R}^d \to \mathbb{R}_+$ be continuous with $\int_{\mathbb{R}^d} p(y)dy = 1$ and define a probability P on $X \times \mathbb{R}^d$ by $dP(x, y) := p(y)dm(x)dy$; then $(X \times \mathbb{R}^d, \mathcal{B}(X \times \mathbb{R}^d), P, T_\phi)$ is a nonsingular transformation with Frobenius-Perron operators given by

$$P_{T_\phi^n, P} f(x, y) = \frac{1}{p(y)} P_{T_\phi^n}(f \cdot 1 \otimes p)(x, y)$$

where $P_{T_\phi^n} := P_{T_\phi^n, m \times m_{\mathbb{R}^d}}$.

Consider the map $\pi : X \times \mathbb{R}^d \to X$ defined by $\pi(x, y) = x$. This is a factor map as it satisfies $\pi^{-1}\mathcal{B}(X) \subset \mathcal{B}(X \times \mathbb{R}^d)$, $\pi \circ T_\phi = T \circ \pi$, $P \circ \pi^{-1} = m$.

The fibre expectation of π is given by

$$E(f|\pi)(x) = \int_{\mathbb{R}^d} f(x, y)p(y)dy \quad (f \in\ L^1(X \times \mathbb{R}^d, \mathcal{B}(X \times \mathbb{R}^d), P)).$$

By corollary 1.3 and exactness of T, it suffices to show that π is relatively exact.

To do this, we show that

$$\int_{\mathbb{R}^d} f(x,y)p(y)dy = 0 \text{ a.e.} \quad \Longrightarrow$$
$$\int_{X\times\mathbb{R}^d} |P_{T_\phi^n,P} f| dP = \int_{X\times\mathbb{R}^d} |P_{T_\phi^n}(f\cdot 1\otimes p)| d(m\times m_{\mathbb{R}^d}) \to 0$$

as $n\to\infty$; equivalently (taking $F(x,y) := f(x,y)p(y)$),

$$(\star) \qquad \int_{\mathbb{R}^d} F(x,y)dy = 0 \text{ a.e.} \quad \Longrightarrow \quad \int_{X\times\mathbb{R}^d} |P_{T_\phi^n} F| d(m\times m_{\mathbb{R}^d}) \to 0$$

as $n\to\infty$.

To prove $(\star)$, we first claim that
¶1 for $\lambda > 1$, $h\in L^1(m)$ and $f\in L^1(\mathbb{R}^d)$,

$$\|P_{T_\phi^{n_k}}(h\otimes f)\|_1 \le C\lambda^{\frac{n_k d}{2}} \|P_{T_\phi^{n_k}}(h\otimes f)\|_2 + o(1)$$

as $k\to\infty$ where $C = 2^{\frac{d}{2}} m(B(0,1))$ and $\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ a.e..
Proof As can be checked,

$$P_{T_\phi^n}(h\otimes f)(x,y) = P_{T^n}(h(\cdot)f(y-\phi_n(\cdot)))(x) \quad (h\in L^1(m),\ f\in L^1(\mathbb{R}^d)).$$

Denoting $E(H) := \int_X H dm$ for $H\in L^1(m)$, we have

$$\|P_{T_\phi^{n_k}}(h\otimes f)\|_1 = \int_{\mathbb{R}^d} |E(P_{T^{n_k}}(h(\cdot)f(y-\phi_{n_k}(\cdot))))| dy \le \int_{|y|\le 2\lambda^{n_k}} + \int_{|y|>2\lambda^{n_k}}. \tag{2}$$

By the Cauchy-Schwartz inequality,

$$\int_{|y|\le 2\lambda^{n_k}} \le \sqrt{m_{\mathbb{R}^d}(B(0,2\lambda^{n_k}))} \|P_{T_\phi^{n_k}}(h\otimes f)\|_2 = C\lambda^{\frac{n_k d}{2}} \|P_{T_\phi^{n_k}}(h\otimes f)\|_2 \tag{3}$$

whereas

$$\int_{|y|>2\lambda^{n_k}} \le \int_{|y|>2\lambda^{n_k}} |E(P_{T^{n_k}}(h(\cdot)f(y-\phi_{n_k}(\cdot))1_{[|\phi_{n_k}(\cdot))|\le\lambda^{n_k}]})| dy$$
$$+ \int_{|y|>2\lambda^{n_k}} |E(P_{T^{n_k}}(h(\cdot)f(y-\phi_{n_k}(\cdot))1_{[|\phi_{n_k}(\cdot)|>\lambda^{n_k}]}))| dy = I + II.$$

Here as $k\to\infty$:

$$II \le \|f\|_1 E(|h| 1_{[|\phi_{n_k}(\cdot)|>\lambda^{n_k}]}) \to 0 \tag{4}$$

since $\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0$ a.e.; and

$$\begin{aligned} I &\le \int_{|y|>2\lambda^{n_k}} E(|h||f(y-\phi_{n_k})|1_{[|\phi_{n_k}(\cdot)|\le\lambda^{n_k}]})dy \\ &= E\Big(|h|1_{[|\phi_{n_k}|\le\lambda^{n_k}]}\int_{|y|>2\lambda^{n_k}}|f(y-\phi_{n_k})|dy\Big) \\ &\le E(|h|)\int_{|y|>\lambda^{n_k}}|f(y)|dy \to 0, \end{aligned} \tag{5}$$

Substituting (3),(4) and (5) into (2) proves ¶1. □

To complete the proof of $(\star)$, let $F \in L^1(m\times m_{\mathbb{R}^d})$ satisfy $\int_{\mathbb{R}^d} F(x,y)dy = 0$ for m-a.e. $x \in X$ and fix $\epsilon > 0$. We show that

$$\limsup_{n\to\infty}\int_{X\times\mathbb{R}^d}|P_{T_\phi^n}F|d(m\times m_{\mathbb{R}^d}) < \epsilon. \tag{$\star_\epsilon$}$$

Standard approximation techniques show that $\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N},\ h_1,\dots,h_N \in L,\ g_1,\dots,g_N \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g_k(y)dy = 0\ \ (1 \le k \le N)$ and

$$\Big\|F - \sum_{k=1}^N h_k\otimes g_k\Big\|_{L^1(m\times m_{\mathbb{R}^k})} < \frac{\epsilon}{2}.$$

Next, it follows from theorems 1.6.3 and 1.6.4 in [Rud74] that

$\exists\ f_1,\dots,f_N \in L^1\cap L^2$ such that

- $[\widehat{f_k} \ne 0]$ is compact and bounded away from 0 $\ (1 \le k \le N)$; and
- $\|f_k - g_k\|_{L^1(m_{\mathbb{R}^d})} < \frac{\epsilon}{2N\|h_k\|_{L^1(m)}}$ $\ (1 \le k \le N)$, whence

$$\Big\|\sum_{k=1}^N h_k\otimes f_k - \sum_{k=1}^N h_k\otimes g_k\Big\|_{L^1(m\times m_{\mathbb{R}^d})} \le \sum_{k=1}^N \|h_k\|_{L^1(m)}\cdot\|f_k-g_k\|_{L^1(\mathbb{R}^d)} < \frac{\epsilon}{2},$$

$$\Big\|F - \sum_{k=1}^N h_k\otimes f_k\Big\|_{L^1(m\times m_{\mathbb{R}^d})} < \epsilon.$$

We claim

¶2 If $h \in L$ and $f \in L^1\cap L^2$ is such that $[\widehat{f} \ne 0]$ is compact and bounded away from 0, then $\exists\ 0 < \rho < 1$ such that

$$\|P_{T_\phi^n}(h\otimes f)\|_2 = O(\rho^n) \text{ as } n\to\infty. \tag{6}$$

Proof

Let $[\widehat{f} \ne 0] \subset B(0,M)\setminus B(0,\delta)$. By (ii) (above), $\exists\ K > 0,\ 0 < \rho < 1$ such that

$$|P_\gamma^n h(x)| \le K\rho^n \quad \forall\ x \in X,\ n \ge 1,\ \delta \le |\gamma| \le M,$$

whence using the fact that the Fourier transform of $y \mapsto P_{T_\phi}^n(h\otimes f)(x,y)$ is $\gamma \mapsto \widehat{f}(\gamma)P_\gamma^n h(x)$ and Plancherel's formula, we have

$$\begin{aligned}\|P_{T_\phi^n}(h\otimes f)\|_2^2 &= \int_X\left(\int_{\mathbb{R}^d}|P_{T_\phi^n}(h\otimes f)(x,y)|^2dy\right)dm(x)\\ &= \int_X\left(\int_{\mathbb{R}^d}|\widehat{f}(\gamma)|^2|P_\gamma^n h(x)|^2d\gamma\right)dm(x)\\ &= \int_{\mathbb{R}^d}|\widehat{f}(\gamma)|^2\|P_\gamma^n h\|_2^2d\gamma \le K^2\rho^{2n}\int_{\mathbb{R}^d}|\widehat{f}(\gamma)|^2d\gamma\end{aligned}$$

proving ¶2. □

To finish the proof of theorem 2, we claim

¶3 if (6) holds for $h\in L$ and $f\in L^1\cap L^2$, then

$$\|P_{T_\phi^n}(h\otimes f)\|_1\to 0. \tag{7}$$

Proof

Fix $\lambda>1$ such that $\lambda^{\frac{d}{2}}\rho<1$. Suppose that $\frac{\phi_{n_k}}{\lambda^{n_k}}\to 0$ a.e.. Using (6), we have by ¶1,

$$\|P_{T_\phi^{n_k}}(h\otimes f)\|_1\le C\lambda^{\frac{n_kd}{2}}\|P_{T_\phi^{n_k}}(h\otimes f)\|_2+o(1)=O(\lambda^{\frac{n_kd}{2}}\rho^{n_k})+o(1)\to 0$$

as $k\to\infty$; establishing (7) since $\|P_{T_\phi^n}(h\otimes f)\|_1\downarrow$. □

This completes the proof of theorem 2.

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